# A NOTE ON 3-FREE PERMUTATIONS 

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#### Abstract

Let $\theta(n)$ denote the number of permutations of $\{1,2, \ldots, n\}$ that do not contain a 3 -term arithmetic progression as a subsequence. Such permutations are known as 3 -free permutations. We present a dynamic programming algorithm to count all 3 -free permutations of $\{1,2, \ldots, n\}$. We use the output to extend and correct enumerative results in the literature for $\theta(n)$ from $n=20$ out to $n=90$ and use the new values to inductively improve existing bounds on $\theta(n)$.


Keywords: 3-free permutation; Costas array

## 1. Introduction and Results

Let $n$ be a positive integer and let $[n]$ denote the set $\{1,2, \ldots, n\}$. Let $\alpha=$ $\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ be a permutation of $[n]$. Then $\alpha$ is a 3 -free permutation if and only if, for every index $j(1 \leq j \leq n)$, there do not exist indices $i<j$ and $k>j$ such that $a_{i}+a_{k}=2 a_{j}$. Let $\theta(n)$ be the function that gives the number of 3 -free permutations of $[n]$. Of course the value of $\theta(n)$ will be unchanged if we replace $[n]$ with any set of $n$ integers in arithmetic progression so we will hereafter use $[n]$ when referring to $\theta(n)$. In 1973 Entringer and Jackson initiated the study of 3 -free permutations by posing

Problem 1 (Entringer and Jackson [5]). Does every permutation of $\{0,1, \ldots, n\}$ contain an arithmetic progression of at least three terms?

Three solutions (see [7], [8], [12]) to Problem 1 showing that the answer is "No" along with comments [11] containing a table of values of $\theta(n)$ for $1 \leq n \leq 20$ were published. The solutions of Odda [8] and Thomas [12] contained the first
constructions for 3 -free permutations. Odda describes how to construct one 3free permutation for each $n$. Thomas devised a method to generate $2^{n-1} 3$-free permutations for each $n$. Thomas's examples show that the sets of permutations his method generates aren't exhaustive.

The purpose of this note is to present an algorithm that counts the number of 3 -free permutations of $n$ consecutive integers for each $n$. We correct and extend the tables of known values of $\theta(n)$ out to $n=90$ and improve upper and lower bounds by proving the following four results.
Theorem 2. For positive integers $n \geq 45$,

$$
\begin{equation*}
\theta(n) \geq \frac{c_{1}^{n}}{2}, c_{1}=\sqrt[80]{2 \theta(80)}=2.201 \ldots \tag{1}
\end{equation*}
$$

Theorem 3. For positive integers $n \geq 36$,

$$
\begin{equation*}
\theta(n) \leq \frac{c_{2}^{n}}{21}, c_{2}=\sqrt[64]{21 \theta(64)}=2.364 \ldots \tag{2}
\end{equation*}
$$

Theorem 4. For positive integers $k \geq 6$ and $n=2^{k}$,

$$
\begin{equation*}
\theta(n) \geq \frac{c_{3}^{n}}{2}, c_{3}=\sqrt[64]{2 \theta(64)}=2.279 \ldots \tag{3}
\end{equation*}
$$

Theorem 5. For all positive integers n,

$$
\begin{equation*}
\theta(n) \geq \frac{n c_{4}^{n}}{40}, c_{4}=\sqrt[40]{\theta(40)}=2.156 \ldots \tag{4}
\end{equation*}
$$

The existence of $\lim \theta(n)^{1 / n}$ as $n \rightarrow \infty$ was identified in [9] as a key problem in the study of $\theta(n)$. It remains an open question although Theorems 2,3 imply that the limit lies within the interval $\left[c_{1}, c_{2}\right]$ if it exists. The first author explored connections between 3 -free permutations and Costas arrays in [2], where slightly weaker versions of Theorems 2 and 3 were stated without proof.

