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THREE-DIMENSIONAL MAPS AND SUBGROUP GROWTH

LAURA CIOBANU & ALEXANDER KOLPAKOV

Abstract. Firstly, we derive a generating series for the number of free subgroups of finite index in $\Delta^+ = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ by using a connection between free subgroups of Δ^+ and certain three dimensional maps known as pavings, and show that this generating series is non-holonomic. We also provide a non-linear recurrence relation for its coefficients.

Secondly, we study the generating series for conjugacy classes of free subgroups of finite index in Δ^+, which correspond to isomorphism classes of pavings. Asymptotic formulas are provided for the numbers of free subgroups of given finite index, conjugacy classes of such subgroups, and the equivalent types of pavings and their isomorphism classes.

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1. Introduction

In this note we explore the connections between free subgroups of given index n in the free product $\Delta^+ = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ and the number of rooted three-dimensional maps, or pavings, on n darts, as introduced in [2, 15, 29]. For any surface or higher-dimensional manifold that has been triangulated or otherwise subdivided into cells (not necessarily simplices), combinatorial maps are a way of recording the neighbouring relations between cells (vertices, edges, faces, etc), such as incidence or adjacency. The number of *darts* (defined in Sections 2.1 and 2.2), which are essentially edges or half-edges, is for us the key parameter in quantifying the number of maps, and can be seen as an "elementary particle" from which the combinatorial objects in this paper are assembled.

The connection we establish is between a free subgroup H of index n in Δ^+ (more precisely, its embedding in Δ^+) and the complexity of a paving associated to H. Thus we obtain the means of classifying both kinds of objects, the geometric ones and the algebraic ones, with respect to some natural measure of their intrinsic complexity. We also count the conjugacy classes of free subgroups of index n in Δ^+, and investigate the link between these and isomorphism classes of pavings; the connections between free subgroups (and their conjugacy classes) of finite index in certain Fuchsian triangle groups and two-dimensional maps have been previously exploited by a number of authors (c.f. [5, 12, 18, 20, 21, 26, 32] and more).

General subgroup growth is the subject of the book [16], and further information on counting the number of subgroups in free products of cyclic groups of prime orders can be found in the papers [22, 23, 24]. There the general theory of subgroup structure in free products of (finite and infinite) cyclic groups is enhanced by using the methods of representation theory, analytic number theory and probability theory, among other tools.

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In this note we use the species theory initiated by Joyal [13] (c.f. the monographs [4, 9]) as our main computational tool, which allows us to derive the generating series for the number of rooted pavings in Theorem 4.1 (or free subgroups of finite index in Theorem 4.3) and the number of isomorphism classes of hypermaps in Theorem 5.1 (or conjugacy classes of said subgroups in Theorem 5.3) in a relatively simple form suitable for routine calculation and computer experiments. We are able to associate the generating series for the number of rooted pavings to solutions of the classical Riccati equations, which shows they are non-holonomic by a result of [14]. C.f. [1] for other connections between map enumeration and the Riccati equation.

Throughout the paper we give several concrete and illustrative examples, as well as a sample of our SAGE code Monty (see Appendix) which supports our computations.

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2. Preliminaries

2.1. **Two-dimensional maps.** A two-dimensional orientable combinatorial map or, simply, a combinatorial map, is a triple $H = \langle D; \alpha, \sigma \rangle$, where $D = \{1, 2, \dots, n\}$ is a finite set of $n \geq 0$ *darts* (to be defined below), α, $\sigma \in \mathfrak{S}_n$ are permutations of D, and α is an involution. A map H is *connected* if the group $G_H = \langle \alpha, \sigma \rangle$ acts transitively on D.

Any combinatorial map has a topological realisation Γ_H as a family of graphs, each embedded into a connected orientable surface. In order to construct Γ_H, one may proceed as follows. Let $\phi = \sigma^{-1}\alpha$, and for each cycle of ϕ consider a polygon, called a *face* of Γ_H, whose edges are oriented anticlockwise.

Then two edges i and j of the newly produced faces are identified in accordance with the transpositions of α, that is, if $\alpha(i) = j$ then i is identified with j, and each new edge becomes the union of the now two half-edges or *darts* i and j, pointing in opposite directions. This ensures that the resulting topological space Γ_H is orientable. The ordered sequence of darts pointing towards a vertex of Γ_H is now described by a suitable cycle of σ. Thus the vertices of Γ_H correspond to the disjoint cycles of σ.

By construction, the topological space that we obtain after performing the procedure above is an orientable surface without boundary, which is connected if G_H acts transitively on D. However, we do not always assume connectivity/transitivity.

The above argument establishes a bijection between combinatorial maps and topological maps, i.e. graphs embedded into orientable (possible disconnected) surfaces, where for each connected component $\langle \Sigma_g; \Gamma, \iota \rangle$ with Σ_g a genus g surface, and Γ embedded in Σ_g via the map ι, the complement $\Sigma_g \setminus \iota(\Gamma)$ is a union of topological discs. Each edge of such a Γ is split into a pair of labelled half-edges pointing in opposite directions. The darts D are exactly those oriented half-edges.

The permutations α, σ and $\phi = \sigma^{-1}\alpha$ defining H can be read off the labelled topological map Γ_H as follows:

1) the cycles of α correspond to the darts forming entire edges of Γ_H,
2) the cycles of σ correspond to the sequences of darts around vertices read in an anticlockwise direction,
3) the cycles of ϕ correspond to the sequences of darts obtained by moving around faces in an anticlockwise direction.

Two combinatorial maps $H_1 = \langle \alpha_1, \sigma_1 \rangle$ and $H_2 = \langle \alpha_2, \sigma_2 \rangle$ are isomorphic if there exists $\pi \in \mathfrak{S}_n$ such that $\pi\,\alpha_1 = \alpha_2\,\pi$ and $\pi\,\sigma_1 = \sigma_2\,\pi$, which for the associated topological maps translates into the existence of an orientation-preserving homeomorphism between Γ_{H_1} and Γ_{H_2} that respects dart adjacencies.

