

# The Geometry of Gaussoids

Tobias Boege, Alessio D’Alì, Thomas Kahle and Bernd Sturmfels

## Abstract

A gaussoid is a combinatorial structure that encodes independence in probability and statistics, just like matroids encode independence in linear algebra. The gaussoid axioms of Lněnička and Matúš are equivalent to compatibility with certain quadratic relations among principal and almost-principal minors of a symmetric matrix. We develop the geometric theory of gaussoids, based on the Lagrangian Grassmannian and its symmetries. We introduce oriented gaussoids and valuated gaussoids, thus connecting to real and tropical geometry. We classify small realizable and non-realizable gaussoids. Positive gaussoids are as nice as positroids: they are all realizable via graphical models.

## 1 Introduction

Gaussoids are combinatorial structures that arise in statistics and are reminiscent of matroids. They were introduced by Lněnička and Matúš [23] to represent conditional independence relations among  $n$  Gaussian random variables. The theory of matroids is ubiquitous in the mathematical sciences, as it captures the combinatorial essence of many objects in algebra and geometry. Matroids of rank  $d$  on  $[n] = \{1, 2, \dots, n\}$  are possible supports of Plücker coordinates on the Grassmannian of  $d$ -dimensional linear subspaces in a vector space  $K^n$ .

This article develops the geometric theory of gaussoids, with focus on parallels to matroid theory. The role of the Grassmannian is played by a natural projection of the Lagrangian Grassmannian, namely the variety of principal and almost-principal minors of a symmetric  $n \times n$ -matrix  $\Sigma$ . Gaussoids aim to characterize which almost-principal minors can simultaneously vanish provided  $\Sigma$  is positive definite. This issue is important in statistics, where  $\Sigma$  is the covariance matrix of a Gaussian distribution on  $\mathbb{R}^n$ , and almost-principal minors measure partial correlations. The sign of a minor indicates whether the partial correlation is positive or negative. The minor is zero if and only if conditional independence holds.

Our goal in this paper is to carry out the program that was suggested in [34, §4]. We assume that our readers are familiar with the geometric approach to matroids, including oriented matroids and valuated matroids, as well as basic concepts in algebraic statistics. Introductory books for the former include [3, 5, 24]. Sources for the latter include [9, 33, 34].

Let  $\Sigma = (\sigma_{ij})$  be a symmetric  $n \times n$ -matrix whose  $\binom{n+1}{2}$  entries are unknowns. A *minor* of  $\Sigma$  is the determinant of a square submatrix. The projective variety parametrized by all minors of  $\Sigma$  is the *Lagrangian Grassmannian*  $\text{LGr}(n, 2n)$ . It is obtained by intersecting the

usual Grassmannian  $\text{Gr}(n, 2n)$  in its Plücker embedding in  $\mathbb{P}^{\binom{2n}{n}-1}$  with a linear subspace. An affine chart of  $\text{LGr}(n, 2n)$  consists of all row spaces of rank  $n$  matrices of the form

$$\left( \text{Id}_n \ \Sigma \right) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \sigma_{11} & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1n} \\ 0 & 1 & 0 & \cdots & 0 & \sigma_{12} & \sigma_{22} & \sigma_{23} & \cdots & \sigma_{2n} \\ 0 & 0 & 1 & \cdots & 0 & \sigma_{13} & \sigma_{23} & \sigma_{33} & \cdots & \sigma_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \sigma_{1n} & \sigma_{2n} & \sigma_{3n} & \cdots & \sigma_{nn} \end{pmatrix}. \quad (1)$$

The right  $n \times n$ -block is symmetric. The quadratic Plücker relations for  $\text{Gr}(n, 2n)$  restrict to quadrics that define  $\text{LGr}(n, 2n)$ . It is known that those quadrics form a Gröbner basis. For more information we refer to Oeding's dissertation [27, § III.A] and the references therein.

A minor of  $\Sigma$  is *principal* if its row indices and its column indices coincide, and it is *almost-principal* if its row and column indices differ in exactly one element. We introduce unknowns that represent the  $2^n$  principal minors and the  $2^{n-2}\binom{n}{2}$  almost-principal minors:

$$\mathcal{P} = \{p_I : I \subseteq [n]\} \quad \text{and} \quad \mathcal{A} = \{a_{ij|K} : i, j \in [n] \text{ distinct and } K \subseteq [n] \setminus \{i, j\}\}.$$

To simplify notation, we write  $p$  for  $p_\emptyset$ ,  $p_{12}$  for  $p_{\{1,2\}}$ ,  $a_{12|3}$  for  $a_{12|\{3\}}$ , etc. These unknowns correspond respectively to the vertices and 2-faces of the  $n$ -cube, as shown in Figure 1.

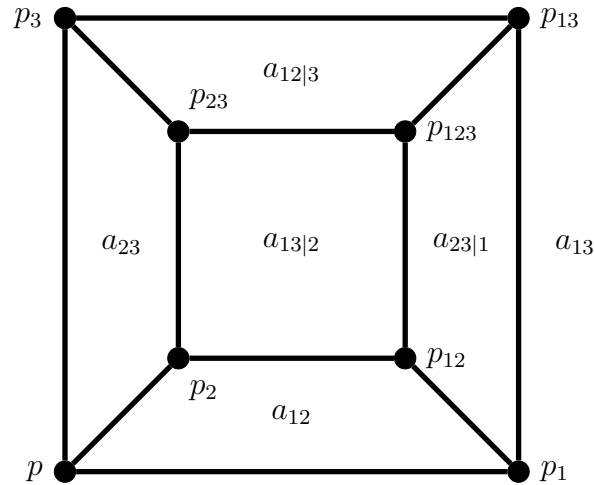


Figure 1: The vertices and 2-faces of the  $n$ -cube are labeled by the set of unknowns  $\mathcal{P} \cup \mathcal{A}$ .

Consider the homomorphism  $\mathbb{R}[\mathcal{P} \cup \mathcal{A}] \rightarrow \mathbb{R}[\Sigma]$  from a polynomial ring in  $2^{n-2}(4 + \binom{n}{2})$  unknowns to a polynomial ring in  $\binom{n+1}{2}$  unknowns, where  $p_I$  is mapped to the minor of  $\Sigma$  with row indices  $I$  and column indices  $I$ , and  $a_{ij|K}$  is mapped to the minor of  $\Sigma$  with row indices  $\{i\} \cup K$  and column indices  $\{j\} \cup K$ . Here, the row indices are sorted so that  $i$  comes first and is followed by  $K$ , and the column indices are sorted so that  $j$  comes first and is followed by  $K$ , where the elements of  $K$  are listed in increasing numerical order. For instance,  $a_{12|3}$  maps to  $\sigma_{12}\sigma_{33} - \sigma_{13}\sigma_{23}$  whereas  $a_{13|2}$  maps to  $-(\sigma_{12}\sigma_{23} - \sigma_{13}\sigma_{22})$ . Maintaining this sign convention is important to keep the algebra consistent with its statistical interpretation.

Let  $J_n$  denote the ideal generated by all homogeneous polynomials in the kernel of the map above. This defines an irreducible variety  $V(J_n)$  of dimension  $\binom{n+1}{2}$  in the projective space  $\mathbb{P}^{2^{n-2}(4+\binom{n}{2})-1}$  whose coordinates are  $\mathcal{P} \cup \mathcal{A}$ . There is a natural projection from  $\text{LGr}(n, 2n)$  onto  $V(J_n)$ , obtained by deleting all minors that are neither principal nor almost-principal. This is analogous to [27, Observation III.12], where the focus was on principal minors  $p_I$ .

**Proposition 1.1.** *The degree of the projective variety of principal and almost-principal minors coincides with the degree of the Lagrangian Grassmannian. For  $n \geq 2$ , it equals*

$$\text{degree}(V(J_n)) = \text{degree}(\text{LGr}(n, 2n)) = \frac{\binom{n+1}{2}!}{1^n \cdot 3^{n-1} \cdot 5^{n-2} \cdots (2n-1)}.$$

*Proof.* The degree of  $\text{LGr}(n, 2n)$  is due to Hiller [15, Corollary 5.3]. We learned this formula from Totaro's comment on the sequence A005118 in the OEIS: 2, 16, 768, 292864, ...

It suffices to show that the birational map from  $\text{LGr}(n, 2n)$  onto  $V(J_n)$  is base-point free. Indeed, the center of the map  $\mathbb{P}^{2^{n-2}(4+\binom{n}{2})-1} \dashrightarrow \mathbb{P}^{\binom{n+2}{2}}$  consists of the points whose coordinates indexed by principal and almost-principal minors are all zero. No such point (1) can arise from a nonzero matrix  $\Sigma = (\sigma_{ij})$ . Hence the center is disjoint from  $\text{LGr}(n, 2n)$ .  $\square$

There are two natural symmetry classes of trinomials in  $J_n$ . First, there is one trinomial for each 2-face of the  $n$ -cube. The cardinality of that class is  $2^{n-2}\binom{n}{2}$ . A representative is

$$a_{12}^2 - p_1 p_2 + p_{12} p. \quad (2)$$

Second, there is one trinomial for each inclusion of an edge in a 3-cube, in the boundary of the  $n$ -cube. The number of these *edge trinomials* is  $12 \cdot 2^{n-3}\binom{n}{3}$ . One representative is

$$p a_{23|1} - p_1 a_{23} + a_{12} a_{13}. \quad (3)$$

In Section 2 we review the axiom system for gaussoids found in [23], and we show in Theorem 2.4 that these axioms are equivalent to compatibility with the edge trinomials (3). In Section 3 we examine a natural action of the group  $G = \text{SL}_2(\mathbb{R})^n$  on the polynomial ring  $\mathbb{R}[\mathcal{P} \cup \mathcal{A}]$ . This fixes the ideal  $J_n$ . Certain finite subgroups of  $G$  serve as symmetry groups for the combinatorial structures in this paper. In Section 4 we classify gaussoids up to  $n = 5$ , taking into account the various symmetry groups in  $G$ . Our computations make extensive use of state-of-the-art *SAT solvers*. In Section 5 we introduce and classify oriented gaussoids. Theorem 5.6 asserts that every positive gaussoid is realizable by an undirected graphical model. In Section 6 we determine all quadrics in  $J_n$ , and we conjecture that they generate. Section 7 focuses on valuated gaussoids and tropical geometry, and Section 8 addresses the realizability problem for gaussoids and oriented gaussoids. Our supplementary materials website [www.gaussoids.de](http://www.gaussoids.de) contains various classifications reported in this paper.

## 2 Gaussians and Axioms

A symmetric  $n \times n$ -matrix  $\Sigma = (\sigma_{ij})$  is the covariance matrix of an  $n$ -dimensional normal distribution (or *Gaussian*) if  $\Sigma$  is positive definite, i.e., if the  $2^n$  principal minors  $p_I$  of  $\Sigma$  are

all positive. Let  $X_1, X_2, \dots, X_n$  be random variables whose joint distribution is Gaussian with covariance matrix  $\Sigma$ . For any subset  $K \subseteq [n]$  we write  $X_K$  for the random vector  $(X_i : i \in K)$  in  $\mathbb{R}^{|K|}$ . The variable  $X_i$  is independent of the variable  $X_j$  given the variable  $X_K$  if and only if the almost-principal minor  $a_{ij|K}$  of  $\Sigma$  is zero. See [9, Proposition 3.1.13]. This *conditional independence (CI) statement* is usually denoted by  $X_i \perp\!\!\!\perp X_j \mid X_K$  and also known as an *elementary CI statement*. Restriction to only these statements is justified in [33, § 2.2.3]. Other notations found in the literature include  $i \perp j \mid K$ ,  $\langle i, j \mid K \rangle$ , and  $(ij \mid K)$ . We shall keep things simple by identifying all of these symbols with our unknown  $a_{ij|K} \in \mathcal{A}$ .

Reasoning and inference with conditional independence statements plays a fundamental role in statistics, especially in the study of graphical models [10, 23, 26, 33, 36]. A guiding problem has been to characterize collections of conditional independence statements that can hold simultaneously within some class of distributions. This led to the theory of *semigraphoids*; see e.g. [26, § 2]. We here focus on the class of Gaussian distributions on  $\mathbb{R}^n$ . The guiding problem now takes the following algebraic form: which sets of almost-principal minors  $a_{ij|K}$  can be simultaneously zero for a positive definite symmetric  $n \times n$ -matrix  $\Sigma$ ?

To study this question, Lněnička and Matúš [23] introduced the following axiom system, which we present here in our notation. As before,  $\mathcal{A}$  is the set of all symbols  $a_{ij|K}$  where  $i, j$  are distinct elements in  $[n] = \{1, 2, \dots, n\}$  and  $K$  is a subset of  $[n] \setminus \{i, j\}$ . Thus the set  $\mathcal{A}$  consists of  $\binom{n}{2} 2^{n-2}$  symbols  $a_{ij|K}$ . We identify these symbols with the 2-faces of the  $n$ -cube.

Following [23, Definition 1], a subset  $\mathcal{G}$  of  $\mathcal{A}$  is a *gaussoid* on  $[n]$  if it satisfies:

- (G1)  $\{a_{ij|L}, a_{ik|jL}\} \subset \mathcal{G}$  implies  $\{a_{ik|L}, a_{ij|kL}\} \subset \mathcal{G}$ ,
- (G2)  $\{a_{ij|kL}, a_{ik|jL}\} \subset \mathcal{G}$  implies  $\{a_{ij|L}, a_{ik|L}\} \subset \mathcal{G}$ ,
- (G3)  $\{a_{ij|L}, a_{ik|L}\} \subset \mathcal{G}$  implies  $\{a_{ij|kL}, a_{ik|jL}\} \subset \mathcal{G}$ ,
- (G4)  $\{a_{ij|L}, a_{ij|kL}\} \subset \mathcal{G}$  implies  $(a_{ik|L} \in \mathcal{G} \text{ or } a_{jk|L} \in \mathcal{G})$ .

Axiom (G1) is the definition of a *semigraphoid*, and (G2) is known as the *intersection axiom*. Axiom (G3) is a *converse to intersection*, and axiom (G4) is called *weak transitivity*.

Being a gaussoid is a necessary condition for a subset  $\mathcal{G} \subseteq \mathcal{A}$  to comprise the vanishing almost-principal minors of a positive definite symmetric  $n \times n$ -matrix  $\Sigma$ . The gaussoid  $\mathcal{G}$  is called *realizable* if such a matrix  $\Sigma$  exists. All gaussoids are realizable for  $n = 3$ . This is no longer true for  $n \geq 4$ , as shown in [10, 23]. For an explicit example see Remark 4.3 below.

**Example 2.1.** Let  $n = 3$ . The set  $\mathcal{A}$  has 6 elements, and hence it has  $2^6 = 64$  subsets. Among these 64 subsets, precisely 11 are gaussoids. They are

$$\begin{aligned} & \{\}, \{a_{12}\}, \{a_{13}\}, \{a_{23}\}, \{a_{12|3}\}, \{a_{13|2}\}, \{a_{23|1}\}, \{a_{12}, a_{12|3}, a_{13}, a_{13|2}\}, \\ & \{a_{12}, a_{12|3}, a_{23}, a_{23|1}\}, \{a_{13}, a_{13|2}, a_{23}, a_{23|1}\}, \{a_{12}, a_{12|3}, a_{13}, a_{13|2}, a_{23}, a_{23|1}\}. \end{aligned} \quad (4)$$

Each of these gaussoids  $\mathcal{G}$  is realizable by a positive definite symmetric  $3 \times 3$ -matrix. Equivalently, the variety  $V(J_3)$  contains a real point  $(p, a)$  whose coordinates  $p_I$  are all positive and whose coordinates that vanish are precisely the elements  $a_{ij|K}$  in  $\mathcal{G}$ . We invite the reader to check that all 11 gaussoids  $\mathcal{G}$  arise from an appropriate point  $(p, a)$  in the variety  $V(J_3)$ .  $\diamond$

Gaussoids are analogous to matroids. In matroid theory, one asks which sets of maximal minors of a rectangular matrix can be simultaneously zero. Being a matroid is necessary but not sufficient for this to hold. Vámos [39] showed that there is no finite axiom system for realizability of matroids. Sullivant [36] established the same result for gaussoids.

One of the many axiom systems for matroids is the combinatorial compatibility with the quadratic Plücker relations that define the Grassmannian [8, § 4]. Our aim is to derive the analogous result for gaussoids. The role of the Grassmannian  $\text{Gr}(n, 2n)$  is now played by a projection of the Lagrangian Grassmannian  $\text{LGr}(n, 2n)$ , namely the variety  $V(J_n)$ .

Let  $f \in J_n$  be any polynomial relation among principal and almost-principal minors. A subset  $\mathcal{G}$  of  $\mathcal{A}$  is *incompatible with  $f$*  if precisely one monomial in  $f$  has no unknown in  $\mathcal{G}$ . Otherwise  $\mathcal{G}$  is *compatible with  $f$* . Hilbert's Nullstellensatz implies that  $\mathcal{G}$  is compatible with all  $f$  in  $J_n$  if and only if it is realizable by a symmetric  $n \times n$ -matrix  $\Sigma$  with complex entries.

The ideal  $J_n$  contains two classes of distinguished trinomials of degree two, namely the *square trinomials* (2), one for each 2-cube in the  $n$ -cube, and the *edge trinomials* (3), one for each 1-cube in a 3-cube in the  $n$ -cube. The total number of these trinomials equals

$$2^{n-2} \binom{n}{2} + 12 \cdot 2^{n-3} \binom{n}{3} = 2^{n-3} n(n-1)(2n-3). \quad (5)$$

To represent the gaussoid axioms algebraically, we use the  $12 \cdot 2^{n-3} \binom{n}{3}$  edge trinomials.

