

ON THE NUMBER OF PRIMES UP TO THE n TH RAMANUJAN PRIME

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ABSTRACT. The n th Ramanujan prime is the smallest positive integer R_n such that for all $x \geq R_n$ the interval $(x/2, x]$ contains at least n primes. In this paper we undertake a study of the sequence $(\pi(R_n))_{n \in \mathbb{N}}$, which tells us where the n th Ramanujan prime appears in the sequence of all primes. In the first part we establish new explicit upper and lower bounds for the number of primes up to the n th Ramanujan prime, which imply an asymptotic formula for $\pi(R_n)$ conjectured by Yang and Togbé. In the second part of this paper, we use these explicit estimates to derive a result concerning an inequality involving $\pi(R_n)$ conjectured by of Sondow, Nicholson and Noe.

1. INTRODUCTION

Let $\pi(x)$ denotes the number of primes not exceeding x . In 1896, Hadamard [6] and de la Vallée-Poussin [15] proved, independently, the asymptotic formula $\pi(x) \sim x/\log x$ as $x \rightarrow \infty$, which is known as the *Prime Number Theorem*. Here, $\log x$ is the natural logarithm of x . In his later paper [16], where he proved the existence of a zero-free region for the Riemann zeta-function $\zeta(s)$ to the left of the line $\operatorname{Re}(s) = 1$, de la Vallée-Poussin also estimated the error term in the Prime Number Theorem by showing

$$(1.1) \quad \pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

The prime counting function and the asymptotic formula (1.1) play an important role in the definition of Ramanujan primes, which have their origin in Bertrand's postulate.

Bertrand's Postulate. *For each $n \in \mathbb{N}$ there is a prime number p with $n < p \leq 2n$.*

In terms of the prime counting function, Bertrand's postulate states that $\pi(2n) - \pi(n) \geq 1$ for every $n \in \mathbb{N}$. Bertrand's postulate was first proved by Chebyshev [4] in 1850. In 1919, Ramanujan [8] proved an extension of Bertrand's postulate by investigating inequalities of the form $\pi(x) - \pi(x/2) \geq n$ for $n \in \mathbb{N}$. In particular, he found that

$$\pi(x) - \pi\left(\frac{x}{2}\right) \geq 1 \quad (\text{respectively } 2, 3, 4, 5, \dots)$$

for every

$$(1.2) \quad x \geq 2 \quad (\text{respectively } 11, 17, 29, 41, \dots).$$

Using the fact that $\pi(x) - \pi(x/2) \rightarrow \infty$ as $x \rightarrow \infty$, which follows from (1.1), Sondow [10] introduced the notation R_n to represent the smallest positive integer for which the inequality $\pi(x) - \pi(x/2) \geq n$ holds for every $x \geq R_n$. In (1.2), Ramanujan calculated the numbers $R_1 = 2$, $R_2 = 11$, $R_3 = 17$, $R_4 = 29$, and $R_5 = 41$. All these numbers are prime, and it can easily be shown that R_n is actually prime for every $n \in \mathbb{N}$. In honor of Ramanujan's proof, Sondow [10] called the number R_n the n th Ramanujan prime. A legitimate question is, where the n th Ramanujan prime appears in the sequence of all primes. Letting p_k denotes the k th prime number, we have $R_n = p_{\pi(R_n)}$, and it seems natural to study the sequence $(\pi(R_n))_{n \in \mathbb{N}}$. The first few values of $\pi(R_n)$ for $n = 1, 2, 3, \dots$ are

$$\pi(R_n) = 1, 5, 7, 10, 13, 15, 17, 19, 20, 25, 26, 28, 31, 35, 36, 39, 41, 42, 49, 50, 51, 52, 53, \dots$$

For further values of $\pi(R_n)$, see [9]. Since both R_n for large n and $\pi(x)$ for large x are hard to compute, we are interested in explicit upper and lower bounds for $\pi(R_n)$. Sondow [10, Theorem 2] found a first lower bound for $\pi(R_n)$ by showing that the inequality

$$(1.3) \quad \pi(R_n) > 2n$$

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holds for every positive integer $n \geq 2$. Combined with [10, Theorem 3] and the Prime Number Theorem, we get the asymptotic relation

$$(1.4) \quad \pi(R_n) \sim 2n \quad (n \rightarrow \infty).$$

This, together with (1.3), means, roughly speaking, that the probability of a randomly chosen prime being a Ramanujan prime is slight less than $1/2$. The first upper bound for $\pi(R_n)$ is also due to Sondow [10, Theorem 2]. He found that the upper bound $\pi(R_n) < 4n$ holds for every positive integer n , and conjectured [10, Conjecture 1] that the inequality $\pi(R_n) < 3n$ holds for every positive integer n . This conjecture was proved by Laishram [7, Theorem 2] in 2010. Applying Theorem 4 from the paper of Sondow, Nicholson and Noe [11], we get a refined upper bound for the number of primes less or equal to $\pi(R_n)$, namely that the inequality $\pi(R_n) \leq \pi(41p_{3n}/47)$ holds for every positive integer n with equality at $n = 5$. Srinivasan [12, Theorem 1.1] proved that for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that

$$(1.5) \quad \pi(R_n) < \lfloor 2n(1 + \varepsilon) \rfloor$$

for every positive integer $n \geq N$ and conclude [12, Corollary 2.1] that $\pi(R_n) \leq 2.6n$ for every positive integer n . The present author [1, Theorem 3.22] showed independently that for each $\varepsilon > 0$ there is a computable positive integer $N = N(\varepsilon)$ so that $\pi(R_n) \leq \lfloor 2n(1 + \varepsilon) \rfloor$ for every positive integer $n \geq N$ and conclude that

$$(1.6) \quad \pi(R_n) \leq \lceil tn \rceil$$

for every positive integer n , where t is a arbitrary real number satisfying $t > 48/19$. The inequality (1.5) was improved by Srinivasan and Nicholson [14, Theorem 1]. They proved that

$$\pi(R_n) \leq 2n \left(1 + \frac{3}{\log n + \log \log n - 4} \right)$$

for every positive integer $n \geq 242$. Later, Srinivasan and Arés [13, Theorem 1.1] found a more precise result by showing that for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that

$$(1.7) \quad \pi(R_n) < 2n \left(1 + \frac{\log 2 + \varepsilon}{\log n + j(n)} \right)$$

for every positive integer $n \geq N$, where j is any positive function satisfying $j(n) \rightarrow \infty$ and $nj'(n) \rightarrow 0$ as $n \rightarrow \infty$. Setting $\varepsilon = 0.5$ and $j(n) = \log \log n - \log 2 - 0.5$, they found [13, Corollary] that the inequality (1.7) holds for every positive integer $n \geq 44$. In 2016, Yang and Togbé [17, Theorem 1.2] established the following current best upper and lower bound for $\pi(R_n)$ when n satisfies $n > 10^{300}$.

Proposition 1.1 (Yang, Togbé). *Let n be a positive integer with $n > 10^{300}$. Then*

$$\beta < \pi(R_n) < \alpha,$$

where

$$\alpha = 2n \left(1 + \frac{\log 2}{\log n} - \frac{\log 2 \log \log n - \log^2 2 - \log 2 - 0.13}{\log^2 n} \right),$$

$$\beta = 2n \left(1 + \frac{\log 2}{\log n} - \frac{\log 2 \log \log n - \log^2 2 - \log 2 + 0.11}{\log^2 n} \right).$$

The proof of Proposition 1.1 is based on explicit estimates for the k th prime number p_k obtained by Dusart [5, Proposition 6.6 and Proposition 6.7] and on Srinivasan's lemma [12, Lemma 2.1] concerning Ramanujan primes. Instead of using Dusart's estimates, we use the estimates obtained in [3, Corollary 1.2 and Corollary 1.4] to get the following improved upper bound for $\pi(R_n)$.