For permutations $\pi_i \in \mathfrak{S}_n$, $i = 1, \ldots, l$, let $\zeta(\pi_1, \ldots, \pi_l)$ be the number of orbits of the group $\langle \pi_1, \ldots, \pi_l \rangle$ acting on $D = \{1, 2, \ldots, n\}$. Then the connected components of $H = \langle D; \alpha, \sigma \rangle$ are represented by the orbits of $\langle \alpha, \sigma \rangle$, the faces of H are the orbits of $\langle \sigma^{-1}\alpha \rangle$, and its edges and vertices are the orbits of $\langle \alpha \rangle$ and $\langle \sigma \rangle$, respectively. Thus the Euler characteristic of H can be defined as $\chi(H) = \zeta(\sigma^{-1}\alpha) - \zeta(\alpha) + \zeta(\sigma)$.

2.2. **Three-dimensional maps.** A three-dimensional orientable combinatorial map or, simply, a (combinatorial) paving is a quadruple $P = \langle D; \alpha, \sigma, \varphi \rangle$, where D is an n-element set ($n \geq 0$) and $\alpha, \sigma, \varphi \in \mathfrak{S}_n$ are permutations of D such that $H = \langle D; \alpha, \sigma \rangle$ is a map (not necessarily connected), and they satisfy:

(I-1) the product $\alpha\varphi$ is an involution,
(I-2) the product $\varphi\sigma^{-1}$ is an involution,
(FP) neither of the above involutions has fixed points.

A paving P is connected if $G_P = \langle \alpha, \sigma, \varphi \rangle$ acts transitively on D. Given a paving $P = \langle D; \alpha, \sigma, \varphi \rangle$, the map $H = \langle D; \alpha, \sigma \rangle$ is called the *underlying map* of P.

We may also think of P as a quadruple $P = \langle D; \alpha, \beta, \gamma \rangle$ where D is an n-element set ($n \geq 0$) and $\alpha, \beta, \gamma \in \mathfrak{S}_n$ are involutions without fixed points. In this case it is easy to see that letting $\varphi = \alpha\beta$ and $\sigma = \gamma\alpha\beta$ produces the initial definition.

As in the case of two-dimensional maps, a combinatorial paving P has a topological realisation M_P which, however, is not always a three-dimensional manifold. Such an example can be delivered by Thurston's figure-eight glueing from [30, Ch. 1, p. 4], described below.

Example 2.1. Let $D = \{1, 2, \ldots, 12\}$ be a set. Let α and σ be permutations of D such that $\alpha = (1, 2)(3, 4)(5, 6) \ldots (9, 10)(11, 12)$ and $\sigma = (1, 5, 3)(2, 9, 8)(4, 11, 10)(6, 7, 12)$. We define $D' = \{-1, -2, \ldots, -12\}$, and $\alpha'(i) = -\alpha(-i)$, $\sigma'(i) = -\sigma(-i)$ for all $i \in D'$. Also, let $\varphi = (1, -3)(2, -11)(3, -12)(4, -2)(5, -7)(6, -5)(7, -9)(8, -6)(9, -10)(10, -8)(11, -4)(12, -1)$.

Consider a paving $P = \langle D \cup D'; \alpha\alpha', \sigma\sigma', \varphi \rangle$, whose underlying map consists of two tetrahedra depicted in Figure 2.1. After glueing their faces with respect to φ we obtain a cellular space with Euler characteristic $+1$, which has two 3-cells, four 2-cells, two 1-cells, and a single 0-cell. The link of the 0-cell is a torus and not a sphere; therefore one does not obtain a manifold.

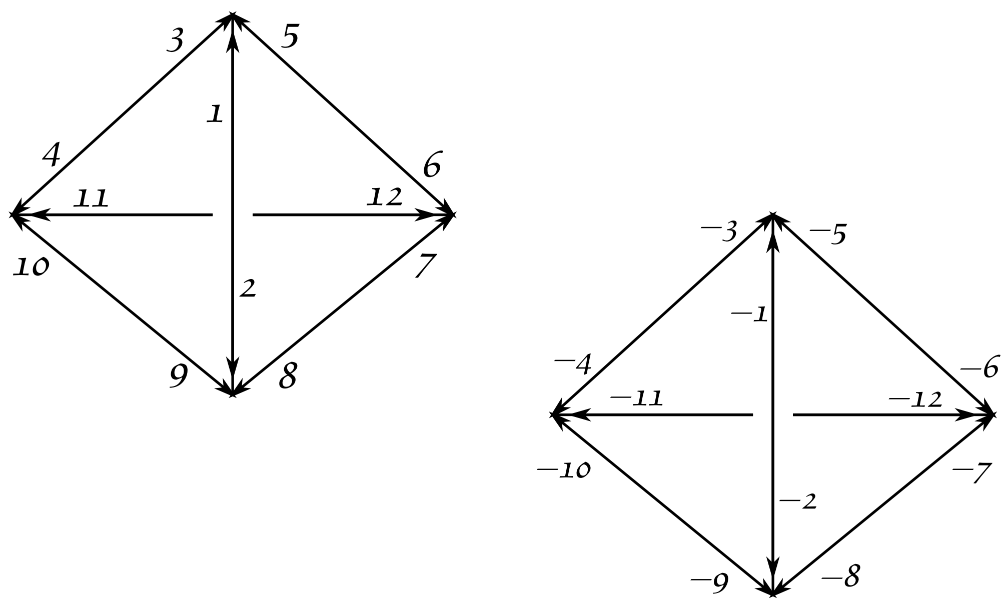

Figure 1. Two tetrahedra used in Thurston's figure-eight glueing. Here, they do not need to be geometrically realisable.

Let $H = \langle D; \alpha, \sigma \rangle$ be the underlying map for a paving $P = \langle D; \alpha, \sigma, \varphi \rangle$, and let us realise each connected component of H as a topological map, i.e. as a surface Σ^i with an embedded graph Γ^i, $i = 1, 2, \ldots, m$, having labelled half-edges as described in Section 2.1. Each surface Σ^i represents the boundary of a handle-body B^i, and then the handle-bodies B^i become identified along their boundaries in order to produce a labelled orientable cellular complex representing P topologically. Indeed, the faces of Σ^i's defined by the permutation $\sigma^{-1}\alpha$ are identified in accordance with the permutation φ, and the conditions I-1, I-2, FP ensure that one face cannot be identified to multiple disjoint counterparts (implied by conditions I-1 and I-2), and edges or faces cannot bend onto themselves (implied by condition FP). Also, conditions I-1 and I-2 ensure that the faces of two disjoint handle-bodies come together with coherent orientations, thus resulting in an orientable topological space M_P.

The definitions of isomorphism for combinatorial and topological pavings are absolutely analogous to those for combinatorial and topological maps.

