TREE SHIFT COMPLEXITY

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Abstract. We prove for dyadic trees labeled by a finite alphabet an analogue of the Nivat Conjecture: the complexity function is bounded if and only if the labeling is eventually periodic in an appropriate sense. The labeled trees of minimal unbounded complexity are exactly those which have the same Sturmian sequence along each infinite path from the root. We give a definition of topological entropy for tree shifts, prove that the limit in the definition exists, and show that it dominates the topological entropy of the associated one-dimensional shift of finite type when the labeling of the tree shares the same restrictions.

1. Introduction

Tree shifts were introduced by Aubrun and Béal [1–5] as interesting objects of study, since they are more complicated than one-dimensional subshifts while preserving some directionality, but perhaps not so hard to analyze as multidimensional subshifts. They have been studied further by Ban and Chang [6–10]. We consider here the complexity of tree shifts and labeled trees in general, especially minimal complexity and two variations of topological entropy. We prove that the labeled dyadic trees of minimal unbounded complexity are exactly those which are labeled with a single Sturmian sequence along each infinite path from the root. We define the topological entropy of a tree shift in a different manner than Ban and Chang [8], prove that the limit in the definition exists, and consider ways to estimate it. In particular, if the tree shift consists of all trees labeled by a finite alphabet subject to adjacency restrictions given by a $0, 1$ matrix that also defines a one-dimensional shift of finite type (SFT), we prove that the entropy of the tree shift is bounded below by that of the SFT.

2. Terminology and notation

Although much of what we say extends to general trees, we focus on labelings of the standard infinite dyadic tree, which corresponds to the set of all finite words on a two-element alphabet, for us $\Sigma = \{L, R\}$. As usual, Σ^0 is the empty word ϵ, for $n \geq 1$ we denote by Σ^n the set of all words of length n on the alphabet Σ, and $\Sigma^* = \cup_{n \geq 0} \Sigma^n$. A word $w \in \Sigma^*$ corresponds uniquely to a path in the tree from the root and to the vertex which is at the end of that path. We denote by $|w|$ the length of the word w. There are one-to-one correspondences between elements of Σ^*, nodes of the tree, and finite paths starting at the root.

Let $A = \{0, 1\}$ be a labeling alphabet. Then a *labeled tree* is a function $\tau : \Sigma^* \to A$. For each $w \in \Sigma^*$, $\tau(w)$ is thought of as the label attached to the node at the end of the path determined by w. The two shifts on Σ^* are defined by $\sigma_i(w) = iw, i = L, R$. For $w = w_0 w_1 \dots w_{n-1} \in \Sigma^*$, define $\sigma_w = \sigma_{n-1} \dots \sigma_1 \sigma_0$ (note the reverse order). On a labeled tree τ, define $(\sigma_i \tau)(w) = \tau(iw), i = L, R$.

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For each $n \geq 0$ let $\Delta_n = \cup_{0 \leq i \leq n} \Sigma^i$ denote the initial n-subtree of the dyadic tree. Then Δ_n has $n+1$ "levels" and $2^{n+1}-1$ nodes. For any $x \in \Sigma^*$, the *shift to* x of Δ_n is the n-subtree $x\Delta_n = \sigma_x \Delta_n$. An *$n$-block* is a function $B : \Delta_n \to A$, i.e., a labeling of the nodes of Δ_n, or a *configuration* on Δ_n. We may write a 1-block B for which $B(\epsilon) = a, B(L) = b, B(R) = c$ as a_b^c. We say that an n-block B *appears* in a labeled tree τ if there is a node $x \in \Sigma^*$ such that $\tau(xw) = B(w)$ for all $w \in \Delta_n$. Denote by p_τ the *complexity function* of the labeled tree τ: for each $n \geq 0$, $p_\tau(n)$ is the number of distinct n-blocks that appear in τ. A *tree shift* X is the set of all labeled trees which omit all of a certain set (possibly infinite) of forbidden blocks. These are exactly the closed shift-invariant subsets of the full tree shift space $T(A) = A^{\Sigma^*}$. We deal here only with *transitive* tree shifts X, those for which there exists $\tau \in X$ such that every block that appears in X appears in τ.