Theorem 1.2. *Let n be a positive integer satisfying $n \geq 5225$ and let*

$$(1.8) \quad U(x) = \frac{\log 2 \log x (\log \log x)^2 - c_1 \log x \log \log x + c_2 \log x - \log^2 2 \log \log x + \log^3 2 + \log^2 2}{\log^4 x + \log^3 x \log \log x - \log^3 x \log 2 - \log^2 x \log 2},$$

where $c_1 = 2 \log^2 2 + \log 2$ and $c_2 = \log^3 2 + 2 \log^2 2 + 0.565$. Then

$$\pi(R_n) < 2n \left(1 + \frac{\log 2}{\log n} - \frac{\log 2 \log \log n - \log^2 2 - \log 2}{\log^2 n} + U(n) \right).$$

With the same method, we used for the proof of Theorem 1.2, we get the following more precised lower bound for the number of primes not exceeding the n th Ramanujan prime.

Theorem 1.3. *Let n be a positive integer satisfying $n \geq 1\,245$ and let*

$$(1.9) \quad L(x) = \frac{\log 2 \log x (\log \log x)^2 - d_1 \log x \log \log x + d_2 \log x - \log^2 2 \log \log x + \log^3 2 + \log^2 2}{\log^4 x + \log^3 x \log \log x - \log^3 x \log 2 - \log^2 x \log 2},$$

where $d_1 = 2 \log^2 2 + \log 2 + 1.472$ and $d_2 = \log^3 2 + 2 \log^2 2 - 2.51$. Then

$$\pi(R_n) > 2n \left(1 + \frac{\log 2}{\log n} - \frac{\log 2 \log \log n - \log^2 2 - \log 2}{\log^2 n} + L(n) \right).$$

A direct consequence of Theorem 1.2 and Theorem 1.3 is the following result, which implies the correctness of a conjecture stated by Yang and Togbé [17, Conjecture 5.1] in 2015.

Corollary 1.4. *Let $n \geq 2$ be a positive integer. Then*

$$\pi(R_n) = 2n \left(1 + \frac{\log 2}{\log n} - \frac{\log 2 \log \log n - \log^2 2 - \log 2}{\log^2 n} + \frac{\log 2 (\log \log n)^2}{\log^3 n} + O \left(\frac{\log \log n}{\log^3 n} \right) \right).$$

The initial motivation for writing this paper, was the following conjecture stated by Sondow, Nicholson and Noe [11, Conjecture 1] involving $\pi(R_n)$.

Conjecture 1.5 (Sondow, Nicholson, Noe). *For $m = 1, 2, 3, \dots$, let $N(m)$ be given by the following table:*

m	1	2	3	4	5	6	7, 8, ..., 19	20, 21, ...
$N(m)$	1	1245	189	189	85	85	10	2

Then we have

$$(1.10) \quad \pi(R_{mn}) \leq m\pi(R_n) \quad \forall n \geq N(m).$$

Note that the inequality (1.10) clearly holds for $m = 1$ and every positive integer n . In the cases $m = 2, 3, \dots, 20$, the inequality (1.10) has been verified for every positive integer n with $R_{mn} < 10^9$. For any fixed positive integer m , we have, by (1.4), $\pi(R_{mn}) \sim 2mn \sim m\pi(R_n)$ as $n \rightarrow \infty$. A first result in the direction of Conjecture 1.5 is due to Yang and Togbé [17, Theorem 1.3]. They used Proposition 1.1 to find the following result, which proves Conjecture 1.5 when n satisfies $n > 10^{300}$.

Proposition 1.6 (Yang, Togbé). *For $m = 1, 2, 3, \dots$, and $n > 10^{300}$, we have*

$$\pi(R_{mn}) \leq m\pi(R_n).$$

Using the same method, we apply Theorem 1.2 and Theorem 1.3 to get the following result.

Theorem 1.7. *The Conjecture 1.5 of Sondow, Nicholson and Noe holds except for $(m, n) = (38, 9)$.*

2. PRELIMINARIES

Let n be a positive integer. For the proof of Theorem 1.2 and Theorem 1.3, we need sharp estimates for the n th prime number. The current best upper and lower bound for the n th prime number were obtained in [3, Corollary 1.2 and Corollary 1.4] and are given as follows.

Lemma 2.1. *For every positive integers $n \geq 46\,254\,381$, we have*

$$p_n < n \left(\log n + \log \log n - 1 + \frac{\log \log n - 2}{\log n} - \frac{(\log \log n)^2 - 6 \log \log n + 10.667}{2 \log^2 n} \right).$$

Lemma 2.2. *For every positive integer $n \geq 2$, we have*

$$p_n > n \left(\log n + \log \log n - 1 + \frac{\log \log n - 2}{\log n} - \frac{(\log \log n)^2 - 6 \log \log n + 11.508}{2 \log^2 n} \right).$$

3. PROOF OF THEOREM 1.2

To prove Theorem 1.2, we use the method investigated by Yang and Togbé [17] for the proof of the upper bound for $\pi(R_n)$ given in Proposition 1.1. First, we note following result, which was obtained by Srinivasan [12, Lemma 2.1]. Although it is a direct consequence of the definition of a Ramanujan prime, it plays an important role in the proof of the upper bound for $\pi(R_n)$ in Proposition 1.1.

Lemma 3.1 (Srinivasan). *Let $R_n = p_s$ be the n th Ramanujan prime. Then we have $2p_{s-n} < p_s$ for every positive integer $n \geq 2$.*

Now, let n be a positive integer. We define for each real x with $2n < x < 2.6n$ the functions

$$(3.1) \quad G(x) = x \left(\log x + \log \log x - 1 + \frac{\log \log x - 2}{\log x} - \frac{(\log \log x)^2 - 6 \log \log x + 10.667}{2 \log^2 x} \right)$$

and

$$(3.2) \quad H(x) = x \left(\log x + \log \log x - 1 + \frac{\log \log x - 2}{\log x} - \frac{(\log \log x)^2 - 6 \log \log x + 11.508}{2 \log^2 x} \right),$$

and consider the function $F_1 : (2n, 2.6n) \rightarrow \mathbb{R}$ defined by

$$(3.3) \quad F_1(x) = G(x) - 2H(x - n).$$

In the following proposition, we note a first property of the function $F_1(x)$ concerning its derivative.

