RECURRENCE RELATIONS FOR BINOMIAL-EULERIAN POLYNOMIALS

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Abstract. Binomial-Eulerian polynomials were introduced by Postnikov, Reiner and Williams. In this paper, properties of the binomial-Eulerian polynomials, including recurrence relations and generating functions are studied. We present three constructive proofs of the recurrence relations for binomial-Eulerian polynomials. Moreover, we give a combinatorial interpretation of the Betti number of the complement of the k-equal real hyperplane arrangement.

Keywords: Binomial-Eulerian polynomials; Eulerian polynomials; Recurrence relations

1. Introduction

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]$, where $[n]=\{1,2,\ldots,n\}$. For each $\pi\in\mathfrak{S}_n$, an index i is called *a descent* (resp. *an ascent*) of π if $\pi(i)>\pi(i+1)$ (resp. $\pi(i)<\pi(i+1)$), where $i\in[n-1]$. Define

$$\mathrm{Des}\,(\pi)=\{\pi(i)\mid \pi(i)>\pi(i+1), i\in[n-1]\},\ \mathrm{des}\,(\pi)=|\mathrm{Des}\,(\pi)|,$$
$$\mathrm{Asc}\,(\pi)=\{\pi(i)\mid \pi(i)<\pi(i+1), i\in[n-1]\},\ \mathrm{asc}\,(\pi)=|\mathrm{Asc}\,(\pi)|,$$

where $|S|$ denote the cardinality of the set S. The classical *Eulerian polynomials* $A_n(x)$ are defined by

$$A_n(x)=\sum_{\pi\in\mathfrak{S}_n}x^{\mathrm{des}\,(\pi)}=\sum_{\pi\in\mathfrak{S}_n}x^{\mathrm{asc}\,(\pi)}. \tag{1}$$

Let $A_n(x)=\sum_{k=0}^{n-1}\left\langle{n\atop k}\right\rangle x^k$, where $\left\langle{n\atop k}\right\rangle$ are called the Eulerian numbers. The numbers $\left\langle{n\atop k}\right\rangle$ satisfy the recurrence relation

$$\left\langle{n\atop k}\right\rangle=(k+1)\left\langle{n-1\atop k}\right\rangle+(n-k)\left\langle{n\atop k}\right\rangle,$$

with the initial conditions $\left\langle{1\atop 0}\right\rangle=1$ and $\left\langle{1\atop k}\right\rangle=0$ for $k\geq 1$ (see [18, A008292]). In [4], Chung, Graham and Knuth noted that if we set $\left\langle{0\atop 0}\right\rangle=0$, then the following symmetrical identity holds:

$$\sum_{k\geq 0}\binom{a+b}{k}\left\langle{k\atop a-1}\right\rangle=\sum_{k\geq 0}\binom{a+b}{k}\left\langle{k\atop b-1}\right\rangle, \tag{2}$$

where a,b are positive integers. Subsequently, the q-generalizations of the identity (2) have been pursued by several authors. See, e.g., [5, 10, 13, 17].

Let $G=K_{1,n}$ be the n-star graph with the central node $n+1$ connected to the nodes $1,\cdots,n$. The associated polytope $P_{\mathcal{B}(K_{1,n})}$ is called the *stellohedron*. Following [16, Section 10.4], the h-polynomial of the n-dimensional stellohedron is given by

$$h_{\mathcal{B}(K_{1,n})}(x)=1+x\sum_{k=1}^{n}\binom{n}{k}A_k(x), \tag{3}$$

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which is named as the *binomial-Eulerian polynomial* (see [17]). As usual, let

$$\widetilde{A}_n(x) = h_{\mathcal{B}(K_{1,n})}(x).$$

The γ-positivity of $\widetilde{A}_n(x)$ follows from a general result of Postnikov, Reiner and Williams [16, Theorem 11.6]. As an application of the γ-positivity, we see that $\widetilde{A}_n(x)$ is symmetric. Very recently, Shareshian and Wachs [17] further studied γ-positivity of the binomial-Eulerian and q-binomial-Eulerian polynomials, and noticed that the identity (2) is equivalent to the symmetry of $\widetilde{A}_n(x)$. The reader is referred to [1] for a survey of the theory of γ-positivity.

Definition 1.1. *Let $\mathcal{Q}_n$ be the set of permutations of $[n]$ with the restriction that the entry n appears as the first descent. For convenience, let the identity permutation $12\cdots n$ be an element of $\mathcal{Q}_n$ and we say that the entry n appears as the first descent of $12\cdots n$ (In fact, the identity permutation has no descent).*

For example, $\mathcal{Q}_1=\{1\}, \mathcal{Q}_2=\{12,21\}$ and $\mathcal{Q}_3=\{123,132,231,312,321\}$. Postnikov, Reiner and Williams [16, Section 10.4] discovered that

$$\widetilde{A}_n(x)=\sum_{\pi\in\mathcal{Q}_{n+1}} x^{\operatorname{des}(\pi)}.$$

The first few of $\widetilde{A}_n(x)$ are given as follows:

$$\widetilde{A}_0(x)=1, \widetilde{A}_1(x)=1+x, \widetilde{A}_2(x)=1+3x+x^2, \widetilde{A}_3(x)=1+7x+7x^2+x^3.$$

It is clear that the ascent and descent statistics are equidistributed on $\mathfrak{S}_n$, since reversing an element of $\mathfrak{S}_n$ turns ascents into descents and vice versa. It is less obvious that ascent and descent statistics are equidistributed on $\mathcal{Q}_n$, since reversing an element of $\mathcal{Q}_n$ may leads to an element of $\mathfrak{S}_n\backslash\mathcal{Q}_n$.

This paper is motivated by the following problem.

Problem 1.2. *Is there a bijective proof of the symmetry of $\widetilde{A}_n(x)$ by using the descent and ascent statistics on $\mathcal{Q}_n$?*

This paper is organized as follows. In Section 2, we present three constructive proofs of the recurrence relations for $\widetilde{A}_n(x)$. In Theorem 2.11, as a combination of the first two constructive proofs, we give a solution to Problem 1.2. In Section 3, we study the generating function of a kind of multivariable binomial-Eulerian polynomials. As an application, in Theorem 3.5, we give a combinatorial interpretation of the Betti number of the complement of the k-equal real hyperplane arrangement.