**Example 2.2.** Fix  $n = 3$ . There are 12 edge trinomials, one for each edge in Figure 1:

$$\begin{aligned} & p_1 a_{23} - p a_{23|1} - a_{12} a_{13}, \quad p_2 a_{13} - p a_{13|2} - a_{12} a_{23}, \quad p_3 a_{12} - p a_{12|3} - a_{23} a_{13}, \\ & p_{12} a_{13} - p_1 a_{13|2} - a_{12} a_{23|1}, \quad p_{12} a_{23} - p_2 a_{23|1} - a_{12} a_{13|2}, \quad p_{13} a_{12} - p_1 a_{12|3} - a_{13} a_{23|1}, \\ & p_{13} a_{23} - p_3 a_{23|1} - a_{13} a_{12|3}, \quad p_{23} a_{12} - p_2 a_{12|3} - a_{23} a_{13|2}, \quad p_{23} a_{13} - p_3 a_{13|2} - a_{23} a_{12|3}, \\ & p_{123} a_{12} - p_{12} a_{12|3} - a_{23|1} a_{13|2}, \quad p_{123} a_{13} - p_{13} a_{13|2} - a_{23|1} a_{12|3}, \quad p_{123} a_{23} - p_{23} a_{23|1} - a_{12|3} a_{13|2}. \end{aligned}$$

The subsets  $\mathcal{G}$  of  $\mathcal{A}$  that are compatible with these quadrics are precisely the sets in (4). The full list of all 21 generators of  $J_3$ , clustered by symmetry class, appears in Example 3.3.  $\diamond$

The edge trinomials for  $n \geq 4$  are obtained by replicating these 12 quadrics on every 3-face of the  $n$ -cube. We can replace the indices 1, 2, 3 in the first quadric by  $i, j, k$  and then add any set  $L \subseteq [n] \setminus \{i, j, k\}$  to get the trinomial  $p_{iL} \cdot a_{jk|L} - p_L \cdot a_{jk|iL} - a_{ij|L} \cdot a_{ik|L}$  in  $J_n$ .

**Example 2.3.** Fix  $n = 4$ . The 4-cube has 24 two-dimensional faces, so  $\mathcal{A}$  has  $2^4 = 16777216$  subsets. Only 679 of these are gaussoids. This was found in [23, Remark 6]. The gaussoids on [4] are precisely the subsets  $\mathcal{G}$  of  $\mathcal{A}$  that are compatible with the 96 edge trinomials.  $\diamond$

The following is our main result in Section 2. It generalizes the previous two examples.

**Theorem 2.4.** *The following conditions are equivalent for a set  $\mathcal{G}$  of 2-faces of the  $n$ -cube:*

- (a)  $\mathcal{G}$  is a gaussoid, i.e. the four axioms (G1)–(G4) are satisfied for  $\mathcal{G}$ ;
- (b)  $\mathcal{G}$  is compatible with all edge trinomials (3).

*Proof.* We begin by showing the implication from (b) to (a). For each of the four gaussoid axioms we list either one or two of the edge trinomials that are relevant:

- (G1)  $a_{ij|L}a_{jk|iL} + a_{ik|jL}p_{iL} - a_{ik|L}p_{ijL}$  and  $a_{ij|L}p_{ijkL} - a_{ik|jL}a_{jk|iL} - a_{ij|kL}p_{ijL}$ ,
- (G2)  $a_{ij|kL}p_{ijL} + a_{ik|jL}a_{jk|iL} - a_{ij|L}p_{ijkL}$  and  $a_{ik|jL}p_{ikL} + a_{ij|kL}a_{jk|iL} - a_{ik|L}p_{ijkL}$ ,
- (G3)  $a_{ij|L}p_{kL} - a_{ik|L}a_{jk|L} - a_{ij|kL}p_L$  and  $a_{ij|L}a_{jk|L} - a_{ik|L}p_{jL} + a_{ik|jL}p_L$ ,
- (G4)  $a_{ij|L}p_{kL} - a_{ij|kL}p_L - a_{ik|L}a_{jk|L}$ .

Compatibility with these quadrics implies the axiom. For instance, consider axiom (G1). Suppose that  $a_{ij|L}$  and  $a_{ik|jL}$  are in  $\mathcal{G}$ . Then the first two terms of  $a_{ij|L}a_{jk|iL} + a_{ik|jL}p_{iL} - a_{ik|L}p_{ijL}$  have an unknown in  $\mathcal{G}$ . Since  $p_{ijL}$  cannot be an element of  $\mathcal{G}$ , we conclude that  $a_{ik|L}$  is in  $\mathcal{G}$ . Similarly, if the set  $\mathcal{G}$  is compatible with the edge trinomial  $a_{ij|L}p_{ijkL} - a_{ik|jL}a_{jk|iL} - a_{ij|kL}p_{ijL}$  then we can conclude that  $a_{ij|kL}$  is in  $\mathcal{G}$ . The other three axioms are shown similarly.

For the implication from (a) to (b) we first note that the statement was already shown for  $n = 3$ . Namely, each of the 11 gaussoids is compatible with the 12 edge trinomials. Now, suppose  $n \geq 4$ . Each of the gaussoid axioms only refers to collections of unknowns  $a_{ij|K}$  that lie within a particular 3-face of the  $n$ -cube. This means that a subset  $\mathcal{G}$  of  $\mathcal{A}$  is a gaussoid if and only if the restriction of  $\mathcal{G}$  to any 3-face is one of the 11 gaussoids on 3 symbols. The same restriction property holds for compatibility with the edge trinomials.  $\square$

Among the 679 gaussoids for  $n = 4$ , precisely 629 are realizable. The other 50 are eliminated by the higher axioms in [10, Lemma 2.4] and [23, Lemma 10]. In Section 8 we initiate a similar analysis for  $n = 5$ . Of course, by [36], we cannot hope for a complete axiom system for Gaussian realizability, and it makes sense to focus on gaussoids and their relation to the combinatorics of quadrics in  $J_n$ . This relation has a striking similarity to matroid theory. It can be derived from the combinatorics of the the Grassmann–Plücker relations. This approach was initiated thirty years ago by Dress and Wenzel [8] and extended recently by Baker and Bowler [2]. The extent to which matroid theory and gaussoid theory can be further developed in parallel remains to be investigated. It seems promising to study *gaussoids over hyperfields*. Here is one concrete conjecture that points in such a direction.

**Conjecture 2.5.** *Every gaussoid is compatible with all quadrics in  $J_n$ , not just trinomials.*

A proof for  $n \leq 4$  is given in Corollary 4.4, but that proof technique does not generalize. To prove Conjecture 2.5 for  $n \geq 5$ , it suffices to check compatibility with those quadrics that are circuits in the subspace  $(J_n)_2$  of the space of all quadrics. Each circuit lies in one of the weight components described in Section 3. However, that check would amount to a prohibitive computation, even for  $n = 5$ , because there are too many circuits.

As support for Conjecture 2.5 we verified compatibility with the quadrics in Theorem 6.4 for  $n = 5, 6$ . In general, quadrics with two or more terms that are products of only  $p$  variables, such as the square trinomials in (2), need not be checked, as every subset of  $\mathcal{A}$  is compatible with them. This situation changes for the valuated gaussoids of Section 7.

Minors and duality play an important role in matroid theory. The minors of a matroid are obtained by the iterated application of deletions and contractions. These two operations

are reversed under matroid duality. For gaussoids, the roles of deletion and contraction are played by *marginalization* and *conditioning*. These statistical operations are also swapped by the duality  $\Sigma \leftrightarrow \Sigma^{-1}$ . Let  $\mathcal{G}$  be any gaussoid on  $[n]$ . The *dual gaussoid*  $\mathcal{G}^*$  of  $\mathcal{G}$  is

$$\mathcal{G}^* = \{ a_{ij|[n]\setminus(K\cup\{i,j\})} : a_{ij|K} \in \mathcal{G} \}. \quad (6)$$

For any element  $u \in [n]$ , the *marginal gaussoid*  $\mathcal{G}\setminus u$  is the gaussoid on  $[n]\setminus\{u\}$  given by

$$\mathcal{G}\setminus u = \{ a_{ij|K} \in \mathcal{G} : u \notin \{i,j\} \cup K \}.$$

Similarly, the *conditional gaussoid*  $\mathcal{G}/u$  is the gaussoid on  $[n]\setminus\{u\}$  given by

$$\mathcal{G}/u = \{ a_{ij|K\setminus\{u\}} \in \mathcal{G} : u \in K \}.$$

We have the following basic result relating these minors and duality:

**Proposition 2.6.** *If  $\mathcal{G}$  is a gaussoid on  $[n]$  and  $u \in [n]$  then both  $\mathcal{G}\setminus u$  and  $\mathcal{G}/u$  are gaussoids. If  $\mathcal{G}$  is realizable then so are  $\mathcal{G}^*$ ,  $\mathcal{G}\setminus u$ , and  $\mathcal{G}/u$ . The following duality relation holds:*

$$(\mathcal{G}\setminus u)^* = \mathcal{G}^*/u \quad \text{and} \quad (\mathcal{G}/u)^* = \mathcal{G}^*\setminus u. \quad (7)$$

*Proof.* The set of edge trinomials in  $J_n$  is invariant under the duality operation that swaps  $p_K$  with  $p_{[n]\setminus K}$  and also swaps  $a_{ij|K}$  with  $a_{ij|[n]\setminus(K\cup\{i,j\})}$ . Theorem 2.4 hence ensures that  $\mathcal{G}^*$  is a gaussoid. The duality operation preserves realizability: if a positive definite matrix  $\Sigma$  realizes  $\mathcal{G}$ , then its inverse  $\Sigma^{-1}$  realizes  $\mathcal{G}^*$  by [23, Corollary 1 and Lemma 2].

A similar argument works for marginalization and conditioning. The edge trinomials for  $[n]\setminus\{u\}$  appear among those for  $[n]$ , and similarly if we augment the index set  $K$  with  $u$ . That realizability is preserved under these operations is [31, Lemma 1]. Indeed, if  $\Sigma$  realizes  $\mathcal{G}$ , then we obtain a realization of  $\mathcal{G}\setminus u$  by deleting row  $u$  and column  $u$  from  $\Sigma$ , and we obtain a realization of  $\mathcal{G}/u$  by taking the Schur complement of  $\Sigma$  with respect to  $u$ .

The duality relations (7) are verified by a direct check, bearing in mind that two of the duals in this formula are taken with respect to the index set  $[n]\setminus\{u\}$  instead of  $[n]$ .  $\square$

Kenyon and Pemantle [21] initiated the study of the ideal  $J_n$  from the perspective of cluster algebras. They conjectured a formula for the entries of  $\Sigma$  in terms of principal and almost-principal minors whose index sets are connected. That conjecture was proved by Sturmfels, Tsukerman, and Williams in [35]. As explained in [35, § 5], this is closely related to formulas for partial correlations in statistics [18]. If  $\Sigma$  is a covariance matrix, the associated *correlation matrix* has ones on the diagonal and off-diagonal entries  $\rho_{ij} = a_{ij}/\sqrt{p_i p_j}$ . More generally, the *partial correlations* of the Gaussian distribution given by  $\Sigma$  are the quantities

$$\rho_{ij|K} = \frac{a_{ij|K}}{\sqrt{p_{iK} p_{jK}}}. \quad (8)$$

Joe and his collaborators discuss the algebraic relations among the  $\rho_{ij|K}$  and construct subsets that serve as convenient transcendence bases modulo these relations. Their *d-vines* in [18] correspond precisely to the *standard networks* of Kenyon and Pemantle in [21]. Our results on gaussoids and the ideal  $J_n$  immediately imply new constraints on partial correlations.



### 3 Symmetry

We are interested in the ideal  $J_n$  of algebraic relations among the  $2^n$  principal minors  $p_L$  and the  $\binom{n}{2}2^{n-2}$  almost-principal minors  $a_{ij|K}$  of a symmetric  $n \times n$ -matrix of unknowns. The analogous problem for principal minors alone was solved (set-theoretically) by Oeding [27, 28]. He showed that the variety of the elimination ideal  $J_n \cap \mathbb{R}[\mathcal{P}]$  is defined by quartics.

**Example 3.1.** Eliminating the six unknowns in  $\mathcal{A}$  from  $J_3$ , we obtain the principal ideal

$$J_3 \cap \mathbb{R}[\mathcal{P}] = \langle p^2 p_{123}^2 + p_1^2 p_{23}^2 + p_2^2 p_{13}^2 + p_3^2 p_{12}^2 + 4pp_{12}p_{13}p_{23} + 4p_1p_2p_3p_{123} - \cdots - 2p_1p_3p_{12}p_{23} \rangle.$$

The quartic generator is the  $2 \times 2 \times 2$  hyperdeterminant. This fact was first found in [16].  $\diamond$

Oeding's result is based on the representation theory of the group  $G = \mathrm{SL}_2(\mathbb{R})^n$ . We aim to understand  $J_n$  by using this technique. The point of departure is the observation that  $G$  acts on the space  $W$  spanned by the principal and almost-principal minors. This action is induced by the  $G$ -action on the space of  $n \times 2n$ -matrices. Here, the group  $\mathrm{SL}_2(\mathbb{R})$  in the  $i$ th factor acts by replacing columns  $i$  and  $n+i$  by linear combinations of these two columns. If we apply this to (1) and then multiply by the inverse of the left  $n \times n$ -block then the right  $n \times n$ -block is again symmetric. See [16, Lemma 13] for a proof of this crucial observation.

In this section we study the structure of the  $G$ -module  $W$ . Let  $V_i \simeq \mathbb{R}^2$  denote the defining representation of the  $i$ -th factor  $\mathrm{SL}_2(\mathbb{R})$ . Let  $W_{\mathrm{pr}}$  be the space spanned by all principal minors and  $W_{\mathrm{ap}}^{ij}$  the space spanned by the almost-principal minors  $a_{ij|K}$  where  $i, j$  are fixed and  $K$  runs over subsets of  $[n] \setminus \{i, j\}$ . The following is similar to [28, Theorem 1.1]:

**Lemma 3.2.** *We have the following isomorphisms of irreducible  $G$ -modules:*

$$W_{\mathrm{pr}} \simeq \bigotimes_{i=1}^n V_i \quad \text{and} \quad W_{\mathrm{ap}}^{ij} \simeq \bigotimes_{k \in [n] \setminus \{i, j\}} V_k \quad \text{for } 1 \leq i < j \leq n.$$

We use the unknown  $x_i$  to refer to the highest weight of the  $G$ -module  $V_i$ . The highest weight of a tensor product of such modules is the product of the corresponding  $x_i$ . For instance,  $\mathrm{Sym}_2(V_1) \otimes V_2$  has highest weight  $x_1^2 x_2$ . The *formal character* of a  $G$ -module is the sum of the Laurent monomials representing the weights in a weight basis. Let  $W = W_{\mathrm{pr}} \oplus \bigoplus_{i,j} W_{\mathrm{ap}}^{ij}$  be the  $G$ -module of principal and almost-principal minors. The set  $\mathcal{A} \cup \mathcal{P}$  is a distinguished weight basis of  $W$ . By Lemma 3.2, the formal character of  $W$  equals

$$\prod_{i=1}^n (x_i + x_i^{-1}) + \sum_{1 \leq i < j \leq n} \prod_{k \in [n] \setminus \{i, j\}} (x_k + x_k^{-1}). \quad (9)$$

Our prime ideal  $J_n$  lives in the polynomial ring  $\mathrm{Sym}_*(W) = \bigoplus_{d=0}^{\infty} \mathrm{Sym}_d(W) = \mathbb{R}[\mathcal{P} \cup \mathcal{A}]$ . It is invariant under the  $G$ -action. The weight of a monomial in  $\mathbb{R}[\mathcal{P} \cup \mathcal{A}]$  is a vector in  $\mathbb{Z}^n$ , namely, the exponent vector of the corresponding Laurent monomial in  $x_1, \dots, x_n$ .

We focus on the  $G$ -module of all quadrics,  $\mathrm{Sym}_2(W)$ . Its dimension equals

$$\dim(\mathrm{Sym}_2(W)) = (2^n + 2^{n-2} \binom{n}{2} + 1) \cdot (2^{n-1} + 2^{n-3} \binom{n}{2}).$$



The formal character of  $\text{Sym}_2(W)$  is the sum of all pairs of products (with repetition allowed) of the  $2^n + 2^{n-2} \binom{n}{2}$  Laurent monomials that appear in the expansion of (9).

Each irreducible  $G$ -module has the form

$$S_{d_1 d_2 \dots d_n} = \bigotimes_{i=1}^n \text{Sym}_{d_i}(V_i),$$

where  $d_1, d_2, \dots, d_n$  are nonnegative integers. In Oeding's work [27, 28], this module was written as  $S_{d_1 0} S_{d_2 0} \dots S_{d_n 0}$ . The formal character of the irreducible  $G$ -module  $S_{d_1 d_2 \dots d_n}$  equals

$$\prod_{i=1}^n \sum_{\ell=0}^{d_i} x_i^{d_i-2\ell} = x_1^{d_1} x_2^{d_2} \dots x_n^{d_n} + \text{lower terms.} \quad (10)$$

Our task is to express the formal character of  $\text{Sym}_2(W)$  as a sum of Laurent polynomials (10), and to identify the submodule  $(J_n)_2$  in terms of the irreducible  $G$ -modules in  $\text{Sym}_2(W)$ .

