Triangular fractal approximating graphs and their covering paths and cycles

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Abstract. We observed and described the *generalized Sierpiński Arrowhead Curve* in our previous paper [K17a]. Now we focus on its background structure. In *Section 1* we summarize our previous results on the triangular grid and supplement them with Hamiltonian-cycles, tiling-cycles and a new kind of path on the possible largest trapezoid grid which are needed for the following sections. We describe the basic rule of the transformability of the paths and the cycles into each other and extend our grids to larger graphs. In *Sections 2* and *3* we define two kinds of graphs related to a checked fractal pattern on the *generalized Sierpiński Gasket*. We continue our observations with the basic properties of these triangular fractal approximating graphs independently of the recursive curves. We will describe the numbers of their vertices and edges, and their covering paths and cycles in general case with recursive and explicit formulas. Some of their cardinality specify new integer sequences. We also find the bijective relations between these formations.

1 Paths and cycles on the triangular grid

In this section we summarize and supplement our previous definitions [K17a] and show a table with our results, the cardinality of these formations on simple triangular grids. We describe the basic rules of the transformability of the paths and cycles into each other and extend our grids to larger graphs.

1.1 Checked generator pattern

First we make a checked pattern on a *triangular grid* of order n by colouring the tiles that face upwards dark and colouring the rest of the subtriangles white. Then we substitute all the dark tiles with the contracted copy of this generator pattern. Our pattern is related to the two-dimensional *generalized Sierpiński Gasket* $SG_{2,n}(k)$. We will be referring to them as F_n generator pattern and as $F_n(k)$ fractal approximating pattern, if $k > 1$.

1.2 Triangular grids and their paths and cycles

Our generator pattern F_n contains n^2 subtriangles, and T_{n-1} white tiles, T_n dark tiles and T_{n+1} grid points as three consecutive triangular numbers. The centroids of the dark tiles form the *inscribed grid*, and their corners form the *overall grid*. All of our paths originate from the leftmost node and terminate in the rightmost node of these grids.

Let us consider a self-avoiding tiling-path called *S-path* (referring to Sierpiński), denoted by S_n, and a self-avoiding tiling-cycle called *D-cycle*, denoted by D_n, on the overall grid. Both consist of T_n edges. All of the edges must be lying on different dark subtriangles. For practical reasons we will be using the notation of McKenna: marking the tiles with little ticks in the middle of the edges [McK94]. See the left side of *Figure 1*.

We denote the Hamiltonian-paths (H-paths) by H_n, and the Hamiltonian-cycles (C-cycles) by C_n on the inscribed grid. They have a subset in which all edges have well-formed turns. We will describe this well-formed property later. These paths and cycles are bijective pairs and they have the same cardinality. They are unambiguously transformable into each other [K17a]. We call the well-formed Hamiltonian-paths W-path, and denote them by W_n. You can see a well-formed Hamiltonian-cycle on the middle of *Figure 1*.

The so-called Z-paths (Z_n) on the possible largest trapezoid grid without the uppermost node of the inscribed grid are also needed for *Section 3*. See the right side of *Figure 1*.

We enumerated these cardinality with our computer program, which is a smart backtrack algorithm. H-paths and Z-paths appear in [SEH05], W-paths first appear in [K17a], C-cycles appear in [P14] and [OEIS1], and D-cycles, which specify a new integer sequence, first appear here. See *Table 1*.

n	T_n	H_n	$W_n = S_n$	Z_n	C_n	D_n
2	3	1	1	1	1	1
3	6	2	2	3	1	1
4	10	10	4	11	3	3
5	15	92	16	112	26	8
6	21	1852	68	2286	474	42
7	28	78032	464	94696	17214	240
8	36	6846876	3828	8320626	1371454	2120
9	45	1255156712	44488	1527633172	231924780	22724

Table 1. **Cardinality of H-, W-, S-, Z-paths and C- and D-cycles on the generator pattern F_n consisting of T_n dark tiles.**