2. Recurrence relations

2.1. The descent statistic on $\mathcal{Q}_n$.

It is well known that the Eulerian polynomials $A_n(x)$ satisfy the recurrence relation

$$A_{n+1}(x)=(1+nx)A_n(x)+x(1-x)A_n'(x),$$

with the initial values $A_0(x) = A_1(x) = 1$ (see [3] for instance), and they can be defined by the exponential generating function

$$A(x,z) = \sum_{n\geq 0} A_n(x)\frac{z^n}{n!} = \frac{x-1}{x-e^{z(x-1)}}.$$

It is easy to verify that

$$(1-xz)\frac{\partial}{\partial z}A(x,z) = A(x,z) + x(1-x)\frac{\partial}{\partial x}A(x,z). \tag{4}$$

Set $\widetilde{A}_0(x) = 1$. We define $\widetilde{A}(x,z) = \sum_{n\geq 0}\widetilde{A}_n(x)\frac{z^n}{n!}$. It follows from (3) that

$$\widetilde{A}(x,z) = e^{xz}A(x,z). \tag{5}$$

Combining (4) and (5), we obtain

$$(1-xz)\frac{\partial}{\partial z}\widetilde{A}(x,z) = (1+x-xz)\widetilde{A}(x,z) + x(1-x)\frac{\partial}{\partial x}\widetilde{A}(x,z). \tag{6}$$

Let $\widetilde{A}_n(x) = \sum_{k=0}^{n}\widetilde{A}(n,k)x^k$. Equating the coefficients of $x^k z^n/n!$ in both sides of (6) leads to the following result.

Theorem 2.1. *For $n \geq 1$, we have*

$$\widetilde{A}(n+1,k) = (k+1)\widetilde{A}(n,k) + (n-k+2)\widetilde{A}(n,k-1) - n\widetilde{A}(n-1,k-1), \tag{7}$$

with the initial conditions $\widetilde{A}(0,0) = 1$ and $\widetilde{A}(0,k) = 0$ for $k \neq 0$.

In the following, we present a constructive proof of the recurrence relation (7). Let $\alpha_i(\pi)$ be the permutation in $\mathfrak{S}_{n-1}$ obtained from π by the following two steps:

- Step 1. Delete the entry i from π;
- Step 2. Every entry in π, which is larger than i, is decreased by 1.

Let $\beta_{i,j}(\pi)$ be the permutation in $\mathfrak{S}_{n+1}$ obtained from π by the following two steps:

- Step 1. Every entry in π, which is larger than or equal to i, is increased by 1;
- Step 2. Insert the entry i between j-st and $(j+1)$-st elements of π.

In the sequel, we define

$$\mathrm{Des}^*(\pi) = \{0\} \cup \mathrm{Des}(\pi),$$

$$QD_{n,k} = \{\pi \in \mathcal{Q}_n \mid \mathrm{des}(\pi) = k\}.$$

Denote by $FD_{n+1,k}$ the set of pairs $[\pi, i]$ such that $\pi \in QD_{n+1,k}$ and $i \in \{0,1,2,\ldots,k\}$. Hence

$$|FD_{n+1,k}| = (k+1)\widetilde{A}(n,k).$$

We use $RD_{n+2,k}$ to denote the set of permutations π of $[n+2]$ which satisfy the following three conditions:

(1) the entry $n+2$ appears as the first descent of π from left to right;
(2) π has k descents;
(3) Either $a = 1$ or $\pi(a-1) > \pi(a+1)$, where $a = \pi^{-1}(1)$.

Lemma 2.2. *There is a bijection $\phi = \phi_{n,k}$ from $RD_{n+2,k}$ to $FD_{n+1,k}$.*

Proof. For any $\pi \in RD_{n+2,k}$, let $a = \pi^{-1}(1)$ and $\sigma = \alpha_1(\pi)$. Clearly, $\sigma \in QD_{n+1,k}$. Suppose that

$$\mathrm{Des}^*(\sigma) = \{j_0, j_1, \ldots, j_k\}$$

with $j_0 < j_1 < \ldots < j_{k-1}$ and $j_k = 0$. Note that $a - 1 \in \mathrm{Des}^*(\sigma)$. Suppose that $j_i = a - 1$ for some $i \in \{0, 1, \ldots, k\}$. Define a map $\phi : RD_{n+2,k} \mapsto FD_{n+1,k}$ by letting $\phi(\pi) = [\alpha_1(\pi), i]$.

Conversely, for any $[\sigma, i] \in FD_{n+1,k}$, suppose that $\mathrm{Des}^*(\sigma) = \{j_0, j_1, \ldots, j_k\}$ with $j_0 < j_1 < \ldots < j_{k-1}$, $j_k = 0$ and $a = j_i$. Let us consider the permutation $\pi = \beta_{1,a}(\sigma)$. Then $\pi(1) = 1$ if $a = 0$; otherwise, $\pi(a+1) = 1$ and $\pi(a) > \pi(a+2)$ since $\sigma(a) > \sigma(a+1)$. So, $\pi \in RD_{n+2,k}$. Thus, for any $[\sigma, i] \in FD_{n+1,k}$, the inverse ϕ^{-1} of the map ϕ is given by $\phi^{-1}(\sigma, i) = \beta_{1,a}(\sigma)$. □

Let $HD_{n+1,k-1}$ be the set of pairs $[\pi, i]$ such that $\pi \in QD_{n+1,k-1}$ and $i \in \{1, 2, \ldots, n-k+2\}$. Then

$$|HD_{n+1,k-1}| = (n-k+2)\widetilde{A}(n, k-1).$$

Denote by $RHD_{n+1,k-1}$ the set of pairs $[\pi, i]$ such that $[\pi, i] \in HD_{n+1,k-1}$ and $i > \pi^{-1}(n+1)-1$. We use $\overline{RD}_{n+2,k}$ to denote the set of permutations π of $[n+2]$ which satisfy the following three conditions:

(1) the entry $n+2$ appears as the first descent of π from left to right;
(2) π has k descents;
(3) Either $a = n+2$ or $\pi(a-1) < \pi(a+1)$, where $a = \pi^{-1}(1)$.

Lemma 2.3. *There is a bijection $\theta = \theta_{n,k}$ from $\overline{RD}_{n+2,k}$ to $RHD_{n+1,k-1}$.*

Proof. For any $\pi \in \overline{RD}_{n+2,k}$, let $a = \pi^{-1}(1)$ and $\sigma = \alpha_1(\pi)$. Clearly, $\sigma \in QD_{n+1,k-1}$ and $\mathrm{asc}\,(\sigma) = n-k+1$. Suppose that

$$\mathrm{Asc}^*(\sigma) = \{j_1, j_2, \ldots, j_{n-k+2}\}$$

with $j_1 < j_2 < \ldots < j_{n-k+2} = n+1$. Note that $a-1 \in \mathrm{Asc}^*(\sigma)$. Suppose that $j_i = a-1$ for some $i \in \{1, 2, \ldots, n-k+2\}$. Then $i > \sigma^{-1}(n+1)-1$. Define a map $\theta : \overline{RD}_{n+2,k} \mapsto RHD_{n+1,k-1}$ by letting $\theta(\pi) = [\alpha_1(\pi), i]$.

