# A Short Combinatorial Proof of Derangement Identity

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#### Abstract

The *n*-th rencontres number with the parameter r is the number of permutations having exactly r fixed points. In particular, a derangement is a permutation without any fixed point. We presents a short combinatorial proof for a weighted sum derangement identities.

**Keywords:** derangements, rencontres numbers, recurrence relation, factorial, binomial coefficient

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### **1** Introduction

Having a permutation  $\sigma \in S_n$ ,  $\sigma : [n] \to [n]$  where  $[n] := \{1, 2, ..., n\}$ , it is said that  $k \in [n]$  is a *fixed point* if it is mapped to itself,  $\sigma(k) = k$ . Permutations without fixed points are of particular interest and are usually called *derangements*. We let  $D_n$  denote the number of derangements of the set [n],  $D_n = |S_n^{(0)}|$ ,

$$S_n^{(0)} := \{ \sigma \in S_n : \sigma(k) \neq k, k = 1, \dots, n \}.$$

Derangements are usually introduced in the context of inclusion-exclusion principle [1, 6, 10], since this principle is used to provide an interpretation of  $D_n$  as a subfactorial,

$$D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}.$$
 (1)

The numbers  $D_0, D_1, D_2, \ldots, D_n, \ldots$  form recursive sequence  $(D_n)_{n\geq 0}$  defined by the recurrence formulae

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$
(2)

and initial terms  $D_0 = 1, D_1 = 0$ . There is a counting argument to prove this. Let the number k be mapped by  $\sigma$  to the number  $j, j = 1, \dots, k-1, k+1, \dots, n$ . Note that there are (n-1) such permutations  $\sigma$ . Now, we separate the set of permutations  $\sigma$  into two disjoint sets  $\mathcal{A}$  and  $\mathcal{B}$ , such that

$$\begin{aligned} \mathcal{A} &:= & \{ \sigma \in S_n^{(0)} : \sigma(j) \neq k, \sigma(k) = j \} \\ \mathcal{B} &:= & \{ \sigma \in S_n^{(0)} : \sigma(j) = k, \sigma(k) = j \}. \end{aligned}$$

This means that

$$D_n = (n-1)(|\mathcal{A}| + |\mathcal{B}|).$$

The set  $\mathcal{A}$  counts  $D_{n-1}$  elements while the set  $\mathcal{B}$  counts  $D_{n-2}$  elements. The fact that the number k in this reasoning is chosen without losing generality, completes the proof of (2).

By a simple algebraic manipulation with (2) we obtain another recurrence for the sequence  $(D_n)_{n>0}$ ,

$$D_n = nD_{n-1} + (-1)^n. (3)$$

Namely, it holds true

$$nD_{n-2} - D_{n-1} - D_{n-2} = -nD_{n-3} + D_{n-2} + 2D_{n-3} = \dots = (-1)^n$$

and this immediately gives the above recurrence from (2).

When we iteratively apply recurrence (3) to the derangement number on the r.h.s. of this relation we get

$$nD_{n-1} + (-1)^n = n[(n-1)D_{n-2} + (-1)^{n-1}] + (-1)^n$$

which finally results with

$$n(n-1)(n-2)\cdots 3(-1)^2 + n(n-1)(n-2)\cdots 4(-1)^3 + \dots + (-1)^n.$$
(4)

on the r.h.s. of (3), which completes the proof of (1).

A few identities for the sequence  $(D_n)_{n\geq 0}$  are known [2, 5, 8]. In [2] Deutsche and Elizalde give a nice identity

$$D_n = \sum_{k=2}^{n} (k-1) \binom{n}{k} D_{n-k}.$$
 (5)

Recently, Bhatnagar presents families of identities for some sequences including the shifted derangement numbers [3], deriving it using an Euler's identity [4]. In what follows we demonstrate a combinatorial proof for that derangement identity, with weighted sum.

### 2 A pair of weighted sums for derangements

We define the rencontres number  $D_n(r)$  as the number of permutations  $\sigma \in S_n$  having exactly r fixed points. Thus,  $D_n(0) = D_n$ . For a given  $r \in \mathbb{N}$ , we define the sequence  $D_0(r), D_1(r), \ldots, D_n(r), \ldots$ , denoted  $(D_n(r))_{n \geq r}$ .

Applying an analogue counting argument that we used when proving relation (2), one can represents rencontres numbers by the derangement numbers,

$$D_n(r) = \binom{n}{r} D_{n-r}.$$
(6)

On the other hand, relation (6) follows immediately from the fact that fixed points here are r-combinations over the set of n elements.