**Example 3.3.** Let  $n = 3$ . The 8 principal and 6 almost-principal minors span the  $G$ -module

$$W = S_{111} \oplus S_{100} \oplus S_{010} \oplus S_{001}.$$

This space of quadrics has dimension 105. It decomposes into irreducible  $G$ -modules as

$$\text{Sym}_2(W) = S_{222} \oplus S_{211} \oplus S_{121} \oplus S_{112} \oplus 2S_{200} \oplus 2S_{020} \oplus 2S_{002} \oplus 2S_{110} \oplus 2S_{101} \oplus 2S_{011}.$$

The ring  $\text{Sym}_*(W) = \mathbb{R}[\mathcal{P} \cup \mathcal{A}]$  has 8 unknowns  $p_I$  and 6 unknowns  $a_{ij|K}$ . They are identified with the vertices and facets of the 3-cube (cf. Figure 1). The weights of the 14 unknowns are

unknown	$a_{12}$	$a_{12 3}$	$\dots$	$a_{23 1}$	$p$	$p_1$	$\dots$	$p_{123}$
weight	$(0, 0, 1)$	$(0, 0, -1)$	$\dots$	$(-1, 0, 0)$	$(1, 1, 1)$	$(-1, 1, 1)$	$\dots$	$(-1, -1, -1)$

The 21-dimensional space of quadrics in  $J_3$  generates the ideal. As a  $G$ -module,

$$(J_3)_2 = S_{200} \oplus S_{020} \oplus S_{002} \oplus S_{110} \oplus S_{101} \oplus S_{011}.$$

We display an explicit weight basis for each summand, beginning with the 12 edge trinomials:

$$\begin{array}{l}
S_{110} \\
S_{101}
\end{array}
\begin{array}{l}
\left[ \begin{array}{ll}
(1, 1, 0) & a_{13}a_{23} + a_{12|3}p - a_{12}p_3 \\
(1, -1, 0) & a_{13|2}a_{23} + a_{12|3}p_2 - a_{12}p_{23} \\
(-1, 1, 0) & a_{13}a_{23|1} + a_{12|3}p_1 - a_{12}p_{13} \\
(-1, -1, 0) & a_{13|2}a_{23|1} + a_{12|3}p_{12} - a_{12}p_{123}
\end{array} \right] \\
\left[ \begin{array}{ll}
(1, 0, 1) & a_{12}a_{23} + a_{13|2}p - a_{13}p_2 \\
(1, 0, -1) & a_{12|3}a_{23} + a_{13|2}p_3 - a_{13}p_{23} \\
(-1, 0, 1) & a_{12}a_{23|1} + a_{13|2}p_1 - a_{13}p_{12} \\
(-1, 0, -1) & a_{12|3}a_{23|1} + a_{13|2}p_{13} - a_{13}p_{123}
\end{array} \right]
\end{array}$$

$$S_{011} \begin{bmatrix} (0, 1, 1) & a_{12}a_{13} + a_{23|1}p - a_{23}p_1 \\ (0, 1, -1) & a_{12|3}a_{13} + a_{23|1}p_3 - a_{23}p_{13} \\ (0, -1, 1) & a_{12}a_{13|2} + a_{23|1}p_2 - a_{23}p_{12} \\ (0, -1, -1) & a_{12|3}a_{13|2} + a_{23|1}p_{23} - a_{23}p_{123} \end{bmatrix}$$

See Example 2.2. The last three  $G$ -modules account for the square trinomials:

$$S_{200} \begin{bmatrix} (2, 0, 0) & a_{23}^2 + pp_{23} - p_2p_3 \\ (0, 0, 0) & 2a_{23}a_{23|1} + pp_{123} + p_1p_{23} - p_2p_{13} - p_{12}p_3 \\ (-2, 0, 0) & a_{23|1}^2 + p_1p_{123} - p_{12}p_{13} \end{bmatrix}$$

$$S_{020} \begin{bmatrix} (0, 2, 0) & a_{13}^2 + pp_{13} - p_1p_3 \\ (0, 0, 0) & 2a_{13}a_{13|2} + pp_{123} + p_2p_{13} - p_1p_{23} - p_{12}p_3 \\ (0, -2, 0) & a_{13|2}^2 + p_2p_{123} - p_{12}p_{23} \end{bmatrix}$$

$$S_{002} \begin{bmatrix} (0, 0, 2) & a_{12}^2 + pp_{12} - p_1p_2 \\ (0, 0, 0) & 2a_{12}a_{12|3} + pp_{123} + p_3p_{12} - p_1p_{23} - p_{13}p_2 \\ (0, 0, -2) & a_{12|3}^2 + p_3p_{123} - p_{13}p_{23} \end{bmatrix}$$

The case  $n = 3$  is so small that every minor of  $\Sigma$  is either principal or almost-principal. Hence  $J_3$  is the ideal defining the Lagrangian Grassmannian  $\text{LGr}(3, 6) \subset \mathbb{P}^{13}$ . This variety has dimension 6 and degree  $16 = 6!/(1^3 3^2 5)$ . The following code in `Macaulay2` [14] computes the 21 quadrics from the 35 quadrics that cut out the Grassmannian  $\text{Gr}(3, 6)$  in  $\mathbb{P}^{19}$ :

```
R = QQ[p123,p124,p134,p234,p125,p135,p235,p145,p245,p345,
      p126,p136,p236,p146,p246,p346,p156,p256,p356,p456];
I = Grassmannian(2,5,R) + ideal(p124-p236,p125+p136,p134+p235,
      p346+p245,p356-p145,p256+p146);
J3 = eliminate({p124,p134,p125,p356,p256,p346},I)
```

From the free resolution (computed with `res J3`) it can be verified that  $J_3$  is Gorenstein.

Each of the 20 generators of the polynomial ring  $R$  equals (up to sign) one of the 14 variables in  $\mathcal{P} \cup \mathcal{A}$ . The precise identification is given by the following ordered list of length 20:

$$\begin{array}{cccccccccc} p & a_{13} & -a_{12} & p_1 & a_{23} & -p_2 & a_{12} & a_{13|2} & -a_{23|1} & p_{12} \\ p_3 & -a_{23} & a_{13} & a_{12|3} & -p_{13} & a_{23|1} & p_{23} & -a_{12|3} & a_{13|2} & p_{123} \end{array} \quad (11)$$

One comment for algebraic geometers: canonical curves of genus 9 are obtained by intersecting  $V(J_3) = \text{LGr}(3, 6)$  with subspaces  $\mathbb{P}^8$  in  $\mathbb{P}^{13}$ . This was shown by Mukai and further developed by Iliev and Ranestad [17], who derive the 21 quadrics explicitly in [17, § 2.3].  $\diamond$

**Example 3.4.** Let  $n=4$ . There are 16 principal and 24 almost-principal minors. They span

$$W = S_{1111} \oplus S_{1100} \oplus S_{1010} \oplus S_{1001} \oplus S_{0110} \oplus S_{0101} \oplus S_{0011}.$$

The space of quadrics has dimension 820. It decomposes into irreducible  $G$ -modules as

$$\begin{aligned} \text{Sym}_2(W) = & S_{2222} \oplus S_{2211} \oplus S_{2121} \oplus S_{2112} \oplus S_{1221} \oplus S_{1212} \oplus S_{1122} \oplus 2S_{2200} \oplus 2S_{2020} \\ & \oplus 2S_{2002} \oplus 2S_{0220} \oplus 2S_{0202} \oplus 2S_{0022} \oplus 2S_{2110} \oplus 2S_{2101} \oplus 2S_{2011} \oplus 2S_{1210} \\ & \oplus 2S_{1201} \oplus 2S_{0211} \oplus 2S_{1120} \oplus 2S_{1021} \oplus 2S_{0121} \oplus 2S_{1102} \oplus 2S_{1012} \oplus 2S_{0112} \\ & \oplus 3S_{1111} \oplus 3S_{1100} \oplus 3S_{1010} \oplus 3S_{1001} \oplus 3S_{0110} \oplus 3S_{0101} \oplus 3S_{0011} \oplus 7S_{0000}. \end{aligned}$$

The 226-dimensional submodule of quadrics that vanishes on our variety equals

$$\begin{aligned}
(J_4)_2 = & S_{2200} \oplus S_{2020} \oplus S_{2002} \oplus S_{0220} \oplus S_{0202} \oplus S_{0022} \oplus S_{2110} \oplus S_{2101} \oplus S_{2011} \\
& \oplus S_{1210} \oplus S_{1201} \oplus S_{0211} \oplus S_{1120} \oplus S_{1021} \oplus S_{0121} \oplus S_{1102} \oplus S_{1012} \\
& \oplus S_{0112} \oplus S_{1100} \oplus S_{1010} \oplus S_{1001} \oplus S_{0110} \oplus S_{0101} \oplus S_{0011} \oplus 4S_{0000}.
\end{aligned} \tag{12}$$

Each of the four copies of the one-dimensional module  $S_{0000}$  is spanned by a  $G$ -invariant quadric in the ideal  $J_4$ . Here is one such invariant that involves none of the principal minors:

$$a_{14}a_{14|23} - a_{14|2}a_{14|3} - a_{23}a_{23|14} + a_{23|1}a_{23|4}.$$

This quadric can be derived from the quadrics in Theorem 6.4 (iv). It is instructive to locate the 24 square trinomials and the 96 edge trinomials inside the summands seen in (12).  $\diamond$

In Section 6 we study the quadrics in  $J_n$ . This uses the action of the Lie algebra  $\mathfrak{g}$  of the group  $G$ . The situation differs from that in [27, 28]. Oeding's hyperdeterminantal ideal is generated by a single irreducible module for the action of  $G \rtimes S_n$  on  $\text{Sym}_4(W_{\text{pr}})$ . In our case, there are many irreducibles even modulo the action of  $S_n$ . The space  $(J_3)_2$  in Example 3.3 decomposes into two irreducible  $G \rtimes S_3$ -modules, and  $(J_4)_2$  in Example 3.4 decomposes into five irreducible  $G \rtimes S_4$ -modules. This complexity accounts for the difficulties in Section 6.

We now shift gears and discuss a collection of finite groups that act on the combinatorial structures studied in this paper. These finite groups arise from the following inclusions:

$$S_n \subset (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n \subset (\mathbb{Z}/4\mathbb{Z})^n \rtimes S_n \subset G \rtimes S_n. \tag{13}$$

The symmetric group  $S_n$  acts by permuting indices in the unknowns  $p_I$  and  $a_{ij|K}$ , and by simultaneously permuting the rows and columns of the matrix  $\Sigma$  in (1). The third group in (13) is obtained by taking the following cyclic subgroup in each factor  $\text{SL}_2(\mathbb{R})$ :

$$\mathbb{Z}/4\mathbb{Z} \simeq \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

**Remark 3.5.** The action of  $(\mathbb{Z}/4\mathbb{Z})^n$  on the Lagrangian Grassmannian  $\text{LGr}(n, 2n)$  takes symmetric matrices to symmetric matrices, but it changes their signatures. It thus does not preserve the property that  $\Sigma$  is positive definite. In fact, already the induced action by the hyperoctahedral group (discussed below) does not preserve realizability of gaussoids.

Consider the subgroup  $(\mathbb{Z}/2\mathbb{Z})^n = \{(\pm \text{Id}_2, \pm \text{Id}_2, \dots, \pm \text{Id}_2)\}$  of  $\text{SL}_2(\mathbb{R})^n$ . Each element in this group corresponds to an  $n \times n$ -diagonal matrix  $D$  with entries in  $\{-1, +1\}$ . *Reorientation* is the action of  $(\mathbb{Z}/2\mathbb{Z})^n$  that maps  $\Sigma$  to  $D\Sigma D$ . This does not change the principal minors of  $\Sigma$ . In particular, if  $\Sigma$  is positive definite, then so is  $D\Sigma D$ . Under this action, the almost-principal minor  $a_{ij|K}$  transforms into  $D_{ii}D_{jj}a_{ij|K} = \pm a_{ij|K}$ . This action is trivial for gaussoids, but it is non-trivial for oriented gaussoids, as we shall see in Section 5.

In order to get a faithful action on the set of gaussoids we need to take the quotient of  $(\mathbb{Z}/4\mathbb{Z})^n \rtimes S_n$  modulo its normal subgroup  $(\mathbb{Z}/2\mathbb{Z})^n \rtimes \{\text{Id}\}$ . The resulting group  $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$  is the *hyperoctahedral group*. It acts on the set of gaussoids as the symmetry group of the

$n$ -cube. The quotient  $(\mathbb{Z}/2\mathbb{Z})^n$  acts by swapping indices in and out from the index sets  $I$  and  $K$  in the quantities  $p_I$  and  $a_{i_j|K}$ . When expressed in terms of  $\Sigma$ , the latter action looks like a fusion of matrix inversion and Schur complements. Consider the subgroup of the hyperoctahedral group given by  $\mathbb{Z}/2\mathbb{Z} = \{(\text{Id}_2, \dots, \text{Id}_2), ((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}), \dots, (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}))\}$  inside  $(\mathbb{Z}/2\mathbb{Z})^n$ :

$$S_n \subset (\mathbb{Z}/2\mathbb{Z}) \rtimes S_n \subset (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n. \quad (14)$$

The group  $\mathbb{Z}/2\mathbb{Z}$  in the middle of (14) acts on the set of gaussoids by the *duality* in (6). Algebraically, this is the involution on  $\text{LGr}(n, 2n)$  that maps  $\Sigma$  to its negative inverse  $-\Sigma^{-1}$ . In summary, these finite group actions are subtle. In particular, the distinction between reorientation and the hyperoctahedral group (both isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ ) is crucial.

## 4 Census of Small Gaussoids

In this section we derive and discuss the following result. The proof for  $n = 5$  is by computation. It rests on using state-of-the-art software from the field of *SAT solvers* [37, 38].

**Theorem 4.1.** *For  $n = 3, 4, 5$ , the number of gaussoids  $\mathcal{G}$  is as follows, up to symmetries:*

$n$	all gaussoids	orbits for $S_n$	$\mathbb{Z}/2\mathbb{Z} \rtimes S_n$	$(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$
3	11	5	4	4
4	679	58	42	19
5	60,212,776	508,817	254,826	16,981

The second, third, and fourth column report the number of orbits under the group actions described in (14). Theorem 4.1 for  $n = 3$  is Example 2.1, where the 11 gaussoids are listed. There are five orbits under permuting the indices 1, 2, 3. The two singleton orbits fuse to a single orbit under the group  $\mathbb{Z}/2\mathbb{Z} \rtimes S_3$ . For instance, the gaussoids  $\{a_{12}\}$  and  $\{a_{12|3}\}$  are swapped under duality. The same four orbits persist under the action of the hyperoctahedral group  $(\mathbb{Z}/2\mathbb{Z})^3 \rtimes S_3$ , since  $|\mathcal{G}|$  is an invariant of that action. For  $n = 4$ , Lněnička and Matúš [23] showed that there are 679 gaussoids, of which 629 are realizable. Their computation was confirmed by Drton and Xiao [10]. The action by the hyperoctahedral group  $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_4$  was not used in [10, 23], but we find this to be natural in our setting.

**Lemma 4.2.** *The 679 gaussoids for  $n = 4$  are organized into orbits as follows:*

- *The symmetric group  $S_4$  of order 24 acts on the gaussoids by permuting indices. There are 58 orbits under that action. Five of these orbits consist of non-realizable gaussoids.*
- *The twisted symmetric group  $\mathbb{Z}/2\mathbb{Z} \rtimes S_4$  of order 48 acts on the gaussoids by duality and permuting indices. This action preserves realizability, and it has 42 orbits. Five of these orbits consist of non-realizable gaussoids.*
- *Under the action of the hyperoctahedral group  $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_4$  of order 384, there are 19 orbits. Three of the orbits contain non-realizable gaussoids.*

The difference between the three group actions can already be seen for the 24 singleton gaussoids. These correspond to the 2-faces of the 4-cube. The symmetric group  $S_4$  has three distinct orbits on the set  $\mathcal{A}$ : the six  $1 \times 1$ -minors  $a_{ij}$ , the twelve  $2 \times 2$ -minors  $a_{ij|k}$ , and the six  $3 \times 3$ -minors  $a_{ij|kl}$ . The  $1 \times 1$ -minors and the  $3 \times 3$ -minors are swapped under duality. So, there are two orbits of size 12 for the group  $\mathbb{Z}/2\mathbb{Z} \rtimes S_4$ . Finally, the full symmetry group of the 4-cube acts transitively on the 2-faces. Hence that group has only one orbit of size 24.

The following 19 items are the orbits of the hyperoctahedral group  $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_4$ . The symbol  $\ell_m$  at the beginning indicates that the orbit consists of  $m$  gaussoids  $\mathcal{G}$ , each of cardinality  $|\mathcal{G}| = \ell$ . This is followed by a list of  $\mathbb{Z}/2\mathbb{Z} \rtimes S_4$ -orbits, each given by its lexicographically first representative. For instance, the fourth item, marked  $2_{48}$ , is a hyperoctahedral orbit of size 48 that consists of two-element gaussoids  $\mathcal{G}$ . If we restrict to permuting indices and duality then this orbit breaks into four smaller orbits, of cardinalities 6, 6, 12 and 24.