Conversely, for any $[\sigma, i] \in RHD_{n+1,k-1}$, we have $\mathrm{asc}\,(\sigma) = n-k+1$. Suppose that

$$\mathrm{Asc}^*(\sigma) = \{j_1, j_2, \ldots, j_{n-k+2}\}$$

with $j_1 < j_2 < \ldots < j_{n-k+2} = n+1$ and $a = j_i$. Let us consider the permutation $\pi = \beta_{1,a}(\sigma)$. Thus, $\pi^{-1}(1) = n+2$ if $a = n+1$; otherwise, $\pi(a+1) = 1$ and $\pi(a) < \pi(a+2)$ since $\sigma(a) < \sigma(a+1)$. Hence $\pi \in \overline{RD}_{n+2,k}$ since $i > \sigma^{-1}(n+1)-1$. Therefore, the inverse θ^{-1} of the map θ is $\theta^{-1}(\sigma, i) = \beta_{1,a}(\sigma)$ for any $[\sigma, i] \in RHD_{n+1,k-1}$. □

Let $\overline{HD}_{n,k-1}$ be the set of pairs $[\pi, a]$ such that $\pi \in QD_{n,k-1}$ and $a \in \{1, 2, \ldots, n\}$. Then

$$|\overline{HD}_{n,k-1}| = n\widetilde{A}(n-1, k-1).$$

Let $\overline{RHD}_{n+1,k-1} = HD_{n+1,k-1} \setminus RHD_{n+1,k-1}$. In fact, $\overline{RHD}_{n+1,k-1}$ is the set of pairs $[\pi, i]$ such that $[\pi, i] \in HD_{n+1,k-1}$ and $i \in \{1, 2, \ldots, \pi^{-1}(n+1)-1\}$.

Lemma 2.4. *There is a bijection $\psi = \psi_{n,k}$ from $\overline{HD}_{n,k-1}$ to $\overline{RHD}_{n+1,k-1}$.*

Proof. For any $[\sigma,a]\in\overline{HD}_{n,k-1}$, suppose $p=\sigma^{-1}(n)$, then $0=\sigma(0)<\sigma(1)<\sigma(2)<\ldots<\sigma(p)=n$ since the entry n appears as the first descent of σ from left to right. There exists a unique index $i\in\{0,1,\ldots,p-1\}$ such that $\sigma(i)<a\le\sigma(i+1)$ since $a\in\{1,2,\ldots,n\}$. Then $\beta_{a,i}(\sigma)\in QD_{n+1,k-1}$ and $[\beta_{a,i}(\sigma),i+1]\in\overline{RHD}_{n+1,k-1}$. Define a map $\psi:\overline{HD}_{n,k-1}\mapsto\overline{RHD}_{n+1,k-1}$ by letting $\psi(\sigma,a)=[\beta_{a,i}(\sigma),i+1]$.

Conversely, for any $[\sigma,i]\in\overline{RH}_{n+1,k-1}$, suppose $a=\sigma(i)$, then $a\in\{1,2,\ldots,n\}$ since the entry $n+1$ appears as the first descent of σ from left to right and $i<\sigma^{-1}(n+1)$. Moreover, $\alpha_a(\sigma)\in QD_{n,k-1}$ and $\alpha_a(\sigma)(i-1)<a\le\alpha_a(\sigma)(i)$. The inverse ψ^{-1} of the map ψ is

$$\psi^{-1}(\sigma,i)=[\alpha_a(\sigma),a].$$

□

The proof of the recurrence relation (7)*:*

Note that

$$QD_{n+2,k}=RD_{n+2,k}\cup\overline{RD}_{n+2,k}.$$

So

$$\widetilde{A}(n+1,k)=|QD_{n+2,k}|=|RD_{n+2,k}|+|\overline{RD}_{n+2,k}|.$$

Lemma 2.2 implies that $|RD_{n+2,k}|=|FD_{n+1,k}|=(k+1)\widetilde{A}(n,k)$. Lemmas 2.3 and 2.4 tell us that

$$\begin{aligned}|\overline{RD}_{n+2,k}| &= |RHD_{n+1,k-1}|\\ &= |HD_{n+1,k-1}|-|\overline{RHD}_{n+1,k-1}|\\ &= |HD_{n+1,k-1}|-|\overline{HD}_{n,k-1}|\\ &= (n-k+2)\widetilde{A}_{n,k-1}-n\widetilde{A}_{n-1,k-1}.\end{aligned}$$

Hence, $\widetilde{A}(n+1,k)=(k+1)\widetilde{A}(n,k)+(n-k+2)\widetilde{A}(n,k-1)-n\widetilde{A}(n-1,k-1)$. □

Corollary 2.5. *The polynomials $\widetilde{A}_n(x)$ satisfy the recurrence relation*

$$\widetilde{A}_{n+1}(x)=(1+(n+1)x)\,\widetilde{A}_n(x)+x(1-x)\widetilde{A}'_n(x)-nx\widetilde{A}_{n-1}(x),$$

with the initial value $\widetilde{A}_0(x)=1$.

Based on empirical evidence, we propose the following conjecture.

Conjecture 2.6. *For any $n\ge1$, the polynomial $\widetilde{A}_n(x)$ has only real zeros.*

2.2. The ascent statistic on $\mathcal{Q}_n$.

Theorem 2.7. *We have $\widetilde{A}(n,k)=|\{\pi\in\mathcal{Q}_{n+1}:\mathrm{asc}\,(\pi)=k\}|$.*

Along the same lines of the proof of Theorem 2.1, we shall present a constructive proof of Theorem 2.7.

For any $n\ge1$ and $\pi\in\mathfrak{S}_n$, we define

$$\mathrm{Asc}^*(\pi)=\{n\}\cup\mathrm{Asc}\,(\pi),$$

$$QA_{n,k}=\{\pi\in\mathcal{Q}_n\mid\mathrm{asc}\,(\pi)=k\}.$$

Suppose that the number of permutations in $\mathcal{Q}_{n+1}$ with k ascents is $\widetilde{B}(n,k)$. Let $HA_{n+1,k-1}$ be the set of pairs $[\pi, i]$ such that $\pi \in QA_{n+1,k-1}$ and $i \in \{1,2,\ldots,n-k+2\}$. Then

$$|HA_{n+1,k-1}| = (n-k+2)\widetilde{B}(n,k-1).$$

We use $RA_{n+2,k}$ to denote the set of permutations π of $[n+2]$ which satisfy the following three conditions:

(1) the entry $n+2$ appears as the first descent of π from left to right;
(2) π has k ascents;
(3) Either $a=1$ or $\pi(a-1) > \pi(a+1)$, where $a = \pi^{-1}(1)$.

