# A Short Combinatorial Proof of Derangement Identity 

Ivica Martinjak<br>Faculty of Science, University of Zagreb<br>Bijenička cesta 32, HR-10000 Zagreb, Croatia<br>and<br>Dajana Stanić<br>Department of Mathematics, University of Osijek<br>Trg Ljudevita Gaja 6, HR-31000 Osijek, Croatia


#### Abstract

The $n$-th rencontres number with the parameter $r$ is the number of permutations having exactly $r$ fixed points. In particular, a derangement is a permutation without any fixed point. We presents a short combinatorial proof for a weighted sum derangement identities.


Keywords: derangements, rencontres numbers, recurrence relation, factorial, binomial coefficient
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## 1 Introduction

Having a permutation $\sigma \in S_{n}, \sigma:[n] \rightarrow[n]$ where $[n]:=\{1,2, \ldots, n\}$, it is said that $k \in[n]$ is a fixed point if it is mapped to itself, $\sigma(k)=k$. Permutations without fixed points are of particular interest and are usually called derangements. We let $D_{n}$ denote the number of derangements of the set $[n], D_{n}=\left|S_{n}^{(0)}\right|$,

$$
S_{n}^{(0)}:=\left\{\sigma \in S_{n}: \sigma(k) \neq k, k=1, \ldots, n\right\} .
$$

Derangements are usually introduced in the context of inclusion-exclusion principle [1, 6, 10], since this principle is used to provide an interpretation of $D_{n}$ as a subfactorial,

$$
\begin{equation*}
D_{n}=n!\sum_{k=0}^{n} \frac{(-1)^{k}}{k!} . \tag{1}
\end{equation*}
$$

The numbers $D_{0}, D_{1}, D_{2}, \ldots, D_{n}, \ldots$ form recursive sequence $\left(D_{n}\right)_{n \geq 0}$ defined by the recurrence formulae

$$
\begin{equation*}
D_{n}=(n-1)\left(D_{n-1}+D_{n-2}\right) \tag{2}
\end{equation*}
$$

and initial terms $D_{0}=1, D_{1}=0$. There is a counting argument to prove this. Let the number $k$ be mapped by $\sigma$ to the number $j, j=1, \ldots, k-1, k+1, \ldots, n$. Note
that there are $(n-1)$ such permutations $\sigma$. Now, we separate the set of permutations $\sigma$ into two disjoint sets $\mathcal{A}$ and $\mathcal{B}$, such that

$$
\begin{aligned}
\mathcal{A} & :=\left\{\sigma \in S_{n}^{(0)}: \sigma(j) \neq k, \sigma(k)=j\right\} \\
\mathcal{B} & :=\left\{\sigma \in S_{n}^{(0)}: \sigma(j)=k, \sigma(k)=j\right\}
\end{aligned}
$$

This means that

$$
D_{n}=(n-1)(|\mathcal{A}|+|\mathcal{B}|)
$$

The set $\mathcal{A}$ counts $D_{n-1}$ elements while the set $\mathcal{B}$ counts $D_{n-2}$ elements. The fact that the number $k$ in this reasoning is chosen without losing generality, completes the proof of (2).

By a simple algebraic manipulation with (2) we obtain another recurrence for the sequence $\left(D_{n}\right)_{n \geq 0}$,

$$
\begin{equation*}
D_{n}=n D_{n-1}+(-1)^{n} \tag{3}
\end{equation*}
$$

Namely, it holds true

$$
n D_{n-2}-D_{n-1}-D_{n-2}=-n D_{n-3}+D_{n-2}+2 D_{n-3}=\cdots=(-1)^{n}
$$

and this immediately gives the above recurrence from (2).
When we iteratively apply recurrence (3) to the derangement number on the r.h.s. of this relation we get

$$
n D_{n-1}+(-1)^{n}=n\left[(n-1) D_{n-2}+(-1)^{n-1}\right]+(-1)^{n}
$$

which finally results with

$$
\begin{equation*}
n(n-1)(n-2) \cdots 3(-1)^{2}+n(n-1)(n-2) \cdots 4(-1)^{3}+\cdots+(-1)^{n} \tag{4}
\end{equation*}
$$

on the r.h.s. of (3), which completes the proof of (11).
A few identities for the sequence $\left(D_{n}\right)_{n \geq 0}$ are known [2, 5, 8]. In [2] Deutsche and Elizalde give a nice identity

$$
\begin{equation*}
D_{n}=\sum_{k=2}^{n}(k-1)\binom{n}{k} D_{n-k} \tag{5}
\end{equation*}
$$