Five of the small orbits are distinguished by double-brackets  $[[ \ ]]$  instead of single curly brackets  $\{ \}$ . The  $50 = 8 + 6 + 6 + 6 + 24$  elements in these five  $\mathbb{Z}/2\mathbb{Z} \rtimes S_4$ -orbits are the non-realizable gaussoids. For instance, the eight triple gaussoids in the orbit  $[[a_{12|3}, a_{13|4}, a_{14|2}]]_8$  are non-realizable. We discuss the issue of realizability in more detail after our list.

- $0_1 : \emptyset_1$
- $1_{24} : \{a_{12}\}_{12} \quad \{a_{12|3}\}_{12}$
- $2_{12} : \{a_{12}, a_{12|34}\}_6 \quad \{a_{12|3}, a_{12|4}\}_6$
- $2_{48} : \{a_{12}, a_{34}\}_6 \quad \{a_{12}, a_{34|12}\}_6 \quad \{a_{12|3}, a_{34|1}\}_{12} \quad \{a_{12}, a_{34|1}\}_{24}$
- $2_{96} : \{a_{12}, a_{13|24}\}_{24} \quad \{a_{12}, a_{13|4}\}_{48} \quad \{a_{12|3}, a_{13|4}\}_{24}$
- $3_{32} : \{a_{12}, a_{13|24}, a_{14|3}\}_{24} \quad [[a_{12|3}, a_{13|4}, a_{14|2}]]_8$
- $3_{48} : \{a_{12}, a_{12|34}, a_{34}\}_{12} \quad \{a_{12}, a_{12|34}, a_{34|1}\}_{12} \quad \{a_{12}, a_{34|1}, a_{34|2}\}_{12} \quad \{a_{12|3}, a_{12|4}, a_{34|1}\}_{12}$
- $3_{48} : \{a_{12}, a_{12|34}, a_{34}\}_{12} \quad \{a_{12}, a_{12|34}, a_{34|1}\}_{12} \quad \{a_{12}, a_{34|1}, a_{34|2}\}_{12} \quad \{a_{12|3}, a_{12|4}, a_{34|1}\}_{12}$
- $3_{192} : \{a_{12}, a_{13|24}, a_{24|13}\}_{24} \quad \{a_{12}, a_{13|4}, a_{24|3}\}_{24} \quad \{a_{12}, a_{13|4}, a_{34|12}\}_{24}$   
 $\{a_{12|3}, a_{13|4}, a_{24|1}\}_{24} \quad \{a_{12}, a_{13|24}, a_{24|3}\}_{48} \quad \{a_{12}, a_{13|4}, a_{34|2}\}_{48}$
- $4_{12} : \{a_{12}, a_{12|34}, a_{34}, a_{34|12}\}_3 \quad \{a_{12|3}, a_{12|4}, a_{34|1}, a_{34|2}\}_3 \quad [[a_{12}, a_{12|34}, a_{34|1}, a_{34|2}]]_6$
- $4_{24} : \{a_{12}, a_{12|3}, a_{13}, a_{13|2}\}_{24}$
- $4_{48} : \{a_{12}, a_{13|4}, a_{24|3}, a_{34|12}\}_{12} \quad [[a_{12}, a_{13|24}, a_{24|13}, a_{34}]]_6$   
 $[[a_{12|3}, a_{13|4}, a_{24|1}, a_{34|2}]]_6 \quad [[a_{12}, a_{13|24}, a_{24|3}, a_{34|1}]]_{24}$
- $5_{48} : \{a_{12}, a_{12|3}, a_{13}, a_{13|2}, a_{23|14}\}_{24} \quad \{a_{12}, a_{12|3}, a_{13}, a_{13|2}, a_{23|4}\}_{24}$
- $6_8 : \{a_{12}, a_{12|3}, a_{13}, a_{13|2}, a_{23}, a_{23|1}\}_8$
- $7_{48} : \{a_{12}, a_{12|3}, a_{12|34}, a_{13}, a_{13|2}, a_{24|13}, a_{24|3}\}_{24} \quad \{a_{12}, a_{12|3}, a_{12|4}, a_{13}, a_{13|2}, a_{24}, a_{24|1}\}_{24}$
- $12_4 : \{a_{12}, a_{12|3}, a_{12|34}, a_{12|4}, a_{13}, a_{13|2}, a_{13|24}, a_{13|4}, a_{14}, a_{14|2}, a_{14|23}, a_{14|3}\}_4$
- $14_{24} : (\mathcal{A} \setminus \{a_{23|14}, a_{23|4}, a_{24}, a_{24|1}, a_{24|13}, a_{24|3}, a_{34}, a_{34|1}, a_{34|12}, a_{34|2}\})_{24}$
- $16_3 : (\mathcal{A} \setminus \{a_{14}, a_{14|2}, a_{14|23}, a_{14|3}, a_{23}, a_{23|1}, a_{23|14}, a_{23|4}\})_3$
- $20_6 : (\mathcal{A} \setminus \{a_{34}, a_{34|1}, a_{34|12}, a_{34|2}\})_6$
- $24_1 : (\mathcal{A})_1$

It is instructive to look at the list above through the lens of Remark 3.5. The action of  $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_4$  on the variety  $V(J_4)$  and on the 679 gaussoids can be understood via the Lagrangian Grassmannian  $\text{LGr}(4, 8) \subset \mathbb{P}^{39}$ . Here a symmetric  $4 \times 4$ -matrix  $\Sigma$  corresponds to the  $4 \times 8$ -matrix  $(\text{Id}_4 \ \Sigma)$ , where  $\text{Id}_4$  is the  $4 \times 4$  identity matrix. The group  $S_4$  acts on this  $4 \times 8$ -matrix by simultaneously permuting rows and columns of  $\Sigma$  and of  $\text{Id}_4$ . Each factor of  $(\mathbb{Z}/2\mathbb{Z})^4$  switches a column of  $\text{Id}_4$  with the corresponding column of  $\Sigma$  and changes the sign of one of the columns. If one multiplies the  $4 \times 8$ -matrix on the left by the inverse of its left  $4 \times 4$ -block, then the result is a matrix  $(\text{Id}_4 \ \Sigma')$ , where  $\Sigma'$  is symmetric by [16, Lemma 13].

After swapping one column and switching the sign, the signatures of the symmetric matrices  $\Sigma'$  and  $\Sigma$  differ by one. Thus, if  $\Sigma$  is positive definite, then  $\Sigma'$  is not positive definite. After having performed this operation for all four columns, the resulting matrix  $\Sigma'$  is negative definite. We then replace  $\Sigma'$  by its negative  $-\Sigma'$  to get a positive definite matrix. Including this last step, this group action represents gaussoid duality, which retains realizability.

Because of this change in signature, the action of  $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_4$  on gaussoids does not preserve realizability in the Gaussian sense where all  $p_I$  are to remain positive. However it does retain a weaker notion of realizability which only requires that the  $p_I$  remain nonzero.

**Remark 4.3.** The non-realizability of the five  $\mathbb{Z}/2\mathbb{Z} \rtimes S_4$ -orbits that were highlighted above can be certified by polynomials in the ideal  $J_4$ . The existence of such certificates is guaranteed by the *Real Nullstellensatz*. For example, to show that the gaussoid  $\mathcal{G} = \{a_{12|3}, a_{13|4}, a_{14|2}\}$  has no Gaussian realization, we can use the following algebraic relation which lies in  $J_4$ :

$$a_{14}(a_{34}^2 p_2 p_4 p_{23} + a_{23}^2 a_{34}^2 p_{24} + p_2^2 p_3 p_4 p_{34}) - (a_{23} a_{24} a_{34} + p_2 p_3 p_4)(a_{24} p_4 \underline{a_{12|3}} + a_{24} a_{23} \underline{a_{13|4}} + p_3 p_4 \underline{a_{14|2}}).$$

Indeed, in any realization the second summand is zero. However, the first summand is nonzero because the three terms in the left parenthesis are strictly positive. Starting from the proofs in [23, Corollary 4], we can derive similar certificates for the other four gaussoids.

**Corollary 4.4.** *Conjecture 2.5 is true for  $n = 4$ .*

*Proof.* Each of the five non-realizable gaussoids  $\mathcal{G}$  has a realizable gaussoid in its orbit under the group  $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_4$ . Hence  $\mathcal{G}$  admits a realization where all  $p_I$  are nonzero, but some are negative. The existence of such a non-Gaussian realization shows that  $\mathcal{G}$  is compatible with every polynomial in  $J_4$ . In particular,  $\mathcal{G}$  is compatible with every quadric in  $J_4$ .  $\square$

We now come to the census of gaussoids for  $n = 5$ . This is the main result in Theorem 4.1. It is derived by direct computation using SAT solvers. Here SAT stands for the satisfiability problem of propositional logic. This problem is NP-complete. However, there have been considerable advances in solving SAT problems in practice. We believe that these techniques can be useful for a wide range of problems in applied algebraic geometry.

All SAT solvers use the same input: a Boolean formula in conjunctive normal form (cnf). A *cnf formula* is a conjunction of clauses, where a clause is a disjunction of variables or negated variables. Every Boolean formula can be brought into cnf. A standard file format is the DIMACS cnf file format. There are three natural problems for a given cnf formula:

**SAT:** Is the formula satisfiable?

**#SAT:** How many satisfying assignments are there?

**AISAT**: Enumerate all satisfying assignments.

The three problems are listed by increasing difficulty. The third is the most relevant for us. For example, gaussoid enumeration is an instance of **AISAT**. To show this, we introduce one Boolean variable for each element  $a_{ij|K}$  of  $\mathcal{A}$ . The gaussoid  $\mathcal{G}$  consists of those variables that take the value zero. This convention is consistent with the assignment of zero to the variables in the gaussoid, when checking compatibility with the edge trinomials. The gaussoid axioms can be formulated as Boolean formulas. Specifically, (G1)-(G3) can be written as  $A \wedge B \Rightarrow C \wedge D$  where  $A, B, C, D$  are statements of the form  $a_{ij|K} \in \mathcal{G}$ , or  $a_{ij|K} = 0$ . The implication above can be brought into cnf with two disjunctions as follows:

$$\{A \wedge B \Rightarrow C \wedge D\} \iff (C \vee \neg A \vee \neg B) \wedge (D \vee \neg A \vee \neg B).$$

The weak transitivity axiom (G4) is of the form  $A \wedge B \Rightarrow C \vee D$ . It has the simple cnf

$$C \vee D \vee \neg A \vee \neg B.$$

These axioms in cnf need to be specified for all possible choices of  $i, j, k, L$  in (G1)-(G4).

**Lemma 4.5.** *The enumeration of all gaussoids on  $[n]$  is an instance of the AISAT problem, where the Boolean formula in conjunctive normal form (cnf) has  $7 \binom{n}{3} 2^{n-3} 3!$  clauses.*

*Proof.* For each choice of an ordered triple  $(i, j, k)$  from  $[n]$  and a subset  $L$  of  $[n] \setminus \{i, j, k\}$ , we have introduced seven clauses: two for each axiom (G1)-(G3) and one for (G4).  $\square$

*Proof of Theorem 4.1.* For  $n \leq 4$  see the discussion above. The proof for  $n = 5$  consists of the following computation. Using Lemma 4.5, we expressed the gaussoid axioms as an AISAT instance with 1680 clauses. The formula was then solved using the solver `bdd_minisat_all` due to Toda and Takehide [38]. The output is the list of all 60212776 gaussoids. This count was verified independently using Thurley's #SAT solver `sharpSAT` [37] on the same input. To group the gaussoids into orbits under the actions of the three finite groups in (14) we wrote our own code in `sage` [30]. The numbers of orbits we found are those in the table. Our homepage [www.gaussoids.de](http://www.gaussoids.de) contains this data and the code to reproduce it.  $\square$

## 5 Oriented Gaussoids and Positivity

Theorem 2.4 establishes a strong parallel between matroids and gaussoids. An important feature of matroid theory is its numerous extensions, notably to oriented matroids [3], positroids [1], and valuated matroids [7]. The analogues in our setting are oriented gaussoids, positive gaussoids, and valuated gaussoids. This section is devoted to the first two of these.

Given any gaussoid  $\mathcal{G} \subset \mathcal{A}$ , we can assign orientations  $+$  or  $-$  to the unknowns  $a_{ij|K}$  in  $\mathcal{A} \setminus \mathcal{G}$ . These represent inequalities  $a_{ij|K} > 0$  and  $a_{ij|K} < 0$  that express the sign of the partial correlation (8) among the random variables  $X_i$  and  $X_j$  given  $X_K$ . Not all assignments are compatible with the edge relations, which is a necessary condition for representability.

An *oriented gaussoid* is a map  $\mathcal{A} \rightarrow \{0, \pm 1\}$  with the following property for each edge trinomial: after setting elements in  $\mathcal{P}$  to  $+1$  and elements in  $\mathcal{A}$  to their images, the set of



resulting terms is either  $\{0\}$  or  $\{-1, +1\}$  or  $\{-1, 0, +1\}$ . A *positive gaussoid* is an assignment  $\mathcal{A} \rightarrow \{0, 1\}$  satisfying the same compatibility requirement. For any oriented gaussoid, the inverse image of 0 is a gaussoid  $\mathcal{G}$ . This is analogous to the *chirotope axiom* for oriented matroids [3, § 1.2], which expresses compatibility with the Grassmann–Plücker relations. An oriented gaussoid with  $\mathcal{G} = \emptyset$ , so that  $\mathcal{A} \rightarrow \{\pm 1\}$ , is called a *uniform oriented gaussoid*.

Positroids are oriented matroids all of whose bases are positively oriented. They play an important role in representation theory and algebraic combinatorics, and they have desirable geometric properties. Ardila, Rincón, and Williams proved a longstanding conjecture of Da Silva by showing that all positroids are realizable [1]. In Theorem 5.6 we prove the same fact for gaussoids. Positive gaussoids are important for statistics, because they correspond to the  $\text{MTP}_2$  distributions, which have received a lot of attention in the recent literature [11, 22].

We begin by discussing the enumeration of oriented gaussoids. We start with an ordinary gaussoid  $\mathcal{G}$ . The aim is to list all of its orientations. According to Theorem 2.4, when setting  $\mathcal{P}$  to 1 and  $\mathcal{G}$  to 0 in the edge trinomials, each trinomial either vanishes, stays a trinomial, or becomes a binomial. The resulting nonzero polynomials are the *mutilated edge relations*. They combinatorially constrain the possible orientations. Here is a simple example:

**Example 5.1.** Fix  $n = 4$  and consider the singleton gaussoid  $\mathcal{G} = \{a_{34|2}\}$ . The edge trinomial  $p_{12}a_{34|2} - p_2a_{34|12} - a_{13|2}a_{14|2}$  is mutilated to  $-a_{34|12} - a_{13|2}a_{14|2}$ . This binomial forbids four of the eight possible assignments to its three unknowns. In particular, assigning all  $+$  is forbidden. Hence  $\mathcal{G}$  is not positively orientable. Still,  $\mathcal{G}$  has 576 orientations.  $\diamond$

Enumerating all orientations of a gaussoid  $\mathcal{G}$  can be formulated as an **AIISAT** instance. We use one binary variable  $V_a$  for each element  $a \in \mathcal{A} \setminus \mathcal{G}$ . We set  $V_a = 1$  when  $a \mapsto -1$  and  $V_a = 0$  when  $a \mapsto +1$ . With this convention, the addition  $V_a \oplus V_b$  in the field  $\mathbb{F}_2$  gives the sign of the product  $ab$ . Consider a non-mutilated edge trinomial  $a - b - cd$ , where  $a, b, c, d$  are elements in  $\mathcal{A}$ . Compatibility means: whenever one term is positive, another term must be negative, and vice versa. This translates into the following Boolean formula:

$$(\neg V_a \vee V_b \vee (V_c \oplus V_d)) \Leftrightarrow (V_a \vee \neg V_b \vee \neg(V_c \oplus V_d)). \quad (15)$$

The formula (15) has a fairly short conjunctive normal form (cnf):

$$(V_a \vee \neg V_b \vee V_c \vee \neg V_d) \wedge (V_a \vee \neg V_b \vee \neg V_c \vee V_d) \wedge (\neg V_a \vee V_b \vee V_c \vee V_d) \wedge (\neg V_a \vee V_b \vee \neg V_c \vee \neg V_d).$$

If the trinomial is mutilated, then we omit from (15) all variables which appear no longer.

**Example 5.2.** Consider the cnf above for the empty gaussoid  $\mathcal{G} = \emptyset$  with  $n = 4$ . Applying a **#SAT** solver yields the number 5376 for the uniform oriented gaussoids on  $n = 4$ .  $\diamond$

Here is our main result on the classification of small oriented gaussoids.

**Theorem 5.3.** *For  $n = 3, 4, 5$ , the numbers of oriented gaussoids are as follows:*

$n$	ordinary	oriented	positive	uniform
3	11	51	8	20
4	679	34,873	64	5,376
5	60,212,776	54,936,241,913	1,024	878,349,984

*Proof.* Our count of oriented gaussoids is the result of a #SAT computation. Each variable in  $\mathcal{A}$  can assume three values 0, +1, or -1. We modeled one such ternary variable with two Boolean variables  $V_a^1, V_a^2$  and the surjection  $\eta : \mathbb{F}_2^2 \rightarrow \{0, +, -\}$ , forbidding one configuration of  $(V_a^1, V_a^2)$  so that  $\eta$  becomes a bijection on all allowed configurations. Formula (15) can be adapted to describe all oriented gaussoids. The results are summarized in the table.  $\square$

The symmetries of oriented gaussoids differ in two ways from the symmetries of gaussoids. On the one hand, there are fewer symmetries coming from the groups in (14). The two groups on the right change the signs of the principal minors of  $\Sigma$ , so their action on gaussoids does not lift to oriented gaussoids. Only the action by the permutation group  $S_n$  survives.