Proposition 3.2. *Let n be a positive integer with $n \geq 16$. Then $F_1(x)$ is a strictly decreasing function on the interval $(2n, 2.6n)$.*

Proof. Setting

$$q_1(x) = \frac{\log \log x - 2}{\log x} - \frac{(\log \log x)^2 - 4 \log \log x + 4.667}{2 \log^2 x} + \frac{(\log \log x)^2 - 7 \log \log x + 13.667}{\log^3 x}$$

and

$$r_1(x) = -\frac{2(\log \log(x - n) - 1)}{\log(x - n)} + \frac{(\log \log(x - n))^2 - 4 \log \log(x - n) + 5.508}{2 \log^2(x - n)} - \frac{2(\log \log(x - n))^2 - 14 \log \log(x - n) + 29.016}{\log^3(x - n)},$$

a straightforward calculation shows that the derivative of $F_1(x)$ is given by

$$F_1'(x) = \log x - 2 \log(x - n) + \log \log x - 2 \log \log(x - n) + \frac{1}{\log x} + q_1(x) + r_1(x).$$

Note that $\log \log(x - n) \geq 1$, $t^2 - 4t + 4.667 > 0$ and $2t^2 - 14t + 29.016 > 0$ for every $t \in \mathbb{R}$. Hence

$$F_1'(x) < \log x - 2 \log(x - n) + \log \log x - 2 \log \log(x - n) + \frac{1}{\log x} + \frac{\log \log x - 2}{\log x} + \frac{(\log \log x)^2 - 7 \log \log x + 13.667}{\log^3 x} + \frac{(\log \log(x - n))^2 - 4 \log \log(x - n) + 5.508}{\log^2(x - n)}.$$

The function $t \mapsto (\log \log t - 2)/\log t$ has a global maximum at $t = \exp(\exp(3))$. Together with $32 \leq 2n < x < 2.6n$, and the fact that the functions $t \mapsto ((\log \log t)^2 - 7 \log \log t + 13.667)/\log^3 t$ and $t \mapsto ((\log \log t)^2 - 4 \log \log t + 5.508)/\log^3 t$ are monotonic decreasing for every $t > 1$, we obtain that

$$F_1'(x) < 1.772 - \log n + \log \log(2.6n) - \log(\log^2 n).$$

Finally, we use the fact that $t \log^2 t > e^{1.772} \log(2.6t)$ for every $t \geq 6$ to get $F_1'(x) < 0$ for every $x \in (2n, 2.6n)$, which means that $F_1(x)$ is a strictly decreasing function on the interval $(2n, 2.6n)$. \square

Next, we define the function $\gamma : \mathbb{R}_{\geq 4} \rightarrow \mathbb{R}$ by

$$(3.4) \quad \gamma(x) = \frac{\log 2 + \log 2 / \log x + 0.565 / \log^2 x}{\log x + \log \log x - \log 2 - \log 2 / \log x}.$$

A simple calculation shows that

$$(3.5) \quad \gamma(x) = \frac{\log 2}{\log x} - \frac{\log 2 \log \log x - \log^2 2 - \log 2}{\log^2 x} + U(x),$$

where $U(x)$ is defined as in (1.8). In the following lemma, we note some useful properties of $\gamma(x)$.

Lemma 3.3. *Let $\gamma(x)$ be defined as in (3.4). Then the following hold:*

- (a) $\gamma(x) > 0$ for every $x \geq 8$,
- (b) $\gamma(x) < \log 2 / \log x$ for every $x \geq 10\,734$,
- (c) $\gamma(x) < 1/4$ for every $x \geq 10\,734$.

Proof. The statement in (a) is clear. To prove (b), we first note that $U(x) < \log 2(\log \log x)^2 / \log^3 x$ for every $x \geq 230 \geq \exp(\exp(1 + \log 2))$. Now we use (3.5) and the fact that $(\log \log x - \log 2 - 1) \log x \geq (\log \log x)^2$ for every $x \geq 10\,734$, to conclude (b). Finally, (c) is a direct consequence of (b). \square

Now, we give a proof of Theorem 1.2.

Proof of Theorem 1.2. First, we consider the case where n is a positive integer with $n \geq 528\,491\,312 \geq \exp(\exp(3))$. By (1.3) and (1.6), we have $2n < \pi(R_n) < 2.6n$. Hence $\pi(R_n) \geq 2n \geq 1\,056\,982\,624$ and $\pi(R_n) - n \geq 528\,491\,312$. Now we apply Lemma 2.1 and Lemma 2.2 to get that $F_1(\pi(R_n)) > p_{\pi(R_n)} - 2p_{\pi(R_n)-n}$, where F_1 is defined as in (3.3). Since $R_n = p_{\pi(R_n)}$, Srinivasan's Lemma 3.1 yields

$$(3.6) \quad F_1(\pi(R_n)) > 0.$$

For convenience, we write in the following $\gamma = \gamma(n)$ and $\alpha = 2n(1 + \gamma)$. Now, by (3.5), we need to show that $\pi(R_n) < \alpha$. For this, we first show that $F_1(\alpha) < 0$. By Lemma 3.3, we have $2n < \alpha < 2.6n$. Further,

$$(3.7) \quad \frac{F_1(\alpha)}{2n} = (1 + \gamma) \log 2 - \gamma \log n + \gamma + A_1 + B_1 + C_1 + D_1 + (1 + 2\gamma) \frac{0.841}{2 \log^2(n + 2n\gamma)},$$

where

$$\begin{aligned} A_1 &= (1 + \gamma) \log(1 + \gamma) - (1 + 2\gamma) \log(1 + 2\gamma), \\ B_1 &= (1 + \gamma) \log \log(2n + 2n\gamma) - (1 + 2\gamma) \log \log(n + 2n\gamma), \\ C_1 &= (1 + \gamma) \frac{\log \log(2n + 2n\gamma) - 2}{\log(2n + 2n\gamma)} - (1 + 2\gamma) \frac{\log \log(n + 2n\gamma) - 2}{\log(n + 2n\gamma)}, \\ D_1 &= -(1 + \gamma) \frac{(\log \log(2n + 2n\gamma))^2 - 6 \log \log(2n + 2n\gamma) + 10.667}{2 \log^2(2n + 2n\gamma)} \\ &\quad + (1 + 2\gamma) \frac{(\log \log(n + 2n\gamma))^2 - 6 \log \log(n + 2n\gamma) + 10.667}{2 \log^2(n + 2n\gamma)}. \end{aligned}$$

In the following, we give upper bounds for the quantities A_1 , B_1 , C_1 and D_1 . We start with A_1 . We use the inequalities

$$(3.8) \quad t - \frac{t^2}{2} < \log(1 + t) < t,$$

which hold for every real $t > 0$, and Lemma 3.3(c) to get

$$(3.9) \quad A_1 < (1 + \gamma)\gamma - (1 + 2\gamma)(2\gamma - 2\gamma^2) = -\gamma - \gamma^2 + 4\gamma^3 < -\gamma.$$

Next, we estimate B_1 . Using the right-hand side inequality of (3.8), we easily get

$$(3.10) \quad B_1 < \frac{(1 + \gamma) \log 2}{\log n} - \gamma \log \log n.$$

To find an upper bound for C_1 , we note that $t \mapsto (\log \log t - 2)/\log t$ is a decreasing function on the interval $(\exp(\exp(3)), \infty)$. Together with Lemma 3.3(a), we obtain that the inequality

$$(3.11) \quad C_1 < 0$$

holds. Finally, we estimate D_1 . For this purpose, we consider the function $f : (1, \infty) \rightarrow \mathbb{R}$ defined by

$$f(x) = \frac{(\log \log x)^2 - 6 \log \log x + 10.667}{2 \log^2 x}.$$

By the mean value theorem, there exists a real number $\xi \in (n + 2n\gamma, 2n + 2n\gamma)$ such that $f(2n + 2n\gamma) - f(n + 2n\gamma) = nf'(\xi)$. Since $f''(x) \geq 0$ for every $x > 1$, we get $f'(\xi) \geq f'(n + 2n\gamma) \geq f'(n)$. Hence we get

$$f(n + 2n\gamma) - f(2n + 2n\gamma) = -nf'(\xi) \leq -nf'(n) = \frac{(\log \log n)^2 - 7 \log \log n + 13.667}{\log^3 n}.$$