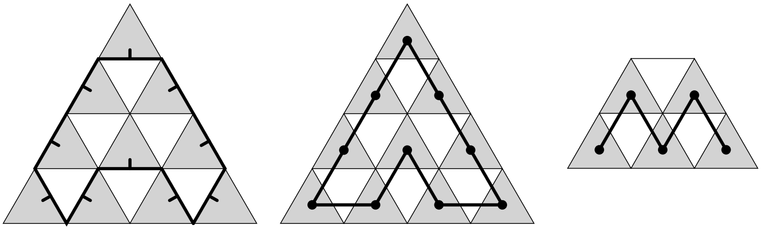

Figure 1. A tiling-cycle (D_4) (left side), the corresponding well-formed Hamiltonian-cycle (C_4) (middle), and a Z-path (Z_3) with a wrong turn of the edges (right side).

1.3 Untransformable wrong turns of the edges

Let us consider all edges of the paths and cycles on the inscribed grid described by a string of their absolute direction code consisting of 0 to 5 values in counterclockwise from the right. The direction right $\pm 120°$ means an even value and the direction left $\pm 120°$ means an odd value of the string.

If a path or a cycle on the inscribed grid consists of only well-formed turns, then it is unambiguously transformable into a tiling-path or a tiling-cycle.

The forbidden turns as (d_i, d_{i+1}) number pairs are the following:

$$d_{i+1} \not\equiv \begin{cases} (d_i + 4) \mod 6 & \text{if } d_i \text{ is even} \\ (d_i + 2) \mod 6 & \text{if } d_i \text{ is odd} \end{cases}$$

which means that the next edge cannot turn $120°$ to the right after an even direction and it cannot turn $120°$ to the left after an odd direction. Naturally, turning back by $180°$ is also forbidden.

For example on the right side of *Figure 1*, Z-path contains a wrong turn of the edges (the middle edge-pair), therefore it is an untransformable path. These three dark tiles have only one contact point instead of two, therefore we cannot connect them with three consecutive edges of a tiling-path.

1.4 Extending our grids to larger graphs

The *generalized Sierpiński Gasket* fractal family contains two kinds of *triangular fractal approximating graphs* as the background structure of our recursive curves. They are the extended version of the *inscribed grid* and the *overall grid* in larger approximations, where $k > 1$.

We observe triangular graphs based on the k-th power of the n-th *Triangular number*, denoted by T_n^k. By connecting the centroids of the neighbouring dark tiles of $F_n(k)$ we get a graph that we call the *Inscribed Graph* denoted by I_n^k. By connecting the corners of the neighbouring dark tiles of $F_n(k)$ we get a graph that we call the *Overall Graph* denoted by $O_n(k)$. We will observe and describe their properties in the rest of this paper.

2 The Overall Graph $O_n(k)$

Let us define the *Overall Graph* ($O_n(k)$), related to the $F_n(k)$ fractal approximating pattern where we replace all the dark tiles with their corners as the nodes of the graph and with their sides as the edges of the graph.

It consists of T_n^{k-1} simple T_{n+1} sized triangular grids which share their corners with their neighbour grids. See *Figure 2*. We will describe the numbers of their nodes and edges, and the numbers of their paths and cycles in this section.

$O_2(k)$ with all its possible connecting edges is also known as the *Sierpiński Sieve Graph*.

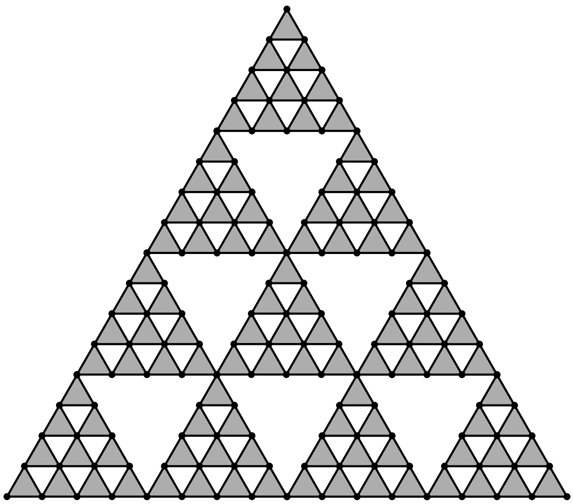

Figure 2. The Overall Graph $O_4(2)$
has $|O_4(2)| = 135$ nodes, $E(O_4(2)) = 300$ edges
and $S_{4,2} = 4^{11} = 4194304$ possible S-paths.

