FACTORIZATION THEOREMS FOR HADAMARD PRODUCTS AND HIGHER-ORDER DERIVATIVES OF LAMBERT SERIES GENERATING FUNCTIONS

MAXIE D. SCHMIDT
GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL OF MATHEMATICS
117 SKILES BUILDING
686 CHERRY STREET NW
ATLANTA, GA 30332
USA
MAXIEDS@GMAIL.COM
MSCHMIDT34@GATECH.EDU

Abstract. We first summarize joint work on several preliminary canonical Lambert series factorization theorems. Within this article we establish new analogs to these original factorization theorems which characterize two specific primary cases of the expansions of Lambert series generating functions: factorizations for Hadamard products of Lambert series and for higher-order derivatives of Lambert series. The series coefficients corresponding to these two generating function cases are important enough to require the special due attention we give to their expansions within the article, and moreover, are significant in that they connect the characteristic expansions of Lambert series over special multiplicative functions to the explicitly additive nature of the theory of partitions. Applications of our new results provide new exotic sums involving multiplicative functions, new summation-based interpretations of the coefficients of the integer-order j^{th} derivatives of Lambert series generating functions, several new series for the Riemann zeta function, and an exact identity for the number of distinct primes dividing n.

1. Introduction

1.1. **Lambert series factorization theorems.** In the references we have proved several variants and generalized expansions of *Lambert series factorization theorems* of the form [3, 5, 6, 9]

$$\sum_{n\geq 1}\frac{a_n q^n}{1-q^n} = \frac{1}{(q;q)_\infty}\sum_{n\geq 1}\sum_{k=1}^{n} s_{n,k} a_k \cdot q^n,$$

and of the form

$$\sum_{n\geq 1}\frac{\bar{a}_n q^n}{1-q^n} = \frac{1}{(q;q)_\infty}\sum_{n\geq 1}\sum_{k=1}^{n} \widetilde{s}_{n,k} \widetilde{a}_k \cdot q^n,$$

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for $\widehat{a}_n$ and $\bar{a}_n$ depending on an arbitrary arithmetic function a_n and where the lower-triangular sequence $s_{n,k} := [q^n](q;q)_\infty q^k/(1-q^k)$, which we typically require to be independent of the a_n, is the difference of two partition functions counting the number of k's in their respective odd (even) distinct partitions.

In the concluding remarks to [3] we gave several specific examples of other constructions of related Lambert series factorization theorems. In the reference, we also proved a few new properties of the factorizations of Lambert series generating functions over the convolution of two arithmetic functions, $f * g$ expanded by

$$\sum_{n\geq 1} \frac{(f*g)(n)q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \sum_{n\geq 1} \sum_{k=1}^{n} \widetilde{s}_{n,k}(g) f(k) \cdot q^n,$$

which we cite in this article. The Lambert series factorizations of these two forms and their variations, two to three of which we consider in this article, also imply matrix factorizations of these expansions which are dictated by the corresponding typically invertible matrix of $s_{n,k}$ or $\widetilde{s}_{n,k}(g)$. The matrix-based interpretation of these factorization theorems is perhaps the most intuitive way to explore how these expansions "*factor*" into distinct matrices applied to vectors of special sequences.

1.2. **Significance of our results.** Our new results are rare and important because they provide a mechanism that effectively translates the divisor sums of the coefficients of a Lambert series generating function into ordinary sums which similarly generate the same prescribed arithmetic function, say a_n. Moreover, these factorization theorems connect the special functions in multiplicative number theory which are typically tied to a particular Lambert series expansion with the additive theory of partitions and partition functions in unusual and unexpected new ways. We are one of the first authors to examine such relations between the additive and multiplicative in detail (see also [4, 3, 5]). We note that we are not the first to consider the derivatives of Lambert series generating functions [8], though our perspective on the connections afforded by these factorizations is distinctly new.

1.3. **Focus within this article.** Within this article we explore the expansions of factorization theorems for two primary additional special case variants which are distinctive and important enough in their applications to require special attention here: Hadamard products of two Lambert series generating functions and the higher-order integer derivatives of Lambert series generating functions. Section 2 proves several new properties of the first case, where the results proved in Section 3 consider the second case in detail. The significance of these two particular factorizations is that they have a broad range of applications to expanding special and classical arithmetic functions from multiplicative number theory. In Section 4 we tie up loose ends by offering two other related variants of the Lambert series factorizations. Namely, we prove factorization theorems for generating function convolutions and provide a purely matrix-based proof of a new formula for the coefficients enumerated by a Lambert series generating function.

New results and characterizations. The Hadamard product generating function cases lead to several forms of new so-termed "*exotic*" sums for classical special functions as illustrated in the explicit corollaries given in Example 2.4 of the next section. For example, if we form the Hadamard product of generating functions of the two Lambert series over Euler's totient function, $\phi(n)$, we obtain the following more exotic-looking sum for our multiplicative function of interest:

$$\phi(n) = \sum_{k=1}^{n} \sum_{d|n} \frac{p(d-k)}{d} \mu(n/d) \left[k^2 + \sum_{b=\pm 1} \sum_{j=1}^{\left\lfloor \frac{\sqrt{24k-23}-b}{6} \right\rfloor} (-1)^j \left(k - \frac{j(3j+b)}{2} \right)^2 \right].$$

The importance of obtaining new formulas and identities for the derivatives of a series whose coefficients we are interested in studying should be obvious, though our factorization results also provide new alternate characterizations of these derivatives apart from typical ODEs which we can form involving the derivatives of any sequence ordinary generating functions. Our results for the factorization theorems for derivatives of Lambert series generating functions in Section 3 provides two particular factorizations which characterize these expansions.