For clarity, we comment here that we are not presenting any results on the related problem of evaluating and bounding the function $r(n)$ giving the longest 3 -free subsequence of the sequence $1,2, \ldots, n$. The latest developments in solving that problem currently appear in [4].

## 2. Some Results From the Literature on $\theta(n)$

Davis, Entringer, Graham, and Simmons [3] established a number of bounds on the growth of $\theta(n)$ including the following:
Theorem 6 (Davis, Entringer, Graham, and Simmons, [3]). For positive integers $n$,

$$
\begin{align*}
\theta(2 n) & \geq 2 \theta^{2}(n)  \tag{5}\\
\theta(2 n+1) & \geq 2 \theta(n) \theta(n+1) \tag{6}
\end{align*}
$$

Theorem 7 (Davis, Entringer, Graham, and Simmons, [3]). For $n=2^{k}, k \geq 4$,

$$
\begin{equation*}
\theta(n) \geq \frac{c^{n}}{2}, c=\sqrt[16]{2 \theta(16)}=2.248 \ldots \tag{7}
\end{equation*}
$$

Sharma's dissertation [10] is noteworthy in that it established the long-conjectured result that $\theta(n)$ has an exponential upper bound. Sharma used parity arguments to prove

Theorem 8 (Sharma, [9]). For each $n \geq 3$,

$$
\begin{equation*}
\theta(n) \leq 21 \theta\left(\left\lceil\frac{n}{2}\right\rceil\right) \theta\left(\left\lfloor\frac{n}{2}\right\rfloor\right) \tag{8}
\end{equation*}
$$

The key result in [10] (and in the follow-up journal paper [9] as well as the book [13]) he obtains from Theorem 8 is
Theorem 9 (Sharma, [9]). For $n \geq 11$,

$$
\begin{equation*}
\theta(n) \leq \frac{2.7^{n}}{21} \tag{9}
\end{equation*}
$$

Sharma also improved Thomas's strict, constructive lower bound of $2^{n-1}$ for $n>5$ by showing that:

Theorem 10 (Sharma, [9]). For all positive integers n,

$$
\begin{equation*}
\theta(n) \geq \frac{n 2^{n}}{10} \tag{10}
\end{equation*}
$$

but LeSaulnier and Vijay were able to establish
Theorem 11 (LeSaulnier and Vijay [6]). For $n \geq 8$,

$$
\begin{equation*}
\theta(n) \geq \frac{1}{2} c^{n}, \text { where } c=(2 \theta(10))^{\frac{1}{10}}=2.152 \ldots \tag{11}
\end{equation*}
$$

In Section 5 we use Theorems 6 and 8 in inductive proofs of Theorems 2, 3, and 4 to improve Theorems 11, 9, and 7, respectively. We also rework Sharma's proof of Corollary 3.2 .1 of [9] relying on Theorem 6 using our additional computed values of $\theta(n)$ to improve upon Theorem 10 for $n \geq 19$.

## 3. Algorithm Descriptions

Recall from Section 1 that we write $\alpha=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ for a permutation of $[n]$. For $j=1,2, \ldots, n$, if we define

$$
\begin{equation*}
T_{j} \equiv\left\{a_{k} \mid j \leq k \leq n\right\} \tag{12}
\end{equation*}
$$