On the other hand, certain new symmetries arise, namely those given by *reorientations*. We discussed this point after Remark 3.5. They are analogous to reorientations of oriented matroids [3, § 1.2]. Reorientations act on the signs of the almost-principal minors  $a_{ij|K}$  as follows. If  $\phi : \mathcal{A} \rightarrow \{0, \pm 1\}$  is an oriented gaussoid, and  $L$  is any subset of  $[n]$ , then the *reorientation of  $\phi$  along  $L$*  is the oriented gaussoid  $\phi_L : \mathcal{A} \rightarrow \{0, \pm 1\}$  given by  $\phi_L(a_{ij|K}) = (-1)^{|\{i,j\} \cap L|} \cdot \phi(a_{ij|K})$ . The group of all reorientations has order  $2^{n-1}$  since  $\phi_L = \phi_{[n] \setminus L}$ .

The *symmetry classes of oriented gaussoids* are the orbits of oriented gaussoids under the semidirect product of  $S_n$  and the group of reorientations. That group has order  $2^{n-1}n!$ .

**Example 5.4.** Let  $n = 3$  and consider the  $S_3$ -orbit of gaussoids  $\{\{a_{12}\}, \{a_{13}\}, \{a_{23}\}\}$ . Each gaussoid  $\{a_{ij}\}$  is the support of four oriented gaussoids that are related by reorientation. Altogether, this results in a symmetry class of size  $12 = 3 \times 4$ . We display each of these 12 oriented gaussoids by listing the six signs for  $\mathcal{A}$  in the order  $a_{12}, a_{13}, a_{23}, a_{12|3}, a_{13|2}, a_{23|1}$ :

$$\begin{array}{cccc} 0 - - - - - & 0 - + + - + & 0 + - + + - & 0 + + - + + \\ - 0 - - - - & + 0 - + + - & - 0 + - + + & + 0 + + - + \\ - - 0 - - - & - + 0 - + + & + - 0 + - + & + + 0 + + - \end{array}$$

The first oriented gaussoid  $0 - - - - -$  is realized by the symmetric  $3 \times 3$ -matrix  $\Sigma$  with  $(p_1, p_2, p_3, a_{12}, a_{13}, a_{23}) = (2, 2, 2, 0, -1, -1)$ . Matrices for the other 11 oriented gaussoids in this class are obtained by relabeling and setting  $\Sigma \mapsto D\Sigma D$ , where  $D = \text{diag}(\pm 1, \pm 1, \pm 1)$ .  $\diamond$

**Corollary 5.5.** *For  $n = 3$  there are 51 oriented gaussoids, in 7 symmetry classes. These are all realizable. Among them are 20 uniform oriented gaussoids, in 3 symmetry classes. Among these 20, there are 8 positive gaussoids.*

The following table exhibits the seven classes. The first column gives a positive definite symmetric  $3 \times 3$ -matrix  $\Sigma$  that realizes the first oriented gaussoid in that symmetry class.

$(p_1, p_2, p_3, a_{12}, a_{13}, a_{23})$	Symmetry class of oriented gaussoids	Size
$(2, 2, 2, 1, 1, 1)$	$+++++, +---+--, --+---+, -+---+-$	4
$(3, 5, 1, 1, 1, 2)$	$+++ - ++, +-----, --+ + - +, \dots, --+ ----$	12
$(6, 9, 6, -1, -1, -7)$	$-----, +-+ + + -, -+ + - + +, +-+ + - +$	4
$(4, 3, 3, 2, 2, 1)$	$+++++0, ++++0+, ++++0+, \dots, --+ -- 0$	12
$(2, 2, 2, 0, -1, -1)$	$0-----, 0-+ + - +, \dots$ (Example 5.4)	12
$(3, 2, 2, 0, 0, 1)$	$00+00+, 00-00-, -00-00, +00+00, 0-00-0, 0+00+0$	6
$(1, 1, 1, 0, 0, 0)$	$000000$	1

See Theorem 8.1 for the classification in the  $n = 4$  case.

We now shift gears and focus on positive gaussoids. In analogy to the situation for positroids [1], all positive gaussoids are realizable and their realization spaces are very nice.

Let  $\Gamma = ([n], E)$  be an undirected simple graph with vertex set  $[n] = \{1, 2, \dots, n\}$ . This defines a gaussoid  $\mathcal{G}_\Gamma$  by taking all the conditional independence statements that hold for the graphical model  $\Gamma$ . To be precise, an unknown  $a_{ij|K}$  lies in  $\mathcal{G}_\Gamma$  if and only if every path from vertex  $i$  to vertex  $j$  in  $\Gamma$  uses at least one of the vertices in  $K$ . Thus  $a_{ij} \in \mathcal{G}_\Gamma$  when  $i$  and  $j$  are in separate connected components of  $\Gamma$ , and  $a_{ij|[n]\setminus\{i,j\}} \in \mathcal{G}_\Gamma$  when  $\{i, j\} \notin E$ .

**Theorem 5.6.** *Fix a positive integer  $n$ . There are exactly  $2^{\binom{n}{2}}$  positive gaussoids  $\mathcal{G}_\Gamma$ , one for each of the graphs  $\Gamma = ([n], E)$ . These gaussoids are all realizable. The space of covariance matrices  $\Sigma$  that realize  $\mathcal{G}_\Gamma$  is homeomorphic to an open ball of dimension  $|E| + n$ .*

*Proof.* We first show that  $\mathcal{G}_\Gamma$  supports a positive gaussoid. Our argument follows [11, Proposition 6.3]. Let  $A = (a_{ij})$  be the adjacency matrix of  $\Gamma$ , with  $a_{ij} = 1$  if  $\{i, j\} \in E$  and  $a_{ij} = 0$  otherwise. Take  $\Sigma = (t \cdot \text{Id}_n - A)^{-1}$  for sufficiently large  $t > 0$ . Then  $\Sigma$  is positive definite and all its almost-principal minors are nonnegative. Indeed,  $\Sigma^{-1}$  is an  $M$ -matrix, i.e. it is a positive definite matrix whose off-diagonal entries are nonpositive. By [19, Theorem 2], all partial correlations of the associated Gaussian distribution are nonnegative. Following [22], this is precisely what it means for a distribution to be  $\text{MTP}_2$ . Hence, all  $a_{ij|K}$  are nonnegative for our matrix  $\Sigma$ . Moreover, by [11, Theorem 6.1], the distribution given by  $\Sigma$  is faithful to the graph  $\Gamma$ , i.e. a principal minor  $a_{ij|K}$  is zero if and only if it lies in  $\mathcal{G}_\Gamma$ .

Using the same line of reasoning, we can show that the realization space of  $\mathcal{G}_\Gamma$  is homeomorphic to an open ball of dimension  $|E| + n$ . Indeed, if  $\Sigma$  is any covariance matrix with gaussoid  $\mathcal{G}_\Gamma$ , then  $\Sigma^{-1}$  is an  $M$ -matrix with support  $\Gamma$ . The set of all such matrices is a (relatively open) convex cone of dimension  $|E| + n$ . It is the face indexed by  $\Gamma$  of the cone of all  $M$ -matrices. That cone is denoted by  $\mathcal{M}^p$  in [22, § 2]. It has dimension  $\binom{n}{2} + n$ , and it is open in the ambient space of symmetric matrices. Note that the cone  $\mathcal{M}^p$  is the realization space of the strictly positive gaussoid  $++++ \cdots +$ , for the complete graph  $\Gamma = K_n$ .

Matrix inversion defines a homeomorphism from the aforementioned relatively open face of  $\mathcal{M}^p$  onto a subset of  $\mathbb{R}^{\mathcal{P} \cup \mathcal{A}}$  that is topologically a ball of dimension  $|E| + n$ . Its image in the positive part of the variety  $V(J_n)$  is a semialgebraic stratum of dimension  $|E| + n$ .

To complete the proof, let us now assume that  $\mathcal{G}$  is an arbitrary positive gaussoid. A priori we do not know that  $\mathcal{G}$  is realizable. We must prove that  $\mathcal{G}$  equals  $\mathcal{G}_\Gamma$  for some graph  $\Gamma = ([n], E)$ . By [29, Theorem 1], it suffices to check that  $\mathcal{G}$  is a singleton-transitive compositional graphoid. Equivalently, by [29, Corollary 7], we must verify that the edge trinomials imply the three axioms *singleton-transitivity*, *intersection*, and *upward-stability*. Singleton-transitivity is equivalent to the gaussoid axiom (G4). Intersection is (G2) and thus these two axioms hold for  $\mathcal{G}$ . Upward stability says that  $a_{ij|L} \in \mathcal{G}$  implies  $a_{ij|kL} \in \mathcal{G}$ . This follows from the trinomial  $a_{ij|L}p_{kL} - a_{ik|L}a_{jk|L} - a_{ij|kL}p_L$  in  $J_n$ . Indeed, since  $p_{kL}$  and  $p_L$  are positive and the middle product is nonnegative, we see that  $a_{ij|L} = 0$  implies  $a_{ij|kL} = 0$ .  $\square$

**Remark 5.7.** The oriented gaussoids that result from positive ones by reorientation correspond to *signed  $\text{MTP}_2$  distributions*. We refer to [22, § 5] and the references given there.

## 6 Quadratic Relations

In this section we return to the ideal  $J_n$  of relations among principal and almost-principal minors, and we derive a conjectural characterization of its minimal generators. We begin by discussing the extent to which the trinomials suffice to generate. Let  $T_n$  denote the ideal in  $\mathbb{R}[\mathcal{P} \cup \mathcal{A}]$  that is generated by all square trinomials (2) and all edge trinomials (3).

**Example 6.1.** The ideal  $T_3$  is generated by  $18 = 6 + 12$  quadratic trinomials, displayed in Example 3.3. It is radical and its prime decomposition has five components:

$$T_3 = J_3 \cap P_{\emptyset,123} \cap P_{1,23} \cap P_{2,13} \cap P_{3,12}.$$

Each associated prime  $P_{I,J}$  is generated by 12 of the 14 unknowns in  $\mathcal{P} \cup \mathcal{A}$ . The two unknowns not in  $P_{I,J}$  are  $p_I$  and  $p_J$ . The variety  $V(P_{I,J})$  is a coordinate line  $\mathbb{P}^1$  in  $\mathbb{P}^{13}$ .  $\diamond$

At present we do not know the primary decomposition of  $T_n$  for  $n \geq 4$ . However, we next show that  $T_n$  becomes equal to the prime ideal  $J_n$  after inverting all unknowns in  $\mathcal{P}$ .

**Proposition 6.2.** *The ideal  $J_n$  is an associated prime of the trinomial ideal  $T_n$ . Every other associated prime of  $T_n$  contains at least one of the  $2^n$  unknowns  $p_I \in \mathcal{P}$ .*

*Proof.* Let  $R = \mathbb{R}[\mathcal{A} \cup \mathcal{P}^{\pm 1}]$  denote the partial Laurent polynomial ring obtained from  $\mathbb{R}[\mathcal{A} \cup \mathcal{P}]$  by adjoining  $p_I^{-1}$  for all  $I \subseteq [n]$ . Consider the ideal  $T_n R$  in  $R$ . Modulo this ideal,

$$p_{i j K} = p_{i K} p_{j K} p_K^{-1} - a_{i j | K}^2 p_K^{-1} \quad \text{and} \quad a_{i j | k L} = p_{k L} a_{i j | L} p_L^{-1} - a_{i k | L} a_{j k | L} p_L^{-1}.$$

These relations express each principal or almost-principal minor of size  $\geq 2$  as a Laurent polynomial in the entries of the symmetric matrix  $\Sigma$ . This shows that  $R/T_n R$  is isomorphic to a partial Laurent polynomial ring in  $\binom{n+1}{2} + 1$  unknowns. The same reduction argument works for the ideal  $J_n$ . In symbols, we have the following isomorphism of  $\mathbb{R}$ -algebras:

$$R/T_n R \simeq R/J_n R \simeq \mathbb{R}[p^{\pm 1}, p_1^{\pm 1}, p_2^{\pm 1}, \dots, p_n^{\pm 1}, a_{12}, a_{13}, \dots, a_{n-1, n}].$$

We conclude that  $T_n R$  equals the prime ideal  $J_n R$  in  $R$ , and this proves the assertion.  $\square$

In Theorem 6.4 we describe all quadrics in the ideal  $J_n$ , which we also conjecture to generate  $J_n$ . The proof of Theorem 6.4 appears after Example 6.6.

**Conjecture 6.3.** *The ideal  $J_n$  is generated by its quadrics, listed explicitly in Theorem 6.4.*

$n$	2	3	4	5	6	7	8
# quadrics in $J_n$	1	21	226	1810	12261	74613	421716

Table 1: The number of quadratic generators of  $J_n$ .

**Theorem 6.4.** *The space of all quadrics in the ideal  $J_n$  is a  $G$ -module of dimension*

$$\dim((J_n)_2) = 3^{n-2} \binom{n}{2} + 4 \sum_{m=3}^n 3^{n-m} \binom{n}{m} \binom{m}{2} + \sum_{k=2}^{\lfloor \frac{n}{2} \rfloor} 2k \cdot 3^{n-2k} \binom{n}{2k}. \quad (16)$$

*The following four classes of quadrics and their images under  $S_n$  are the highest weight vectors for the distinct irreducible representations occurring in the  $G$ -module  $(J_n)_2$ :*

$$p_{12}p - p_1p_2 + a_{12}^2 \quad (i)$$

$$\sum_{L \subseteq [m] \setminus \{1,2\}} (-1)^{|L|} p_L a_{12|L^c} + \sum_{j=3}^m \sum_{K \subseteq [m] \setminus \{1,2,j\}} (-1)^{|K|} a_{1j|K} a_{2j|K^c} \quad \text{for } 3 \leq m \leq n, m \text{ odd} \quad (ii)$$

$$\sum_{j=3}^m \sum_{K \subseteq [m] \setminus \{1,2,j\}} (-1)^{|K|} a_{1j|K} a_{2j|K^c} \quad \text{for } 4 \leq m \leq n, m \text{ even} \quad (iii)$$

$$\sum_{\substack{(L,L') \text{ partition} \\ \text{of } [m]}} (-1)^{|L|} p_L p_{L'} + 2 \cdot \sum_{j=2}^m \sum_{\substack{(K,K') \text{ partition} \\ \text{of } [m] \setminus \{1,j\}}} (-1)^{|K|} a_{1j|K} a_{1j|K'} \quad \text{for } 3 < m \leq n, m \text{ even.} \quad (iv)$$

To find these quadrics, we used the Lie algebra  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})^{\oplus n}$  of the group  $G = \mathrm{SL}_2(\mathbb{R})^n$ . As  $n$  increases, so does the number of quadrics. However, just a small fraction is new: most come from earlier ones via lowering operators in  $\mathfrak{g}$ . These are described in [27, Remark III.16].

The  $k$ -th *lowering operator*  $\ell_k$  is the following endomorphism of  $W = W_{\mathrm{pr}} \oplus \bigoplus_{i,j} W_{\mathrm{ap}}^{ij}$ :

$$p_L \mapsto \begin{cases} p_{L \cup \{k\}} & \text{if } k \notin L \\ 0 & \text{otherwise} \end{cases} \quad a_{ij|L} \mapsto \begin{cases} a_{ij|L \cup \{k\}} & \text{if } k \notin L \cup \{i, j\} \\ 0 & \text{otherwise} \end{cases}$$

Similarly, the  $k$ -th *raising operator*  $r_k$  acts on  $W$  as follows:

$$p_L \mapsto \begin{cases} p_{L \setminus \{k\}} & \text{if } k \in L \\ 0 & \text{otherwise} \end{cases} \quad a_{ij|L} \mapsto \begin{cases} a_{ij|L \setminus \{k\}} & \text{if } k \in L \\ 0 & \text{otherwise} \end{cases}$$

These operators are extended to  $\mathrm{Sym}_2(W)$  by the Leibniz rule [13, §8.1]:

$$\ell_k(vw) = \ell_k(v) \cdot w + v \cdot \ell_k(w) \quad \text{and} \quad r_k(vw) = r_k(v) \cdot w + v \cdot r_k(w).$$

These endomorphisms of  $\mathrm{Sym}_2(W)$  restrict to the  $G$ -submodule  $(J_n)_2$ .

**Remark 6.5.** A polynomial of weight  $\omega$  is a highest (respectively, lowest) weight vector precisely when it lies in the kernel of all possible raising (respectively, lowering) operators.