The approach to pavings described above is largely due to Spehner, c.f. [29]. Another, dual, approach is due to Arquès and Koch [2], and these two approaches to pavings are shown to be equivalent in [15].

Arquès and Koch's approach is as follows. Let $P = \langle D; \alpha, \sigma, \varphi \rangle$ be a combinatorial paving. Then we assemble an orientable cellular complex M_P in such a way that the underlying map $H_P = \langle D; \alpha, \sigma \rangle$ produces (possibly disjoint) links of vertices in M_P. Each link is a map whose edges are intersections of the two-dimensional angular segments (or, simply, labelled corners of its two-faces [2, Définition 2.2, 1) & 2)], c.f. discussion in [15, p. 71]) representing the darts D and emanating from each vertex, with the respective link surface. In this case, the latter should be thought of as the boundary of a sufficiently small neighbourhood of said vertex. Then φ brings angular segments belonging to the same two-cell of M_P together, which finalises the construction. We may also think of taking $H_P^* = \langle D; \alpha, \varphi \rangle$ as the underlying map and performing Spehner's construction as previously described (with the only difference that the rôles of σ and φ are interchanged). Finally, M_P is a topological presentation for P.

If each component of H_P is planar, that is, a connected map on the two-sphere $\mathbb{S}^2$, then all the vertex links in M_P are homeomorphic to $\mathbb{S}^2$, and M_P is a three-dimensional manifold.

Given a paving M_P with labelled angular segments, we can easily read off the corresponding combinatorial data. Thus, we can compose the permutations α, σ and φ that constitute its combinatorial presentation P.

For a paving $P = \langle D; \alpha, \sigma, \varphi \rangle$, the number of connected components of its underlying map $H = \langle D; \alpha, \sigma \rangle$ is $f_3 = \zeta(\alpha, \sigma)$, which is also the number of connected three-dimensional handlebodies constituting M_P, or the number of "pieces" as described in [29, Definition 1.5]. The number of two-dimensional faces of P equals $f_2 = \zeta(\sigma^{-1}\alpha, \varphi^{-1}\sigma)$. The number of edges and vertices of P is $f_1 = \zeta(\alpha, \varphi)$, resp. $f_0 = \zeta(\sigma, \varphi)$.

The f-vector of P is $\mathrm{f}(P) = (f_0, f_1, f_2, f_3)$. The complexity of P equals $\mathrm{c}(P) = f_3 - f_2 + f_1 - f_0$. In general, this quantity does not coincide with the Euler characteristic of P, unless the underlying map H is planar (i.e. all the connected components of H are spheres).

2.3. **Formal power series.** Here we follow [6]. A *hypergeometric sequence* $(c_k)_{k\geq 0}$ has $c_0 = 1$ and enjoys the property that the ratio of its any two consecutive terms is a rational function in k, i.e. there exist monic polynomials $P(k)$ and $Q(k)$ such that

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)}.$$

Moreover, if P and Q are factored as

$$\frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)\dots(k+a_p)}{(k+b_1)(k+b_2)\dots(k+b_q)(k+1)},$$

then we use the notation

$${}_pF_q\left[\begin{array}{c} a_1 \dots a_p \\ b_1 \dots b_q \end{array} ; z\right]$$

for the formal series $F(z) = \sum_{k\geq 0} c_k z^k$, c.f. [27, §3.2]. Here, the factor $(k+1)$ belongs to the denominator for historical reasons. Such a hypergeometric series satisfies the differential equation

$$\Big(\vartheta(\vartheta + b_1 - 1)\cdots(\vartheta + b_q - 1) - z(\vartheta + a_1)\cdots(\vartheta + a_p)\Big)\ {}_pF_q(z) = 0, \tag{1}$$

where $\vartheta = z\frac{d}{dz}$, c.f. [7, §16.8(ii)]. Among numerous differential equations related to (1) is the *classical Riccati equation*, which plays an important rôle later on. It is a first order non-linear equation with variable coefficients $f_i(x)$, of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f_1(x) + f_2(x)y + f_3(x)y^2. \tag{2}$$

The *Pocchammer symbol* is connected to hypergeometric series and defined as

$$(a)_n = a(a+1)\dots(a+n-1).$$

As $n \to \infty$, it has the following asymptotic expansion

$$(a)_n \propto \frac{\sqrt{2\pi}}{\Gamma(a)}\, e^{-n}\, n^{a+n-\frac{1}{2}}, \tag{3}$$

where $\Gamma(a)$ is the Gamma function of a, defined as $\Gamma(a) = (a-1)!$ for a a positive integer, and $\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$ for all the non-integer real positive numbers.

A formal power series $y = f(x)$ is called *D-finite*, or *differentiably finite*, or *holonomic*, if there exist polynomials $p_0, \dots, p_m$ (not all zero) such that $p_m(x) y^{(m)} + \cdots + p_0(x) y = 0$, where $y^{(m)}$ denotes the m-th derivative of y with respect to x. All algebraic power series are holonomic, but not vice versa, c.f. [9, Appendix B.4].

Finally, we recall that the *Hadamard product* of two formal single-variable series $A(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!}$ and $B(z) = \sum_{n\geq 0} b_n \frac{z^n}{n!}$ is denoted $(A \odot B)(z)$ and given by $(A \odot B)(z) := \sum_{n\geq 0} a_n b_n \frac{z^n}{n!}$.

Let $\lambda = (n_1, \dots, n_m)$ be a partition of a natural number $n \geq 0$, i.e. $n = \sum_{i\geq 1} i n_i$. We write $\lambda \vdash n$ and define $\lambda! := 1^{n_1} n_1! 2^{n_2} n_2! \dots m^{n_m} n_m!$. Let $\mathbf{z}^\lambda := z_1^{n_1} z_2^{n_2} \dots z_m^{n_m}$ for some collection of variables $z_1, z_2, \dots$. Then for two multi-variable series $A(\mathbf{z}) = \sum_{n\geq 0} \sum_{\lambda \vdash n} a_\lambda \frac{\mathbf{z}^\lambda}{\lambda!}$ and $B(\mathbf{z}) = \sum_{n\geq 0} \sum_{\lambda\vdash n} b_\lambda \frac{\mathbf{z}^\lambda}{\lambda!}$ we have $(A \odot B)(\mathbf{z}) := \sum_{n\geq 0} \sum_{\lambda\vdash n} a_\lambda b_\lambda \frac{\mathbf{z}^\lambda}{\lambda!}$.