then the 3 -free property of a permutation can be restated as saying that there do not exist $a_{i} \notin T_{j}$ and $a_{k} \in T_{j}-\left\{a_{j}\right\}$ such that $a_{i}+a_{k}=2 a_{j}$. Further inspection of the 3 -free property allows us to replace $T_{j}-\left\{a_{j}\right\}$ with $T_{j}$, because $a_{k}=a_{j}$ would imply $a_{i}=a_{j}$ (from $a_{i}+a_{k}=2 a_{j}$ ), but then $a_{i}$ would be in $T_{j}$. If the 3 -free property holds for all $1 \leq j \leq n$, then $\alpha$ is a 3 -free permutation. This suggests the following algorithm to generate 3 -free permutations:

```
Backtracking algorithm to enumerate 3-free permutations
Subroutine Enumerate
Input: A (possibly empty) sequence \(\rho=\left(p_{1}, p_{2}, \ldots, p_{k}\right)\) of distinct integers \(p_{k} \in[n]\)
Output: All 3 -free permutations of \([n]\) that begin with \(\rho\)
    procedure Enumerate \((\rho)\)
        if \(|\rho|=n\) then
            print \(\rho\)
        else
            \(P=\left\{\rho_{1}, \rho_{2}, \ldots, \rho_{|\rho|}\right\}\)
            for \(1 \leq j \leq n\) do
                if \(j \notin P\) and \(\nexists i \in P, k \in[n] \backslash P\) such that \(i+k=2 j\) then
                Enumerate \(((\rho, j))\)
                end if
            end for
        end if
    end procedure
Main Backtracking Algorithm
Input: Positive integer \(n\)
Output: List of all 3-free permutations of \([n]\)
    procedure EnumerateMain \((n)\)
        Enumerate([]) (ロ) Empty sequence
    end procedure
```

In the subroutine Enumerate, the notation $(\rho, j)$ on line 8 denotes the sequence obtained from appending the integer $j$ to the sequence $\rho$. The main backtracking algorithm recursively generates the 3 -free permutations one by one, so it could be of use in generating data from which new structural properties of 3 -free permutations could be deduced. Clearly its running time is bounded below by $\theta(n)$. If we are interested in the number $\theta(n)$ of 3 -free permutations and not the permutations themselves, we can speed up the counting process by using dynamic programming. Dynamic programming algorithms (see, for instance, [1]) solve programs by combining solutions to subproblems. The subproblems can be dependent in that they have common subsubproblems.

A key observation is that the 3 -free property of a permutation depends on the set of elements that have been used so far in building up that permutation. The exact
ordering of those elements is not relevant. The dynamic programming algorithm recursively evaluates $\theta(n)$ using dynamic programming. It uses bitsets to keep track of which integers have not been placed in an effort to build up a 3-free permutation (a bitset is a sequence of zeros and ones.)

```
Dynamic programming algorithm to count 3-free permutations
Subroutine Count
Input: A bitset \(b\) of length \(n\)
Output: The number \(\theta(b)\) of 3 -free permutations of \([n]\) that begin with \(\rho\), where
\(\rho\) is any valid initial sequence that uses exactly the integers that \(b\) maps to 0 . Note
that if there is more than one such sequence, then they must give the same number,
due to the 3 -free property
function Count (b)
        if \(\exists(b, v) \in C\) for some \(v\) then
            return \(v\)
        else if \(\max \{b[1], b[2], \ldots, b[n]\}=0\) then
            return 1
        else
            ans \(\leftarrow 0\)
            for \(1 \leq j \leq n\) do
                    if \(b[j]=1\) and \(\nexists 1 \leq i, k \leq n\) such that \(b[i]=0\) and \(b[k]=1\) and
\(i+k=2 j\) then
                \(b^{\prime} \leftarrow b\)
                \(b^{\prime}[j] \leftarrow 0\)
                ans \(\leftarrow a n s+\operatorname{Count}\left(b^{\prime}\right)\)
            end if
        end for
        \(C \leftarrow C \cup\{(b, a n s)\}\)
        return ans
        end if
end function
Main Dynamic Programming Algorithm
Input: Positive integer \(n\)
Output: \(\theta(n)\)
function CountMain \((n)\)
        \(C \leftarrow \emptyset\)
        return \(\operatorname{Count}((1,1, \ldots, 1)) \quad(\triangleright)\) Bitset of \(n\) ones
end function
```

In the above algorithm, $C$ denotes a set of pairs $(b, v)$, where $b$ is a bitset of length $n$ and $v$ is a non-negative integer. The set $C$ is intended to be implemented by a data structure known as a "map". In our usage of $C$, the value of $v$ for each b is $\theta(b)$.