**Example 6.6.** Consider the square trinomial  $p_{12}p - p_1p_2 + a_{12}^2$  in (i). If  $n = 2$ , then it has weight 00. For  $n > 2$  the weight is 0022...2. The quadric (i) is a highest weight vector since it is annihilated by the raising operators  $r_k$ . It generates the  $3^{n-2}$ -dimensional  $G$ -module  $S_{002\dots 2}$  inside  $(J_n)_2$ . To get an  $\mathbb{R}$ -basis of this  $G$ -module, we apply all lowering operators, which yields  $3^{n-2}$  quadrics. For instance, if  $n = 4$  then lowering via  $\ell_3$  yields  $p_3p_{12} + pp_{123} - p_{13}p_2 - p_1p_{23} + 2a_{12}a_{12|3}$ . Taking into account all  $\binom{n}{2}$  permutations of 002...2, we find  $3^{n-2} \binom{n}{2}$  quadrics originating from (i). This explains the first summand in (16).  $\diamond$

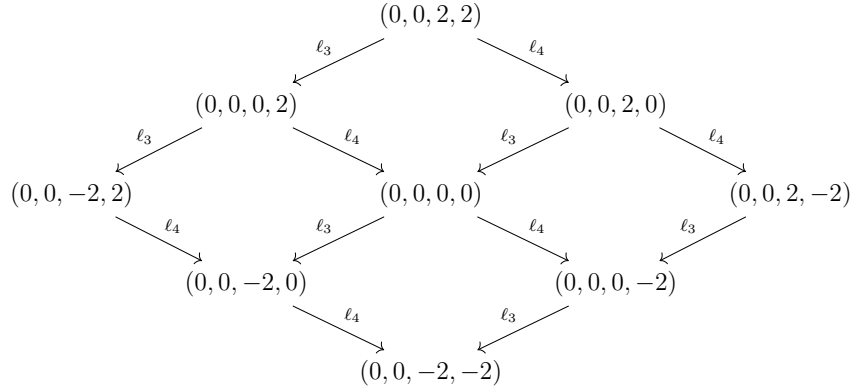


Figure 2: A visualization of the  $G$ -module  $S_{0022}$  inside  $(J_4)_2$ . Arrows pointing down and left represent lowerings via  $\ell_3$ , while arrows pointing down and right represent lowerings via  $\ell_4$ .

*Proof of Theorem 6.4.* The proposed quadrics lie in the kernel of the raising operators and hence are highest weight vectors by Remark 6.5. The count in (16) is explained by working through the action of the lowering operators on (i)-(iv). Each of the  $\binom{n}{2}$  quadrics in (i) contributes  $3^{n-2}$  quadrics to  $J_n$ . For fixed  $m$ , each of the  $\binom{n}{m}\binom{m}{2}$  quadrics of types (ii) and (iii) contributes  $4 \cdot 3^{n-m}$  quadrics. Similarly, for fixed  $m$ , each of the  $m\binom{n}{m}$  highest weight quadrics of type (iv) gives rise to  $3^{n-m}$  linearly independent quadrics in  $J_n$ .

We next prove that quadrics of type (iv) lie in  $J_n$ , i.e. they map to zero in  $\mathbb{R}[\Sigma] = \mathbb{R}[\sigma_{11}, \sigma_{12}, \dots, \sigma_{nn}]$ . The proofs for (ii) and (iii) are similar, but simpler. Without loss of generality we assume that  $m = n$  (and hence  $n$  is even). Any monomial in the quadrics (iv) maps to some monomial of  $\det \Sigma$ . Specifically,  $p_L p_{L'}$  and each  $a_{1j|K} a_{1j|K'}$  map to monomials  $\sigma_\pi := \sigma_{1,\pi(1)} \sigma_{2,\pi(2)} \cdots \sigma_{n,\pi(n)}$  where  $\pi \in S_n$ . To show this for the  $a$  monomials, it is helpful to arrange the rows of  $\Sigma$  as  $(1, K, j, K')$  and the columns as  $(j, K, 1, K')$ . With this arrangement,

$$\Sigma = \begin{pmatrix} \Sigma_{1K \times jK} & \Sigma_{1K \times 1K'} \\ \Sigma_{jK' \times jK} & \Sigma_{jK' \times 1K'} \end{pmatrix} \quad (17)$$

Then  $a_{1j|K} a_{1j|K'}$  maps to a monomial in the expansion of  $\det \Sigma_{1K \times jK} \cdot \det \Sigma_{jK' \times 1K'}$ .

Fix a monomial  $\sigma_\pi$  and let  $C_1 C_2 \cdots C_s$  be the cycle decomposition of  $\pi$ . Assume  $1 \in C_1$ . Let  $r$  be the number of cycles of length  $\geq 3$ . Since  $\Sigma$  is symmetric, replacing a cycle  $C_i$  in  $\pi$  by its inverse does not change  $\sigma_\pi$ . The monomial  $\sigma_\pi$  appears in the image of  $p_L p_{L'}$  whenever  $(C_1, \dots, C_s)$  refines the partition  $(L, L')$ , and it appears with the same sign as in  $\det \Sigma$ . The monomial  $\sigma_\pi$  appears in the image of  $a_{1j|K} a_{1j|K'}$  only if  $C_1$  contains  $j$ , as seen from (17). Additionally, each of  $C_2, C_3, \dots, C_s$  must be contained in either  $K$  or  $K'$ . Finally, if  $C_1 = (1, i_2, \dots, i_l, j, i_{l+2}, \dots, i_t)$ , then  $\{i_2, \dots, i_l\} \subseteq K$  and  $\{i_{l+2}, \dots, i_t\} \subseteq K'$ . These three properties together characterize the monomials  $\sigma_\pi$  that appear in the image of (iv). According to our sign convention for the  $a_{ij|K}$ , the product  $a_{1j|K} a_{1j|K'}$  has a global minus with respect to  $\det \Sigma$ , again visible from (17) as the columns 1 and  $j$  have been exchanged.

Assume first that  $\pi$  has an even cycle  $\tilde{C}$ . Since  $n$  is even, there is another even cycle in

$\pi$ , and we can assume  $1 \notin \tilde{C}$ . Consider the following matching of terms of (iv):

$$\begin{aligned} & (p_L p_{L'}, p_{L \cup \tilde{C}} p_{L' \setminus \tilde{C}}) \text{ if } \tilde{C} \not\subseteq L, \quad (p_L p_{L'}, p_{L \setminus \tilde{C}} p_{L' \cup \tilde{C}}) \text{ otherwise,} \\ & (a_{1j|K} a_{1j|K'}, a_{1j|K \cup \tilde{C}} a_{1j|K' \setminus \tilde{C}}) \text{ if } \tilde{C} \not\subseteq K, \quad (a_{1j|K} a_{1j|K'}, a_{1j|K \setminus \tilde{C}} a_{1j|K' \cup \tilde{C}}) \text{ otherwise.} \end{aligned}$$

Since  $\tilde{C}$  has odd cardinality, the matched terms differ in signs and cancel in the image.

Let now  $\pi$  be a product of odd cycles and denote  $C_1 = \{1, i_2, i_3, \dots, i_{2w}\}$ . We can again produce cancellations by matching the following terms for any  $1 < u \leq w$ :

$$(a_{1i_{2u-1}|K} a_{1i_{2u-1}|K'}, a_{1i_{2u}|K \cup \{i_{2u-1}\}} a_{1i_{2u}|K' \setminus \{i_{2u-1}\}}).$$

After subtraction of the matched terms, the remaining terms are of the form  $a_{1i_2|K} a_{1i_2|K'}$ . We now count the occurrences of  $\sigma_\pi$  in the images of these. For the  $p$ -part, there are  $2^{s-1}$  partitions  $(L, L')$  of  $[n]$  that coarsen the cycles of  $\pi$ . For each of these, there are  $2^r$  copies of  $\sigma_\pi$  because there are  $r$  cycles with cardinality  $\geq 3$ . In total, the coefficient of  $\sigma_\pi$  in the image of  $\sum_{(L, L')} p_L p_{L'}$  is  $2^{s-1+r}$ . For the  $a$ -part, we distinguish two cases. First, if  $C_1$  is a transposition, there are  $2^{s-2}$  partitions  $(K, K')$  that coarsen  $(C_2, \dots, C_s)$ . Again,  $\sigma_\pi$  appears  $2^r$  times from reorientations of cycles of length  $\geq 3$ . Thus the total count is  $2^{(s-2)+r} = 2^{s+r-2}$ . Now, if  $C_1$  is not a transposition, then there are  $r-1$  cycles in  $C_2, \dots, C_s$  with cardinality  $\geq 3$ . Then  $2^{r-1}$  copies of  $\sigma_\pi$  appear for each of the  $2^{s-1}$  coarsenings. Again, the total count is  $2^{(r-1)+(s-1)} = 2^{s+r-2}$ . In (iv), the coefficient 2 corrects the count, and the sign of the monomials in the  $a$  terms has a global minus relative to the determinant of  $\Sigma$ .

We now show that the  $G$ -module  $(J_n)_2$  is spanned by the quadrics (i)-(iv). Let  $f$  be any quadric in  $J_n$ . We can assume that  $f \notin J_m$  for  $m < n$  and  $f$  is a highest weight vector. A priori the weight of  $f$  is an element of  $\{-2, -1, 0, 1, 2\}^n$ . We claim that it is in  $\{0, 1\}^n$ . To see this, assume first that  $-2$  or  $-1$  appears in the weight. In this case, raising  $f$  at the corresponding index yields a nonzero quadric of higher weight. Moreover, an entry 2 can only appear if the corresponding index appears in no variable of  $f$  and thus  $f \in J_{n-1}$ .

Given the weights of  $a_{ij|K}$  and  $p_L$ , the only possible weights for  $f$  (up to permutation) are 111100...0, 1100...0, and 00...0. The following are general quadrics for these weights:

$$\begin{aligned} 111100\dots 0: & \sum_{K \subseteq [n] \setminus \{1,2,3,4\}} d_K \cdot a_{12|K} a_{34|K^c} \\ 1100\dots 0: & \sum_{L \subseteq [n] \setminus \{1,2\}} c_L \cdot p_L a_{12|L^c} + \sum_{j=3}^n \sum_{K \subseteq [n] \setminus \{1,2,j\}} d_K^{(j)} \cdot a_{1j|K} a_{2j|K^c} \\ 00\dots 0: & \sum_{\substack{(L, L') \text{ partition} \\ \text{of } [n]}} c_L \cdot p_L p_{L'} + \sum_{\substack{i, j \in [n] \\ i < j}} \sum_{\substack{(K, K') \text{ partition} \\ \text{of } [n] \setminus \{i, j\}}} d_K^{(ij)} \cdot a_{ij|K} a_{ij|K'}. \end{aligned}$$

That  $f$  lies in the kernel of all raising operators imposes conditions on the coefficients  $c, d$ . In particular, all coefficients in an inner sum (like  $c_L$  or  $d_K^{(j)}$  for a fixed  $j$ ) differ by at most a sign, in an alternating fashion. More precisely, for each  $i \in L$  one has  $c_{L \setminus \{i\}} + c_L = 0$  and hence, inductively,  $c_L = (-1)^{|L|} c_\emptyset$ . This implies that  $c_\emptyset = c_{[n]} = (-1)^n c_\emptyset$ , and a similar



statement holds for each  $d_\emptyset^{(ij)}$ . We conclude that, when  $n$  is odd, there can be no quadric of weight  $00\dots 0$  that lies in  $J_n$  and satisfies our hypotheses.

By the first part of the proof, the quadrics of types (i)-(iv) do arise. It is therefore enough to prove that there are no further linearly independent quadrics in each weight. To do this, we look at the image of  $f$  in  $\mathbb{R}[\Sigma]$  and there at the coefficients of certain monomials:

- $111100\dots 0$ : The monomial  $\sigma_{1,2}\sigma_{3,5}(\prod_{i=5}^{n-1} \sigma_{i,i+1})\sigma_{n,4}$  is among the terms in the image of  $a_{12}a_{34}|5\dots n$  only and thus  $d_\emptyset = 0$ . Hence no quadrics arise.
- $1100\dots 0$ ,  $n$  odd: Similarly,  $\sigma_{1,3}\sigma_{2,3}(\prod_{i=4}^{n-1} \sigma_{i,i+1})\sigma_{n,4}$  yields  $c_\emptyset = d_\emptyset^{(3)}$ . Permuting suitably we find  $c_\emptyset = d_\emptyset^{(j)}$  for each  $j$ , and hence there is at most one quadric of this weight.
- $1100\dots 0$ ,  $n$  even:  $\sigma_{1,2}(\prod_{i=3}^{n-1} \sigma_{i,i+1})\sigma_{n,3}$  and  $\sigma_{1,3}\sigma_{3,4}\sigma_{4,2}(\prod_{i=5}^{n-1} \sigma_{i,i+1})\sigma_{n,5}$  give that  $c_\emptyset = 0$  and  $d_\emptyset^{(3)} = d_\emptyset^{(4)}$ ; permuting the indices suitably we get that  $d_\emptyset^{(i)} = d_\emptyset^{(j)}$  for all  $i, j$  and hence there is at most one quadric with this weight.
- $00\dots 0$ ,  $n \geq 4$  even: When  $n \geq 4$ , the preimage of the monomial  $\prod_{i=1}^{n/2} \sigma_{2i-1,2i}^2$  gives that  $2c_\emptyset = d_\emptyset^{(12)} + d_\emptyset^{(34)} + \dots + d_\emptyset^{(n-1,n)}$ . From this relation and its permutations one derives that  $d_\emptyset^{(ij)} + d_\emptyset^{(kl)} = d_\emptyset^{(ik)} + d_\emptyset^{(jl)}$  for any four distinct indices  $i, j, k, l$ . Consequently, all coefficients in  $f$  can be expressed in terms of  $d_\emptyset^{(1j)}$  (where  $j$  ranges from 2 to  $n$ ) and  $d_\emptyset^{(23)}$ . Thus, the dimension of the associated vector subspace of  $(J_n)_2$  is at most  $n$ .  $\square$

## 7 Tropical Geometry

In recent years, the theory of matroids has been linked tightly to the emerging field of tropical geometry [2, 24]. Every matroid defines a tropical linear space, and conversely, every tropical linear space corresponds to a *valuated matroid*. First introduced by Dress and Wenzel [7, 8] as a generalization of matroids, valuated matroids are now best understood as vectors of tropical Plücker coordinates. For a textbook introduction to this topic see [24, Chapter 4].

Tropical geometry is a combinatorial shadow of algebraic geometry over a field with valuation. The field of real Puiseux series,  $\mathbb{R}\{\{\epsilon\}\}$ , is our primary example. This field is ordered and it contains the rational functions  $\mathbb{R}(\epsilon)$ . The unknown  $\epsilon$  is positive but smaller than any positive real number. Covariance matrices with entries that contain  $\epsilon$  can be found in the realizations of gaussoids by Lněnička and Matúš [23, Table 1]. Indeed, statisticians frequently consider Gaussian distributions that depend on a perturbation parameter  $\epsilon$ . The development in this section represents a systematic approach to the analysis of such distributions.

A *valuated gaussoid* on  $[n]$  is a map  $\nu : \mathcal{P} \cup \mathcal{A} \rightarrow \mathbb{R}$  such that the minimum of  $\nu(m_1), \nu(m_2), \nu(m_3)$  is attained at least twice for every quadratic trinomial  $m_1 + m_2 + m_3$  in  $T_n$ . Here  $\nu(m_i)$  is the sum of the values of  $\nu$  on the two terms in  $m_i$ . In other words, a valuated gaussoid is a point  $\nu$  in the tropical prevariety defined by the trinomials (2) and (3). Recall that  $V(J_n) = V(T_n)$  in the torus by Proposition 6.2. Every point  $\nu$  in the tropical variety  $\text{trop}(V(J_n)) = \text{trop}(V(T_n))$  is a *realizable* valuated gaussoid. The distinction between valuated gaussoids and those that are realizable mirrors the distinction in [24, § 4.4] between

tropical linear spaces and tropicalized linear spaces. The former are parametrized by the Dressian whereas the latter are parametrized by the tropical Grassmannian. This is the distinction between the tropical prevariety and tropical variety defined by our trinomials.

In Section 8 we encounter non-realizable valuated gaussoids for  $n \geq 5$ . Here we focus on the case  $n = 3$ . The variety  $V(J_3)$  equals the Lagrangian Grassmannian  $\text{LGr}(3, 6) \subset \mathbb{P}^{13}$ . That 6-dimensional variety has a 3-dimensional torus action. Modulo lineality, the tropical variety  $\text{trop}(\text{LGr}(3, 6))$  is a 3-dimensional fan, hence a 2-dimensional polyhedral complex.

Recall that  $\text{LGr}(3, 6)$  is a linear section of the classical Grassmannian  $\text{Gr}(3, 6) \subset \mathbb{P}^{19}$ . That 9-dimensional variety has a 5-dimensional torus action. Modulo its lineality space, the *tropical Grassmannian*  $\text{trop}(\text{Gr}(3, 6))$  is a 4-dimensional fan, hence a 3-dimensional polyhedral complex. It is glued from 990 tetrahedra and 15 bipyramids [24, Example 4.3.15]. This complex is well-known to tropical geometers. A detailed description is found in [24, § 5.4].

The following is our main result in this section. In the course of proving it, we also describe the inclusion of  $\text{trop}(\text{LGr}(3, 6))$  inside  $\text{trop}(\text{Gr}(3, 6))$ , and we compute *Khovanskii bases* and *Newton–Okounkov bodies* as in [20]. In Corollary 7.3 we connect to statistics by explaining the  $\text{MTP}_2$  distributions encoded in the *positive tropical variety*  $\text{trop}_+(\text{LGr}(3, 6))$ .