2. Factorization Theorems for Hadamard Products

2.1. Hadamard products of generating functions. The *Hadamard product* of two ordinary generating functions $F(q)$ and $G(q)$, respectively enumerating the sequences of $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$ is defined by

$$(F \circ G)(q) := \sum_{n\geq 0} f_n g_n \cdot q^n, \quad \text{for } |q| < \sigma_F \sigma_G,$$

where σ_F and σ_G denote the radii of convergence of each respective generating function. Analytically, we have an integral formula and corresponding coefficient extraction formula for the Hadamard product of two generating functions when $F(q)$ is expandable in a fractional series respectively given by [1, §1.12(V); Ex. 1.30, p. 85] [11, §6.3]

$$(F \circ G)(q^2) = \frac{1}{2\pi} \int_0^{\pi} F(qe^{\imath t}) G(qe^{-\imath t}) dt$$
$$(F \circ G)(q) = [x^0] F\left(\frac{q}{x}\right) G(x).$$

In the context of the factorization theorems we consider in this article and in the references, we consider the Hadamard products of two Lambert series generating functions for special arithmetic functions f_n and g_n which we define coefficient-wise to enumerate the product of the divisor sums over each sequence corresponding to the coefficients of the individual Lambert series over the two functions. This subtlety is discussed shortly in Definition 2.1. As we prove below, it turns out that we can formulate analogous factorization theorems for the cases of these Hadamard products as well. The next definition makes the expansion of the particular Hadamard product functions we consider in this section more precise.

Definition 2.1 (Hadamard Products for Lambert Series Generating Functions)**.** For any fixed arithmetic functions f and g, we define the Hadamard product of the two Lambert series over f and g to be the auxiliary Lambert series generating function over the composite function $a_{\mathrm{fg}}(n)$ whose coefficients are given by

$$\sum_{d|n} a_{\mathrm{fg}}(d) = [q^n] \sum_{m\geq 1} \frac{a_{\mathrm{fg}}(m) q^m}{1-q^m} := \underbrace{\left(\sum_{d|n} f_d \right) \left(\sum_{d|n} g_d \right)}_{:=\mathrm{fg}(n)},$$

so that by Möbius inversion we have that

$$a_{\mathrm{fg}}(n) = \sum_{d|n} \mathrm{fg}(d)\mu(n/d) = (\mathrm{fg} * \mu)(n).$$

We note that this definition of our Hadamard product generating functions is already slightly different than the one given above in that we define the Hadamard product of two Lambert series generating functions by the expansion of a third composite Lambert series which corresponds to the particular expansion of the factorization in (1) below.

2.2. **Main results and applications.** The next theorems in this section define the key matrix sequences, $s_{n,k}(f)$ and $s_{n,k}^{(-1)}(f)$, in terms of the next factorization of the Lambert series over $a_{\mathrm{fg}}(n)$ from the definition above in the form of

$$\sum_{n\geq 1} \frac{a_{\mathrm{fg}}(n)q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \sum_{n\geq 1} \sum_{k=1}^{n} s_{n,k}(f) g_k \cdot q^n, \tag{1}$$

where $s_{n,k}(f)$ is independent of the function g, which is equivalent to defining the factorization expansion by the inverse matrix sequences as

$$g_n = \sum_{k=1}^{n} s_{n,k}^{(-1)} \cdot [q^k] \left((q;q)_\infty \times \sum_{n\geq 1} \frac{a_{\mathrm{fg}}(n)q^n}{1-q^n} \right). \tag{2}$$

We also define the following function to expand divisor sums over arithmetic functions as ordinary sums for any integers $1 \leq k \leq n$:

$$T_{\mathrm{div}}(n,k) := \begin{cases} 1, & \text{if } k|n; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.2. *For all integers $1 \leq k \leq n$, we have the following definition of the factorization matrix sequence defining the expansion on the right-hand-side of* (1) *where we adopt the notation $\widetilde{f}(n) := \sum_{d|n} f_d$:*

$$\begin{aligned} s_{n,k}(f) &= T_{\mathrm{div}}(n,k)\widetilde{f}(n) \\ &\quad + \sum_{b=\pm 1} \sum_{j=1}^{\left\lfloor \frac{\sqrt{24(n-k)+1}-b}{6} \right\rfloor} (-1)^j T_{\mathrm{div}}\left(n - \frac{j(3j+b)}{2}, k\right) \cdot \widetilde{f}\left(n - \frac{j(3j+b)}{2}\right). \end{aligned}$$

Proof. By the factorization in (1) and the definition of $a_{\mathrm{fg}}(n)$ given above, we have that for $\widetilde{f}(n) = \sum_{d|n} f_d$

$$\begin{aligned} s_{n,k}(f) &= [g_k] \left(\sum_{d|n} f_d \right) \times \sum_{d=1}^{n} g_d T_{\mathrm{div}}(n,d) \\ &= [q^n](q;q)_\infty \times \sum_{n\geq 1} T_{\mathrm{div}}(n,k)\widetilde{f}(n) q^n, \end{aligned}$$

which equals the stated expansion of the sequence by Euler's pentagonal number theorem which provides that

$$(q;q)_\infty = 1 + \sum_{j\geq 1} (-1)^j \left(q^{j(3j-1)/2} + q^{j(3j+1)/2} \right). \qquad \square$$