Let $T \subseteq[n]$. It takes $O(n)$ time to check if the 3 -free property is violated and it takes $O(n)$ time to iterate over every element $t$ in $T$. On the surface Algorithm 3 appears to require $O\left(2^{n}\right)$ memory to store $\theta(T)$ for every subset $T$ of $[n]$ and the running time appears to be $O\left(n^{2} 2^{n}\right)$. However, it turns out that only a small percentage of the subsets of $[n]$ are needed in the recurrence because most of them are not reachable due to a violation of the 3-free property. This helped us to tabulate $\theta(n)$ out to $n=90$. The value of $\theta(90)$ has 31 digits.

## 4. Computational Enumerative Results

We pushed a Java implementation of the dynamic programming algorithm out to $n=90$ and updated entry A003407 of the Online Encyclopedia of Integer Sequences (http://www.oeis.org/A003407) with the values in Table 1. For $n=90$, the fraction of subsets that had to be visited was only

$$
\begin{equation*}
254931123 /\left(2^{90}\right) \approx 2.059\left(10^{-19}\right) \tag{13}
\end{equation*}
$$

Our Java implementation ran out of memory for $n=91$. Algorithm 3 does not lend itself to parallelization due to the way it uses memory. Additional values of $\theta(n)$ can be obtained on computing platforms having additional memory, support for arbitrarily long integers, and adequate processing power.

Before our computations, there were at least 4 published tables of values of $\theta(n)$ for $1 \leq n \leq 20$ although only two of these tables are correct. The very first table to appear is in [11] and claims that 73904 is the value of $\theta(15)$ but the correct value is $\theta(15)=74904$. For $n=17$ the table in [9] claims that 360016 is the value of $\theta(17)$ but the correct value is $\theta(17)=368016$. The first twenty values of $\theta(n)$ listed above do agree with the table in [3]. The first 20 entries in entry A003407 were correct at the time we extended them.

## 5. Proofs

Theorems 2, 3, and 4 can be proven by induction:
Proof. To prove Theorem 2 it suffices, by Theorem 6, to prove $\theta(n) \geq \frac{c^{n}}{2}$ for $42 \leq n \leq 83$ and some constant $c$. Computation shows that the maximal such $c$ is $\min (2 \theta(n))^{1 / n}=c_{1}$ and occurs for $n=42$.

Proof. To prove Theorem 3, we observe that, for $42 \leq n \leq 83$, $\max (21 \theta(n))^{\frac{1}{n}}=c_{2}$, and occurs for $n=64$ so (3) holds for all $n \in[42,83]$. That inequality (3) holds for