Lemma 2.8. *There is a bijection $\hat{\theta} = \hat{\theta}_{n,k}$ from $RA_{n+2,k}$ to $HA_{n+1,k-1}$.*

Proof. For any $\pi \in RA_{n+2,k}$, let $a = \pi^{-1}(1)$ and $\sigma = \alpha_1(\pi)$. Clearly, $\sigma \in QA_{n+1,k-1}$ and $\operatorname{des}(\sigma) = n-k+1$. Suppose that

$$\operatorname{Des}^*(\sigma) = \{j_1, j_2, \ldots, j_{n-k+2}\}$$

with $j_1 < j_2 < \ldots < j_{n-k+1}$ and $j_{n-k+2} = 0$. Note that $a-1 \in \operatorname{Des}^*(\sigma)$. Suppose that $j_i = a-1$ for some i. Define a map $\hat{\theta} : RA_{n+2,k} \mapsto HA_{n+1,k-1}$ by letting $\hat{\theta}(\pi) = [\alpha_1(\pi), i]$.

Conversely, for any $[\sigma, i] \in HA_{n+1,k-1}$, we have $\operatorname{des}(\sigma) = n-k+1$. Suppose that

$$\operatorname{Des}^*(\sigma) = \{j_1, j_2, \ldots, j_{n-k+2}\}$$

with $j_1 < j_2 < \ldots < j_{n-k+1}$, $j_{n-k+2} = 0$ and $a = j_i$. Let us consider the permutation $\pi = \beta_{1,a}(\sigma)$. Then $\pi^{-1}(1) = 1$ if $a = 0$; otherwise, $\pi(a+1) = 1$ and $\pi(a) > \pi(a+2)$ since $\sigma(a) > \sigma(a+1)$. Hence $\pi \in RA_{n+2,k}$. Therefore, the inverse $\hat{\theta}^{-1}$ of the map $\hat{\theta}$ is

$$\hat{\theta}^{-1}(\sigma, i) = \beta_{1,a}(\sigma)$$

for any $[\sigma, i] \in HA_{n+1,k-1}$. □

Denote by $FA_{n+1,k}$ the set of pairs $[\pi, i]$ such that $\pi \in QA_{n+1,k}$ and $i \in \{1,\ldots,k\} \cup \{n+1\}$. Hence

$$|FA_{n+1,k}| = (k+1)\widetilde{B}(n,k).$$

Let $RFA_{n+1,k}$ be the set of pairs $[\pi, i]$ in $FA_{n+1,k}$ such that $i > \pi^{-1}(n+1) - 1$. Use $\overline{RA}_{n+2,k}$ to denote the set of permutations π of $[n+2]$ which satisfy the following three conditions:

(1) the entry $n+2$ appears as the first descent of π from left to right;
(2) π has k ascents;
(3) Either $a = n+2$ or $\pi(a-1) < \pi(a+1)$, where $a = \pi^{-1}(1)$.

Lemma 2.9. *There is a bijection $\hat{\phi} = \hat{\phi}_{n,k}$ from $\overline{RA}_{n+2,k}$ to $RFA_{n+1,k}$.*

Proof. For any $\pi \in \overline{RA}_{n+2,k}$, let $a = \pi^{-1}(1)$ and $\sigma = \alpha_1(\pi)$. Clearly, $\sigma \in QA_{n+1,k}$. Suppose that

$$\operatorname{Asc}^*(\sigma) = \{j_0, j_1, \ldots, j_k\}$$

with $j_0 < j_1 < \ldots < j_{k-1} < j_k = n+1$. Note that $a-1 \in \operatorname{Asc}^*(\sigma)$. Moreover, suppose that $j_i = a-1$ form some i. Then $i > \sigma^{-1}(n+1) - 1$ since $a > \pi^{-1}(n+2)$. Define a map $\hat{\phi} : \overline{RA}_{n+2,k} \mapsto RFA_{n+1,k}$ by letting $\hat{\phi}(\pi) = [\alpha_1(\pi), i]$.

Conversely, for any $[\sigma, i] \in RFA_{n+1,k}$, suppose that

$$\mathrm{Asc}^*(\sigma) = \{j_0, j_1, \ldots, j_k\}$$

with $j_0 < j_1 < \ldots < j_{k-1} < j_k = n+1$ and $a = j_i$. Let us consider the permutation $\pi = \beta_{1,a}(\sigma)$. Then $\pi(n+2) = 1$ if $a = n+1$; otherwise, $\pi(a+1) = 1$ and $\pi(a) < \pi(a+2)$ since $\sigma(a) < \sigma(a+1)$. So, $\pi \in \overline{RA}_{n+2,k}$ since $i > \sigma^{-1}(n+1) - 1$. Thus, for any $[\sigma, i] \in RFA_{n+1,k}$, the inverse $\hat{\phi}^{-1}$ of the map $\hat{\phi}$ is given by $\hat{\phi}^{-1}(\sigma, i) = \beta_{1,a}(\sigma)$. □

Let $\overline{HA}_{n,k-1}$ be the set of pairs $[\pi, a]$ such that $\pi \in QA_{n,k-1}$ and $a \in \{1, 2, \ldots, n\}$. Then

$$|\overline{HA}_{n,k-1}| = n\widetilde{B}(n-1, k-1).$$

Let $\overline{RFA}_{n+1,k} = FA_{n+1,k} \setminus RFA_{n+1,k}$. Note that $\overline{RFA}_{n+1,k}$ is the set of pairs $[\pi, i]$ such that $[\pi, i] \in FA_{n+1,k}$ and $i \in \{1, 2, \ldots, \pi^{-1}(n+1) - 1\}$.

Lemma 2.10. *There is a bijection $\hat{\psi} = \hat{\psi}_{n,k}$ from $\overline{HA}_{n,k-1}$ to $\overline{RFA}_{n+1,k}$.*

Proof. For any $[\sigma, a] \in \overline{HA}_{n,k-1}$, suppose $p = \sigma^{-1}(n)$, then $0 = \sigma(0) < \sigma(1) < \sigma(2) < \ldots < \sigma(p) = n$ since the entry n appears as the first descent of σ from left to right. There exists a unique index $i \in \{0, 1, \ldots, p-1\}$ such that $\sigma(i) < a \leq \sigma(i+1)$ since $a \in [n]$. Then $\beta_{a,i}(\sigma) \in QA_{n+1,k}$ and $[\beta_{a,i}(\sigma), i+1] \in \overline{RFA}_{n+1,k}$. Define a map $\hat{\psi} : \overline{HA}_{n,k-1} \mapsto \overline{RFA}_{n+1,k}$ by letting

$$\hat{\psi}(\sigma, a) = [\beta_{a,i}(\sigma), i+1].$$

Conversely, for any $[\sigma, i] \in \overline{RFA}_{n+1,k}$, suppose $a = \sigma(i)$, then $a \in [n]$ since the entry $n+1$ appears as the first descent of σ from left to right and $i \leq \sigma^{-1}(n+1) - 1$. Moreover, $\alpha_a(\sigma) \in QA_{n,k-1}$ and $\alpha_a(\sigma)(i-1) < a \leq \alpha_a(\sigma)(i)$. The inverse $\hat{\psi}^{-1}$ of the map $\hat{\psi}$ is $\hat{\psi}^{-1}(\sigma, i) = [\alpha_a(\sigma), a]$. □