Recently, Bhatnagar presents families of identities for some sequences including the shifted derangement numbers [3], deriving it using an Euler's identity [4]. In what follows we demonstrate a combinatorial proof for that derangement identitiy, with weighted sum.

## 2 A pair of weighted sums for derangements

We define the rencontres number $D_{n}(r)$ as the number of permutations $\sigma \in S_{n}$ having exactly $r$ fixed points. Thus, $D_{n}(0)=D_{n}$. For a given $r \in \mathbb{N}$, we define the sequence $D_{0}(r), D_{1}(r), \ldots, D_{n}(r), \ldots$, denoted $\left(D_{n}(r)\right)_{n \geq r}$.

Applying an analogue counting argument that we used when proving relation (2), one can represents rencontres numbers by the derangement numbers,

$$
\begin{equation*}
D_{n}(r)=\binom{n}{r} D_{n-r} . \tag{6}
\end{equation*}
$$

On the other hand, relation (6) follows immediately from the fact that fixed points here are $r$-combinations over the set of $n$ elements.

A few other notable properties of the rencontres numbers is also known. The difference between numbers in the sequences $\left(D_{n}\right)_{n \geq 0}$ and $\left(D_{n}(1)\right)_{n \geq 1}$ alternate for the value 1 , which follows from (3). According to the definition of rencontres numbers, the sum of the $n$-th row in the array of numbers $\left(D_{n}(r)\right)_{n \geq r}$ is equal to $n$ !,

$$
\begin{equation*}
n!=\sum_{k=0}^{n} D_{n}(k) . \tag{7}
\end{equation*}
$$

Moreover, identity (6) shows that $D_{n}$ can be interpreted as a weighted sum of rencontres numbers in the $n$-th row of the array, by means of relation (5),

$$
\begin{equation*}
D_{n}=\sum_{k=2}^{n}(k-1) D_{n}(k) . \tag{8}
\end{equation*}
$$

The number $D_{n} /(n-1)$ is also a weighted sum of previous consecutive derangement numbers. For example, $24+12 D_{2}+4 D_{3}+D_{4}=\frac{D_{6}}{5}$. In general we have

$$
\begin{equation*}
n!+\sum_{k=2}^{n} \frac{n!}{k!} D_{k}=\frac{D_{n+2}}{n+1}, \tag{9}
\end{equation*}
$$

as follows from Theorem 1 .
Theorem 1. For $n \in \mathbb{N}$ and the sequence of derangement numbers $\left(D_{n}\right)_{n \geq 0}$ we have

$$
\begin{equation*}
1+\sum_{k=1}^{n} \frac{D_{k}}{k!}=\frac{D_{n+2}}{(n+1)!} . \tag{10}
\end{equation*}
$$

Proof. Within a derangement $\sigma$, the number $k, k=1, \ldots, n$ can be mapped to any $j, j=1, \ldots, k-1, k+1, \ldots, n$. We let $\mathcal{A}_{n}$ denote the set of derangements with $\sigma(k)=j$, where $j \neq k$,

$$
\mathcal{A}_{n}:=\left\{\sigma \in S_{n}^{(0)}: \sigma(k)=j\right\} .
$$

Obviously, cardinality of the set $\mathcal{A}_{n}$ is invariant to the choice of $j, j \neq k$. More precisely,

$$
\left|\mathcal{A}_{n}\right|=\frac{D_{n}}{n-1} .
$$

Furthermore, we separate the set $\mathcal{A}_{n}$ into two disjoint sets of derangements, sets $\mathcal{B}_{n}$ and $\mathcal{C}_{n}$,

$$
\begin{aligned}
\mathcal{B}_{n} & :=\left\{\sigma \in \mathcal{A}_{n}: \sigma(j)=k\right\} \\
\mathcal{C}_{n} & :=\left\{\sigma \in \mathcal{A}_{n}: \sigma(j) \neq k\right\} .
\end{aligned}
$$