Also, for a multiple Hadamard product of a series $A(\mathbf{z})$ with itself, i.e. $B(\mathbf{z}) = (A \odot \cdots \odot A)(\mathbf{z})$, we shall write $B(\mathbf{z}) = A^{\odot n}(\mathbf{z})$, with a suitable $n \geq 0$.

2.4. **Species theory.** Species theory (théorie des espèces) is initially due to A. Joyal [13] and is a powerful way to describe and count labelled discrete structures. Since it requires a lengthy and formal setup, we give here only the basic ideas and refer the reader to [4, 9] for further details.

A *species of structures* is a rule (or functor) F which produces

i) for each finite set U (of labels), a finite set $F[U]$ of structures on U,
ii) for each bijection $\sigma : U \to V$, a function $F[\sigma] : F[U] \to F[V]$.

The functions $F[\sigma]$ should further satisfy the following functorial properties:

i) for all bijections $\sigma : U \to V$ and $\tau : V \to W$, $F[\tau \circ \sigma] = F[\tau] \circ F[\sigma]$,
ii) for the identity map $Id_U : U \to U$, $F[Id_U] = Id_{F[U]}$.

Let $[n] = \{1, 2, \dots, n\}$ be an n-element set, and assume that $[0] = \emptyset$. A species F of *labelled structures* has a generating function $F(z) = \sum_{n\geq 0} \operatorname{card} F[n] \frac{z^n}{n!}$.

For a species of *unlabelled structures* (i.e. structures up to isomorphism) we write $\widetilde{F}$, and its generating function is a specialisation of the cycle index series, in the sense that $\widetilde{F}(z) = \mathcal{Z}_F(z, z^2, z^3 \dots)$, where the *cycle index series* (see [4, §1.2.3]) is defined as:

$$\mathcal{Z}_F(z_1, z_2, \dots) = \sum_{n\geq 0} \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{card} Fix(F[\sigma])\, \mathbf{z}^\sigma.$$

Here $Fix(F[\sigma])$ is the set of elements of $F[n]$ having $F[\sigma]$ as automorphism, and $\mathbf{z}^\sigma = z_1^{c_1} z_2^{c_2} \dots z_m^{c_m}$ if the cycle type of σ is $c(\sigma) = (c_1, c_2, \dots, c_m)$ (i.e. c_k is the number of cycles of length k in the decomposition of σ into disjoint cycles).

3. Maps and subgroups

We will assume that all pavings are connected and, if rooted, have root 1. Let $P = \langle D; \alpha, \beta, \gamma \rangle$ be a rooted paving from $\mathcal{P}_r(n)$. Then there is an epimorphism ψ from $\Delta^+ = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \cong \langle a | a^2 = \varepsilon \rangle * \langle b | b^2 = \varepsilon \rangle * \langle c | c^2 = \varepsilon \rangle$ onto the group $G_P = \langle \alpha, \beta, \gamma \rangle \subset \mathfrak{S}_n$ given by $\psi : a \mapsto \alpha, b \mapsto \beta, c \mapsto \gamma$. Moreover, Δ^+ acts transitively on D via this epimorphism. By taking $\Gamma = Stab(1)$ with respect to this action, we observe that the action of Δ^+ on D is isomorphic to the action of Δ^+ on the set of cosets $\Delta^+ \diagdown \Gamma$.

If we consider the isomorphism class of P or, equivalently, consider $P \in \mathcal{P}_r(n)$ as a representative from $\mathcal{P}(n)$, a change of root in P from 1 to i corresponds to conjugation of Γ by an element $w \in \Delta^+$ such that $\omega = \psi(w)$ has the property $\omega(1) = i$.

By an argument analogous to that of [6, Lemmas 3.1-3.2], we can prove the following.

Lemma 3.1. *There exists a bijection between the set $\mathcal{P}_r(n)$ of rooted connected pavings with n darts and the set of free subgroups of index n in $\Delta^+ = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$.*

Lemma 3.2. *There exists a bijection between the set $\mathcal{P}(n)$ of isomorphisms classes of connected pavings with n darts and the set of conjugacy classes of free subgroups of index n in $\Delta^+ = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$.*

4. Counting subgroups of free products

In this section we shall count the number of transitive triples $\langle \alpha, \beta, \gamma \rangle \subset \mathfrak{S}_n$ such that α, β and γ are fixed point free involutions. Let S_2 be the species of such fixed point free involutions in $\mathfrak{S}_n$. Then since pavings correspond to triples of such involutions, for the species P^* of labelled pavings (not necessarily connected) on n darts we have

$$P^* = S_2 \times S_2 \times S_2, \tag{4}$$

while the species P of labelled connected pavings on n darts is related to P^* by the Hurwitz equation

$$P^* = E(P). \tag{5}$$

The species P° of rooted connected pavings on n darts can be expressed in terms of the derivative of P as

$$P^\circ = Z \cdot P', \tag{6}$$

where Z is the singleton species with exponential generating function $Z(z) = z$.

The above relations between species can be translated into relations between the corresponding exponential and ordinary generating functions.

The exponential generating functions for S_2, P^* and P are given by

$$S_2(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{2^k k!}, \tag{7}$$

$$P^*(z) = S_2(z) \odot S_2(z) \odot S_2(z) = \sum_{k=0}^{\infty} \frac{((2k)!)^2}{2^{3k}(k!)^3} z^{2k}, \tag{8}$$

$$P(z) = \log P^*(z) = \log \left(\sum_{k=0}^{\infty} \frac{((2k)!)^2}{2^{3k}(k!)^3}\, z^{2k} \right). \tag{9}$$

The ordinary generating function for the number of rooted connected pavings with n darts coincides with $P^\circ(z)$ since the species P° is rigid and every root assignment corresponds to $(n-1)!$ non-isomorphic labellings of the remaining darts:

$$P^\circ(z) = z\, \frac{d}{dz} \log P^*(z) = z\, \frac{d}{dz} \log \left(\sum_{k=0}^{\infty} \frac{((2k)!)^2}{2^{3k}(k!)^3}\, z^{2k} \right). \tag{10}$$