The proof of the theorem 2.7:

Note that $QA_{n+2,k} = RA_{n+2,k} \cup \overline{RA}_{n+2,k}$. So $\widetilde{B}(n+1, k) = |QA_{n+2,k}| = |RA_{n+2,k}| + |\overline{RA}_{n+2,k}|$. Lemma 2.8 implies that $|RA_{n+2,k}| = |HA_{n+1,k}| = (n-k+2)\widetilde{B}(n, k-1)$. Lemmas 2.9 and 2.10 tell us that

$$\begin{aligned} |\overline{RA}_{n+2,k}| &= |RFA_{n+1,k}| \\ &= |FA_{n+1,k}| - |\overline{RFA}_{n+1,k}| \\ &= |FA_{n+1,k}| - |\overline{HA}_{n,k-1}| \\ &= (k+1)\widetilde{B}_{n,k} - n\widetilde{B}_{n-1,k-1}. \end{aligned}$$

Thus $\widetilde{B}(n+1, k) = (k+1)\widetilde{B}(n, k) + (n-k+2)\widetilde{B}(n, k-1) - n\widetilde{B}(n-1, k-1)$ and so $\widetilde{B}(n, k)$ has the same recursion as $\widetilde{A}(n, k)$. It is easy to check that $\widetilde{B}(0, 0) = \widetilde{A}(0, 0) = 1$, $\widetilde{B}(1, 0) = \widetilde{A}(1, 0) = 1$ and $\widetilde{B}(1, 1) = \widetilde{A}(1, 1) = 1$. Hence $\widetilde{B}(n, k) = \widetilde{A}(n, k)$. □

Theorem 2.11. *There is a bijection Ω_n from $\mathcal{Q}_n$ to itself such that* $\mathrm{des}(\pi) = \mathrm{asc}(\Omega_n(\pi))$.

Proof. we can give a recursive definition of the bijection Ω_n. For $n = 1$, we have $\mathcal{Q}_1 = \{1\}$. Let $\Omega_1(1) = 1$. For $n = 2$, we have $\mathcal{Q}_2 = \{12, 21\}$. Let $\Omega_2(12) = 21$ and $\Omega_2(21) = 12$.

For any $m = 1, 2, \ldots, n+1$, suppose that Ω_m is a bijection from $\mathcal{Q}_m$ to itself such that $\operatorname{des}(\pi) = \operatorname{asc}(\Omega_m(\pi))$ for any $\pi \in \mathcal{Q}_m$. Furthermore, for any pair $[\pi, i]$ with $\pi \in \mathcal{Q}_m$ and a nonnegative integer i, we let

$$\hat{\Omega}_m(\pi, i) = [\Omega_m(\pi), i] \text{ and } \hat{\Omega}_m^{-1}(\pi, i) = [\Omega_m^{-1}(\pi), i].$$

For any $\pi \in \mathcal{Q}_{n+2}$, suppose that $\pi \in \mathcal{QD}_{n+2,k}$ for some k. Note that

$$QD_{n+2,k} = RD_{n+2,k} \cup \overline{RD}_{n+2,k}.$$

Combing the bijections in Lemmas 2.2, 2.3, 2.4, 2.8, 2.9 and 2.10 and the induction hypothesis, we give the bijection Ω_{n+2} from $\mathcal{Q}_{n+2}$ to itself as follows:

(c_1) If $\pi \in RD_{n+2,k}$ and $\hat{\Omega}_{n+1} \circ \phi(\pi) \in RFA_{n+1,k}$, then let

$$\Omega_{n+2}(\pi) = \hat{\phi}^{-1} \circ \hat{\Omega}_{n+1} \circ \phi(\pi);$$

(c_2) If $\pi \in RD_{n+2,k}$ and $\hat{\Omega}_{n+1} \circ \phi(\pi) \in \overline{RFA}_{n+1,k}$, then let

$$\Omega_{n+2}(\pi) = \hat{\theta}^{-1} \circ \hat{\Omega}_{n+1} \circ \psi \circ \hat{\Omega}_n^{-1} \circ \hat{\psi}^{-1} \circ \hat{\Omega}_{n+1} \circ \phi(\pi);$$

(c_3) If $\pi \in \overline{RD}_{n+2,k}$, then let

$$\Omega_{n+2}(\pi) = \hat{\theta}^{-1} \circ \hat{\Omega}_{n+1} \circ \theta(\pi).$$

□

By Theorems 2.1 and 2.7, we get

$$\sum_{\sigma \in \mathcal{Q}_{n+1}} x^{\operatorname{asc}(\sigma)} = \sum_{\sigma \in \mathcal{Q}_{n+1}} x^{n-\operatorname{des}(\sigma)} = \sum_{\sigma \in \mathcal{Q}_{n+1}} x^{\operatorname{des}(\sigma)}.$$

Hence

$$\widetilde{A}_n(x) = x^n \widetilde{A}_n\left(\frac{1}{x}\right),$$

which implies that $\widetilde{A}_n(x)$ is symmetric.

2.3. The nth-order recurrence relations.

Recall the following recurrence relation which is attributed to Euler (see [11] for instance):

$$A_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} (x-1)^{n-k-1} A_k(x) \quad \text{for } n \geq 1. \tag{8}$$

As an analog of (8), we now present the following result.

Theorem 2.12. *The polynomials $\widetilde{A}_n(x)$ satisfy the recurrence relation*

$$\widetilde{A}_n(x) = \sum_{j=1}^{n} \binom{n}{j} (x-1)^{j-1} \widetilde{A}_{n-j}(x) + x^n \tag{9}$$

for $n \geq 1$, with the initial value $\widetilde{A}_0(x) = 1$. Equivalently, we have

$$\widetilde{A}_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} (x-1)^{n-k-1} \widetilde{A}_k(x) + x^n. \tag{10}$$

Proof. Let x be a positive integer. For any $n \geq 0$, let $\mathcal{Q}_{n+1}(x)$ be the set of pairs (π, ϕ) such that $\pi \in \mathcal{Q}_{n+1}$ and ϕ is a map from $\mathrm{Des}\,(\pi)$ to $\{0, 1, \ldots, x-1\}$. Thus

$$\widetilde{A}_n(x) = \sum_{\pi \in \mathcal{Q}_{n+1}} x^{\mathrm{des}\,(\pi)} = |\mathcal{Q}_{n+1}(x)|.$$

For any $(\pi, \phi) \in \mathcal{Q}_{n+1}(x)$, there is a unique index $k \geq 1$ which satisfies $\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1) > \cdots > \pi(n+1)$. For the sequence $\pi(k), \pi(k+1), \ldots, \pi(n+1)$, if $\phi(\pi(i)) = 0$ for some $k \leq i \leq n+1$, then let k' be the largest index in $\{k, k+1, \ldots, n+1\}$ such that $\phi(\pi(k')) = 0$; otherwise, let $k' = k$. Let

$$\sigma = \pi(1), \pi(2), \ldots, , \pi(k')$$

and

$$B = \{\pi(k'+1), \ldots, \pi(n+1)\}.$$

Then σ is a permutation defined on the set $\{1, 2, \ldots, n+1\} \setminus B$ and the entry $n+1$ appears as the first descent of σ from left to right.