Therefore

$$D_1 < (1 + \gamma) \frac{(\log \log n)^2 - 7 \log \log n + 13.667}{\log^3 n} + \gamma f(n + 2n\gamma).$$

Since $f(x)$ is a strictly decreasing function on the interval $(1, \infty)$, it follows that the inequality

$$(3.12) \quad D_1 < (1 + \gamma) \frac{(\log \log n)^2 - 7 \log \log n + 13.667}{\log^3 n} + \gamma \frac{(\log \log n)^2 - 6 \log \log n + 10.667}{2 \log^2 n}$$

holds. Combining (3.7) with (3.9)-(3.12), we get

$$\begin{aligned} \frac{F_1(\alpha)}{2n} &< (1 + \gamma) \log 2 - \gamma \log n + \frac{(1 + \gamma) \log 2}{\log n} - \gamma \log \log n + (1 + \gamma) \frac{r_1(\log \log n)}{\log^3 n} \\ &\quad + \gamma \frac{r_2(\log \log n)}{2 \log^2 n} + (1 + 2\gamma) \frac{0.841}{2 \log^2 n}, \end{aligned}$$

where $r_1(t) = t^2 - 7t + 13.667$ and $r_2(t) = t^2 - 6t + 10.667$. The functions $t \mapsto r_1(\log \log t)/\log t$, $t \mapsto r_1(\log \log t)/\log^2 t$ and $t \mapsto r_2(\log \log t)/\log t$ are decreasing on the interval $(1, \infty)$. Hence $r_1(\log \log n) \leq r_1(3)$ and $r_2(\log \log n) \leq r_2(3)$. Together with Lemma 3.3(a), Lemma 3.3(b) and $n \geq \exp(\exp(3))$, we obtain that

$$\frac{F_1(\alpha)}{2n} < (1 + \gamma) \log 2 - \gamma \log n + \frac{(1 + \gamma) \log 2}{\log n} - \gamma \log \log n + \frac{0.565}{\log^2 n}.$$

Now we use (3.4) to get that the right-hand side of the last inequality is equal to 0. Hence $F_1(\alpha) < 0$. Together with $2n < \pi(R_n)$, $\alpha < 2.6n$, the inequality (3.6) and Proposition 3.2, we get $\pi(R_n) < \alpha$. We conclude by direct computation. \square

We get the following weaker but more compact upper bounds for the parameter s .

Corollary 3.4. *For every positive integer $n \geq 2$, we have*

$$\pi(R_n) < 2n \left(1 + \frac{\log 2}{\log n} - \frac{\log 2 \log \log n - \log^2 2 - \log 2}{\log^2 n} + \frac{\log 2 (\log \log n)^2}{\log^3 n} \right).$$

Proof. If $n \geq 5225$, the corollary follows directly from Theorem 1.2, since $U(x) \leq \log 2 (\log \log x)^2 / \log^3 x$ for every $x \geq 230$. For the remaining cases of n , we use a computer. \square

In the next corollary, we reduce the number 10^{300} in Proposition 1.1 as follows.

Corollary 3.5. *For every positive integer n satisfying $n \geq 4\,842\,763\,560\,306$, we have*

$$\pi(R_n) < 2n \left(1 + \frac{\log 2}{\log n} - \frac{\log 2 \log \log n - \log^2 2 - \log 2 - 0.13}{\log^2 n} \right).$$

Proof. Note that $U(x) \leq 0.13/\log^2 x$ for every $x \geq 4\,842\,763\,560\,306$. Now we can use Theorem 1.2. \square

Corollary 3.6. *Let n be a positive integer satisfying $n \geq 640$. Then*

$$\pi(R_n) < 2n \left(1 + \frac{\log 2}{\log n} \right).$$

Proof. For every positive integer $n \geq 10\,734$, we have

$$\frac{\log 2 \log \log n - \log^2 2 - \log 2}{\log^2 n} - \frac{\log 2 (\log \log n)^2}{\log^3 n} > 0$$

and it suffices to apply Corollary 3.4. We conclude by direct computation. \square

4. PROOF OF THEOREM 1.3

Using a similar argument as in the proof of Lemma 3.1, Yang and Togbé [17, p. 248] derived the following result.

Lemma 4.1 (Yang, Togbé). *Let $R_n = p_s$ be the n th Ramanujan prime. Then we have $p_s < 2p_{s-n+1}$ for every positive integer n .*

Next, we define for each positive integer n the function $F_2 : (2n, 2.6n) \rightarrow \mathbb{R}$ by

$$(4.1) \quad F_2(x) = H(x) - 2G(x - n + 1),$$

where the functions $G(x)$ and $H(x)$ are given by (3.1) and (3.2), respectively. In Proposition 3.2, we showed that for every positive integer $n \geq 16$, the function $F_1(x)$ is decreasing on the interval $(2n, 2.6n)$. In the following proposition, we get a similar result for the function $F_2(x)$.

Proposition 4.2. *Let n be a positive integer with $n \geq 15$. Then $F_2(x)$ is a strictly decreasing function on the interval $(2n, 2.6n)$.*

Proof. A straightforward calculation shows that the derivative of $F_2(x)$ is given by

$$\begin{aligned} F_2'(x) &= \log x - 2 \log(x - n + 1) + \log \log x - 2 \log \log(x - n + 1) + \frac{1}{\log x} + \frac{\log \log x - 2}{\log x} \\ &\quad - \frac{(\log \log x)^2 - 4 \log \log x + 5.508}{2 \log^2 x} + \frac{(\log \log x)^2 - 7 \log \log x + 14.508}{\log^3 x} \\ &\quad - \frac{2(\log \log(x - n + 1) - 1)}{\log(x - n + 1)} + \frac{(\log \log(x - n + 1))^2 - 4 \log \log(x - n + 1) + 4.667}{2 \log^2(x - n + 1)} \\ &\quad - \frac{2(\log \log(x - n + 1))^2 - 14 \log \log(x - n + 1) + 27.334}{\log^3(x - n + 1)}. \end{aligned}$$

Now we argue as in the proof of Proposition 3.2 to obtain that the inequality

$$F_2'(x) < 1.717 - \log n + \log \log(2.6n) - \log(\log^2 n)$$

holds for every real x such that $2n < x < 2.6n$. Since $t \log^2 t > e^{1.717} \log(2.6t)$ for every $t \geq 6$, we get that $F_2(x)$ is a strictly decreasing function on the interval $(2n, 2.6n)$. \square

Now, we define the function $\delta : \mathbb{R}_{\geq 4} \rightarrow \mathbb{R}$ by

$$(4.2) \quad \delta(x) = \frac{\log 2 + \log 2 / \log x - (1.472 \log \log x + 2.51) / \log^2 x}{\log x + \log \log x - \log 2 - \log 2 / \log x}.$$

A simple calculation shows that

$$(4.3) \quad \delta(x) = \frac{\log 2}{\log x} - \frac{\log 2 \log \log x - \log^2 2 - \log 2}{\log^2 x} + L(x),$$

where $L(x)$ is given by (1.9). In the following lemma, we note two properties of the function $\delta(x)$, which will be useful in the proof of Theorem 1.3.

Lemma 4.3. *Let $\delta(x)$ be defined as in (4.2). Then the following two inequalities hold:*

- (a) $\delta(x) > 0.638 / \log x$ for every $x \geq \exp(\exp(3))$,
- (b) $\delta(x) < \log 2 / \log x$ for every $x \geq 230$.