Now let us write $P^*(z) = f(2z^2)$, where $f(\xi) = \sum_{k=0}^{\infty} \frac{f_k}{k!} \xi^k$ and $f_k = \frac{1}{2^{4k}} \left(\frac{(2k)!}{k!} \right)^2$. Then

$$\frac{f_{k+1}}{f_k} = \left(k + \frac{1}{2} \right)^2. \tag{11}$$

Combining equality (11) with the fact that $f(0) = P^*(0) = 1$, we obtain that the function $f(\xi)$ is hypergeometric, can be written as

$$f(\xi) = {}_2F_0 \left(\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ \ldots & \end{matrix} ; \xi \right), \tag{12}$$

and is represented by an everywhere divergent (i.e. convergent only at $z = 0$) series. As a formal series, $f(\xi)$ satisfies

$$\vartheta f(\xi) = \xi \left(\vartheta + \frac{1}{2} \right)^2 f(\xi), \tag{13}$$

where $\vartheta = \xi \frac{d}{d\xi}$. c.f. [7, Section 16.8(ii)]. From equality (10) we get that

$$P^\circ(z) = 2\xi\, \frac{f'(\xi)}{f(\xi)} = 2w(\xi), \tag{14}$$

and by combining (13) and (14) we see that $w(\xi)$ satisfies a Riccati type equation:

$$w'(\xi) = \frac{(1-\xi)w(\xi) - \xi w^2(\xi) - \frac{1}{4}\xi}{\xi^2}. \tag{15}$$

By [14, Theorem 5.2] the function $w(\xi)$ is not holonomic, and therefore neither is $P^\circ(z)$.

Theorem 4.1. *The generating series $P^\circ(z) = \sum_{n=0}^{\infty} pav_r(n)\, z^n$ for the number $pav_r(n)$ of connected orientable rooted pavings with n darts is non-holonomic. Its general term $pav_r(n)$ vanishes for odd values of n and its asymptotic behaviour for even values of n is:*

$$pav_r(2k) \sim 2 \sqrt{\frac{2}{\pi}} \left(\frac{2}{e} \right)^k k^{k+1/2}.$$

Proof. The above discussion contains the proof of non-holonomy. It remains to deduce the asymptotic value of $pav_r(2k)$ as $k \to \infty$. We recall that

$$pav_r(2k) = [z^{2k}]\, P^\circ(z) = [z^{2k}] \left(z\, \frac{d}{dz} \log P^*(z) \right) = \tag{16}$$

$$= [z^{2k}] \left(z \, \frac{d}{dz} \log f(2z^2) \right), \tag{17}$$

where

$$f(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{2} \right)_k^2, \tag{18}$$

according to equality (12).

Let $f(\xi) = \sum_{k=0}^{\infty} \frac{f_k}{k!} \xi^k$ (necessarily with $f_0 = 1$) and let $\log f(\xi) = \sum_{k=1}^{\infty} g_k \xi^k$. Then by [8, Theorem 4.1] (also c.f. [3] and [25, Theorem 7.2]), we get that $g_k \sim \frac{f_k}{k!}$, as $k \to \infty$.

Thus, according to the above computation

$$pav_r(2k) = 2^k \cdot 2k \cdot g_k \sim \frac{2^{k+1}}{(k-1)!} \left(\frac{1}{2} \right)_k^2. \tag{19}$$

Recalling the asymptotic behaviour of the Pocchammer symbol $(a)_k$ from (3) and Stirling's asymptotic formula $k! \sim \sqrt{2\pi k}\, e^{-k}\, k^k$ we obtain the desired asymptotic expression for $pav_r(2k)$ as $k \to \infty$. □

Example 4.2. Since the generating series $P^\circ(z)$ (up to a multiple of 2) satisfies the Riccati equation (15), we obtain a recurrence relation by substituting $P^\circ(z) = \sum_{n=0}^{\infty} pav_r(n)\, z^n$ in it and equating the general term to zero:

$$pav_{2n+2} = 2(n+1)\, pav_n + \sum_{i=0}^{n} pav_{2i}\, pav_{2n-2i}, \text{ for } n \geq 2, \tag{20}$$

with initial conditions $pav_0 = 0$, $pav_2 = 1$ and $pav_d = 0$ for all odd numbers $d \geq 1$. By using Monty, we compute $P^\circ(z) = z^2 + 4z^4 + 25z^6 + 208z^8 + 2146z^{10} + 26368z^{12} + 375733z^{14} + 6092032z^{16} + 110769550z^{18} + 2232792064z^{20} + 49426061818z^{22} + 1192151302144z^{24} + \ldots$. The coefficient sequence of $P^\circ(z)$ has number A005411 in the OEIS [31]. The relation (20) also identifies it as the $S(2, -4, 1)$ self-convolutive sequence from [17].

By Lemma 3.1, the above theorem can be reformulated in group-theoretic language:

Theorem 4.3. *The growth series $S_f(z) = \sum_{n=0}^{\infty} s_f(n)\, z^n$ for the number $s_f(n)$ of free subgroups of index n in $\Delta^+ = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ coincides with the series $P^\circ(z)$ from Theorem 4.1.*

5. Counting conjugacy classes of subgroups

In order to compute the generating series $\widetilde{P}(z) = \sum_{n=0}^{\infty} pav(n)\, z^n$ for the number $pav(n)$ of non-isomorphic connected pavings with n darts, we shall employ again the species equations (4)–(6), while replacing generating functions for the respective species with their cycle index series.

Let C_2 be the species of transpositions from $\mathfrak{S}_n$, $n \geq 1$. Its cycle index series can be easily expressed as $\mathcal{Z}_{C_2}(z_1, z_2, \ldots) = \frac{1}{2} z_1^2 + \frac{1}{2} z_2$. The species S_2 of fixed points free involutions in $\mathfrak{S}_n$ can be expressed as $S_2 = E(C_2)$, since every involution is formed by a set of transpositions. It's also known that $\mathcal{Z}_E(z_1, z_2, \ldots) = \exp\left(\sum_{n=1}^{\infty} \frac{z_n}{n} \right)$.