Now, we distinguish between the following two cases:

Case 1.$\pi(k') = n+1$ and $\phi(\pi(k')) \neq 0$.

Then the entry $n+1$ is the unique descent of the permutation π. Thus, we have

$$\phi(\pi(i)) \neq 0$$

for all $\pi(i) \in \mathrm{Des}\,(\pi)$. Note that $1 \leq |B| \leq n$ and there are $\binom{n}{|B|}$ ways to form the set B. Since $\mathrm{Des}\,(\pi) = \{n+1\} \cup (B \setminus \{\pi(n+1)\})$, there are

$$(x-1)^{|\mathrm{Des}\,(\pi)|} = (x-1)^{|B|}$$

ways to form the map ϕ. This provides the term $\sum_{B \subseteq [n]} (x-1)^{|B|} = x^n$.

Case 2. Either (i) $\pi(k') \neq n+1$ or (ii) $\pi(k') = n+1$ and $\phi(\pi(k')) = 0$.

Let

$$\mathrm{red}\,(\sigma) := \mathrm{red}\,(\sigma(1)), \mathrm{red}\,(\sigma(2)), \ldots, \mathrm{red}\,(\sigma(k')) \in \mathfrak{S}_{k'},$$

where red is an increasing map from $\{\sigma(1), \sigma(2), \ldots, \sigma(k')\}$ to $\{1, 2, \ldots, k'\}$ such that $\mathrm{red}\,(\sigma(i)) < \mathrm{red}\,(\sigma(j))$ if $\sigma(i) < \sigma(j)$ for all i, j. Then the entry k' is the first descent of the permutation $\mathrm{red}\,(\sigma)$ from left to right since $\mathrm{red}\,(n+1) = k'$ and $\mathrm{red}\,(\sigma) \in \mathcal{Q}_{k'}$. Define a map $\phi' : \mathrm{Des}\,(\mathrm{red}\,(\sigma)) \mapsto \{0, 1, \ldots, x-1\}$ by letting

$$\phi'(i) = \phi(\mathrm{red}^{\,-1}(i)) \text{ if } \mathrm{red}^{\,-1}(i) \neq \pi(k').$$

Then $(\mathrm{red}\,(\sigma), \phi') \in \mathcal{Q}_{k'}(x)$. Moreover, $\phi(i) \in \{1, 2, \ldots, x-1\}$ for any $i \in B \setminus \{\pi(n+1)\}$. Note that $1 \leq |B| \leq n$, $k' = n+1-|B|$, there are $\binom{n}{|B|}$ ways to form the set B and $\mathcal{Q}_{k'-1}(x)$ ways to form the pair $(\mathrm{red}\,(\sigma), \phi')$. Moreover, we have $\phi(i) \in \{1, 2, \ldots, x-1\}$ for any $i \in B \setminus \{\pi(n+1)\}$. This provides the term

$$\sum_{j=1}^{n} \binom{n}{j} (x-1)^{j-1} \widetilde{A}_{n-j}(x).$$

Hence we derive the recurrence relation (9). Setting $k = n-j$ in (9), we immediately get (10). This completes the proof.

□

Let $a_n = \sum_{\pi \in \mathcal{Q}_{n+1}} 2^{\mathrm{des}\,(\pi)}$. Note that

$$\widetilde{A}(2,z) = \frac{e^{2z}}{2-e^z}.$$

Let $\left\{ n \atop k \right\}$ be the Stirling number of the second kind, which counts partitions of $[n]$ into k nonempty subsets. It is easy to verify that $a_n = 2\sum_{k=0}^{n} k!\left\{ n+1 \atop k+1 \right\} - 1$. In particular, $a_0 = 1, a_1 = 3, a_2 = 11, a_3 = 51$. The numbers a_n have been studied by Gross [9], Nelsen and Schmidt [14]. It should be noted that a_n is the number of chains in power set of $[n]$ (see [18, A007047]).

Corollary 2.13. *For $n \geq 1$, we have*

$$a_n = \sum_{j=1}^{n} \binom{n}{j} a_{n-j} + 2^n.$$

3. Multivariable binomial-Eulerian polynomials

Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. An *excedance* in π is an index i such that $\pi(i) > i$ and a *fixed point* in π is an index i such that $\pi(i) = i$. As usual, let $\mathrm{exc}\,(\pi)$, $\mathrm{fix}\,(\pi)$ and $\mathrm{cyc}\,(\pi)$ denote the number of excedances, fixed points and cycles in π respectively. For example, the permutation $\pi = 3142765$ has the cycle decomposition $(1342)(57)(6)$, so $\mathrm{cyc}\,(\pi) = 3$, $\mathrm{exc}\,(\pi) = 3$ and $\mathrm{fix}\,(\pi) = 1$. There is a large of literature devoted to various generalizations and refinements of the joint distribution of excedances and cycles, see, e.g. [12, 15, 19] and the references therein.

Define

$$A_n(x,y,q) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)} y^{\mathrm{fix}\,(\pi)} q^{\mathrm{cyc}\,(\pi)}.$$

Let $A(x,y,q;z) = 1 + \sum_{n \geq 1} A_n(x,y,q)\frac{z^n}{n!}$. Brenti [3, Proposition 7.3] obtained that

$$A(x,1,q;z) = \left(\frac{1-x}{e^{z(x-1)} - x} \right)^q.$$

Note that each object of $\mathfrak{S}_n$ is a disjoint union of one object counted by $A(x,0,q;z)$ and some fixed points. Since each fixed point contributes no excedance but one cycle, by rules of exponential generating function one has $A(x,1,q;z) = e^{qz}A(x,0,q;z)$ and $A(x,y,q;z) = e^{yqz}A(x,0,q;z)$. Therefore,

$$A(x,y,q;z) = \left(\frac{1-x}{e^{z(x-y)} - xe^{(1-y)z}} \right)^q, \tag{11}$$

which was also obtained by Ksavrelof and Zeng [12, p. 2]. In the rest of this section, we study multivariable binomial-Eulerian polynomials.