Proof. Since $0.055 \log x + 0.812 > 0.638 \log \log x$ for every $x \geq 4.71 \cdot 10^8$, it follows that the inequality

$$(\log 2 - 0.638) \log x + (1 + 0.638) \log 2 - \frac{1.472 \cdot 3 + 2.51 - 0.638 \log 2}{e^3} > 0.638 \log \log x$$

holds for every $x \geq 4.71 \cdot 10^8$. The function $t \mapsto \log \log t / \log t$ is decreasing for $x \geq e^e$. Hence

$$(\log 2 - 0.638) \log x + (1 + 0.638) \log 2 - \frac{1.472 \log \log x + 2.51 - 0.638 \log 2}{\log x} > 0.638 \log \log x$$

for every $x \geq \exp(\exp(3))$. Now it suffices to note that the last inequality is equivalent to $\delta(x) > 0.638 / \log x$. This proves (a). Next, we prove (b). Since $\log 2 \log \log x > \log 2 + \log^2 2$ for every $x \geq 230 \geq \exp(\exp(1 + \log 2))$, we obtain that the inequality

$$\log 2 + \log^2 2 < \log 2 \log \log x + \frac{1.472 + 2.51 - \log^2 2}{\log x}$$

holds for $x \geq 230$. Again, it suffices to note that the last inequality is equivalent to $\delta(x) < \log 2 / \log x$. \square

Finally, we give the proof of Theorem 1.3.

Proof of Theorem 1.3. First, we consider the case where n is a positive integer with $n \geq 528\,491\,312 \geq \exp(\exp(3))$. By (1.3) and (1.6), we have $2n < \pi(R_n) < 2.6n$. Further, $\pi(R_n) > 2n \geq 1\,056\,982\,624$ and $\pi(R_n) - n > 528\,491\,312$. Applying Lemma 2.1 and Lemma 2.2, we get $F_2(\pi(R_n)) < p_\pi(R_n) - 2p_{\pi(R_n)-n+1}$, where F_2 is defined as in (4.1). Note that $R_n = p_{\pi(R_n)}$. Hence, by Lemma 4.1, we get

$$(4.4) \quad F_2(\pi(R_n)) < p_{\pi(R_n)} - 2p_{\pi(R_n)-n+1} < 0.$$

In the following, we use, for convenience, the notation $\delta = \delta(n)$ and write $\beta = 2n(1 + \delta)$. So, by (4.3), we need to prove that $\beta < \pi(R_n)$. For this purpose, we first show that $F_2(\beta) > 0$. From Lemma 4.3, it follows that $2n < \beta < 2.6n$. Furthermore, we have

$$(4.5) \quad \frac{F_2(\beta)}{2n} = (1 + \delta) \log 2 - \delta \log n - \frac{\log n}{n} + \delta + \frac{1}{n} + A_2 + B_2 + C_2 + D_2 - \frac{0.841(1 + \delta)}{2 \log^2(2n + 2n\delta)},$$

where the quantities A_2 , B_2 , C_2 and D_2 are given by

$$\begin{aligned} A_2 &= (1 + \delta) \log(1 + \delta) - \left(1 + 2\delta + \frac{1}{n}\right) \log\left(1 + 2\delta + \frac{1}{n}\right), \\ B_2 &= (1 + \delta) \log \log(2n + 2n\delta) - \left(1 + 2\delta + \frac{1}{n}\right) \log \log(n + 2n\delta + 1), \\ C_2 &= (1 + \delta) \frac{\log \log(2n + 2n\delta) - 2}{\log(2n + 2n\delta)} - \left(1 + 2\delta + \frac{1}{n}\right) \frac{\log \log(n + 2n\delta + 1) - 2}{\log(n + 2n\delta + 1)}, \\ D_2 &= -(1 + \delta) \frac{(\log \log(2n + 2n\delta))^2 - 6 \log \log(2n + 2n\delta) + 10.667}{2 \log^2(2n + 2n\delta)} \\ &\quad + \left(1 + 2\delta + \frac{1}{n}\right) \frac{(\log \log(n + 2n\delta + 1))^2 - 6 \log \log(n + 2n\delta + 1) + 10.667}{2 \log^2(n + 2n\delta + 1)}. \end{aligned}$$

To show that $F_2(\beta) > 0$, we give in the following some lower bounds for the quantities A_2 , B_2 , C_2 and D_2 . To find a lower bound for A_2 , we consider the function $f : (0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) = x \log x$. Then $A_2 = f(1 + \delta) - f(1 + 2\delta + 1/n)$. By the mean value theorem, there exists $\xi \in (1 + \delta, 1 + 2\delta + 1/n)$, so that $A_2 = -(\delta + 1/n)(\log \xi + 1)$. Since $\log \xi \leq \log(1 + 2\delta + 1/n) \leq 2\delta + 1/n$, we get

$$A_2 \geq -\delta - 2\delta^2 - \frac{1}{n} \left(1 + 3\delta + \frac{1}{n}\right).$$

Applying Lemma 4.3(b) to the last inequality, we obtain that

$$(4.6) \quad A_2 \geq -\delta - \frac{2 \log^2 2}{\log^2 n} - \frac{1}{\log^2 n} \frac{(1 + 3\delta + 1/n) \log^2 n}{n} \geq -\delta - \frac{0.961}{\log^2 n}.$$

Our next goal is to estimate B_2 . For this purpose, we use the right-hand side inequality of (3.8), Lemma 4.3(b) and the inequality $1/(x \log x) < 0.0037/\log^2 x$, which holds for every $x \geq 2036$, to get

$$(4.7) \quad \log \log(n + 2n\delta + 1) < \log \log n + \frac{2 \log 2}{\log^2 n} + \frac{1}{n \log n} < \log \log n + \frac{1.39}{\log^2 n}.$$

On the other hand, we have

$$\log \log(2n + 2n\delta) = \log \log n + \log \left(1 + \frac{\log 2 + \log(1 + \delta)}{\log n}\right).$$

Applying the left-hand side inequality of (3.8), we obtain that

$$\log \log(2n + 2n\delta) \geq \log \log n + \frac{\log 2}{\log n} + \frac{\log(1 + \delta)}{\log n} - \frac{(\log 2 + \log(1 + \delta))^2}{2 \log^2 n}.$$

Combined with

$$(\log 2 + \log(1 + \delta))^2 \leq (\log 2 + \delta)^2 \leq \left(\log 2 + \frac{\log 2}{\log n}\right)^2 \leq 0.53,$$

it follows that the inequality

$$\log \log(2n + 2n\delta) \geq \log \log n + \frac{\log 2}{\log n} + \frac{\log(1 + \delta)}{\log n} - \frac{0.265}{\log^2 n}$$

holds. Again, we use the left-hand side inequality of (3.8) to establish

$$\log \log(2n + 2n\delta) \geq \log \log n + \frac{\log 2}{\log n} + \frac{\delta - \delta^2/2}{\log n} - \frac{0.265}{\log^2 n}.$$

Now we apply Lemma 4.3(a) and Lemma 4.3(b) to obtain that

$$\log \log(2n + 2n\delta) \geq \log \log n + \frac{\log 2}{\log n} + \frac{0.361}{\log^2 n}.$$

Together with the definition of B_2 and (4.6), we get

$$B_2 \geq -\delta \log \log n + \frac{(1 + \delta) \log 2}{\log n} - \frac{\log \log n}{n} - \frac{1.029 + 2.419\delta}{\log^2 n} - \frac{1.39}{n \log^2 n}.$$

Finally, we use a computer and Lemma 4.3(b) to get

$$(4.8) \quad B_2 \geq -\delta \log \log n + \frac{(1 + \delta) \log 2}{\log n} - \frac{1.113}{\log^2 n}.$$