2.1 The numbers of the nodes and the edges in $O_n(k)$

The overall grid consists of T_{n+1} grid points, therefore $|O_n(1)| = T_{n+1}$.

In further approximations we substitute the dark tiles with T_n smaller overall grids which share their corners. By summerizing the nodes of the T_n smaller grids we counted the common nodes twice on each side of the overall graph and we counted the common nodes 3 times inside the overall graph, so we have to subtract these values from the result:

$$|O_n(k)| = |O_n(k-1)| \cdot T_n - 3(n-1) - 2T_{n-2} = |O_n(k-1)| \cdot T_n - n^2 + 1$$

We can transform this recursive formula to explicit formula and we get integer sequences for each n.

$$|O_n(k)| = \frac{(n+4)\left(\frac{n(n+1)}{2}\right)^k + 2(n+1)}{n+2}$$

We always get our result in this form: $|O_n(k)| = \frac{aT_n^k + b}{c}$ where a, b and c values can be simplified by 2 for each even values of n.

For example: $|O_3(k)| = \frac{7 \cdot 6^k + 8}{5}$, $\quad |O_4(k)| = \frac{4 \cdot 10^k + 5}{3}$,

$|O_5(k)| = \frac{9 \cdot 15^k + 12}{7}$, $\quad |O_6(k)| = \frac{5 \cdot 21^k + 7}{4}$, $\quad$ etc.

See the first 6 values of these integer sequences in *Table 2.*

$\lvert O_n(k)\rvert$	$k=1$	2	3	4	5	6
$n=2$	6	15	42	123	366	1095
3	10	52	304	1816	10888	65320
4	15	135	1335	13335	133335	1333335
5	21	291	4341	65091	976341	14645091
6	28	553	11578	243103	5105128	107207653

Table 2. **$|O_n(k)|$ = the number of the nodes in the overall graph.**

Remark. *First row of* Table 2 *is known as sequence A067771 [OEIS2]. Second and third rows are also known [CC06]. Our explicit formula gives new integer sequences for $|O_n(k)|$, where $n > 4$.*

The number of the edges in the overall graph is: $\quad E(O_n(k)) = 3T_n^k$.

2.2 S-paths on $O_n(k)$

We denote S-paths on the overall graph by $S_{n,k}$. In *Section 1* we enumerated S_n values on the overall grid therefore $S_{n,1} = S_n$. See *Table 1.*

In the second approximation we can use all the S_n paths at T_n places and we have S_n ways to connect them, therefore $S_{n,2} = S_n^{T_n+1}$.

In general case for $k > 1$: $\quad S_{n,k} = S_n^{(T_n+1)^{k-1}}$

On $O_2(k)$ we get a unique path ($S_{2,k} = 1$) for all k values. This trivial case is the k-th approximation of the edge-rewriting *Sierpiński Arrowhead Curve.*

For $n > 2$ and $k > 1$ we get:

$S_{3,k} = 2^{7^{k-1}} \qquad S_{4,k} = 4^{11^{k-1}} \qquad S_{5,k} = 16^{16^{k-1}} \qquad S_{6,k} = 68^{22^{k-1}}$

2.3 Tiling-cycles on $O_n(k)$

To find the number of the tiling-cycles on the overall graph we have to substitute the connection of the smaller grids with tiling-cycles (D_n) instead of S-paths. By modifying our previous formula we get the following.