**Theorem 7.1.** *For  $n = 3$ , all valuated gaussoid are realizable, so they are precisely the points in the tropical Lagrangian Grassmannian  $\text{trop}(\text{LGr}(3, 6))$ . The underlying 2-dimensional polyhedral complex has 35 vertices, 151 edges, and 153 facets. The facets come in nine symmetry classes: there are  $12 + 8 + 48 + 24 + 6 + 24 + 24 + 1$  triangles and 6 quadrilaterals. Seven of the nine facet classes represent prime cones in the sense of Kaveh–Manon [20, § 5].*

*Proof and Explanation.* These results are obtained by computation. The tropical variety of  $J_3$  is a pure 7-dimensional fan in  $\mathbb{R}^{14}$  whose lineality space  $L$  has dimension 4. One dimension comes from the usual grading, since  $J_3$  is a homogeneous ideal. The others come from the maximal torus of  $G = \text{SL}_2(\mathbb{R})^3$ . Hence  $\text{trop}(V(J_3)) = \text{trop}(\text{LGr}(3, 6))$  is a pure 3-dimensional fan in  $\mathbb{R}^{14}/L$ . The coordinates on  $\mathbb{R}^{14}$  are dual to a distinguished spanning set of  $\mathbb{R}^{14}/L$ :

$$(a_{12}, a_{12|3}, a_{13}, a_{13|2}, a_{23}, a_{23|1}, p, p_1, p_2, p_3, p_{12}, p_{123}, p_{13}, p_2, p_{23}, p_3). \quad (18)$$

With this ordering of the 14 generators, the lineality space of  $\text{trop}(J_3)$  equals

$$L = \text{rowspace} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 \end{pmatrix}. \quad (19)$$

We use the symbols in (18) to denote the corresponding spanning vectors of  $\mathbb{R}^{14}/L \simeq \mathbb{R}^{10}$ .

The  $35 = 6 + 8 + 3 + 12 + 6$  rays of the fan  $\text{trop}(V(J_3))$  come in five symmetry classes:

- 6 of type **a**:  $\{a_{12}, a_{13}, a_{23}, a_{12|3}, a_{13|2}, a_{23|1}\}$
- 8 of type **p**:  $\{p, p_1, p_2, p_3, p_{12}, p_{13}, p_{23}, p_{123}\}$
- 3 of type **A**:  $\{a_{12} + a_{12|3} + a_{13} + a_{13|2}, a_{12} + a_{12|3} + a_{23} + a_{23|1}, a_{13} + a_{13|2} + a_{23} + a_{23|1}\}$

- 12 of type **B**:  $\{ a_{12}+a_{12|3}+a_{13}+a_{13|2}+2p+2p_1, a_{12}+a_{12|3}+a_{23}+a_{23|1}+2p+2p_2, a_{13}+a_{13|2}+a_{23}+a_{23|1}+2p+2p_3, a_{12}+a_{12|3}+a_{23}+a_{23|1}+2p_1+2p_{12}, a_{13}+a_{13|2}+a_{23}+a_{23|1}+2p_1+2p_{13}, a_{12}+a_{12|3}+a_{13}+a_{13|2}+2p_2+2p_{12}, a_{13}+a_{13|2}+a_{23}+a_{23|1}+2p_2+2p_{23}, a_{12}+a_{12|3}+a_{13}+a_{13|2}+2p_3+2p_{13}, a_{12}+a_{12|3}+a_{23}+a_{23|1}+2p_3+2p_{23}, a_{13}+a_{13|2}+a_{23}+a_{23|1}+2p_{12}+2p_{123}, a_{12}+a_{12|3}+a_{23}+a_{23|1}+2p_{13}+2p_{123}, a_{12}+a_{12|3}+a_{13}+a_{13|2}+2p_{23}+2p_{123} \}$
- 6 of type **C**:  $\{ a_{12}+a_{13|2}+2p_2+2p_{12}, a_{23}+a_{12|3}+2p_3+2p_{23}, a_{23}+a_{13|2}+2p_2+2p_{23}, a_{13|2}+a_{23|1}+2p_{12}+2p_{123}, a_{12|3}+a_{13|2}+2p_{23}+2p_{123}, a_{12|3}+a_{23|1}+2p_{13}+2p_{123} \}$

Each of the sums in the lists above is a vector in  $\mathbb{R}^{14}/L$ . For instance, the last sum in type **C** represents the vector  $(0, 1, 0, 0, 0, 1, 0, 0, 0, 2, 2, 0, 0, 0) + L$  if we use the ordering in (18).

The tropical Lagrangian Grassmannian  $\text{trop}(V(J_3))$  is the intersection of the tropical Grassmannian  $\text{trop}(\text{Gr}(3, 6))$  with a linear space. This intersection is computed in the 20 Plücker coordinates with the `Macaulay2` code in Example 3.3. We shall use the identification of the 20 Plücker coordinates with the 14 principal and almost-principal minors given in (11).

The tropical variety  $\text{trop}(V(J_3))$  has a unique coarsest fan structure with 153 facets. These come in 9 orbits under the symmetries of the 3-cube. In what follows we list these orbits. Each facet in eight of the orbits lies in a unique facet of  $\text{trop}(\text{Gr}(3, 6))$ . We name that facet in the notation of [24, § 5.4]. Facets of type **ppp** lie in triangles of  $\text{trop}(\text{Gr}(3, 6))$ . Here is now the list of all  $153 = 12 + 8 + 48 + 24 + 6 + 24 + 24 + 1 + 6$  facets of  $\text{trop}(V(J_3))$ :

- 12 triangles of type **app**, like  $\{ a_{12}, p_3, p_{123} \}$ . They lie in tetrahedra EEEE.
- 8 triangles of type **ppp**, like  $\{ p_1, p_2, p_3 \}$ . They lie in triangles EEE.
- 48 triangles of type **apB**, like  $\{ a_{12|3}, p, a_{12}+a_{12|3}+a_{13}+a_{13|2}+2p+2p_1 \}$ . They lie in tetrahedra EEG of the tropical Grassmannian  $\text{trop}(\text{Gr}(3, 6))$ .
- 24 triangles of type **ppC**, like  $\{ p, p_{12}, a_{13|2}+a_{23|1}+2p_{12}+2p_{123} \}$ . Twelve lie in tetrahedra EEFfa, and others lie in tetrahedra EEFfb.
- 6 triangles of type **aAA**, like  $\{ a_{12}, a_{12}+a_{12|3}+a_{13}+a_{13|2}, a_{12}+a_{12|3}+a_{23}+a_{23|1} \}$ . They lie in tetrahedra EEFfb.
- 24 triangles of type **aAB**, like  $\{ a_{12}, a_{12}+a_{12|3}+a_{13}+a_{13|2}, a_{12}+a_{12|3}+a_{13}+a_{13|2}+2(p_{23}+p_{123}) \}$ . They lie in tetrahedra EEEFG.
- 24 triangles of type **pBC**:  $\{ p, a_{12}+a_{12|3}+a_{13}+a_{13|2}+2(p+p_1), a_{12|3}+a_{13|2}+2(p_{23}+p_{123}) \}$ . They lie in tetrahedra EEEFG.
- 1 triangle of type **AAA**:  $\{ a_{12}+a_{12|3}+a_{13}+a_{13|2}, a_{12}+a_{12|3}+a_{23}+a_{23|1}, a_{13}+a_{13|2}+a_{23}+a_{23|1} \}$ . This triangle lies in a bipyramid FFFGG.
- 6 squares of type **ABCB**, like  $\{ a_{13}+a_{13|2}+a_{23}+a_{23|1}+2(p_1+p_{13}), a_{13}+a_{13|2}+a_{23}+a_{23|1}, a_{13}+a_{13|2}+a_{23}+a_{23|1}+2(p_2+p_{23}), a_{13|2}+a_{23}+2(p_2+p_{23}) \}$ , lying in bipyramids FFFGG.

We conclude that all 7 combinatorial types of valuated matroids in  $\text{trop}(\text{Gr}(3, 6))$  are realized by valuated gaussoids. This is similar to the result of Brodsky, Ceballos, and Labbé in [6].

Each of our 153 facets supports a monomial-free initial ideal  $\text{in}_\nu(J_3)$ . Here  $\nu \in \mathbb{R}^{14}$  is a vector in the relative interior of that 7-dimensional cone, and the initial ideal is understood in the sense of [24, § 2.4]. For the facets of type **ppp** and **ABCB**, the initial ideal  $\text{in}_\nu(J_3)$  is not a prime ideal. For the other seven types, the initial ideal  $\text{in}_\nu(J)$  is toric and hence prime. In those cases the 14 coordinates form a *Khovanskii basis* of our algebra, by the results of [20].

The list of types above thus classifies the *toric degenerations* of the Lagrangian Grassmannian  $V(J_3)$  in  $\mathbb{P}^{13}$ , and from  $\text{in}_\nu(J_3)$  we can identify the corresponding *Newton-Okounkov bodies*. We illustrate this for type **app** in the example that follows.  $\square$

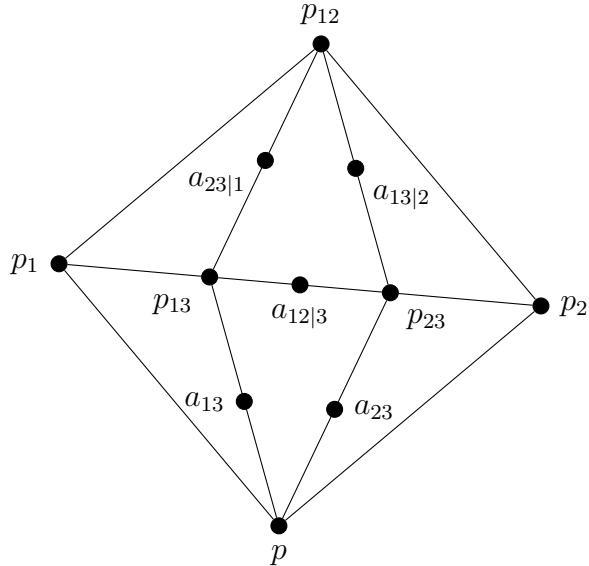


Figure 3: Schlegel diagram of a 3-polytope. Its join with the triangle  $\{a_{12}, p_3, p_{123}\}$  is a 6-polytope. This is the Newton-Okounkov body for a toric degeneration of  $\text{LGr}(3, 6)$  in  $\mathbb{P}^{13}$ .

**Example 7.2.** Consider the **app** triangle  $\{a_{12}, p_3, p_{123}\}$ . The corresponding 7-dimensional cone in  $\text{trop}(V(J_3)) \subset \mathbb{R}^{14}$  consists of all vectors  $\nu = \mu + (a, 0, 0, 0, 0, 0, 0, 0, 0, b, 0, 0, 0, c)$ , where  $a, b, c > 0$  and  $\mu$  is in the subspace  $L$  in (19). Each of these  $\nu$  is a valuated gaussoid.

The initial ideal  $\text{in}_\nu(J_3)$  is obtained by setting  $a_{12}, p_3$  and  $p_{123}$  to zero in all 21 quadrics in Example 3.3. The resulting ideal is generated by binomials and is prime. Hence,  $\text{in}_\nu(J_3)$  is a toric ideal. In the language of [20], the cone indexed by  $\{a_{12}, p_3, p_{123}\}$  is a prime cone.

The vector  $\nu$  defines a degeneration of the Lagrangian Grassmannian  $\text{LGr}(3, 6) = V(J_3)$  to the toric variety  $V(\text{in}_\nu(J_3))$ . Both are 6-dimensional and have degree 16. The corresponding lattice polytope is the Newton-Okounkov body. It has dimension 6 and volume 16. It is the join of the triangle  $\{a_{12}, p_3, p_{123}\}$  with the 3-dimensional polytope shown in Figure 3. This polytope has 6 vertices, 11 edges and 7 facets. Five additional points lie on edges. The toric ideal for this configuration of  $11 = 6 + 5$  lattice points in 3-space is equal to  $\text{in}_\nu(J_3)$ .  $\diamond$

The positive part of the tropical Grassmannian plays an important role in the theory of cluster algebras [6, 32]. Note that  $\text{trop}_+(\text{Gr}(3, 6))$  was worked out in [32, § 6]: it is the boundary of a 4-polytope known as the  $D_4$ -associahedron. In what follows we determine the analogue for the Lagrangian Grassmannian, that is, the space of positive valuated gaussoids

$$\text{trop}_+(\text{LGr}(3, 6)) = \text{trop}_+(\text{Gr}(3, 6)) \cap \text{trop}(\text{LGr}(3, 6)). \quad (20)$$

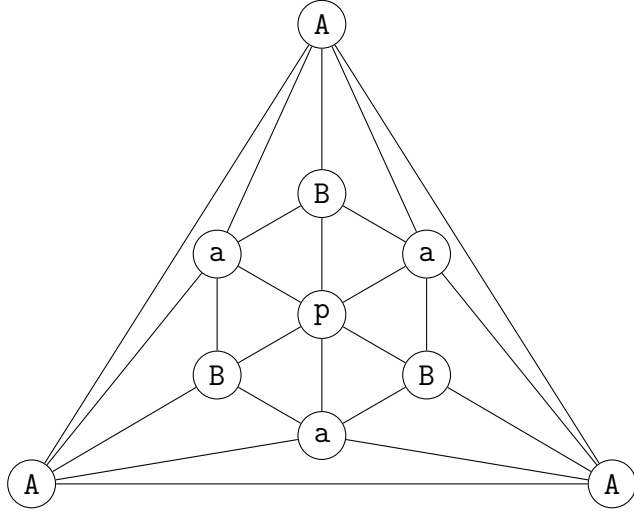


Figure 4: The 3-polytope that represents all positive valuated matroids for  $n = 3$ . Its boundary, a simplicial 2-sphere, is the positive tropical Lagrangian Grassmannian  $\text{trop}_+(\text{LGr}(3, 6))$ .

**Corollary 7.3.** *The intersection (20) corresponds to a triangulated 2-sphere with 10 vertices, 24 edges and 16 facets. It is the boundary of the simplicial 3-polytope shown in Figure 4.*

*Proof and Explanation.* We examined all 153 maximal cones in Theorem 7.1. A cone lies in (20) if and only if its initial ideal  $\text{in}_\nu(J_3)$  is generated by pure difference binomials  $m_1 - m_2$ . This happens for the following 16 cones. For each of them, we list a representative vector  $\nu$ :

apB/EEEG	aAB/EEFG	aAA/EEFFb	AAA/FFFGG
$\begin{bmatrix} 22250084000000 \\ 25220084000000 \\ 22002580000400 \\ 25002280000400 \\ 00222580000004 \\ 00252280000004 \end{bmatrix}$	$\begin{bmatrix} 00666940000004 \\ 00696640000004 \\ 66006940000400 \\ 66690044000000 \\ 69006640000400 \\ 69660044000000 \end{bmatrix}$	$\begin{bmatrix} 44226900000000 \\ 44692200000000 \\ 69442200000000 \end{bmatrix}$	$[77556600000000]$

For instance, the vector  $\nu = (22250084000000)$ , indexed as in (18), is a positive valuated gaussoid. It lies in a cone of type apB, and hence in a cone of type EEEG in  $\text{trop}_+(\text{Gr}(3, 6))$ . Each positive valuated gaussoid  $\nu$  records the  $\epsilon$ -orders of the principal and almost-principal minors of a covariance matrix that defines a Gaussian  $\text{MTP}_2$  distribution over  $\mathbb{R}\{\{\epsilon\}\}$ .

For example,  $\nu = (77556600000000)$  is realized in this sense by the covariance matrix

$$\Sigma = \text{the inverse of } \begin{pmatrix} 1 & -\epsilon^7 & -\epsilon^5 \\ -\epsilon^7 & 1 & -\epsilon^6 \\ -\epsilon^5 & -\epsilon^6 & 1 \end{pmatrix}.$$

Figure 4 is a combinatorial classification of all Gaussian  $\text{MTP}_2$  distributions over  $\mathbb{R}\{\{\epsilon\}\}$ .  $\square$

1	+++++	(1/8, 1/16, 1/4, 1/4, 1/16, 1/8)
2	+++++	(1/8, 1/16, 1/4, -1/2, -1/4, -1/16)
3	+++++	(1/4, 1/2, 1/16, 1/32, -1/128, -1/2)
4	+++++	1/100 · (44, 50, 51, 50, 51, 30)
5	+++++	(1/4, 1/8, 1/512, 1/128, -1/4, 1/8)
6	+++++	(1/4, 1/2, 1/128, 1/64, -1/32, 1/4)
7	+++++	1/100 · (57, 57, 76, 39, 12, 12)
8	+++++	1/100 · (64, 55, 60, 76, 32, 6)
9	+++++	(1/16, 1/32, 1/2, 1/64, 1/128, -1/2)
10	+++++	1/100 · (45, 57, 66, 57, 26, 19)
11	+++++	1/100 · (75, 69, 45, 45, 7, 75)
12	+++++	(1/16, 1/8, 1/2, 1/32, -1/512, -1/2)
13	+++++	(1/16, 1/2, 1/8, 1/512, 1/64, 1/2)
14	+++++	(1/8, 1/2, 1/16, 1/64, -1/64, 1/4)
15	+++++	1/100 · (53, 76, 46, 8, 71, 27)
16	+++++	(1/64, 1/8, 1/128, 1/128, 1/8, 1/8)
17	+++++	(1/2, 1/8, 1/8, 1/64, 1/32, 1/4096)
18	+++++	(1/16, 1/2, 1/2, 1/64, 1/128, 1/8)
19	+++++	(1/2, 1/256, 1/64, 1/256, -1/128, -1/4)
20	+++++	1/100 · (85, 67, 37, 39, 6, 64)
21	+++++	(1/4, 1/16, 1/2, 1/16, 1/32, -1/16)
22	+++++	1/100 · (59, 59, 48, 13, 59, 59)
23	+++++	(1/2, 1/2, 1/8, 1/2, 1/256, -1/8)
24	+++++	1/100 · (81, 84, 39, 43, 2, 49)
25	+++++	1/100 · (60, 85, 39, 21, 55, 27)
26	+++++	(1/8, 1/16, 1/8, 1/16, 1/128, 1/512)
27	+++++	(1/4, 1/2, 1/1024, 1/128, 1/1024, 1/4)
28	+++++	1/100 · (58, 58, 41, 41, 58, 58)
29	+++++	(1/16, 1/8, 1/8, 1/4, 1/4, 1/256)
30	+++++	(1/16, 1/8, 1/16384, -1/8192, -1/4, 1/512)
31	+++++	(1/16, 1/4, 1/8, 1/8, 1/8, -1/32)
32	+++++	(1/8, 1/32, 1/2, 1/32, -1/8, -1/2)
33	+++++	1/100 · (63, 70, 32, 33, 4, 63)
34	+++++	(1/4, 1/32, 1/256, 1/256, 1/2, 1/2)
35	+++++	(1/2, 1/4, 1/4, 1/16, 1/32, -1/4)
36	+++++	bi-quadratic final polynomial
37	+++++	(1/2, 1/2, 1/32, 1/128, -1/2, -1/16)
38	+++++	(1/4, 1/4, 1/16, 1/2, 1/256, 1/1024)
39	+++++	1/100 · (83, 46, 33, 33, 5, 83)
40	+++++	(1/4, 1/2, 1/32, 1/32768, 1/2048, 1/4)
41	+++++	1/100 · (46, 46, 43, 30, 74, 74)
42	+++++	(1/8, 1/256, 1/8, 1/64, 1/2, 1/256)
43	+++++	(1/4, 1/32, 1/512, 1/512, 1/4096, 1/2)
44	+++++	1/100 · (73, 59, 71, 49, 25, 6)
45	+++++	(1/4, 1/2, 1/64, 1/128, -1/4, 1/8)
46	+++++	(1/2, 1/2, 1/1024, 1/256, -1/2, 1/8)

Table 2: The 46 symmetry classes of uniform oriented gaussoids for  $n = 4$ .