Theorem 2.3 (Inverse Sequences)**.** *For all integers $1 \leq k \leq n$, we have the next definition of the inverse factorization matrix sequence which equivalently defines the expansion on the right-hand-side of* (1).

$$s_{n,k}^{(-1)}(f) = \sum_{d|n} \frac{p(d-k)}{\widetilde{f}(d)} \mu(n/d),$$

Proof. We expand the right-hand-side of the factorization in (1) for the sequence $g_n := s_{n,r}^{(-1)}(f)$, i.e., the exact inverse sequence, for some fixed $r \geq 1$ as follows[1]:

$$\begin{aligned}\widetilde{f}(n) \cdot \sum_{d|n} s_{d,r}^{(-1)}(f) &= \sum_{j=0}^{n} \sum_{k=1}^{j} s_{j,k} s_{k,r}^{(-1)} \cdot p(n-j) \\ &= \sum_{j=0}^{n} [j=r]_{\delta}\, p(n-j) = p(n-r).\end{aligned}$$

Then the last equation implies that

$$\sum_{d|n} s_{d,r}^{(-1)}(f) = \frac{p(n-r)}{\widetilde{f}(n)},$$

which by Möbius inversion implies our stated result. □

Example 2.4 (Applications of the Theorem)**.** For the arithmetic function pairs

$$(f, g) := (n^t, n^s), (\phi(n), \Lambda(n)), (n^t, J_t(n)),$$

respectively, and some constants $s, t \in \mathbb{C}$ where $\sigma_\alpha(n)$ denotes the generalized sum-of-divisors function, $\Lambda(n)$ is von Mangoldt's function, $\phi(n)$ is Euler's totient function, and $J_t(n)$ is the Jordan totient function, we employ the equivalent expansions of the factorization result in (2) to formulate the following "*exotic*" sums as consequences of the theorems above:

$$\begin{aligned} n^s &= \sum_{k=1}^{n} \sum_{d|n} \frac{p(d-k)}{\sigma_t(d)} \mu(n/d) \Biggl[\sigma_t(k)\sigma_s(k) \\ &\quad + \sum_{b=\pm 1} \sum_{j=1}^{\left\lfloor \frac{\sqrt{24k+1}-b}{6} \right\rfloor} (-1)^j \sigma_t\left(k - \frac{j(3j+b)}{2}\right) \sigma_s\left(k - \frac{j(3j+b)}{2}\right) \Biggr] \qquad (3) \\ \Lambda(n) &= \sum_{k=1}^{n} \sum_{d|n} \frac{p(d-k)}{d} \mu(n/d) \Biggl[k \log(k) \\ &\quad + \sum_{b=\pm 1} \sum_{j=1}^{\left\lfloor \frac{\sqrt{24k-23}-b}{6} \right\rfloor} (-1)^j \left(k - \frac{j(3j+b)}{2}\right) \log\left(k - \frac{j(3j+b)}{2}\right) \Biggr] \\ J_t(n) &= \sum_{k=1}^{n} \sum_{d|n} \frac{p(d-k)}{d^t} \mu(n/d) \left[k^{2t} + \sum_{b=\pm 1} \sum_{j=1}^{\left\lfloor \frac{\sqrt{24k-23}-b}{6} \right\rfloor} (-1)^j \left(k - \frac{j(3j+b)}{2}\right)^{2t} \right]. \end{aligned}$$

[1] *Notation*: *Iverson's convention* compactly specifies boolean-valued conditions and is equivalent to the *Kronecker delta function*, $\delta_{i,j}$, as $[n=k]_\delta \equiv \delta_{n,k}$. Similarly, $[\texttt{cond} = \texttt{True}]_\delta \equiv \delta_{\texttt{cond},\texttt{True}}$ in the remainder of the article.

By forming a second divisor sum over the divisors of n on both sides of the first equation above, the first more exotic-looking sum for the sum-of-divisors functions leads to an expression for $\sigma_s(n)$ as a sum over the paired product functions, $\sigma_t(n) \cdot \sigma_s(n)$. We do not know of another such identity relating the generalized sum-of-divisors functions existing in the literature or in the references which we cite in this article. However, the following relations between the multiplicative generalized sum-of-divisors functions and special additive partition functions are known where $p_k(n)$ denotes the number of partitions of n into at most k parts and $\mathrm{pp}(n)$ denotes the number of plane, or planar, partitions of n [10, A000219] [7, §26.9, §26.12]:

$$\begin{aligned} n \cdot p(n) &= \sum_{k=0}^{n-1} p(k)\sigma_1(n) \\ n \cdot p_k(n) &= \sum_{t=1}^{n} p_k(n-t) \sum_{\substack{j \mid t \\ j \leq k}} j, \;\; k \geq 1 \\ n \cdot \mathrm{pp}(n) &= \sum_{j=1}^{n} \mathrm{pp}(n-j)\sigma_2(j). \end{aligned}$$