The number of the tiling-cycles on $O_n(k)$ in general case with recursion is:

$$D_{n,k} = D_n \cdot S_{n,k-1}^{T_n^{k-1}}$$

The explicit formula is: $$D_{n,k} = D_n \cdot \left(S_n^{(T_n+1)^{k-2}} \right)^{T_n^{k-1}}$$

We get very large numbers:

$D_{n,k}$	$k=1$	2	3
$n=2$	1	1	1
3	1	2^6	2^{252}
4	3	$3 \cdot 4^{10}$	$3 \cdot 4^{1100}$
5	8	$8 \cdot 16^{15}$	2^{14403}

Table 3. **The number of the tiling-cycles =** $D_{n,k}$ **values on the overall graph.**

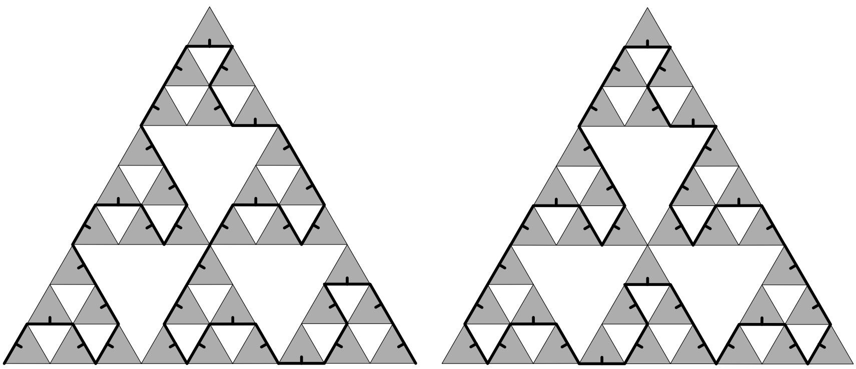

Figure 3. A tiling-path ($S_{3,2}$) **and a tiling-cycle** ($D_{3,2}$) **on the overall graph** $O_3(2)$**.**

3 The Inscribed Graph I_n^k

Let us define the *Inscribed Graph* (I_n^k), related to the $F_n(k)$ fractal approximating pattern, where we replace all the dark tiles with their centroids as the nodes of the graph and by connecting all the centroids between node-neighbour dark subtriangles we get the edges of the graph.

It consists of T_n^{k-1} simple T_n sized independent triangular grids which do not share grid points with each other. Connecting edges from the previous approximations remain among the simple triangular grids, otherwise grid points become simple new triangular grids among the connecting edges. See *Figure 4*.

In this section we will describe the numbers of their nodes and edges, and we will use *Z-paths* and *D-cycles* to calculate the numbers of their Hamiltonian-paths and Hamiltonian-cycles.

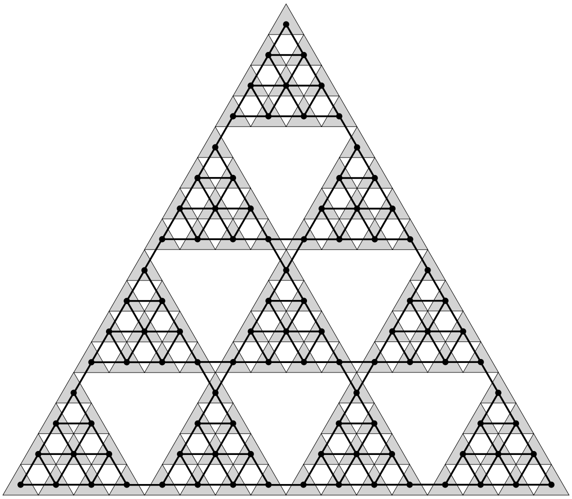

Figure 4. The Inscribed Graph I_4^2

3.1 The numbers of the nodes and the edges in I_n^k

This structure consists of T_n^k nodes in the k-th approximation, the k-th power of a triangular number: $\left|I_n^k\right| = T_n^k$.

The number of the edges:

$$E(I_n^k) = \sum_{i=1}^{k} 3T_{n-1} \cdot T_n^{i-1}$$

because connecting edges from the previous approximations remain among the simple triangular grids.

3.2 Hamiltonian-paths on I_n^k

We denote the Hamiltonian-paths on the Inscribed Grid by $H_{n,k}$. First we observe special cases, then we find the general formula to calculate their cardinality.

On the inscribed grid the number of the Hamiltonian-paths is $H_{n,1} = H_n$.

3.2.1 Hanoi Graph $H_2, k = 1$

I_2^k with all its possible connecting edges is known as the *Hanoi Graph* [H86]. It has a unique Hamiltonian-path (from the leftmost to the rightmost grid point) and a unique Hamiltonian-cycle in any k-th approximation which shows how to solve *Hanoi Tower* puzzle if we have $n+1$ pegs in one row and k discs, and only one disc can be moved at one time to a neighbour peg. Discs can be located only in descending order of the disc-sizes.