3. Minimal complexity

The *complexity function* $p_\omega(n)$ of a sequence ω on a finite alphabet gives for each n the number of distinct blocks (or words) of length n found in the sequence. Hedlund and Morse [14] showed that for a one-sided sequence the following statements are equivalent: (1) there is an n such that $p_\omega(n) \leq n$; (2) there is a k such that $p_\omega(k+1) = p_\omega(k)$; (3) ω is eventually periodic; (4) p_ω is bounded. Moreover, they identified the sequences that have minimal unbounded complexity as exactly the Sturmian sequences. In 1997 M. Nivat conjectured that an analogue of these statements might hold in higher dimensions, specifically for labelings of the integer lattice in $\mathbb{Z}^2$ by a finite alphabet; see for example [12, 13, 17] for the precise statement, some positive progress, and references. We claim that the Hedlund-Morse statements about one-dimensional sequences of minimal complexity extend to labeled trees, with more or less the same proofs.

A finite subset $P \subset \Sigma^*$ is a *complete prefix-free set*, abbreviated CPS, if no word in P is the prefix of another word in P, and if for every $w \in \Sigma^*$ with $|w| \geq \max\{|p| : p \in P\}$ there is $p \in P$ which is a prefix of w. A CPS forms a sort of cross-section of the tree, intersecting each infinite path from the root in a single point.

A labeled tree τ is defined to be *periodic* if there is a CPS Q such that for each $y \in Q$ we have $\sigma_y \tau = \tau$. A labeled tree τ is defined to be *eventually periodic* if there is a CPS P such that for each $x \in P$ the labeled subtree $\sigma_x \tau$ rooted at x is periodic: there is a CPS Q_x such that for all $x \in P$ and all $y \in Q_x$ we have $\sigma_y \sigma_x \tau = \sigma_x \tau$. (Note that we do *not* assume that $\sigma_x \tau = \sigma_{x'} \tau$ for all $x, x' \in P$, which would be a stronger concept of periodicity.)

Theorem 3.1. *The following conditions on any labeled tree τ are equivalent:*

(1) There is an $n \geq 1$ such that $p_\tau(n) \leq n + 1$'.
(2) There is $n \geq 0$ such that $p_\tau(n+1) = p_\tau(n)$.
(3) p_τ is eventually periodic.
(4) p_τ is bounded: there is M such that $p_\tau(n) \leq M$ for all $n \geq 0$.
(5) There is a $k \geq 1$ such that $p_\tau(k+j) = p_\tau(k)$ for all $j \geq 0$.

Proof. (1) implies (2): Clear because $p_\tau(0) = 2$ and p_τ is nondecreasing.

(2) implies (3): Suppose that $k \geq 0$ and $p_\tau(k+1) = p_\tau(k)$. Then each k-block that appears in τ extends uniquely to a $(k+1)$-block by adding a row of length 2^{k+1} of 0's and 1's. Look along any path w from the root in τ of length $m = p_\tau(k)+1$, observing the k-blocks that are rooted at those nodes, $w_0, w_0 w_1, \ldots, w_0 \ldots w_{m-1}$. At least one of those k-blocks occurs twice in this list: there are $0 \leq i < j \leq m-1$ such that the initial k-blocks of $\sigma_{w_0 \ldots w_i} \tau$ and $\sigma_{w_0 \ldots w_j} \tau$ are equal, and hence so are these two labeled subtrees of τ. For each such path w choose the smallest i and then the smallest j as in the preceding sentence. The words $x = w_0 \ldots w_i$ will then form

a CPS P, and so will the words $w_{i+1} \dots w_j$, and the pairs P, Q_x will satisfy the definition of eventual periodicity for τ.

(3) implies (4): Suppose that τ is eventually periodic. Then τ has only finitely many labeled subtrees. (The labeled subtree $\sigma_w \tau$ depends only on the starting node $w \in \Sigma^*$. For each i, j as above, the labeled subtrees rooted at $w_0 \dots w_i z$ and $w_0 \dots w_j z$ will be identical for each $z \in \Sigma^*$. And there are only finitely many nodes above P.) Thus the number of possible initial n-blocks of all labeled subtrees of τ is bounded.