Corollary 2.5 (New Series for the Riemann Zeta Function)**.** *For fixed $s, t \in \mathbb{C}$ such that $\Re(s) > 1$ we have the following infinite sum representations of the Riemann zeta function where we denote the sequence of interleaved pentagonal numbers, $\omega(\pm n)$, by $G_j = \frac{1}{2}\left\lceil \frac{j}{2} \right\rceil \left\lceil \frac{3j+1}{2} \right\rceil$ for $j \geq 0$ [10, A001318]:*

$$\begin{aligned} \zeta(s) &= \sum_{n \geq 1} \sum_{k=1}^{n} \sum_{d \mid n} \frac{p(d-k)}{\sigma_t(d)} \mu(n/d) \times \sum_{j : G_j < k} (-1)^{\lceil j/2 \rceil} \frac{\sigma_t(k-G_j)\sigma_s(k-G_j)}{(k-G_j)^s} \\ \zeta(s) &= \sum_{n \geq 1} \sum_{k=1}^{n} \sum_{d \mid n} \frac{d^t \cdot p(d-k)}{\sigma_t(d)} \mu(n/d) \times \sum_{j : G_j < k} (-1)^{\lceil j/2 \rceil} \frac{\sigma_t(k-G_j)\sigma_s(k-G_j)}{(k-G_j)^{s+t}}. \end{aligned}$$

Proof. These two identities follow as special cases of the theorem in the form of (3) above where we note the identity for the generalized sum-of-divisors functions which provides that $\sigma_{-\alpha}(n) = \sigma_\alpha(n)/n^\alpha$ for all $\alpha \in \mathbb{C}$. We note that the pentagonal number theorem employed in the inner sums depending on j is equivalent to the expansion

$$(q;q)_\infty = \sum_{j \geq 0} (-1)^{\lceil j/2 \rceil} q^{G_j}.$$

The convergence of these infinite series is guaranteed by our hypothesis that $\Re(s) > 1$. □

We compare the results in the previous theorem to the known Dirichlet generating functions for the sum-of-divisors functions which are expanded by [2, Thm. 291] [7, §27.4]

$$\begin{aligned} \zeta(s)\zeta(s-\alpha) &= \sum_{n \geq 1} \frac{\sigma_\alpha(n)}{n^s}, \;\; \Re(s) > 1, \alpha + 1 \\ \frac{\zeta(s)\zeta(s-\alpha)\zeta(s-\beta)\zeta(s-\alpha-\beta)}{\zeta(2s-\alpha-\beta)} &= \sum_{n \geq 1} \frac{\sigma_\alpha(n)\sigma_\beta(n)}{n^s}, \;\; \Re(s) > 1, \alpha+1, \beta+1. \end{aligned}$$

In particular, we note that while the series $\sum_n \sigma_s(n)/n^s$, and similarly for the second series, are divergent, our sums given in Corollary 2.5 do indeed converge for $\Re(s) > 1$.

3. Factorization Theorems for Derivatives of Lambert Series

1	0	0	0	0	0	0	0	0	0	0	0
1	2	0	0	0	0	0	0	0	0	0	0
0	−2	3	0	0	0	0	0	0	0	0	0
−1	2	−3	4	0	0	0	0	0	0	0	0
−2	−4	−3	−4	5	0	0	0	0	0	0	0
−2	2	6	−4	−5	6	0	0	0	0	0	0
−2	−4	−6	0	−5	−6	7	0	0	0	0	0
−1	2	−3	8	0	−6	−7	8	0	0	0	0
0	−2	9	−4	0	0	−7	−8	9	0	0	0
1	2	−6	−8	15	0	0	−8	−9	10	0	0
2	0	−3	4	−10	6	0	0	−9	−10	11	0
3	2	12	12	−5	12	7	0	0	−10	−11	12

(i) $s_{1,n,k}$

1	0	0	0	0	0	0	0	0	0	0	0
$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	0	0	0	0	0
$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0	0	0	0	0	0	0
$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0	0	0	0	0
0	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	0	0	0	0	0	0	0
1	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	0	0	0	0	0	0
$\frac{4}{7}$	1	$\frac{5}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	0	0	0	0	0
$\frac{9}{8}$	$\frac{7}{8}$	$\frac{5}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	0	0	0	0
$\frac{16}{9}$	$\frac{4}{3}$	$\frac{8}{9}$	$\frac{7}{9}$	$\frac{5}{9}$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	0	0	0
$\frac{5}{2}$	$\frac{11}{10}$	$\frac{11}{10}$	$\frac{9}{10}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	0	0
$\frac{31}{11}$	$\frac{30}{11}$	2	$\frac{15}{11}$	1	$\frac{7}{11}$	$\frac{5}{11}$	$\frac{3}{11}$	$\frac{2}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	0
$\frac{13}{4}$	$\frac{8}{3}$	$\frac{7}{4}$	$\frac{5}{4}$	$\frac{13}{12}$	$\frac{3}{4}$	$\frac{7}{12}$	$\frac{5}{12}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$

(ii) $s_{1,n,k}^{(-1)}$

Figure 3.1. *Factorization matrix sequences for the first-order derivatives of an arbitrary Lambert series generating function when* $1 \leq n, k \leq 12$