$H_2 = W_2 = 1$, therefore this is the unique recursive curve on the inscribed graph, the node-rewriting *Sierpiński Arrowhead Curve*, which is also the unique symmetric one.

Hanoi Graphs in general case with more than 3 pegs are also known, but they have other structures than our I_n^k graphs for $n > 2$. Our I_n^k graphs can also be represented as all the numbers in k places in the base T_n numeral system.

3.2.2 Paths on I_n^2 and the v-shaped connecting edges

As we observe the structure of I_n^k for $k > 1$ we can see that we have to use W-paths to connect the smaller neighbouring grids to each other. On the sides of the inscribed graph we always have only one possible connecting edge among the smaller grids.

Inside the graph we always have three possible connecting edges between three neighbour grids which form a little triangle, facing downwards. The whole structure looks like a combination of paths and tiles. We have only one entering and one exiting point on the smaller grids, therefore we have to follow their order as the connecting W-path leads the edges among them.

There are no more passages between the smaller grids, but inside the graph, connecting edges of the W-path can take over a corner point from a neighbouring smaller grid. These corner points change the connecting edge to an edge-pair, forming a little v-shaped connection. It modifies the H-path of the smaller grid to a Z-path. The permutation of the smaller grids stands, and the connecting edges follow the W-path with this little modification. By forgetting v-shapes we calculate the number of the possible covering paths on a smaller grid to get $H_{n,2}$ values, denoted by Y_n, where $Y_n = H_n + Z_n$.

3.2.3 Calculating the Hamiltonian-paths on $H_{3,k}$

We don't have connecting v-shapes and Z-paths on I_2^k. They appear first when $n = 3$, and in this case it is easy to calculate them in any order of k, because on I_3^2

graph our two possible connecting W-paths can always modify exactly one edge to a v-shape, unlike other W-paths in larger orders of n.

$H_{3,k} = W_3^a \cdot H_3^b \cdot Y_3^c = 2^a \cdot 2^b \cdot 5^c$ (by *Table 1*) where $k > 1$, $a = \sum_{i=0}^{k-2} T_3^i$, $b = T_3^{k-1} - a$, $c = a$, therefore we can simplify our formula to

$$H_{3,k} = 10^a \cdot 2^{6^{k-1}-a} \qquad \text{where} \qquad a = \sum_{i=0}^{k-2} 6^i$$

$$(H_{3,2} = 10 \cdot 2^5, \qquad H_{3,3} = 10^7 \cdot 2^{29}, \qquad H_{3,4} = 10^{43} \cdot 2^{173}, \qquad ...)$$

3.2.4 Calculating the Hamiltonian-paths on $H_{n,2}$

We find another problem related to v-shaped connecting edges when $n > 3$. W-paths have different properties in the same order of n. For example, on I_3^2 graph our two possible W-paths can always modify exactly one edge to a v-shape.

On I_5^2 we have 16 possible W-paths, of which 2 can modify 4 inner edges, 8 can modify 5 inner edges and 6 can modify 6 inner edges to a v-shape, therefore it is difficult to calculate the number of the Hamiltonian paths ($H_{n,k}$) when $n > 3$.

See *Table 4*, where W-paths (W_n) for each n can be separated by the possible number of the connecting v-shapes (c_m) to m groups (g_m), where $W_n = \sum_{m=1}^{n-2} g_m$.

n	T_n	g_m	b_m	c_m	W_n
3	6	2	5	1	2
4	10	2	8	2	4
		2	7	3	
5	15	2	11	4	16
		8	10	5	
		6	9	6	
6	21	4	14	7	68
		22	13	8	
		32	12	9	
		10	11	10	
7	28	8	27	11	464
		76	26	12	
		180	25	13	
		160	24	14	
		40	23	15	

***Table 4.* The number of the v-shapes (c_m) for W_n values**

The number of the Hamiltonian-paths on I_n^2 (for $n > 2$) is:

$$H_{n,2} = \sum_{m=1}^{n-2} g_m \cdot H_n^{b_m} \cdot Y_n^{c_m}$$

T_n little triangular grid can be covered by b_m H-paths and c_m triangular or trapezoid covering paths ($Y_n = H_n + Z_n$). $T_n = b_m + c_m$ for all m.