Next, we find an lower bound for C_2 . For this, we apply the inequality

$$\frac{2(1+2\delta+1/n)}{\log(n+2n\delta+1)} \geq \frac{2(1+\delta)}{\log(2n+2n\delta)}$$

to the definition of C_2 to get

$$C_2 \geq (1+\delta) \frac{\log \log(2n+2n\delta)}{\log(2n+2n\delta)} - \left(1+2\delta+\frac{1}{n}\right) \frac{\log \log(n+2n\delta+1)}{\log(n+2n\delta+1)}.$$

We use $2n+2n\delta \geq n+2n\delta+1 \geq n$ to obtain that the inequality

$$C_2 \geq -\log \log(n+2n\delta+1) \frac{(\delta+1/n) \log n + (1+2\delta+1/n)(\log 2 + \log(1+\delta))}{\log(2n+2n\delta) \log(n+2n\delta+1)}$$

holds. Applying the right-hand side inequality of (3.8) and Lemma 4.3(b) to the last inequality, we get

$$C_2 \geq -\log \log(n+2n\delta+1) \frac{(\log 2/\log n + 1/n) \log n + (1+2\log 2/\log n + 1/n)(\log 2 + \log 2/\log n)}{\log(2n+2n\delta) \log(n+2n\delta+1)}.$$

A computation shows that

$$\left(1 + \frac{2\log 2}{\log n} + \frac{1}{n}\right) \left(\log 2 + \frac{\log 2}{\log n}\right) \leq 0.778.$$

Hence

$$C_2 \geq -\frac{(\log 2 + 0.778) \log \log(n+2n\delta+1)}{\log(2n+2n\delta) \log(n+2n\delta+1)} - \frac{\log n \log \log(n+2n\delta+1)}{n \log(2n+2n\delta) \log(n+2n\delta+1)}.$$

Note that the function $t \mapsto \log \log t / \log t$ is a decreasing function for every $t > e^e$, we obtain that

$$(4.9) \quad C_2 \geq -\frac{(\log 2 + 0.778) \log \log n}{\log^2 n} - \frac{\log \log n}{n \log n} \geq -\frac{1.472 \log \log n}{\log^2 n}.$$

Finally, we estimate D_2 . For this purpose, we consider the function $f : (1, \infty) \rightarrow \mathbb{R}$ defined by

$$f(x) = \frac{(\log \log x)^2 - 6 \log \log x + 10.667}{2 \log^2 x}.$$

Note that $f(x)$ is a strictly decreasing function on the interval $(1, \infty)$ and the numerator of $f(x)$ is positive for every real $x > 1$. Together with $2n+2n\delta \geq n+2n\delta+1 \geq n$, we get

$$(4.10) \quad D_2 \geq \left(\delta + \frac{1}{n}\right) \frac{(\log \log n)^2 - 6 \log \log n + 10.667}{2 \log^2 n} > 0.$$

Finally, we combine (4.5) with (4.6) and (4.8)-(4.10) to get that the inequality

$$\begin{aligned} \frac{F_2(\beta)}{2n} &> (1+\delta) \left(\log 2 + \frac{\log 2}{\log n}\right) - \delta \log n - \frac{\log n - 1}{n} - \frac{1.472 \log \log n + 2.4945}{\log^2 n} - \delta \log \log n - \frac{0.841\delta}{2 \log^2 n} \\ &\geq \delta(-\log n - \log \log n + \log 2 + \log 2/\log n) + \log 2 - \frac{1.472 \log \log n + 2.51}{\log^2 n} + \frac{\log 2}{\log n} \end{aligned}$$

holds. Now it suffices to use (4.2) to get that the right-hand side of the last inequality is equal to 0 and it follows that $F_2(\beta) > 0$. Together with $2n < \pi(R_n)$, $\beta < 2.6n$, the inequality (4.4) and Proposition 4.2, we obtain that $\pi(R_n) > \beta$ for every positive integer $n \geq 528\,491\,312$. We conclude by direct computation. \square

Since $L(x) \geq 0$ for every $x \geq 10^{57}$, we use Theorem 1.3 to get the following weaker but more compact lower bound for $\pi(R_n)$.

Corollary 4.4. *Let n be a positive integer satisfying $n \geq 10^{57}$. Then*

$$\pi(R_n) > 2n \left(1 + \frac{\log 2}{\log n} - \frac{\log 2 \log \log n - \log^2 2 - \log 2}{\log^2 n}\right).$$

In the next corollary, we use Theorem 1.3 to find that the lower bound for $\pi(R_n)$ given in Proposition 1.1 also holds for every positive integer n satisfying $51\,396\,214\,158\,824 \leq n \leq 10^{300}$.

Corollary 4.5. *Let n be a positive integer satisfying $n \geq 51\,396\,214\,158\,824$. Then*

$$\pi(R_n) > 2n \left(1 + \frac{\log 2}{\log n} - \frac{\log 2 \log \log n - \log^2 2 - \log 2 + 0.11}{\log^2 n}\right).$$

Proof. The claim follows directly by Theorem 1.3 and the fact that $L(x) \geq -0.11/\log^2 x$ for every $x \geq 51\,396\,214\,158\,824$. \square

Finally, we give the following result concerning a lower bound for $\pi(R_n)$.

Corollary 4.6. *Let n be a positive integer satisfying $n \geq 85$. Then*

$$\pi(R_n) > 2n \left(1 + \frac{\log 2}{\log n} - \frac{\log 2 \log \log n}{\log^2 n} \right).$$

Proof. Since $L(x) + (\log^2 2 + \log 2)/\log^2 x \geq 0$ for every $x \geq 20$, we apply Theorem 1.3 to get the correctness of the corollary for every positive integer $n \geq 1245$. We conclude by direct computation. \square

5. PROOF OF THEOREM 1.7

In this section we give a proof of Theorem 1.7 by using Theorem 3.22 of [1]. For this, we need to introduce the following notations. By [2, Corollary 3.4 and Corollary 3.5], we have

$$(5.1) \quad \frac{x}{\log x - 1 - \frac{1}{\log x}} < \pi(x) < \frac{x}{\log x - 1 - \frac{1.17}{\log x}},$$

where the left-hand side inequality is valid for every $x \geq 468\,049$ and the right-hand side inequality holds for every $x \geq 5.43$. Using the right-hand side inequality of (5.1), we get $p_n > n(\log p_n - 1 - 1.17/\log p_n)$ for every positive integer n . In addition, we set $\varepsilon > 0$ and $\lambda = \varepsilon/2$. Let $S = S(\varepsilon)$ be defined by

$$S = \exp \left(\sqrt{1.17 + \frac{2(1+\varepsilon)}{\varepsilon} \left(0.17 + \frac{\log 2}{\log(2 \cdot 5.43)} \right) + \left(\frac{1}{2} + \frac{(1+\varepsilon)\log 2}{\varepsilon} \right)^2} + \frac{1}{2} + \frac{(1+\varepsilon)\log 2}{\varepsilon} \right)$$

and let $T = T(\varepsilon)$ be defined by $T = \exp(1/2 + \sqrt{1.17 + 0.17/\lambda + 1/4})$. By setting $X_9 = X_9(\varepsilon) = \max\{468\,049, 2S, T\}$, we get the following result.