3.1. **Derivatives of Lambert series generating functions.** We seek analogous factorization theorems for the higher-order t^{th} derivatives for all integers $t \geq 1$ of an arbitrary Lambert series in the form of

$$q^t \cdot D_t \left[\sum_{n \geq t} \frac{a_n q^n}{1-q^n} \right] = \frac{1}{(q;q)_\infty} \sum_{n \geq t} \sum_{k=t}^{n} s_{t,n,k} a_k \cdot q^n. \tag{4}$$

We note that we consider these sums over $n \geq t$ to produce an invertible matrix of the factorization sequences $s_{t,n,k}$. At first computation, the corresponding matrix sequences for these higher-order derivatives do not suggest any immediate intuitions to exact formulas for $s_{t,n,k}$ and $s_{t,n,k}^{(-1)}$ as in the factorizations expanded above. The listings given in Figure 3.1 provide the first several rows of these sequences in the first-order derivative case where $t := 1$ for comparison with our intuition. However, may still prove using the method invoked in the proof of Theorem 2.3 to show that in the first-order case we have that[2]

$$s_{1,n,k}^{(-1)} = \sum_{d|n} \frac{p(d-k)}{d} \mu(n/d),$$

which, in light of the construction of Corollary 2.5 and its notation, leads to the following further convergent series infinite expansion of the Riemann zeta function for $\Re(s) > 1$:

$$\zeta(s) = \sum_{n \geq 1} \sum_{k=1}^{n} \sum_{d|n} \frac{p(d-k)}{d} \mu(n/d) \times \sum_{j: G_j < k} (-1)^{\lceil j/2 \rceil} \frac{\sigma_s(k-G_j)}{(k-G_j)^{s-1}}.$$

If we dig deeper into the expansions of the derivatives of arbitrary Lambert series, we can prove other more natural exact formulas for both matrix sequences which are for the most part actually just restatements of consequences of known factorization theorems already proved in the references [3, §4]. We consider the factorizations of these higher-order derivative cases again as a separate topic in this article due to the significance of the interpretations and the breadth of applications which we can give by explicitly defining the exact factorization expansions in these cases below.

3.2. **Main results.** We will next require the statements of the next two results proved in [8] to state and prove the main results in this section. We refer the reader to the proofs of these two lemmas given in the reference.

Lemma 3.1 (Modified Lambert Series Coefficients)**.** *For any fixed arithmetic function a_n and integers $m, t \geq 1$, $n, k \geq 0$, we have the following identity for the series*

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[2] More generally, if we expand the next mixed series of j^{th} derivatives and initial terms annihilated by the differential operator as

$$q^j \cdot D_j \left[\sum_{n \geq 1} \frac{a_n q^n}{1-q^n} \right] + \sum_{i=1}^{j-1} (a * 1)(i) q^i = \frac{1}{(q;q)_\infty} \sum_{n \geq t} \sum_{k=t}^{n} s_{j,n,k} a_k \cdot q^n,$$

for integers $j \geq 2$, we can easily prove that $s_{j,n,k} = s_{n,k} = [q^n] q^k / (1-q^k)(q;q)_\infty$ for $1 \leq n < j$ and consequently that

$$s_{j,n,k}^{(-1)} = \sum_{d|n} \frac{p(d-k)}{\binom{d}{j} \cdot j! + \delta_{d,1} + \delta_{d,2} + \cdots + \delta_{d,j-1}} \mu(n/d),$$

using the proof method in Theorem 2.3.

coefficients of modified Lambert series generating function expansions:

$$[q^n]\sum_{i\geq t}\frac{a_i q^{mi}}{(1-q^i)^{k+1}} = \sum_{\substack{d|n \\ t\leq d\leq \left\lfloor \frac{n}{m}\right\rfloor}} \binom{\frac{n}{d}-m+k}{k} a_d. \tag{5}$$

Lemma 3.2 (Higher-Order Derivatives of Lambert Series)**.** *For any fixed non-zero $q\in\mathbb{C}$, $i\in\mathbb{Z}^+$, and prescribed integer $s\geq 0$, we have the following two results*[3]*:*

$$q^s D^{(s)}\left[\frac{q^i}{1-q^i}\right] = \sum_{m=0}^{s}\sum_{k=0}^{m}\begin{bmatrix} s \\ m\end{bmatrix}\begin{Bmatrix} m \\ k\end{Bmatrix}\frac{(-1)^{s-k}k!\cdot i^m}{(1-q^i)^{k+1}} \tag{i}$$

$$q^s D^{(s)}\left[\frac{q^i}{1-q^i}\right] = \sum_{r=0}^{s}\left(\sum_{m=0}^{s}\sum_{k=0}^{m}\begin{bmatrix} s \\ m\end{bmatrix}\begin{Bmatrix} m \\ k\end{Bmatrix}\binom{s-k}{r}\frac{(-1)^{s-k-r}k!\cdot i^m}{(1-q^i)^{k+1}}\right)q^{(r+1)i}. \tag{ii}$$

Proposition 3.3. *For integers $n,t\geq 1$, let the function $A_t(n)$ be defined as follows:*

$$A_t(n) := \sum_{\substack{0\leq k\leq m\leq t \\ 0\leq r\leq t}}\sum_{d|n}\begin{bmatrix} t \\ m\end{bmatrix}\begin{Bmatrix} m \\ k\end{Bmatrix}\binom{t-k}{r}\binom{\frac{n}{d}-1-r+k}{k}(-1)^{t-k-r}k!\times$$
$$\times\, d^m a_d\cdot\left[t\leq d\leq\left\lfloor\frac{n}{r+1}\right\rfloor\right]_\delta.$$