Remark. *We have left the order number n from g_m, b_m and c_m for the sake of simplicity. Self-evidently they always belong to the actual W_n values.*

See *Figure 5* for a possible Hamiltonian-path on the inscribed graph I_4^2 and for the explicit formula of $H_{4,2}$.

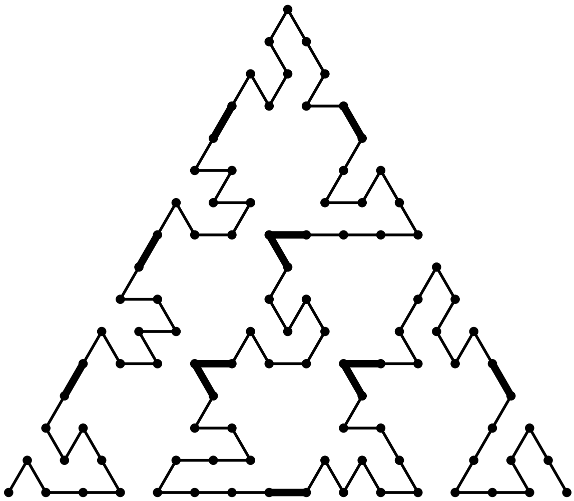

Figure 5. A Hamiltonian-path with v-shaped connecting edge-pairs and Z-paths on I_4^2 graph.

$$H_{4,2} = g_1 \cdot H_4^{b_1} \cdot Y_4^{c_1} + g_2 \cdot H_4^{b_2} \cdot Y_4^{c_2} =$$
$$2 \cdot 10^8 \cdot 21^2 + 2 \cdot 10^7 \cdot 21^3 = 273420000000$$

3.2.5 General formula for any $H_{n,k}$

In the previous subsections we have given the explicit formulas to calculate $H_{n,1}$, $H_{n,2}$, $H_{2,k}$ and $H_{3,k}$ values. Now we give a recursive formula to calculate the Hamiltonian-paths ($H_{n,k}$) on larger Inscribed Graphs I_n^k, where $n > 3$, $k > 2$.

Consider a second order approximation as we can see in *Figure 5.* In the third approximation we can use this second order path like a larger triangular tile, or as in Z-paths by erasing the uppermost grid point as a trapezoid tile, which means we can cover its uppermost first order triangular grid with an H-path or also a Z-path. It modifies our last result (one more first order grid covered by a Y_n path instead of a first order grid covered by an H-path).

We give a recursive formula to calculate the Hamiltonian-paths in general case on I_n^k:

$$H_{n,k} = \sum_{m=1}^{n-2} g_m \cdot H_{n,k-1}^{b_m} \cdot \left(\frac{H_{n,k-1} \cdot Y_n}{H_n} \right)^{c_m}$$

3.3 Hamiltonian-cycles on I_n^k

Here we observe the most complicated structure in this paper, and we give a formula to calculate the number of the Hamiltonian-cycles ($C_{n,k}$) on the Inscribed Graph.

The number of the Hamiltonian-cycles, if $k = 1$ is $C_{n,1} = C_n$. For C_n values see *Table 1*, [P14] or [OEIS1].

For the second approximation ($k = 2$) we have to use well-formed connections between T_n smaller triangular grids, therefore we have to use D-cycles. Like in the previous case, inner connecting edges can be substituted by little v-shaped edge-pairs, but their numbers are different for a constant n, also for D-cycles.

Consider $n = 5$. For $k = 1$, $C_{5,1} = C_5 = 26$. For $k = 2$, $D_5 = 8$, but 6 of the cycles use 4 inner connecting edges, 2 of the cycles use 6 inner connecting edges. Generally on an I_n^2 inscribed graph we have T_n smaller grids. The number of their possible connections is D_n, which can be separated into m groups (f_m), where $D_n = \sum_{m=1}^{n-3} f_m$ and $T_n = r_m + t_m$.