Obviously, the set $\mathcal{B}_{n}$ counts $D_{n-2}$ elements. For derangements in $\mathcal{C}_{n}$ there are now $(n-2)$ equivalent ways to map $j$ (excluding $j$ and $k$ ), as Figure $\mathbb{1}$ illustrates. Thus, we have

$$
\left|\mathcal{C}_{n}\right|=(n-2)\left|\mathcal{A}_{n-1}\right|,
$$

which gives the recurrence relation

$$
\begin{equation*}
\left|\mathcal{A}_{n}\right|=D_{n-2}+(n-2)\left|\mathcal{A}_{n-1}\right| . \tag{11}
\end{equation*}
$$

After repeating usage of (11) we get identity (9) which completes the proof.


Figure 1: In case of derangements in the set $\mathcal{C}_{n}$ there are $(n-2)$ equivalent ways to map $j$.

In order to prove Theorem 1 algebraically, we apply recurrence (2) to get

$$
\begin{aligned}
\frac{D_{n+2}}{(n+1)!} & =\frac{(n+1)\left(D_{n+1}+D_{n}\right)}{(n+1)!}=\frac{D_{n+1}}{n!}+\frac{D_{n}}{n!} \\
& =\frac{n\left(D_{n}+D_{n-1}\right)}{n!}+\frac{D_{n}}{n!}=\frac{D_{n}}{(n-1)!}+\frac{D_{n-1}}{(n-1)!}+\frac{D_{n}}{n!} \\
& =1+\frac{D_{1}}{1!}+\cdots+\frac{D_{n}}{n!}=1+\sum_{k=1}^{n} \frac{D_{k}}{k!} .
\end{aligned}
$$

Theorem 2. For $n \in \mathbb{N}$ and the sequence of derangement numbers $\left(D_{n}\right)_{n \geq 0}$ we have

$$
\begin{equation*}
1+\sum_{k=1}^{n} \frac{(-1)^{k} D_{k+3}}{k+2}=(-1)^{n} D_{n+2} . \tag{12}
\end{equation*}
$$

Proof. By applying recurrence (2) we have

$$
\begin{aligned}
\sum_{k=0}^{n} \frac{(-1)^{k} D_{k+3}}{k+2} & =\frac{2\left(D_{2}+D_{1}\right)}{2}-\frac{3\left(D_{3}+D_{2}\right)}{3}+\cdots(-1)^{n} \frac{(n+2)\left(D_{n+2}+D_{n+1}\right)}{n+2} \\
& =\left(D_{2}+D_{1}\right)-\left(D_{3}+D_{2}\right)+\cdots(-1)^{n}\left(D_{n+2}+D_{n+1}\right) \\
& =(-1)^{n} D_{n+2}
\end{aligned}
$$

which completes the proof.
Once having Theorem [1) substitution of (6) in identity (10) gives the generalization (13).

$$
\begin{equation*}
1+\sum_{k=1}^{n} \frac{D_{k+r}(r)}{k!\binom{k+r}{r}}=\frac{D_{n+r+2}(r)}{(n+1)!\binom{n+r+2}{r}} . \tag{13}
\end{equation*}
$$

The identity (14) follows by substitution of (6) in (12),

$$
\begin{equation*}
1+\sum_{k=1}^{n} \frac{(-1)^{k} D_{k+r+3}(r)}{(k+2)\binom{k++3}{r}}=\frac{(-1)^{n} D_{n+r+2}(r)}{\binom{n+r+2}{r}} . \tag{14}
\end{equation*}
$$

Note that the terms in identity (14) are always integers, which can be seen as a consequence of recurrence relation (22).

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