A few other notable properties of the rencontres numbers is also known. The difference between numbers in the sequences  $(D_n)_{n\geq 0}$  and  $(D_n(1))_{n\geq 1}$  alternate for the value 1, which follows from (3). According to the definition of rencontres numbers, the sum of the *n*-th row in the array of numbers  $(D_n(r))_{n\geq r}$  is equal to n!,

$$n! = \sum_{k=0}^{n} D_n(k).$$
(7)

Moreover, identity (6) shows that  $D_n$  can be interpreted as a weighted sum of rencontres numbers in the *n*-th row of the array, by means of relation (5),

$$D_n = \sum_{k=2}^n (k-1)D_n(k).$$
 (8)

The number  $D_n/(n-1)$  is also a weighted sum of previous consecutive derangement numbers. For example,  $24 + 12D_2 + 4D_3 + D_4 = \frac{D_6}{5}$ . In general we have

$$n! + \sum_{k=2}^{n} \frac{n!}{k!} D_k = \frac{D_{n+2}}{n+1},\tag{9}$$

as follows from Theorem 1.

**Theorem 1.** For  $n \in \mathbb{N}$  and the sequence of derangement numbers  $(D_n)_{n\geq 0}$  we have

$$1 + \sum_{k=1}^{n} \frac{D_k}{k!} = \frac{D_{n+2}}{(n+1)!}.$$
(10)

*Proof.* Within a derangement  $\sigma$ , the number k, k = 1, ..., n can be mapped to any j, j = 1, ..., k - 1, k + 1, ..., n. We let  $\mathcal{A}_n$  denote the set of derangements with  $\sigma(k) = j$ , where  $j \neq k$ ,

$$\mathcal{A}_n := \{ \sigma \in S_n^{(0)} : \sigma(k) = j \}.$$

Obviously, cardinality of the set  $\mathcal{A}_n$  is invariant to the choice of  $j, j \neq k$ . More precisely,

$$|\mathcal{A}_n| = \frac{D_n}{n-1}.$$

Furthermore, we separate the set  $\mathcal{A}_n$  into two disjoint sets of derangements, sets  $\mathcal{B}_n$  and  $\mathcal{C}_n$ ,

$$\begin{aligned} \mathcal{B}_n &:= \{ \sigma \in \mathcal{A}_n : \sigma(j) = k \} \\ \mathcal{C}_n &:= \{ \sigma \in \mathcal{A}_n : \sigma(j) \neq k \}. \end{aligned}$$

Obviously, the set  $\mathcal{B}_n$  counts  $D_{n-2}$  elements. For derangements in  $\mathcal{C}_n$  there are now (n-2) equivalent ways to map j (excluding j and k), as Figure 1 illustrates. Thus, we have

$$|\mathcal{C}_n| = (n-2)|\mathcal{A}_{n-1}|,$$

which gives the recurrence relation

$$|\mathcal{A}_n| = D_{n-2} + (n-2)|\mathcal{A}_{n-1}|. \tag{11}$$

After repeating usage of (11) we get identity (9) which completes the proof.  $\Box$ 

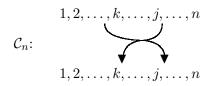


Figure 1: In case of derangements in the set  $C_n$  there are (n-2) equivalent ways to map j.

In order to prove Theorem 1 algebraically, we apply recurrence (2) to get

$$\frac{D_{n+2}}{(n+1)!} = \frac{(n+1)(D_{n+1}+D_n)}{(n+1)!} = \frac{D_{n+1}}{n!} + \frac{D_n}{n!}$$
$$= \frac{n(D_n+D_{n-1})}{n!} + \frac{D_n}{n!} = \frac{D_n}{(n-1)!} + \frac{D_{n-1}}{(n-1)!} + \frac{D_n}{n!}$$
$$= 1 + \frac{D_1}{1!} + \dots + \frac{D_n}{n!} = 1 + \sum_{k=1}^n \frac{D_k}{k!}.$$

**Theorem 2.** For  $n \in \mathbb{N}$  and the sequence of derangement numbers  $(D_n)_{n\geq 0}$  we have

$$1 + \sum_{k=1}^{n} \frac{(-1)^k D_{k+3}}{k+2} = (-1)^n D_{n+2}.$$
 (12)

*Proof.* By applying recurrence (2) we have

$$\sum_{k=0}^{n} \frac{(-1)^{k} D_{k+3}}{k+2} = \frac{2(D_{2}+D_{1})}{2} - \frac{3(D_{3}+D_{2})}{3} + \dots (-1)^{n} \frac{(n+2)(D_{n+2}+D_{n+1})}{n+2}$$
$$= (D_{2}+D_{1}) - (D_{3}+D_{2}) + \dots (-1)^{n} (D_{n+2}+D_{n+1})$$
$$= (-1)^{n} D_{n+2}$$

which completes the proof.

Once having Theorem 1, substitution of (6) in identity (10) gives the generalization (13).

$$1 + \sum_{k=1}^{n} \frac{D_{k+r}(r)}{k!\binom{k+r}{r}} = \frac{D_{n+r+2}(r)}{(n+1)!\binom{n+r+2}{r}}.$$
(13)

The identity (14) follows by substitution of (6) in (12),

$$1 + \sum_{k=1}^{n} \frac{(-1)^{k} D_{k+r+3}(r)}{(k+2)\binom{k+r+3}{r}} = \frac{(-1)^{n} D_{n+r+2}(r)}{\binom{n+r+2}{r}}.$$
(14)

Note that the terms in identity (14) are always integers, which can be seen as a consequence of recurrence relation (2).

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