A *right-to-left maximum* of $\sigma \in \mathcal{Q}$ is an element σ_i such that $\sigma_i > \sigma_j$ for every $j \in \{i+1, i+2, \ldots, n\}$ or $i = n$. Let $\mathrm{RLMAX}(\sigma)$ denote the set of entries of right-to-left maxima of σ. Let $\mathrm{rlmax}(\sigma) = |\,\mathrm{RLMAX}(\sigma)|$. For example, $\mathrm{RLMAX}(163254) = \{4,5,6\}$ and $\mathrm{rlmax}(163254) = 3$. A *block* of σ is a substring which ends with a right-to-left maximum, and contains exactly this one right-to-left maximum; moreover, the substring is maximal, i.e., not contained in any larger such substring. Clearly, any permutation has a unique decomposition as a sequence of blocks. Let $\mathrm{bk}\,(\sigma)$ and $\mathrm{bkone}\,(\sigma)$ be the numbers of blocks and blocks of length one of σ, respectively. Let $\mathrm{fcyc}(\sigma)$ be the length (number of terms) of the first block of σ from left to right. For example, the block decomposition of 163254 is given by [16][325][4], $\mathrm{bk}\,(163254) = 3$, $\mathrm{bkone}\,(163254) = 1$ and $\mathrm{fbk}\,(163254) = 2$.

For any $\sigma \in \mathfrak{S}_n$, we can write σ in standard cycle form satisfying the following conditions:

(i) each cycle is end with its largest element;

(ii) the cycles are written in decreasing order of their largest element.

In the following discussion, we shall always write the cycle structure of $\sigma \in \mathfrak{S}_n$ in standard cycle form.

Definition 3.1. *Let $\widehat{\mathcal{Q}}_n$ be the set of permutations of $[n]$ with the restriction that the sequence in the cycle containing n is increasing.*

For example,

$$\widehat{\mathcal{Q}}_3 = \{(3)(2)(1), (2,3)(1), (3)(1,2), (1,3)(2), (1,2,3)\}.$$

Define $\widehat{\sigma}$ to be the word obtained from $\sigma \in \widehat{\mathcal{Q}}_n$ by writing it in standard cycle form and erasing the parentheses. Then $\widehat{\sigma} \in \mathcal{Q}_n$. Thus, we get a bijection from $\widehat{\mathcal{Q}}_n$ to $\mathcal{Q}_n$. Suppose that

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{i_1})(\sigma_{i_1+1}, \sigma_{i_1+2}, \ldots, \sigma_{i_2}) \cdots (\sigma_{i_{k-1}+1}, \sigma_{i_{k-2}+2}, \ldots, \sigma_{i_k}) \in \widehat{\mathcal{Q}}_n.$$

Then $\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}$ are the largest elements of their cycles, and $\sigma_{i_1} > \sigma_{i_2} > \ldots > \sigma_{i_k}$. Hence $\sigma(\sigma_i) > \sigma_i$ if and only if $\sigma_i < \sigma_{i+1}$. Let $\mathrm{fcyc}(\sigma)$ be the number of elements in the first cycle of σ.

From the above discussion, we can now conclude the following result.

Proposition 3.2. *For any $n \geq 1$, we have*

$$\sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{asc}\,(\sigma)} y^{\mathrm{bkone}\,(\sigma)} q^{\mathrm{bk}\,(\sigma)} p^{\mathrm{fbk}\,(\sigma)} = \sum_{\sigma \in \widehat{\mathcal{Q}}_n} x^{\mathrm{exc}\,(\sigma)} y^{\mathrm{fix}\,(\sigma)} q^{\mathrm{cyc}\,(\sigma)} p^{\mathrm{fcyc}(\sigma)}.$$

Let $\widetilde{A}_n(x,y,q,p) = \sum_{\sigma \in \widehat{\mathcal{Q}}_{n+1}} x^{\mathrm{exc}\,(\sigma)} y^{\mathrm{fix}\,(\sigma)} q^{\mathrm{cyc}\,(\sigma)} p^{\mathrm{fcyc}(\sigma)}$. The first few $\widetilde{A}_n(x,y,q,p)$ are given as follows:

$$\begin{aligned}
\widetilde{A}_0(x,y,q,p) &= ypq, \\
\widetilde{A}_1(x,y,q,p) &= y^2pq^2 + xp^2q, \\
\widetilde{A}_2(x,y,q,p) &= pq^3y^3 + pq^2xy + 2p^2q^2xy + p^3qx^2.
\end{aligned}$$

Theorem 3.3. *Let $\widetilde{A}(x,y,q,p;z) = \sum_{n\geq 0} \widetilde{A}_n(x,y,q,p)\frac{z^n}{n!}$. We have*

$$\widetilde{A}(x,y,q,p;z) = (e^{xpz} + y - 1)\, pqA(x,y,q;z). \tag{12}$$

Proof. Let n be a fixed positive integer. Given $\pi \in \widehat{\mathcal{Q}}_{n+1}$. Suppose the first cycle of π is given by $\sigma = (c_1, c_2, \ldots, c_k, n+1)$. So π can be split into the cycle σ and a permutation τ on the set $\{1, 2, \ldots, n+1\} \setminus \{c_1, c_2, \ldots, c_k, n+1\}$, i.e., $\pi = \sigma \cdot \tau$. When $k = 0$, we have

$$\mathrm{exc}\,(\pi) = \mathrm{exc}\,(\tau), \mathrm{fix}\,(\pi) = \mathrm{fix}\,(\tau) + 1, \mathrm{cyc}\,(\pi) = \mathrm{cyc}\,(\tau) + 1, \mathrm{fcyc}(\pi) = 1.$$

This provides the term $ypqA_n(x,y,q)$. When $1 \leq k \leq n$, there are $\binom{n}{k}$ ways to form the set $\{c_1, c_2, \ldots, c_k\}$. Moreover, we have

$$\mathrm{exc}\,(\pi) = \mathrm{exc}\,(\tau) + k, \mathrm{fix}\,(\pi) = \mathrm{fix}\,(\tau), \mathrm{cyc}\,(\pi) = \mathrm{cyc}\,(\tau) + 1, \mathrm{fcyc}(\pi) = k + 1.$$

This provides the term $\sum_{k=1}^{n}\binom{n}{k}x^kqp^{k+1}A_{n-k}(x,y,q)$. Therefore, we obtain

$$\widetilde{A}_n(x,y,q,p)=ypqA_n(x,y,q)+\sum_{k=1}^{n}\binom{n}{k}x^kqp^{k+1}A_{n-k}(x,y,q). \tag{13}$$

Multiplying both sides of (13) by $z^n/n!$ and summing over all nonnegative integers n, we get that

$$\begin{aligned}\widetilde{A}(x,y,q,p;z)&=ypqA(x,y,q;z)+pq\sum_{n=1}^{\infty}\sum_{k=1}^{n}\binom{n}{k}(xp)^kA_{n-k}(x,y,q)\frac{z^n}{n!}\\&=ypqA(x,y,q;z)+pq\sum_{n=1}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(xp)^kA_{n-k}(x,y,q)\frac{z^n}{n!}-pq\left((A(x,y,q;z)-1\right)\\&=ypqA(x,y,q;z)+pq\left(e^{xpz}A(x,y,q;z)-1\right)-pq\left((A(x,y,q;z)-1\right)\\&=\left(e^{xpz}+y-1\right)pqA(x,y,q;z).\end{aligned}$$