Therefore, by using [4, §1.4, Théorème 2 (c)], the cycle index series for S_2 is

$$\mathcal{Z}_{S_2}(z_1, z_2, z_3, \dots) = \mathcal{Z}_E(\mathcal{Z}_{C_2}(z_1, z_2, \dots), \mathcal{Z}_{C_2}(z_2, z_4, \dots), \mathcal{Z}_{C_2}(z_3, z_6, \dots), \dots) = \tag{21}$$

$$= \exp\left(\frac{z_1^2}{2}\right) \cdot \exp\left(\frac{z_2^2}{4} + \frac{z_2}{2}\right) \cdot \exp\left(\frac{z_3^2}{6}\right) \cdot \dots = \prod_{n=1}^{\infty} T_n(z_n), \tag{22}$$

where

$$T_n(z_n) = \exp\left(\frac{z_n^2}{2n} + \frac{z_n}{n}\right) \text{ for even } n, \text{ and } T_n(z_n) = \exp\left(\frac{z_n^2}{2n}\right) \text{ for odd } n. \tag{23}$$

Thus the cycle index $\mathcal{Z}_{S_2}$ is separable, and the cycle index $\mathcal{Z}_{P^*}$ can be expressed as

$$\mathcal{Z}_{P^*}(z_1, z_2, \dots) = \prod_{n=1}^{\infty} (T_n \odot T_n \odot T_n)(z_n), \tag{24}$$

given that $P^* = S_2 \times S_2 \times S_2$ by equation (4).

By employing [4, §1.4, Exercice 9 (c)] together with equation (5), we obtain the cycle index for the species of pavings:

$$\mathcal{Z}_P(z_1, z_2, \dots) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \mathcal{Z}_{P^*}(z_1, z_2, \dots). \tag{25}$$

It follows from [4, §1.2, Théorème 8 (b)] and equations (24)–(25) that the generating series $\widetilde{P}(z)$ is

$$\widetilde{P}(z) = \mathcal{Z}_P(z, z^2, z^3, \dots) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \mathcal{Z}_{P^*}(z, z^2, z^3, \dots) = \tag{26}$$

$$= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{k=1}^{\infty} \log (T_n \odot T_n \odot T_n)(z_n)|_{z_n = z^{nk}}. \tag{27}$$

Theorem 5.1. *The generating series* $\widetilde{P}(z) = \sum_{n=0}^{\infty} pav(n)\, z^n$ *for the number* $pav(n)$ *of connected orientable pavings with* n *darts is given by formulas* (26) - (27). *Its general term* $pav(n)$ *vanishes for odd values of* n *and has the following asymptotic behaviour for even values of* n:

$$pav(2k) \sim \sqrt{\frac{2}{\pi}} \left(\frac{2}{e}\right)^k k^{k-1/2}.$$

Proof. By an argument analogous to that of [8, Section 7.1], we obtain $pav(2k) \sim \frac{pav_r(2k)}{2k}$ as $k \to \infty$. Now the claim follows from Theorem 4.1. □

Example 5.2. By using Monty, we compute the initial sequence of coefficients for $\widetilde{P}(z)$ and obtain that $\widetilde{P}(z) = z^2 + 4z^4 + 11z^6 + 60z^8 + 318z^{10} + 2806z^{12} + 29359z^{14} + 396196z^{16} + 6231794z^{18} + 112137138z^{20} + \dots$. The coefficient sequence of $\widetilde{P}(z)$ has number A002831 in the OEIS [31], which represents the number of edge-3-coloured trivalent multi-graphs[1] on $2n$

[1] i.e. with multiple edges

vertices, $n \geq 0$, without loops. Let this number be $tri(n)$ and let $\widetilde{G}(z) = \sum_{n\geq 0} tri(n) z^{2n}$. Thus the number of isomorphism classes of transitive triples of fixed-point-free involutions from $\mathfrak{S}_{2n}$ equals both $pav(n)$ (as shown above) and $tri(n)$.

Indeed, in order to create a labelled (not necessarily connected) edge-3-coloured trivalent multigraph without loops, we need to choose three matchings in the set of $2n$ vertices, which we may think of as a set $V = [2n]$. Each matching will consist of edges of same colour, say red (R), green (G) or blue (B). A matching of some colour $c \in \{R, G, B\}$ is then described as a product σ of disjoint transpositions (i, j) corresponding to the two vertices i and j from V joined by an edge. Since there are no loops, each matching has exactly n edges, and σ_c has no fixed points. See [28] for a general approach to enumeration of graphs with "local restrictions".

Let G^* be the species of vertex-labelled edge-3-coloured trivalent multigraph without loops, and let G be its connected counterpart. Then G^* can be described as a species of triples of fixed-point-free involutions $\langle \sigma_R, \sigma_G, \sigma_B \rangle$, and thus $G^* \cong P^*$ and, subsequently, $G \cong P$, as species. From this isomorphism, we get that, in particular, $\widetilde{G}(z) = \widetilde{P}(z)$ and the coefficient sequence of $\widetilde{P}(z)$ coincides with A002831.

Theorem 5.3. *The growth series $C_f(z) = \sum_{n=0}^{\infty} c_f(n)\, z^n$ for the number $c_f(n)$ of conjugacy classes of free subgroups of index n in $\Delta^+ = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ coincides with the series $\widetilde{P}(z)$ from Theorem 5.1.*

Example 5.4. Below we present the non-isomorphic pavings with $n \leq 4$ darts, which provide a classification for all conjugacy classes of free subgroups of index ≤ 4 in Δ^+ in view of Lemma 3.2 and the preceding discussion. The corresponding pavings can easily be classified by hand.

The conjugacy growth series for Δ^+ is given in Example 5.2. An independent computation with GAP [11] by issuing `LowIndexSubgroupsFPGroup` command gives matching results. We may also use `FactorCosetAction` command to observe the action of a conjugacy class representative on its cosets.

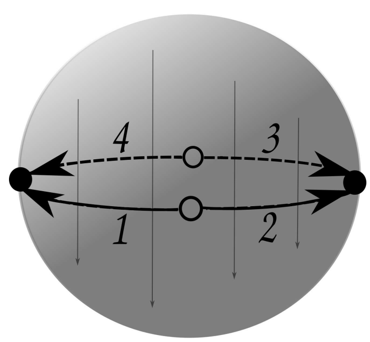

FIGURE 2. Paving P_2 with 4 darts produced by face-glueing. The face identification $(x, y, z) \mapsto (x, y, -z)$ is depicted by arrows.

Let $P = \langle D; \alpha, \beta, \gamma \rangle$ be a paving. For the case of two darts $D = \{1, 2\}$ we obtain only one paving P_1 with

$$(\alpha, \beta, \gamma) \longmapsto ((1, 2), (1, 2), (1, 2)). \tag{28}$$

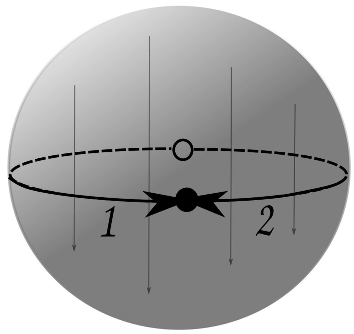

FIGURE 3. Paving P_1 with 2 darts produced by face-glueing. The face identification $(x, y, z) \mapsto (x, y, -z)$ is depicted by arrows.