Table 1: Number of 3-free permutations $\theta(n)$ of $[n]$

| $n$ | $\theta(n)$ | $n$ | $\theta(n)$ | $n$ | $\theta(n)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 31 | 41918682488 | 61 | 1612719155955443585092 |
| 2 | 2 | 32 | 121728075232 | 62 | 4640218386156695178110 |
| 3 | 4 | 33 | 207996053184 | 63 | 13557444070821420327240 |
| 4 | 10 | 34 | 360257593216 | 64 | 39911512393313043466768 |
| 5 | 20 | 35 | 639536491376 | 65 | 67867319248960144994224 |
| 6 | 48 | 36 | 1144978334240 | 66 | 115643050433241064474672 |
| 7 | 104 | 37 | 2362611440576 | 67 | 199272038058617170554928 |
| 8 | 282 | 38 | 4911144118024 | 68 | 344053071167567188894208 |
| 9 | 496 | 39 | 10417809568016 | 69 | 608578303898604406167840 |
| 10 | 1066 | 40 | 22388184630824 | 70 | 1080229099508551381463536 |
| 11 | 2460 | 41 | 50301508651032 | 71 | 1929269192569465070403584 |
| 12 | 6128 | 42 | 113605533519568 | 72 | 3452997322628833453585008 |
| 13 | 12840 | 43 | 265157938869936 | 73 | 7096327095079914521075040 |
| 14 | 29380 | 44 | 622473467900178 | 74 | 14611112240136930804928288 |
| 15 | 74904 | 45 | 1527398824248200 | 75 | 30235147387260979648843264 |
| 16 | 212728 | 46 | 3784420902143392 | 76 | 62757445134327428602306464 |
| 17 | 368016 | 47 | 9503564310606436 | 77 | 132956581436718531491070160 |
| 18 | 659296 | 48 | 23991783779046768 | 78 | 282272593229156186280461264 |
| 19 | 1371056 | 49 | 48820872045382552 | 79 | 605672649054377049472147568 |
| 20 | 2937136 | 50 | 99986771685259808 | 80 | 1302375489530691442230524528 |
| 21 | 6637232 | 51 | 209179575852808848 | 81 | 2914298247043287576460093712 |
| 22 | 15616616 | 52 | 441563057878399888 | 82 | 6537258415569149903366841040 |
| 23 | 38431556 | 53 | 992063519708141728 | 83 | 14713284774210886488265138336 |
| 24 | 96547832 | 54 | 2241540566114243168 | 84 | 33155372641605493828236640928 |
| 25 | 198410168 | 55 | 5185168615770591200 | 85 | 77219028670778815210019118736 |
| 26 | 419141312 | 56 | 12057653703359308256 | 86 | 180104653062631494787580542664 |
| 27 | 941812088 | 57 | 31151270610676979624 | 87 | 421733920870430143234318231648 |
| 28 | 2181990978 | 58 | 81046346414827952010 | 88 | 990082990967384066255452324186 |
| 29 | 5624657008 | 59 | 213208971281274232760 | 89 | 2428249522507620383597702223224 |
| 30 | 14765405996 | 60 | 563767895033816986864 | 90 | 5963505178650560845887322154368 |

all $n \geq 42$ follows from using the fact that it holds for $42 \leq n \leq 83$ as a basis for an inductive argument and from Theorem 8. Straightforward numerical investigation reveals that inequality (3) actually holds for $n \geq 36$ (but not for $n \leq 35$ ).

Proof. A proof of Theorem 4 follows by induction on $k$ using Theorem 6.
To prove Theorem 5, we rework the reasoning of Section 3 of [9] through the proof of Corollary 3.2.1 using an exponential base $\alpha>2$ and the values of $\theta(n)$ in Table 1. We obtain improved variants of Theorems 3.1 and 3.2 of [9] along the way. First note that:
(a) If for some positive integer $n, \theta(n) \geq \alpha^{n}$ and $\theta(n+1) \geq \alpha^{n+1}$ then by Theorem 6 , we have $\theta(2 n) \geq 2 \alpha^{2 n}$ and $\theta(2 n+1) \geq 2 \alpha^{2 n+1}$.

By computer verification and the data in Table 1 , we see that $\theta(n) \geq \alpha^{n}$ for $n \in[40,79]$ for $\alpha=c_{4}$ and that $c_{4}$ is the maximal such value. Thus, by (a), $\theta(n) \geq 2 \alpha^{n}$ for $n \in[80,159]$. Applying (a) to this inequality yields $\theta(n) \geq 8 \alpha^{n}$ for $n \in[160,319]$. An inductive argument allows us to prove the following improvement on Theorem 3.1 of [9].