Then we have the next two Lambert series expansions for the function $A_t(n)$ given by

$$A_t(n) = [q^n]q^t\cdot D_t\left[\sum_{n\geq t}\frac{a_n q^n}{1-q^n}\right] = [q^n]\sum_{n\geq 1}\frac{(A_t*\mu)(n)q^n}{1-q^n}.$$

Proof. The first equation follows from (5) applied to Lemma 3.2 when $s:=t$. To prove the second form of a Lambert series generating function over some sequence c_n enumerating $A_t(n)$, we require that

$$\sum_{d|n}c_d = A_t(n),$$

which is true if and only if

$$c_d = \sum_{d|n}A_t(d)\mu(n/d) = (A_t*\mu)(n),$$

by Möbius inversion. □

Theorem 3.4 (Higher-Order Derivatives of Lambert Series Generating Functions)**.** *Let the notation for the function $A_t(n)$ be defined as in Proposition 3.3. Then we have the next formulas for $A_t(n)$ given by*

$$A_t(n) = [q^n]\frac{1}{(q;q)_\infty}\sum_{n\geq 1}\sum_{k=1}^{n}\widetilde{s}_{n,k}(\mu)A_t(k)\cdot q^n$$

[3] Notation: The bracket notation of $\begin{bmatrix} n \\ k\end{bmatrix}\equiv(-1)^{n-k}s(n,k)$ denotes the unsigned triangle of Stirling numbers of the first kind and $\begin{Bmatrix} n \\ k\end{Bmatrix}\equiv S(n,k)$ denotes the triangle of Stirling numbers of the second kind.

$$A_t(n) = \sum_{k=1}^{n} \widetilde{s}_{n,k}^{(-1)}(\mu) \left[A_t(k) + \sum_{p=\pm 1} \sum_{j=1}^{\left\lfloor \frac{\sqrt{24k-23}-p}{6} \right\rfloor} (-1)^j A_t\left(k - \frac{j(3j+p)}{2}\right)\right],$$

where

$$\widetilde{s}_{n,k}(\mu) = \sum_{j=1}^{n} s_{n,kj} \cdot \mu(j),$$

for $s_{n,k} := s_o(n,k) - s_e(n,k)$ *when the sequences* $s_o(n,k)$ *and* $s_e(n,k)$ *respectively denote the number of* k*'s in all partitions of* n *into an odd (even) number of distinct parts, and where a* n*-fold convolution formula involving* μ *for the inverse matrix sequence* $\widetilde{s}_{n,k}^{(-1)}(\mu)$ *is proved explicitly in the references* [3, §4]. *Moreover, for all* $n \geq 1$ *we have that the full Lambert series* t^{th} *derivative formula is given by*

$$\frac{n!}{(n-t)!} \cdot \sum_{d|n} a_d = \left(\sum_{i=1}^{t-1} \sum_{k=1}^{\left\lfloor \frac{n}{i} \right\rfloor} \widetilde{s}_{n,ik}^{(-1)}(\mu) \cdot \frac{(ik)!a_i}{(ik-t)!}\right) + A_t(n).$$

Proof. The first result follows from the factorizations of the Lambert series over the convolution of two arithmetic functions proved in the reference [3, §4] where our Lambert series expansion in question is provided by Proposition 3.3 above. Similarly, the second result is a consequence of the first whose explicit expansions, i.e., for the inverse sequence are again proved in the reference. The last equation in the theorem follows from the proposition and adding back in the subtracted Lambert series terms when the summation for the series considered for $A_t(n)$ starts from $n \geq t$ instead of from one. The multiples of ik in the last formula reflect that the coefficients of $q^i/(1-q^i)$ and its q^t-scaled derivatives are always zero unless the coefficient index is a multiple of i. $\square$

3.3. **Another related factorization.**

Remark 3.5 (Another Factorization)**.** The first factorization expansion we considered in (4) of this section is obtained by applying Lemma 3.6 in the case where

$$\begin{aligned} b_{n,i} &= [a_i](A_t * \mu)(n) \\ &= \sum_{\substack{0 \leq k \leq m \leq t \\ 0 \leq r \leq t}} \sum_{d|n} \begin{bmatrix} t \\ m \end{bmatrix} \begin{Bmatrix} m \\ k \end{Bmatrix} \binom{t-k}{r} \binom{\frac{d}{i} - 1 - r + k}{k} (-1)^{t-k-r} k! \times \\ &\qquad \times i^m T_{\mathrm{div}}(d,i) \mu(n/d) \cdot \left[t \leq i \leq \left\lfloor \frac{d}{r+1} \right\rfloor \right]_\delta. \end{aligned}$$

In this case, we can obtain a similar expansion of the middle identity in Theorem 3.4 in the form of

$$a_n = \sum_{k=1}^{n} s_{t,n,k}^{(-1)}(b) \left[A_t(k) + \sum_{p=\pm 1} \sum_{j=1}^{\left\lfloor \frac{\sqrt{24k-23}-p}{6} \right\rfloor} (-1)^j A_t\left(k - \frac{j(3j+p)}{2}\right)\right].$$