(4) implies (5): Since p_τ is nondecreasing, if it is bounded it must be eventually constant.

(5) implies (1): Clear. □

4. Trees labeled with Sturmian sequences

The preceding theorem raises the question of labeled trees of unbounded minimal complexity, those for which $p_\tau(n) = n + 2$ for all $n \geq 0$. (We have $n + 2$ here instead of $n + 1$ because we start with $n = 0$ rather than $n = 1$.) A trivial way to make such examples would be to put the same Sturmian sequence for labels down each infinite path from the root. Then for each n there is a one-to-one correspondence between n-blocks in the labeled tree and words of length n in the Sturmian sequence.

Theorem 4.1. *The* only *dyadic trees of unbounded minimal complexity are the ones which have a fixed Sturmian sequence down each infinite path from the root.*

Proof. Suppose that τ is a dyadic tree of unbounded minimal complexity, so that $p_\tau(n) = n+2$ for all $n \geq 0$. Then the one-dimensional words seen along paths must all come from the same Sturmian system. Sturmian sequences are codings of rotations of the unit circle by an irrational (with respect to π) number α via a partition $P = \{[0, 1-\alpha), [1-\alpha, 1)\}$, the symbol 0 being assigned to $P_0 = [0, 1-\alpha)$ and 1 to $P_1 = [1-\alpha, 1)$. Each of the $n+1$ intervals comprising $P_0^{n-1} = P \vee T^{-1}P \vee \cdots \vee T^{-n+1}P$ (here $Tx = x + \alpha \mod 1$) corresponds to a different n-block in the language of the system. Moreover, the frequency of the block equals the length of its corresponding interval.

In our Sturmian system the words 01 and 10 must appear, and exactly one of the words 00 and 11 does not appear; assume that it is 11. Therefore τ must contain the 1-block 1_0^0. We will argue now that τ must also contain 0_0^0 and 0_1^1. From this it follows that τ cannot contain either of 0_0^1 or 0_1^0, since this would give us $p_\tau(1) \geq 4$. Then all levels of τ must be constant, which is to say that the *same* Sturmian sequence labels each infinite path starting at the root.

The intervals of P_0^{n-1} whose union is $T^{-n+1}P_0$ are the ones whose corresponding (usual, one-dimensional) n-blocks end with 0, and there are approximately $(n+1)(1-\alpha)$ of them, since about this many of their endpoints $k\alpha \mod 1, 0 \leq k \leq n$, are in $T^{-n+1}P_0$. When n is large, all the intervals of P_0^{n-1} are short, since the multiples of α are uniformly distributed modulo 1. When we join P_0^{n-1} with $T^{-n}P$, exactly one interval of P_0^{n-1} is split; this is the one whose n-block coding is *right special*—it has two possible continuations to the right as $(n+1)$-blocks. Since $P_0 \cap T^{-1}P_0 \neq \emptyset$, when n is large enough all intervals of P_0^{n-1} have measure much less than the measure of $P_0 \cap T^{-1}P_0$ and $P_0 \cap T^{-1}P_1$. Therefore many of the (short) remaining intervals of P_0^{n-1} whose union is $T^{-n+1}P_0$ which are not split are in $T^{-n}P_0$, meaning that their n-blocks end with 0 and can be continued only with 0. Similarly, since $P_0 \cap T^{-1}P_1 \neq \emptyset$, many of them are in $T^{-n}P_1$, meaning that their n-blocks end with 0 and can be followed only by 1. Therefore τ contains both of the blocks 0_0^0 and 0_1^1. □

5. Definitions of tree shift entropy and existence of the limit

The *complexity function* $p(n)$ of a tree shift gives for each $n \geq 0$ the number of n-blocks among all labeled trees in the tree shift. Recall that we deal only with transitive tree shifts; thus if $\tau \in X$ has the property that every block that appears in X also appears in τ, then $p(n) = p_\tau(n)$ for all n. The *entropy* of the tree shift is defined to be

$$h = \limsup_{n\to\infty} \frac{\log p_\tau(n)}{2^{n+1}-1}, \tag{5.1}$$

the exponential growth rate of the number of different labelings of shifts of Δ_n in τ divided by the number of sites for the labels. (We could equivalently divide by just 2^{n+1} instead of $2^{n+1}-1$.) Ban and Chang use a different definition,

$$h_2 = \limsup_{n\to\infty} \frac{\log\log p(n)}{n}. \tag{5.2}$$

They show that for labelings of dyadic trees consistent with 1-step finite type restrictions on adjacent symbols, h_2 is always either 0 or $\log 2$ [8]. We will see below that the quantity h behaves quite differently.