This completes the proof. □

From (12), we see that

$$\begin{aligned}\widetilde{A}(x,1,-1,-1;z)&=e^{-xz}A(x,1,-1;z)=\frac{e^{-z}-xe^{-xz}}{1-x},\\\widetilde{A}(x,1,-1,1;z)&=-e^{xz}A(x,1,-1;z)=\frac{e^{2xz-z}-xe^{xz}}{x-1}.\end{aligned}$$

It is routine to check that

$$\frac{e^{-z}-xe^{-xz}}{1-x}=\sum_{n=0}^{\infty}(-1)^n\frac{1-x^{n+1}}{1-x}\frac{z^n}{n!},$$

$$\frac{e^{(2x-1)z}-xe^{xz}}{x-1}=\sum_{n=0}^{\infty}\frac{(1-2x)^{2n}-x^{2n+1}}{x-1}\frac{z^{2n}}{(2n)!}+\sum_{n=1}^{\infty}\frac{(1-2x)^{2n-1}+x^{2n}}{1-x}\frac{z^{2n-1}}{(2n-1)!}. \tag{14}$$

Therefore, we get the following corollary.

Corollary 3.4. *For $n\geq 0$, we have*

$$\widetilde{A}_n(x,1,-1,-1)=\sum_{\sigma\in\widehat{\mathcal{Q}}_{n+1}}x^{\mathrm{exc}\,(\sigma)}(-1)^{\mathrm{cyc}\,(\sigma)+\mathrm{fcyc}(\sigma)}=(-1)^n(1+x+x^2+\cdots+x^n);$$

$$\widetilde{A}_n(x,1,-1,1)=\sum_{\sigma\in\widehat{\mathcal{Q}}_{n+1}}x^{\mathrm{exc}\,(\sigma)}(-1)^{\mathrm{cyc}\,(\sigma)}=\sum_{k=0}^{n}x^{n-k}\sum_{i=k}^{n}(-1)^{i-1}2^{n-i}\binom{n}{i}.$$

It would be interesting to present a combinatorial proof of Corollary 3.4.

Let

$$B(n,k)=\sum_{i=k}^{n}(-1)^{k-i}2^{n-i}\binom{n}{i}.$$

It should be noted that the numbers $B(n,k)$ are known as the $(k-2)$-nd *Betti numbers* of the complement of the k-equal real hyperplane arrangement in $\mathbb{R}^n$ (see [7, Theorem 4.1.5] for instance). The Betti number $B(n,i)$ was first studied by Björner and Welker [2], and subsequently studied by Green [7, 8]. The reader is referred to Green [8, page 1038] for various interpretations of the numbers $B(n,i)$.

From Corollary 3.4, we see that

$$\sum_{\sigma\in\widehat{\mathcal{Q}}_{n+1}} x^{\mathrm{exc}\,(\sigma)}(-1)^{\mathrm{cyc}\,(\sigma)} = \sum_{k=0}^{n}(-1)^{k+1}B(n,k)x^{n-k}. \tag{15}$$

An *anti-excedance* in $\pi\in\mathfrak{S}_n$ is an index i such that $\pi(i)\leq i$. Let $\mathrm{aexc}\,(\pi)$ be the number of anti-excedances of π. Clearly, $\mathrm{exc}\,(\pi)+\mathrm{aexc}\,(\pi)=n$ for $\pi\in\mathfrak{S}_n$. For $\pi\in\widehat{\mathcal{Q}}_{n+1}$, if $\mathrm{exc}\,(\pi)=n-k$, then $\mathrm{aexc}\,(\pi)=k+1$. Therefore, using (15), we get the following result.

Theorem 3.5. *For $n\geq 0$, we have*

$$B(n,k)=\sum_{\substack{\pi\in\widehat{\mathcal{Q}}_{n+1}\\ \mathrm{exc}\,(\pi)=n-k}}(-1)^{\mathrm{cyc}\,(\pi)+\mathrm{aexc}\,(\pi)}.$$

Using Theorem (3.5), one may introduce some q-analogs of the Betti numbers $B(n,k)$.

Let $B_n(x)=\sum_{k=0}^n B(n,k)x^k$. Combining (14) and (15), we obtain the following result.

Proposition 3.6. *We have*

$$\sum_{n\geq 0}B_n(x)\frac{z^n}{n!}=\frac{e^z+xe^{(2+x)z}}{1+x}.$$

Define

$$T_n(q)=\sum_{\sigma\in\widehat{\mathcal{Q}}_{n+1}}q^{\mathrm{cyc}\,(\sigma)}=\sum_{k=1}^{n+1}T(n,k)q^k.$$

Let $T(q,z)=\sum_{n\geq 0}T_n(q)\frac{z^n}{n!}$. It follows from (12) that

$$T(q,z)=qe^z\sum_{n\geq 0}\sum_{k=0}^{n}{n\brack k}q^k\frac{z^n}{n!}=\frac{qe^z}{(1-z)^q}, \tag{16}$$

where ${n\brack k}$ is the signless Stirling number of the first kind, i.e., the number of permutations of $\mathfrak{S}_n$ with k cycles. Using (16), we immediately get the following result.

Proposition 3.7. *For $n\geq 2$, we have $T_n(-1)=\sum_{\sigma\in\widehat{\mathcal{Q}}_{n+1}}(-1)^{\mathrm{cyc}\,(\sigma)}=n-1$.*

Let $F_n(q)=\sum_{k=0}^n {n\brack k}q^k$. Combining (16) and the well known recurrence relation $F_n(q)=(n-1+q)F_{n-1}(q)$, one can easily derive that the polynomials $T_n(q)$ satisfy the recurrence relation

$$T_{n+1}(q)=(n+1+q)T_n(q)-nT_{n-1}(q), \tag{17}$$

with the initial conditions $T_0(q)=q,\ T_1(q)=q+q^2$. Equivalently, we have

$$T(n+1,k)=(n+1)T(n,k)+T(n,k-1)-nT(n-1,k).$$

Recall that the Charlier polynomials are defined by

$$C_n^{(a)}(x)=\sum_{k=0}^{n}(-a)^{n-k}\binom{n}{k}\binom{x}{k}k!,\ a\neq 0.$$

These polynomials are generated by $e^{-az}(1+z)^x = \sum_{n\geq 0} C_n^{(a)}(x)\frac{z^n}{n!}$. Hence

$$T_n(q) = (-1)^n q C_n^{(1)}(-q) = q\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{-q}{k}k!.$$

It is well known that Charlier polynomials are orthogonal polynomials and have only real zeros. Hence the polynomial $T_n(q)$ has only real zeros for any $n \geq 0$.

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