Lemma 3.6 (A Related Factorization Result)**.** *If we expand the Lambert series factorization*

$$\sum_{n\geq 1} \frac{\sum_{j=1}^{n} b_{n,j} a_j \cdot q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \sum_{n\geq 1} \sum_{k=1}^{n} s_{n,k}(b) a_k \cdot q^n,$$

then we have the formula

$$s_{n,k}(b) = \sum_{j=1}^{n} s_{n,j} \cdot b_{j,k},$$

where $s_{n,k} := [q^n](q;q)_\infty q^k/(1-q^k) = s_o(n,k) - s_e(n,k)$ *when the sequences* $s_o(n,k)$ *and* $s_e(n,k)$ *respectively denote the number of* k*'s in all partitions of* n *into an odd (even) number of distinct parts.*

Proof. If we take the nested coefficients first of a_k and then of $b_{j,k}$ for some $j, k \geq 1$ on both sides of the factorization cited above, we obtain that

$$\frac{q^j}{1-q^j} \cdot (q;q)_\infty = [b_{j,k}] \sum_{n\geq 1} s_{n,k}(b) \cdot q^n.$$

Then if we take the coefficients of q^n on each side of the previous equation we arrive at the identity

$$s_{n,j} = [b_{j,k}] s_{n,k}(b),$$

for $j = 1, 2, \ldots, n$. Finally, we multiply through both sides of the last equation by $b_{j,k}$ and then sum over all j to conclude that the stated formula for $s_{n,k}(b)$ in the lemma is correct. Equivalently, since both sequences of $b_{n,k}$ and $s_{n,k}(b)$ are lower triangular, we could have deduced this identity by truncating the partial sums up to n and employing a matrix argument to justify the formula above. □

4. Expansions of other special factorization theorems

4.1. A factorization theorem for convolutions of Lambert series.

In what follows we adopt the next notation for the Lambert series over a prescribed arithmetic function h defined by

$$H_L(q) := \sum_{n\geq 1} \frac{h(n) q^n}{1-q^n}.$$

We seek a factorization theorem for the convolution of two Lambert series generating functions, $F_L(q)$ and $G_L(q)$, in the following form:

$$\frac{1}{q} \cdot F_L(q) G_L(q) = \frac{1}{(q;q)_\infty} \sum_{n\geq 1} \sum_{k=1}^{n} s_{n,k}(g) f_k \cdot q^n. \tag{6}$$

Theorem 4.1 (Factorization Theorem for Convolutions)**.** *For the Lambert series factorization defined in* (6)*, we have the following exact expansions of the two matrix sequences characterizing the factorization where the difference of partition functions* $s_{n,k} := [q^n](q;q)_\infty q^k/(1-q^k)$*:*

$$s_{n,k}(g) = \sum_{j=1}^{n+1} s_{j,k} \left(\sum_{d|n+1-j} g(d) \right)$$

$$s_{n,k}^{(-1)}(g) = \sum_{d|n} [q^d] \left(\frac{q^{k+1}}{(q;q)_\infty G_L(q)} \right) \mu(n/d).$$

Proof. We prove our second result along the same lines as the proof of Theorem 2.3 above. Namely, we choose $F_L(q)$ to denote the Lambert series over $s_{n,k}^{(-1)}(g)$ for some fixed $k \geq 1$ and expand the right-hand-side of (6) as

$$[q^n] \frac{q^k}{(q;q)_\infty \cdot G_L(q)/q} = \sum_{d|n} s_{d,k}^{(-1)}(g),$$

which by Möebius inversion implies our stated result. Then to obtain a formula for the sequence $s_{n,k}(g)$ from the identity expanding the inverse sequence, we observe a property of the product of any two inverse matrices which is that

$$[n = k]_\delta = \sum_{j=1}^{n} s_{n,j}^{(-1)}(g) s_{j,k}(g).$$

We then perform a divisor sum over n in the previous equation to obtain that

$$\begin{aligned} [k|n]_\delta &= \sum_{d|n} s_{d,k}^{(-1)}(g) \\ &= \sum_{j=1}^{n} [q^{n-j-1}] \frac{1}{(q;q)_\infty \cdot G_L(q)} \times s_{j,k}(g), \end{aligned}$$

which by another generating function argument implies that

$$\begin{aligned} s_{n,k}(g) &= [q^n] \left(\sum_{n \geq 1} [k|n]_\delta \, q^n \right) (q;q)_\infty \cdot \frac{G_L(q)}{q} \\ &= \frac{q^k}{1-q^k} (q;q)_\infty \cdot \frac{G_L(q)}{q} \\ &= \sum_{j=1}^{n+1} s_{j,k} \left(\sum_{d|n+1-j} g(d) \right), \end{aligned}$$

as claimed. □

We notice that the factorization in (6) together with the theorem imply that we have the two expansions of the following form:

$$\begin{aligned} [q^n] F_L(q) &= \sum_{k=1}^{n} \sum_{d|n} [q^d] \frac{q^{k+1}}{(q;q)_\infty G_L(q)} \times \mu(n/d) \cdot [q^k] (q;q)_\infty F_L(q) G_L(q) \\ [q^n] G_L(q) &= \sum_{k=1}^{n} \sum_{d|n} [q^d] \frac{q^{k+1}}{(q;q)_\infty G_L(q)} \times \mu(n/d) \cdot [q^k] (q;q)_\infty G_L(q)^2. \end{aligned}$$

The special case where $F_L(q) := G_L(q)$ in the last expansion provides a curious new relation between any Lambert series generating function $G_L(q)$, its reciprocal, and its square. This observation can be iterated to obtain even further multiple sum identities involving powers of $G_L(q)$.