## 8 Realizability

In this section we study the realizability problem for gaussoids and oriented gaussoids. There is a substantial literature on the realizability of matroids and oriented matroids. We point to [12] and the references therein. It is our aim to extend this to the setting developed in this paper. Our first result concerns the realizability of uniform oriented gaussoids for  $n = 4$ .

**Theorem 8.1.** *There are 46 symmetry classes of uniform oriented gaussoids for  $n = 4$ , listed in Table 2. All but one of them are realizable. The unique non-realizable class admits a bi-quadratic final polynomial in the sense of Bokowski and Richter [4].*

*Proof and Explanation.* The 46 classes were derived from the list of 5376 uniform oriented gaussoids in Theorem 5.3. The lists of 24 signs in Table 2 is with respect to the ordering

$$a_{12}, a_{12|3}, a_{12|4}, a_{12|34}, a_{13}, a_{13|2}, a_{13|4}, a_{13|24}, a_{14}, a_{14|2}, a_{14|3}, a_{14|23},$$

$$a_{23}, a_{23|1}, a_{23|4}, a_{23|14}, a_{24}, a_{24|1}, a_{24|3}, a_{24|13}, a_{34}, a_{34|1}, a_{34|2}, a_{34|12}.$$

In each realizable case, we list the entries  $(\sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{23}, \sigma_{24}, \sigma_{34})$  of a positive definite symmetric  $4 \times 4$ -matrix  $\Sigma$  with  $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{44} = 1$  for that oriented gaussoid. The

realization space of an oriented gaussoid is a semi-algebraic set. We used random search with values  $2^{-k}$  for small  $k$  and the optimization software SCIP [25] to find realizations.

The oriented gaussoid #36 is of special interest since it has a *bi-quadratic final polynomial*. We review this concept from [4]. The edge trinomials can be written as  $x_1x_2 + x_3x_4 - x_5x_6 = 0$ , where each  $x_i$  is a positive unknown, equal to either some  $p_I$  or some  $a_{ij|K}$  multiplied by its sign. The equation hence implies the inequalities  $x_1x_2 < x_5x_6$  and  $x_3x_4 < x_5x_6$ . After replacing each  $x_i$  by its logarithm,  $y_i = \log(x_i)$ , we get  $y_1 + y_2 < y_5 + y_6$  and  $y_3 + y_4 < y_5 + y_6$ . Using Linear Programming (LP), we can easily decide whether the resulting system of linear inequalities has a solution. If not, then the oriented gaussoid is non-realizable. A solution to the dual LP yields a non-realizability certificate known as bi-quadratic final polynomial.

Here is how it works for type #36. Among the edge trinomials we find the following:

$$\begin{aligned} &(-a_{23|4})(p_{134}) + (a_{13|4})(a_{12|34}) - (-a_{23|14})(p_{34}), \quad (a_{12|3})(p_{134}) + (a_{14|3})(-a_{24|13}) - (a_{12|34})(p_{13}) \\ &(a_{23|1})(p_{134}) + (-a_{23|14})(p_{13}) - (-a_{34|1})(-a_{24|13}), \quad (a_{34})(p_{134}) + (-a_{34|1})(p_{34}) - (a_{13|4})(a_{14|3}). \end{aligned}$$

These are elements of  $J_4$ , written in such a way that each parenthesis is positive for #36. From these four equations we infer the following inequalities among positive quantities:

$$\begin{array}{llll} a_{13|4}a_{12|34} & < & (-a_{23|14})p_{34} & & a_{14|3}(-a_{24|13}) & < & a_{12|34}p_{13} \\ (-a_{23|14})p_{13} & < & (-a_{34|1})(-a_{24|13}) & & (-a_{34|1})p_{34} & < & a_{13|4}a_{14|3} \end{array}$$

The product of the left hand sides equals the product of the right hand sides.  $\square$

We now briefly discuss the case  $n = 5$ . A *complex realization* of a gaussoid  $\mathcal{G}$  on  $[n]$  is a symmetric  $n \times n$ -matrix  $\Sigma$  with entries in  $\mathbb{C}$  whose principal minors are nonzero and whose vanishing almost-principal minors are indexed by  $\mathcal{G}$ . The following example can be viewed as a gaussoid analog to the Vámos matroid, which is the smallest non-realizable matroid.

**Example 8.2.** Let  $n = 5$ . The following collection of ten 2-faces of the 5-cube is a gaussoid:

$$\mathcal{G} = \{a_{12}, a_{13|4}, a_{14|5}, a_{15|23}, a_{23|5}, a_{24|135}, a_{25|34}, a_{34|12}, a_{35|1}, a_{45|2}\}.$$

To see that  $\mathcal{G}$  is not realizable over  $\mathbb{C}$ , consider the ideal in  $\mathbb{Q}[\sigma_{12}, \sigma_{13}, \dots, \sigma_{45}]$  generated by these 10 almost-principal minors, for a symmetric  $5 \times 5$ -matrix with ones on the diagonal and unknowns  $\sigma_{ij}$  off the diagonal. Saturation with respect to  $p_{24} = 1 - \sigma_{24}^2$  yields the maximal ideal  $\langle \sigma_{12}, \sigma_{13}, \dots, \sigma_{45} \rangle$ . So, there is no complex matrix with  $\mathcal{G} = 0$  and  $\sigma_{13}p_{24} \neq 0$ .  $\diamond$

With Example 8.2 it is now easy to define a non-realizable valuated gaussoid.

**Example 8.3.** Fix  $\mathcal{G}$  as in Example 8.2. Let  $\nu$  be the map from  $\mathcal{A} \cup \mathcal{P}$  to  $\mathbb{R}$  that takes  $\mathcal{G}$  to 1 and  $(\mathcal{A} \setminus \mathcal{G}) \cup \mathcal{P}$  to 0. By examining all the edge and square trinomials in  $(J_5)_2$ , we can verify that  $\nu$  is a valuated gaussoid. However, it is not realizable. There is no point in  $V(J_5)$  over the Puiseux series field  $\mathbb{C}\{\{\epsilon\}\}$  whose coordinates have valuation  $\nu$ . Such a point would come from a symmetric matrix  $\Sigma$  whose entries are in  $\mathbb{C}\{\{\epsilon\}\}$  and have valuations  $\geq 0$ . Setting  $\epsilon = 0$  in that matrix gives a complex realization of  $\mathcal{G}$ . But this does not exist.  $\diamond$

We close the paper with some directions for future research on the realizability of gaussoids. We begin by conjecturing that the non-realizable valuated gaussoid above is minimal.



**Conjecture 8.4.** *All tropical gaussoids for  $n = 4$  are realizable, i.e. they lie in  $\text{trop}(V(J_4))$ .*

Realization problems can be formulated as feasibility problems of (semi-)algebraic sets. The following refers to Theorem 4.1. It is a challenge as far as computation goes, but it is also an excellent opportunity for gaining statistical insights on Gaussian random variables.

**Challenge 8.5.** *Classify the 16981  $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ -orbits of gaussoids for  $n = 5$  according to their realizability over  $\mathbb{C}$ . Classify all 254826  $\mathbb{Z}/2\mathbb{Z} \rtimes S_n$ -orbits according to realizability.*

The *Universality Theorem* due to Mnëv [3, §8.6] states, roughly speaking, that any variety arises as the realization space of a matroid, and any semialgebraic set arises as the realization space of an oriented matroid. We wonder whether the same is true for gaussoids.

**Problem 8.6.** *Does universality hold for gaussoids? Can arbitrary varieties and arbitrary semialgebraic sets be the realization spaces of gaussoids and oriented gaussoids respectively?*

**Acknowledgements.** We thank Moritz Firsching, Paul Görlach, Jon Hauenstein, Yue Ren, Caroline Uhler and Charles Wang for help with this project. Bernd Sturmfels was partially supported by the Einstein Foundation Berlin and the US National Science Foundation (DMS-1419018, DMS-1440140). Tobias Boege and Thomas Kahle were partially supported by the Deutsche Forschungsgemeinschaft (314838170, GRK 2297, “MathCoRe”).

## References

- [1] F. Ardila, F. Rincón and L. Williams: *Positively oriented matroids are realizable*, J. Eur. Math. Soc. (JEMS) **19** (2017) 815–833.
- [2] M. Baker and N. Bowler: *Matroids over partial hyperstructures*, arXiv:1709.09707.
- [3] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. Ziegler: *Oriented Matroids*, Cambridge University Press, 1993.
- [4] J. Bokowski and J. Richter: *On the finding of final polynomials*, European J. Combinatorics **11** (1990) 21–34.
- [5] A. Borovik, I.M. Gelfand and N. White: *Coxeter Matroids*, Progress in Mathematics **216**, Birkhäuser, Boston, MA, 2003.
- [6] S. Brodsky, C. Ceballos and J-P. Labbé: *Cluster algebras of type  $D_4$ , tropical planes, and the positive tropical Grassmannian*, Beitr. Algebra Geom. **58** (2017) 25–46.
- [7] A. Dress and W. Wenzel: *Valuated matroids: a new look at the greedy algorithm*, Appl. Math. Letters **3** (1990) 33–35.
- [8] A. Dress and W. Wenzel: *Grassmann–Plücker relations and matroids with coefficients*, Adv. Math. **86** (1991) 68–110.
- [9] M. Drton, B. Sturmfels and S. Sullivant: *Lectures on Algebraic Statistics*, Oberwolfach Seminars, **39**, Birkhäuser Verlag, Basel, 2009.

- [10] M. Drton and H. Xiao: *Smoothness of Gaussian conditional independence models*, Algebraic methods in statistics and probability II, 155–177, *Contemporary Mathematics*, **516**, Amer. Math. Soc., Providence, RI, 2010.
- [11] S. Fallat, S. Lauritzen, K. Sadeghi, C. Uhler, N. Wermuth and P. Zwiernik: *Total positivity in Markov structures*, *Annals of Statistics* **45** (2017) 1152–1184.
- [12] K. Fukuda, H. Miyata and S. Moriyama: *Complete enumeration of small realizable oriented matroids*, *Discrete Comput. Geom.* **49** (2013) 359–381.
- [13] W. Fulton and J. Harris: *Representation Theory*, Graduate Texts in Mathematics **129**, Springer, New York, 1991.
- [14] D. Grayson and M. Stillman: *Macaulay2, a software system for research in algebraic geometry*, available at [www.math.uiuc.edu/Macaulay2/](http://www.math.uiuc.edu/Macaulay2/).
- [15] H. Hiller: *Combinatorics and intersection of Schubert varieties*, *Comment. Math. Helv.* **57** (1982) 41–59.
- [16] O. Holtz and B. Sturmfels: *Hyperdeterminantal relations among symmetric principal minors*, *Journal of Algebra* **316** (2007) 634–648.
- [17] A. Iliev and K. Ranestad: *Geometry of the Lagrangian Grassmannian  $LG(3, 6)$  with applications to Brill–Noether loci*, *Michigan Math. J.* **53**(2) (2005) 383–417.
- [18] H. Joe: *Generating random correlation matrices based on partial correlations*, *J. Multivariate Analysis* **97** (2006) 2177–2189.
- [19] S. Karlin and Y. Rinott: *M-matrices as covariance matrices of multinormal distributions*, *Linear Algebra Appl.* **52** (1983) 419–438.
- [20] K. Kaveh and C. Manon: *Khovanskii bases, higher rank valuations and tropical geometry*, [arXiv:1610.00298](https://arxiv.org/abs/1610.00298).
- [21] R. Kenyon and R. Pemantle: *Principal minors and rhombus tilings*, *J. Phys. A* **47** (2014) 474010, 17 pp.
- [22] S. Lauritzen, C. Uhler and P. Zwiernik: *Maximum likelihood estimation in Gaussian models under total positivity*, [arXiv:1702.04031](https://arxiv.org/abs/1702.04031).
- [23] R. Lněnička and F. Matúš: *On Gaussian conditional independence structures*, *Kybernetika* **43** (2007) 327–342.
- [24] D. Maclagan and B. Sturmfels: *Introduction to Tropical Geometry*, Graduate Studies in Mathematics **161**, American Mathematical Society, Providence, RI, 2015.
- [25] S. J. Maher *et al.*: *The SCIP Optimization Suite 4.0*, Zuse Institute, Berlin.
- [26] F. Mohammadi, C. Uhler, C. Wang and J. Yu: *Generalized permutohedra from probabilistic graphical models*, [arXiv:1606.01814](https://arxiv.org/abs/1606.01814).
- [27] L. Oeding: *G-Varieties and the Principal Minors of Symmetric Matrices*, PhD Dissertation, Texas A&M University, ProQuest LLC, Ann Arbor, MI, 2009.
- [28] L. Oeding: *Set-theoretic defining equations of the variety of principal minors of symmetric matrices*, *Algebra and Number Theory* **5** (2011) 75–109.
- [29] K. Sadeghi: *Faithfulness of probability distributions and graphs*, [arXiv:1701.08366](https://arxiv.org/abs/1701.08366).
- [30] *SageMath, the Sage Mathematics Software System (Version 8.0)*, The Sage Developers, 2017, <http://www.sagemath.org>.

- [31] P. Šimeček: *Gaussian representation of independence models over four random variables* In: Proc. COMPSTAT 2006, World Conference on Computational Statistics 17 (A. Rizzi and M. Vichi, eds.), Rome 2006, pp. 1405–1412.
- [32] D. Speyer and L. Williams: *The tropical totally positive Grassmannian*, J. Algebraic Combinatorics **22** (2005) 189–210.
- [33] M. Studeny: *Probabilistic Conditional Independence Structures*, Information Science and Statistics, Springer, London, 2005.
- [34] B. Sturmfels: *Open problems in algebraic statistics*, Emerging Applications of Algebraic Geometry, edited by M. Putinar and S. Sullivant, Springer, New York (2009), 351–363.
- [35] B. Sturmfels, E. Tsukerman and L. Williams: *Symmetric matrices, Catalan paths, and correlations*, J. Combinatorial Theory, Ser. A **144** (2016) 496–510.
- [36] S. Sullivant: *Gaussian conditional independence relations have no finite complete characterization*, Journal of Pure and Applied Algebra **213** (2009) 1502–1506.
- [37] M. Thurley: *sharpSAT - Counting models with advanced component caching and implicit BCP*, Proc. 9th Int. Conf. Theory and Applications of Satisfiability Testing (SAT 2006), (2006) pp. 424–429.
- [38] T. Toda and S. Takehide: *Implementing efficient all solutions SAT solvers*, J. Experimental Algorithmics **21.1** (2016) 1–12.
- [39] P. Vámos: *The missing axiom of matroid theory is lost forever*, Journal of the London Mathematical Society **18** (1978) 403–408.

**Authors' addresses:**

Tobias Boege, OvGU Magdeburg, Germany, [tboege@st.ovgu.de](mailto:tboege@st.ovgu.de)

Alessio D'Alì, Max-Planck Institute for Math in the Sciences, Leipzig, Germany, [alessio.dali@mis.mpg.de](mailto:alessio.dali@mis.mpg.de)

Thomas Kahle, OvGU Magdeburg, Germany, [thomas.kahle@ovgu.de](mailto:thomas.kahle@ovgu.de)

Bernd Sturmfels, MPI-MiS Leipzig, [bernd@mis.mpg.de](mailto:bernd@mis.mpg.de) and UC Berkeley, [bernd@berkeley.edu](mailto:bernd@berkeley.edu)