See *Table 5* for D_n values grouped by the number of the v-shapes (t_m).

n	T_n	f_m	r_m	t_m	D_n
4	10	3	8	2	3
5	15	6	11	4	8
		2	9	6	
6	21	6	14	7	42
		30	13	8	
		6	11	10	
7	28	24	17	11	240
		108	16	12	
		24	15	13	
		84	14	14	
8	36	72	20	16	2120
		432	19	17	
		932	18	18	
		240	17	19	
		444	16	20	

Table 5. **The number of the v-shapes (t_m) for D_n values**

The Hanoi Graph (I_2^k) has only one Hamiltonian-cycle in any approximations, therefore $C_2, k = 1$.

$C_{3,k} = H^6_{3,k-1}$ because $D_3 = 1$, otherwise there is only one way to connect all the small grids to a cycle, and these cycles do not contain v-shaped edges. The center of the graph is never connected. It is the same as connecting six Hamiltonian-paths ($H_{3,k-1}$) from the previous approximation with each other.

Hamiltonian-cycles on the second order approximations ($C_{n,2}$) can be calculated on the following way, where $n > 3$:

$$C_{n,2} = \sum_{m=1}^{n-3} f_m \cdot H_n^{r_m} \cdot Y_n^{t_m}$$

Remark. *We have left the order number n from f_m, r_m and t_m for the sake of simplicity. Self-evidently they always belong to the actual D_n values.*

See *Figure 6* for a Hamiltonian-cycle on I_4^2.

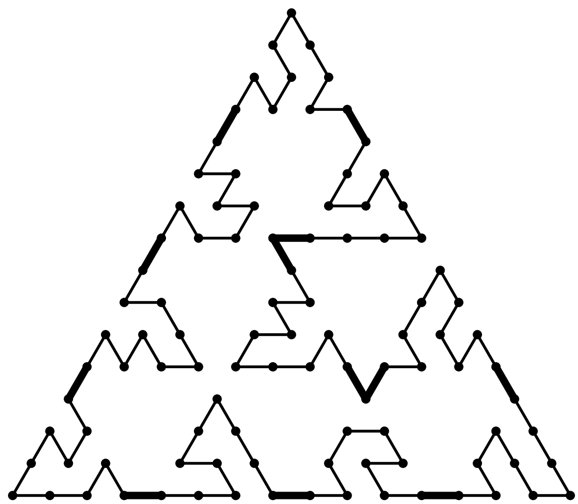

Figure 6. Example of a Hamiltonian-cycle $C_{4,2}$ with v-shaped connecting edge-pairs and Z-paths on I_4^2 graph.

The recursive formula to calculate the Hamiltonian-cycles in general case on I_n^k is the following:

$$C_{n,k} = \sum_{m=1}^{n-3} f_m \cdot H_{n,k-1}^{r_m} \cdot \left(\frac{H_{n,k-1} \cdot Y_n}{H_n} \right)^{t_m}$$

3.4 Well-formed Hamiltonian-paths and -cycles

Hamiltonian-paths and Hamiltonian-cycles on I_n^k have subsets consisting of only well-formed turns (no v-shapes, no Z-paths and H-paths). These paths and cycles on the Inscribed Graph (I_n^k) have the same cardinality as tiling-paths ($S_{n,k}$) and tiling-cycles ($D_{n,k}$) on the Overall Graph ($O_n(k)$).

Summary

In *Section 1* we have summarized and supplemented our previous results, which related to the triangular grids, the *generalized Sierpiński Gasket* and the *generalized Sierpiński Arrowhead Curve*. In *Sections 2* and *3* we have observed the background structures of the same fractal family as two kinds of fractal approximating graphs. We have given explicit and recursive formulas to calculate the cardinality of their nodes and edges in both cases, their tiling-paths and -cycles, which cover all the dark tiles of the *Overall Graph*, and their edge covering paths and cycles on the *Inscribed Graph*. We have found their interesting properties, and we have also found new integer sequences.

Our earlier papers also complete this field with some details *[HK15,HK16,K17a]*. For all of my papers please check my Google Scholar site *[KA]*.

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