Theorem 12. For integers $p \geq 2$ and $\alpha=c_{4}$,

$$
\begin{equation*}
\theta(n) \geq 2^{2^{p-2}-1} \alpha^{n} \text { for all } n \in\left[5 \times 2^{p+1}, 5 \times 2^{p+2}-1\right] \tag{14}
\end{equation*}
$$

Proof. We know the statement is true for $p=2$. Suppose the statement holds for all $p \leq l-1$. Then for $n \in\left[5 \times 2^{l+1}, 5 \times 2^{l+2}-1\right]$ if $n$ is even, applying the inductive hypothesis to $\frac{n}{2}$ and using Theorem 6 verifies the theorem for $n$ (similarly for $n$ odd applying the induction hypothesis to $\frac{n-1}{2}$ and $\frac{n+1}{2}$ ). This verifies the statement for $p=l$ as desired.

Next, we prove the following improvement over Theorem 3.2 of [9].
Theorem 13. For any fixed integer $p \geq 5$ and $\alpha=c_{4}$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\theta(n)}{n^{p} \times \alpha^{n}}=\infty \tag{15}
\end{equation*}
$$

Proof. Consider the sequence $a_{n}=\frac{\theta(n)}{n^{p+1} \times \alpha^{n}}$ for $n \geq 5 \times 2^{p+1}$. Note that $a_{2 n}=$ $\frac{\theta(2 n)}{(2 n)^{p+1} \times \alpha^{2 n}} \geq \frac{2 \times[\theta(n)]^{2}}{(2 n)^{p+1} \times \alpha^{2 n}} \geq \frac{\theta(n)}{n^{p+1} \times \alpha^{n}} \times \frac{\theta(n)}{\alpha^{n+p}}=a_{n} \times \frac{\theta(n)}{\alpha^{n+p}} \geq a_{n} \quad$ (as $\theta(n) \geq$ $2^{2^{p-2}-1} \alpha^{n}$ and $2^{p-2}-1 \geq p \log _{2} \alpha$ for the intervals of $n$ and $p$ values). Similarly $a_{2 n+1} \geq a_{n+1}$ for all such $n$ (proof is identical with the additional step of noting that $\left.(2 n+2)^{p+1} \geq(2 n+1)^{p+1}\right)$. Let $\gamma=\min a_{n}$ for $n \in\left[5 \times 2^{p+1}, 5 \times 2^{p+2}-1\right]$. Using the statements $a_{2 n} \geq a_{n}$ and $a_{2 n+1} \geq a_{n+1}$ recursively implies $a_{n} \geq \gamma$ for all $n \geq 5 \times 2^{p+1}$. Therefore $\frac{\theta(n)}{n^{p} \times \alpha^{n}}=n \times a_{n} \geq n \times \gamma$ for all $n \geq 5 \times 2^{p+1}$ and $\frac{\theta(n)}{n^{p} \times \alpha^{n}}$ clearly tends to $\infty$ as $n \rightarrow \infty$ as desired.

We now prove Theorem 5.

Proof. Let $a_{n}=\frac{\theta(n)}{n \times \alpha^{n}}$. From the values of $\theta(n)$ in Table 1 we note that $a_{n} \geq \frac{1}{40}$ for all $n \in[40,79]$. Since $\theta(n) \geq \alpha^{n}$ for all $n \geq 40$ by Theorem 12, reasoning as in the proof of Theorem 13 lets us prove that $a_{2 n} \geq a_{n}$ and $a_{2 n+1} \geq a_{n+1}$ for all $n \geq 40$. This proves that $a_{n} \geq \frac{1}{40}$ for all positive integers $n$.

## 6. Conjecture

Define the function $h(n)=\log (\theta(n+1))-\log (\theta(n))$. Examining a plot of $h(n)$ suggests

Conjecture 1. The function $h(n)$ is increasing on the intervals $\left[2^{k}, 2^{k}+2^{k-1}-1\right]$ and $\left[2^{k}+2^{k-1}, 2^{k+1}-1\right]$ but is decreasing on $\left[2^{k}+2^{k-1}-1,2^{k}+2^{k-1}\right]$ and the interval $\left[2^{k+1}-1,2^{k+1}\right]$ for $k \geq 2$.

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