This paving is glued from a single 3-ball B_1 with a map H_1 on it, as shown in Figure 3. If we suppose that B_1 is a unit ball centred at the origin of $\mathbb{R}^3$, then the identification of the faces of H_1 can be described by the transformation $(x, y, z) \mapsto (x, y, -z)$. This paving has f-vector $(1, 1, 1, 1)$.

For the case of four darts, that is, $D = \{1, 2, 3, 4\}$, we get four more pavings.

The first one is P_2 with

$$(\alpha, \beta, \gamma) \longmapsto ((1, 2)(3, 4), (1, 2)(3, 4), (1, 3)(2, 4)). \tag{29}$$

Here, P_2 is topologically represented by glueing the boundary of a 3-ball B_2 with a map H_2 on it, as depicted in Figure 2. Again, such a glueing can be described by the transformation $(x, y, z) \mapsto (x, y, -z)$. This paving has f-vector $(2, 2, 1, 1)$

The next paving P_3 has

$$(\alpha, \beta, \gamma) \longmapsto ((1, 2)(3, 4), (1, 3)(2, 4), (1, 2)(3, 4)). \tag{30}$$

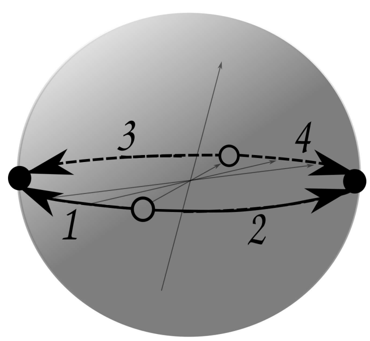

FIGURE 4. Paving P_3 with 4 darts produced by face-glueing. The face identification $(x, y, z) \mapsto (-x, -y, -z)$ is depicted by arrows.

It is depicted in Figure 4, and topologically is a single 3-ball B_3 with a map H_3 on it, whose faces are identified accordingly. The glueing transformation in this case can be described as $(x, y, z) \mapsto (-x, -y, -z)$. This paving has f-vector $(1, 1, 1, 1)$.

An easy computation yields that each of P_i, $i = 1, 2, 3$, has Euler characteristic $\chi(P_i) = 0$, as any three-dimensional manifold [10, Theorem 4.3], and it can be readily seen that in each case the resulting manifold is homeomorphic to the three-sphere $\mathbb{S}^3$.

As for the remaining two pavings P_4 and P_5, both of them correspond topologically to glueing two disjoint balls along their boundaries, and the Euler characteristic for both is 0; thus each is a manifold by [10, Theorem 4.3]. Moreover, each is an orientable manifold of Heegaard genus zero, and thus again homeomorphic to $\mathbb{S}^3$ [10, Ch. 5, §1].

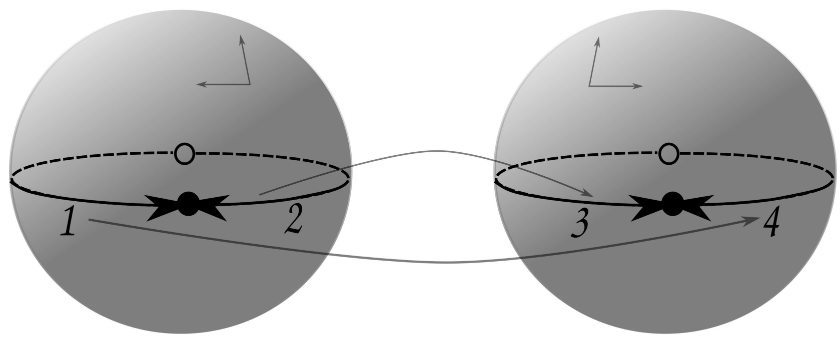

Figure 5. Paving P_4 with 4 darts produced by face-glueing. The face identification is depicted by arrows.

For P_4 we have

$$(\alpha, \beta, \gamma) \longmapsto ((1, 2)(3, 4), (1, 3)(2, 4), (1, 3)(2, 4)), \tag{31}$$

which is a combinatorial description for the two 3-balls $B_{4,1}$ and $B_{4,2}$ shown in Figure 5, each with a connected map $H_{4,1}$, respectively $H_{4,2}$, on it. The faces of those maps are identified by an orientation-reversing transformation on $\partial B_{4,1} \cong \mathbb{S}^2 \cong \partial B_{4,2}$. This paving has f-vector $(1, 1, 2, 2)$.

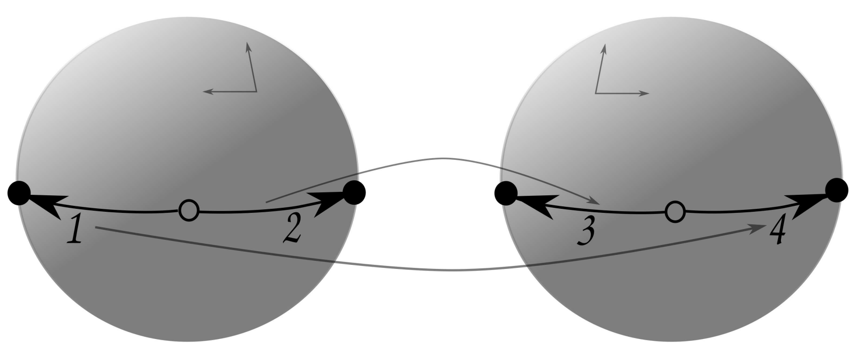

Figure 6. Paving P_5 with 4 darts produced by face-glueing. The face identification is depicted by arrows.

Finally, for P_5 we obtain

$$(\alpha, \beta, \gamma) \longmapsto ((1,2)(3,4), (1,3)(2,4), (1,4)(2,3)). \tag{32}$$

In this case two 3-balls $B_{5,1}$ and $B_{5,2}$ shown in Figure 6 are identified along their boundaries. The identification is described by the glueing of the faces of the corresponding maps $H_{5,1}$ and $H_{5,2}$ on their boundaries. The f-vector of this paving is $(2, 1, 1, 2)$.