Lemma 5.1. *Let $\varepsilon > 0$. For every positive integer n satisfying $n \geq (\pi(X_9) + 1)/(2(1 + \varepsilon))$, we have*

$$R_n \leq p_{\lceil 2(1+\varepsilon)n \rceil}.$$

Proof. This follows from Theorem 3.22 and Lemma 3.23 of [1]. \square

The following proof of Theorem 1.7 consists of three steps. In the first step, we apply Theorem 1.2 and Theorem 1.3 to derive a lower bound for the quantity $m\pi(R_n) - \pi(R_{mn})$, which holds for every positive integers m and n satisfying $m \geq 2$ and $n \geq \max\{\lceil 5225/m \rceil, 1245\}$. Then, in the second step, we use this lower bound and a computer to establish Theorem 1.7 for the cases $m = 2$ and $m \in \{3, 4, \dots, 19\}$. Finally, we consider the case where $m \geq 20$. In this case, we first show that the inequality $\pi(R_{mn}) \leq m\pi(R_n)$ holds for every positive integer $n \geq 1245$. So it suffices to show that the required inequality also holds for every positive integers m and n with $m \geq 20$ and $N(m) \leq n \leq 1244$, where $N(m)$ is defined as in Theorem 1.7, with the only exception $(m, n) = (38, 9)$. For this purpose, note that

$$(5.2) \quad \pi(R_{mn}) \leq m\pi(R_n) \quad \Leftrightarrow \quad R_{mn} \leq p_{m\pi(R_n)}.$$

Now, for each $n \in \{2, \dots, 1244\}$ we use (5.2) and Lemma 5.1 with $\varepsilon = \pi(R_n)/2n - 1$ (note that $\varepsilon > 0$ by (1.3)) to find a positive integer $M(n)$, so that $R_{mn} \leq p_{m\pi(R_n)}$ for every positive integer $m \geq M(n)$. Finally we check with a computer for which $m < M(n)$ the inequality $R_{mn} \leq p_{m\pi(R_n)}$ holds.

Proof of Theorem 1.7. First, we note that the inequality (1.10) holds for $m = 1$. So, we can assume that $m \geq 2$. Let n be a positive integer with $n \geq \max\{\lceil 5225/m \rceil, 1245\}$. By (3.4), (3.5) and Theorem 1.2, we have

$$(5.3) \quad \pi(R_{mn}) < 2mn \left(1 + \frac{\log 2 + \log 2/\log(mn) + 0.565/\log^2(mn)}{\log(mn) + \log \log(mn) - \log 2 - \log 2/\log(mn)} \right)$$

and, by (4.2), (4.3) and Theorem 1.3, we have

$$(5.4) \quad \pi(R_n) > 2n \left(1 + \frac{\log 2 + \log 2/\log n - (1.472 \log \log n + 2.51)/\log^2 n}{\log n + \log \log n - \log 2 - \log 2/\log n} \right).$$

We set $\lambda(x) = \log x + \log \log x - \log 2 - \log 2/\log x$ and $\phi(x) = 1.472 \log \log x + 2.51$. Then, by (5.3) and (5.4), we get

$$(5.5) \quad \frac{m\pi(R_n) - \pi(R_{mn})}{2mn} > \frac{W_m(n)}{\lambda(n)\lambda(mn)},$$

where

$$W_m(n) = \log 2 \log m + \log 2(\log \log(mn) - \log \log n) + \log 2 \left(\frac{\log(mn)}{\log n} - \frac{\log n}{\log(mn)} \right) \\ + \log 2 \left(\frac{\log \log(mn)}{\log n} - \frac{\log \log n}{\log(mn)} \right) - \frac{\phi(n)\lambda(mn)}{\log^2 n} - \frac{0.565\lambda(n)}{\log^2(mn)}.$$

Clearly, it suffices to show that $W_m(n) \geq 0$. Setting $g(x) = \log \log x$, we get, by the mean value theorem, that there exists a real number $\xi \in (n, mn)$ such that $g(mn) - g(n) = (m-1)ng'(\xi)$. Hence

$$(5.6) \quad \log \log(mn) - \log \log n = \frac{(m-1)n}{\xi \log \xi} \geq \frac{m-1}{m \log(mn)} \geq \frac{1}{2 \log(mn)}.$$

Further, we have

$$(5.7) \quad \frac{\log(mn)}{\log n} - \frac{\log n}{\log(mn)} = \frac{\log m}{\log n} + \frac{\log m}{\log(mn)},$$

as well as

$$(5.8) \quad \frac{\log \log(mn)}{\log n} - \frac{\log \log n}{\log(mn)} > \frac{\log m \log \log n}{\log^2(mn)}.$$

Combining (5.6)-(5.8) with the definition of $W_m(n)$, we obtain that the inequality

$$W_m(n) > \log m \left(\log 2 + \frac{\log 2}{\log n} \right) + \log 2 \left(\frac{\log m + 1/2}{\log(mn)} + \frac{\log m \log \log n}{\log^2(mn)} \right) - \frac{\phi(n)\lambda(mn)}{\log^2 n} - \frac{0.565\lambda(n)}{\log^2(mn)}.$$

Since $\lambda(x) < \log x + \log \log x - \log 2 < \log x + \log \log x$, we get

$$W_m(n) > \log m \left(\log 2 + \frac{\log 2}{\log n} - \frac{\phi(n)}{\log^2 n} \right) + \log 2 \left(\frac{\log m + 1/2}{\log(mn)} + \frac{\log m \log \log n}{\log^2(mn)} \right) \\ - \frac{\phi(n)}{\log n} - \frac{\phi(n) \log \log(mn)}{\log^2 n} + \frac{\phi(n) \log 2}{\log^2 n} - \frac{0.565 \log n}{\log^2(mn)} - \frac{0.565 \log \log n}{\log^2(mn)}.$$

Now, we use the right-hand side inequality of (3.8) to get $\log \log(mn) \leq \log \log n + \log m / \log n$. Finally, we have

$$(5.9) \quad W_m(n) > \log m \left(\log 2 + \frac{\log 2}{\log n} - \frac{\phi(n)}{\log^2 n} - \frac{\phi(n)}{\log^3 n} \right) - \frac{\phi(n)}{\log n} - \frac{\phi(n)(\log \log n - \log 2)}{\log^2 n} \\ + \frac{(\log m + 1/2) \log 2 - 0.565}{\log(mn)} + \frac{(\log m \log 2 - 0.565) \log \log n}{\log^2(mn)}$$

for every positive integers m and n satisfying $m \geq 2$ and $n \geq \max\{\lceil 5225/m \rceil, 1245\}$. Next, we use this inequality to prove the theorem. For this purpose, we consider the following three cases:

(i) *Case 1: $m = 2$.*

First, let $n \geq 4903689$. In this case, we have $(\log m + 1/2) \log 2 - 0.565 \geq 0.262$ and $\log m \log 2 - 0.565 > -0.085$. Hence

$$\frac{(\log m + 1/2) \log 2 - 0.565}{\log(mn)} + \frac{(\log m \log 2 - 0.565) \log \log n}{\log^2(mn)} > 0.$$

Applying this inequality to (5.9), we get

$$W_2(n) > \log 2 \left(\log 2 + \frac{\log 2}{\log n} - \frac{\phi(n)}{\log^2 n} - \frac{\phi(n)}{\log^3 n} \right) - \frac{\phi(n)}{\log n} - \frac{\phi(n)(\log \log n - \log 2)}{\log^2 n}.$$

Since $\log 2 - \phi(x)/\log x - \phi(x)/\log^2 x > 0$ for every real $x \geq 10377$, we get

$$W_2(n) > \log^2 2 - \frac{\phi(n)}{\log n} - \frac{\phi(n)(\log \log n - \log 2)}{\log^2 n}.$$

Note that the right-hand side of the last inequality is positive. Combined with (5.5), we get that $\pi(R_{2n}) \leq 2\pi(R_n)$ holds for every positive integer $n \geq 4903689$. A direct computation shows that the inequality $\pi(R_{2n}) \leq 2\pi(R_n)$ also holds for every positive integer n so that $1245 \leq n \leq 4903689$.