4.2. **A matrix-based proof of a factorization for sequences generated by Lambert series.** As a last application of special cases of the Lambert series factorization theorems we have extended in this article, we consider another method of matrix-based proof which provides new formulas for the sequences generated by a Lambert series over a_n: $b(n) = (a * 1)(n)$. This approach is unique because unlike the factorization theorem variants we have explored so far which provide new identities and expansions for the sequence a_n itself, the result in Theorem 4.2 provides useful new inverse sequence expansions exclusively for the terms $b(n)$ whose ordinary generating function is the Lambert series generating function at hand [9, *cf.* Thm. 1.4].

The first factorization theorem expanded in the introduction implicitly provides a matrix-multiplication-based representation of the coefficients $b(n)$ stated in terms of the matrix, $(T_{\mathrm{div}}(i,j))_{n\times n} \equiv (T_{\mathrm{div}})_n$, in the explicit forms of

$$(T_{\mathrm{div}})_n \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \sum_{d|n} a_d \quad \text{and} \quad (T_{\mathrm{div}})_n^{-1} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \sum_{d|n} a_d \cdot \mu(n/d),$$

where the corresponding inverse operation above is Möbius inversion. For example, when $n := 6$ these matrices are given by

$$(T_{\mathrm{div}})_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad (T_{\mathrm{div}})_6^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

Operations with our new definitions of the matrices above allow us to prove the next result.

Theorem 4.2. *For all $n \geq 1$ and a fixed arithmetic function a_n we have the identity*

$$b(n) = \sum_{k=1}^{n} \sum_{j=1}^{k} s_{n,k} C_{k,j} a_j,$$

where the inner matrix entries are given by [10, A000041]

$$C_{n,k} = \sum_{d|n} \sum_{i=1}^{d} p(d-ik)\mu(n/d).$$

Proof. We first note that the theorem is equivalent to showing that we have a desired expansion of the form

$$\sum_{k=1}^{n} s_{n,k}^{(-1)} b(k) = \sum_{k=1}^{n} C_{n,k} a_k \tag{i}$$

The right-hand-side of (i) is equivalent to the expansion of the last row in the matrix-vector product

$$(C_{i,j})_{n\times n}\begin{bmatrix}a_1\\ a_2\\ \vdots\\ a_n\end{bmatrix} = (T_{\mathrm{div}})_n^{-1}\left([q^i]\frac{q^j}{1-q^j}\frac{1}{(q;q)_\infty}\right)_{n\times n}\begin{bmatrix}a_1\\ a_2\\ \vdots\\ a_n\end{bmatrix},$$

and that the left-hand-side of (i) is expanded by

$$(s_{n,k}^{(-1)})_{n\times n}(T_{\mathrm{div}})_n\begin{bmatrix}a_1\\ a_2\\ \vdots\\ a_n\end{bmatrix} = (T_{\mathrm{div}})_n^{-1}\left(p(i-j)\right)_{n\times n}(T_{\mathrm{div}})_n\begin{bmatrix}a_1\\ a_2\\ \vdots\\ a_n\end{bmatrix}.$$

Then we have that the sequence $b(n)$ is expanded by multiplying the left-hand-side of (i) by the matrix $(p(i-j))_{n\times n}^{-1}(T_{\mathrm{div}})_n$ where

$$(p(i-j))_{n\times n}^{-1}\left([q^i]\frac{q^j}{1-q^j}\frac{1}{(q;q)_\infty}\right)_{n\times n}\begin{bmatrix}a_1\\ a_2\\ \vdots\\ a_n\end{bmatrix} = \left([q^n]\frac{q^k}{1-q^k}\right)\begin{bmatrix}a_1\\ a_2\\ \vdots\\ a_n\end{bmatrix}$$

$$\longmapsto [q^n]\sum_{k=1}^{n}\frac{q^k}{1-q^k}\cdot a_k = b(n) \quad \square$$

Corollary 4.3 (An Exact Formula for a Prime Counting Function)**.** *For all $n \geq 1$, we have the exact formula for the function $\omega(n)$ which counts the number of distinct primes dividing n given by* [10, A001221]

$$\omega(n) = \log_2\left[\sum_{k=1}^{n}\sum_{j=1}^{k}\left(\sum_{d|k}\sum_{i=1}^{d}p(d-ji)\right)s_{n,k}\cdot|\mu(j)|\right],$$

where $s_{n,k} = s_o(n,k) - s_e(n,k)$ denotes the difference of the number of k's in all partitions of n into an odd (even) number of distinct parts as in Section 1.

Proof. We select the special case of $(a,b) := (|\mu|, 2^\omega)$ to arrive at the statement in the corollary. $\square$

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E-mail address: maxieds@gmail.com, mschmidt34@gatech.edu

Georgia Institute of Technology, School of Mathematics, 117 Skiles Building, 686 Cherry Street NW, Atlanta, GA 30332, USA