6. Counting pavings of the three-sphere

Let us consider a Heegaard splitting $H \cup H' = \mathbb{S}^3$ of the three-sphere $\mathbb{S}^3$, where the handlebodies H and H' are glued along their common boundary $\Sigma = H \cap H'$. If we suppose that Σ has a map on it, then such a splitting $H \cup H'$ turns into a paving. Indeed, we can split each edge on Σ into two darts, and then double each dart, such that we have two maps Σ and Σ' corresponding to the boundaries of H and H'; then we can write down the permutation representation for each of them. Finally we write down a permutation that pairs the darts of Σ with the darts of Σ': whichever map we choose for Σ will determine the map on Σ'.

We can also think of $\mathbb{S}^3$ as $\mathbb{E}^3 \cup \infty$ and then delete from $\mathbb{E}^3$ a genus g handlebody H. Then the closure H' of the complement $\mathbb{S}^3 \setminus H$ will be a genus g handlebody H', and the surfaces of H and H' will have opposite orientations. Thus, if we choose a map Σ on a genus g surface of a handlebody H, we automatically imprint its chiral (i.e. having inverse orientation) counterpart Σ' on the surface of H'.

More precisely, let us choose a map C on $\Sigma = \partial H$ with a set of darts $D = \{1, 2, \ldots, n\}$, and let its chiral map on $\Sigma' = \partial H'$ be C', with set of darts $D' = \{-1, -2, \ldots, -n\}$, such that $D \cap D' = \emptyset$. We assume that $C = \langle D; \alpha, \sigma \rangle$ and $C' = \langle D'; \alpha', \sigma' \rangle$. Thus, $\alpha'(i) = -\alpha(-i)$, and $\sigma'(i) = -\sigma^{-1}(-i)$ for all $i \in D'$. The glueing of H and H' along their respective boundaries Σ and Σ' provides an involution φ identifying the darts from D to those in D' in pairs. Namely $\varphi(i) = -i$ for all $i \in D \cup D'$. Thus, we have created a paving $P = \langle D \cup D'; \alpha\alpha', \sigma\sigma', \varphi \rangle$ that topologically represents the three-sphere $\mathbb{S}^3$.

If two pavings are isomorphic, then their underlying maps are necessarily isomorphic. By the above construction, we have at least as many non-isomorphic orientable pavings P on $2n$ darts representing $\mathbb{S}^3$ as the total number of non-isomorphic orientable maps H on n darts. Thus, the number of pavings representing $\mathbb{S}^3$ grows super-exponentially with respect to n.

We remark that the complexity of our paving P can be easily computed. If $\mathrm{f}(P) = (f_0, f_1, f_2, f_3)$ then $\chi(H) = f_2 - f_1 + f_0 = 2 - 2g$, where g is the genus of the surface carrying the map H, and $f_3 = 2$. Thus $\mathrm{c}(P) = f_3 - f_2 + f_1 - f_0 = 2 - (2 - 2g) = 2g$, and its value will vary over the set of maps on n darts. This fact motivates the following question.

Question 6.1. Let $\mathcal{P}_c(n)$ be the set of pavings with n darts, all of fixed complexity c. Is it true that $\operatorname{card} \mathcal{P}_c(n) \sim C_1 \exp(C_2 n)$ for some C_1, $C_2 > 0$, if n is great enough?

Monty (a sample SAGE session)

Here we work out Example 4.2. We begin by defining the recurrence relation from (20) in order to produce a list of values $pav_r(2n)$, for $n = 0, \ldots, 20$.

```
from sympy.core.cache import cacheit

#define pav_r(n) which computes
#the number of rooted pavings on 2*n darts

@cacheit
def pav_r(n):
return n if n<2 \
else \
2*n*pav_r(n - 1) + sum([pav_r(k)*pav_r(n - k - 1) for k in xrange(1, n-1)]);
print map(pav_r, xrange(20))
```

Thus we obtain the coefficients sequence of $P^{\circ}(z)$.

```
[0, 1, 4, 25, 208, 2146, 26368, 375733, 6092032, 110769550, 2232792064,
49426061818, 1192151302144, 31123028996164, 874428204384256,
26308967412122125, 843984969276915712, 28757604639850111894,
1037239628039528906752, 39481325230750749160462],
```

which has number A005411 in the OEIS [31]. Then we define the auxiliary function $T_n(z_n)$, and its triple Hadamard product with itself.

```
#defining n, which has to be an even natural number
n = 22;

#defining power series ring over \mathbb{Q}
R.<z> = PowerSeriesRing(QQ, default_prec=2*n);

#defining T_m(z_m)
def T(m):
    sum = 0;
    if (m%2 == 0):
        sum = z^2/(2*m) + z/m;
    else:
        sum = z^2/(2*m);
    return sum.exp(2*n);

#defining the triple Hadamard product of T_m(z_m) with itself
def h_prod_T(m):
    prod = 0;
    T_coeff = T(m).dict();
    for k in T_coeff.keys():
        prod = prod + \
        power(z,k)*power(T_coeff[k], 3)*power(factorial(k), 2)*power(m,2*k);
    return prod;
```

Next, we define the logarithmic term in the expression for $\widetilde{P}(z)$ given by (26) - (27).

```
def log_h_prod_T(m,k):
    return log(h_prod_T(m)).substitute(z=power(z,m*k));

@parallel
def term(m,k):
    return moebius(k)/k*log_h_prod_T(m,k);
```

Finally, we define the series $\widetilde{P}(z)$.

```
def P_tilde(n):
    return sum([t[1] for t in list(term([(m,k) for m in range(1,n) \
                                   for k in range(1,n)]))]).truncate(n);
```

The computation produces the following output (for $n = 22$).

```
P_tilde(n);
> 112137138*z^20 + 6231794*z^18 + 396196*z^16 + 29359*z^14 + 2806*z^12 +
318*z^10 + 60*z^8 + 11*z^6 + 4*z^4 + z^2
```

The coefficient sequence of the series above has number A002831 in the OEIS [31].

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Laura Ciobanu
School of Mathematical and Computer Sciences
Heriot-Watt University
6100 Main Street
Edinburgh EH14 4AS, UK
l.ciobanu@hw.ac.uk

Alexander Kolpakov
Institut de Mathématiques
Université de Neuchâtel
Rue Emile-Argand 11
CH-2000 Neuchâtel
Suisse/Switzerland
kolpakov.alexander(at)gmail.com