(ii) *Case 2:* $m \in \{3, 4, \dots, 19\}$.

First, we consider the case where $n \geq 6675$. By (5.9), we have

$$W_m(n) > \log m \left(\log 2 + \frac{\log 2}{\log n} - \frac{\phi(n)}{\log^2 n} - \frac{\phi(n)}{\log^3 n} \right) - \frac{\phi(n)}{\log n} - \frac{\phi(n)(\log \log n - \log 2)}{\log^2 n}.$$

We set $\delta_2 = 0.003314$ to obtain that the inequality

$$\delta_2 + \frac{\log 2}{\log x} - \frac{\phi(x)}{\log^2 x} - \frac{\phi(x)}{\log^3 x} > 0$$

holds for every real $x \geq 6675$. So we see that

$$W_m(n) > (\log 2 - \delta_2) \log 3 - \frac{\phi(n)}{\log n} - \frac{\phi(n)(\log \log n - \log 2)}{\log^2 n}$$

and since the right-hand side of the last inequality is positive, we use (5.5) to conclude that $\pi(R_{mn}) \leq m\pi(R_n)$ holds for each $m \in \{3, 4, \dots, 19\}$ and every positive integer $n \geq 6675$. For $m \in \{3, 4\}$, we verify with a direct computation that the inequality $\pi(R_{mn}) \leq m\pi(R_n)$ also holds for every positive integer n so that $189 \leq n \leq 6674$. For $m \in \{5, 6\}$, we use a computer to check that the inequality $\pi(R_{mn}) \leq m\pi(R_n)$ is also valid for every positive integer n satisfying $85 \leq n \leq 6674$. Finally, if $m \in \{7, 8, \dots, 19\}$, a computer check shows that the required inequality also holds for every positive integer n with $10 \leq n \leq 6674$.

(iii) *Case 3:* $m \geq 20$.

First, let $n \geq 1245$. Setting $\delta_3 = 0.03$, we obtain, similar to Case 2, that

$$W_m(n) > (\log 2 - \delta_3) \log 20 - \frac{\phi(n)}{\log n} - \frac{\phi(n)(\log \log n - \log 2)}{\log^2 n}.$$

Note that the right-hand side of the last inequality is positive. Together with (5.5), we get that $\pi(R_{mn}) \leq m\pi(R_n)$ holds for all positive integers m and n satisfying $m \geq 20$ and $n \geq 1245$. Now, for each $n \in \{2, \dots, 1244\}$, we use (5.2), Lemma 5.1 with $\varepsilon = \pi(R_n)/2n - 1$ and a C++ version of the following MAPLE code to find positive integer $M(n) \geq 20$, so that $R_{mn} \leq p_{m\pi(R_n)}$ for every positive integer $m \geq M(n)$ and then we check for which m with $20 \leq m < M(n)$ the inequality $R_{mn} \leq p_{m\pi(R_n)}$ holds:

```
> restart: with(numtheory): Digits := 100:
> for n from 1244 by -1 to 2 do
  ep := pi(R[n])/(2*n)-1: # R[n] denotes the nth Ramanujan prime
  lambda := ep/2:
  S := ceil(evalf(exp(sqrt(1.17+2*(1+ep))/ep*(0.17+log(2)/log(2*5.43)))+
    (1/2+(1+ep)*log(2)/ep)^2)+1/2+(1+ep)*log(2)/ep):
  T := ceil(evalf(exp(sqrt(1.17+0.17/lambda+1/4)+1/2))):
  X9 := max(468049, 2*S, T): M := ceil((1+pi(X9))/(2*(1+ep))):
  # Hence pi(R[mn]) <= m*pi(R[n]) for all m >= M by Lemma 5.1
  while M*pi(R[n]) - pi(R[n*M]) >= 0 and M >= 20 do
    M := M-1:
  end do:
  L[n] := M+1:
end do:
```

Since $L[i] = 20$ for every $i \in \{2, \dots, 1244\} \setminus \{9\}$ and $L[9] = 39$, we get that $\pi(R_{mn}) \leq m\pi(R_n)$ for every positive integers n, m with $n \in \{2, \dots, 1244\} \setminus \{9\}$ and $m \geq 20$ and for every positive integers n, m with $n = 9$ and $m \geq 39$. A direct computation shows that the inequality $\pi(R_{9m}) \leq m\pi(R_9)$ holds for every m with $20 \leq m \leq 37$ as well and that $38\pi(R_9) - \pi(R_{9 \cdot 38}) = -2$.

So, we showed that the inequality $\pi(R_{mn}) \leq m\pi(R_n)$ holds for every $m \in \mathbb{N}$ and every positive integer $n \geq N(m)$ with the only exception $(m, n) = (38, 9)$, as desired. \square

We use Theorem 1.7 and a computer to get the following remark.

Remark. The inequality $\pi(R_{mn}) \leq m\pi(R_n)$ fails if and only if $(m, n) \in \mathbb{N}_{\geq 2} \times \{1\}$ (see (1.3)) or

$$(m, n) \in \{(2, 3), (2, 7), (2, 8), (2, 9), (2, 22), (2, 23), (2, 25), (2, 37), (2, 38), (2, 49), (2, 53), (2, 54), (2, 55), (2, 66), (2, 82), (2, 83), (2, 84), (2, 85), (2, 86), (2, 87), (2, 101), (2, 102), (2, 113), (2, 114), (2, 115), (2, 160), (2, 161), (2, 162), (2, 179), (2, 180), (2, 184), (2, 185), (2, 186), (2, 232), (2, 240), (2, 241), (2, 246), (2, 247), (2, 376), (2, 377), (2, 378), (2, 379), (2, 380), (2, 381), (2, 386), (2, 387), (2, 388), (2, 412), (2, 531), (2, 532), (2, 537), (2, 538), (2, 547), (2, 548), (2, 549), (2, 550), (2, 551), (2, 552), (2, 553), (2, 554), (2, 555), (2, 556), (2, 557), (2, 558), (2, 792), (2, 793), (2, 794), (2, 795), (2, 796), (2, 797), (2, 798), (2, 799), (2, 800), (2, 801), (2, 802), (2, 803), (2, 804), (2, 1140), (2, 1141), (2, 1142), (2, 1146), (2, 1147), (2, 1202), (2, 1241), (2, 1242), (2, 1243), (2, 1244), (3, 9), (3, 11), (3, 23), (3, 25), (3, 49), (3, 54), (3, 55), (3, 56), (3, 57), (3, 66), (3, 67), (3, 83), (3, 84), (3, 114), (3, 115), (3, 160), (3, 187), (3, 188), (4, 9), (4, 11), (4, 37), (4, 38), (4, 42), (4, 54), (4, 55), (4, 82), (4, 83), (4, 84), (4, 114), (4, 115), (4, 188), (5, 3), (5, 9), (5, 84), (6, 28), (6, 54), (6, 55), (6, 84), (7, 3), (7, 9), (8, 9), (9, 9), (10, 9), (11, 3), (11, 9), (12, 9), (13, 9), (14, 9), (15, 3), (15, 9), (16, 9), (17, 9), (18, 9), (19, 9), (38, 9)\}.$$

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