Since factor maps between tree shifts are given by sliding block codes, the complexity function is nonincreasing under factor maps, and hence both versions of entropy are invariants of topological conjugacy of tree shifts. Ban and Chang show that for many tree shifts h_2 exists as a limit. For our h we have the following.

Theorem 5.1. *The limit $h = \lim_{n\to\infty} \log p(n)/2^{n+1}$ exists. In fact, for each labeled tree τ the limit $h(\tau) = \lim_{n\to\infty} \log p_\tau(n)/2^{n+1}$ exists.*

Proof. Let τ be a labeled tree as above. We follow the strategy of the standard proof using subadditivity, adapting as necessary to the tree situation. We have, for $m, n \geq 0$,

$$p_\tau(m+n) \leq p_\tau(m) p_\tau(n)^{2^m}. \tag{5.3}$$

This is because Δ_m has 2^m terminal nodes, and Δ_{m+n} is obtained by attaching a shift of Δ_n to each one of these terminal nodes. Thus any configuration on a shift of Δ_{m+n} in τ consists of a configuration on a shift of Δ_m with 2^m configurations on shifts of Δ_n attached to the terminal nodes.

For any (positive) integer k,

$$p_\tau(km) \leq p_\tau(m)^{(2^{km}-1)/(2^m-1)}. \tag{5.4}$$

At the root of any shift of Δ_{km}, we have one Δ_m, with 2^m terminal nodes. Attached to its terminal nodes we have 2^m shifts of Δ_m, bringing the total number of shifts of Δ_m so far to $1 + 2^m$. Next, we have 2^{2m} terminal nodes, with a shift of Δ_m attached to each one, bringing the total so far to $1 + 2^m + 2^{2m}$. Continuing this way, at the last level we have $2^{(k-1)m}$ terminal nodes, with a shift of Δ_m attached to each one, bringing the total number of shifts of Δ_m to $1 + 2^m + 2^{2m} + \cdots + 2^{(k-1)m}$. The labeling in τ of the shift of Δ_{km} gives labelings in τ of each of these shifts of Δ_m (but these latter labelings might not all be independent).

We show now that $\lim \log p_\tau(n)/2^{n+1}$ exists. Let $\alpha = \liminf \log p_\tau(n)/2^{n+1}$. Let $\epsilon > 0$ and choose an integer $r \geq 1$ such that $\log p_\tau(r)/(2^{r+1}-2) < \alpha + \epsilon$. Given an integer n, write $n = i + kr$, with $0 \leq i < r$. Then

$$\begin{aligned} p_\tau(n) &= p_\tau(i+kr) \leq p_\tau(i) p_\tau(kr)^{2^i} \\ &\leq p_\tau(i) [p_\tau(r)^{(2^{kr}-1)/(2^r-1)}]^{2^i} \\ &= p_\tau(i) p_\tau(r)^{2^i(2^{kr}-1)/(2^r-1)}. \end{aligned} \tag{5.5}$$

Now if n is large enough,

$$\begin{aligned}
\alpha - \epsilon &\leq \frac{\log p_\tau(n)}{2^{n+1}} \\
&\leq \frac{\log p_\tau(i)}{2^{n+1}} + \frac{1}{2^{i+kr+1}} \frac{2^i(2^{kr}-1)}{2^r - 1} \log p_\tau(r) \\
&= \frac{\log p_\tau(i)}{2^{n+1}} + \frac{1 - 2^{-kr}}{2^{r+1} - 2} \log p_\tau(r) \\
&< \frac{\log p_\tau(i)}{2^{n+1}} + \frac{\log p_\tau(r)}{2^{r+1} - 2} \\
&< \frac{\log p_\tau(i)}{2^{n+1}} + \alpha + \epsilon \\
&< \alpha + 2\epsilon.
\end{aligned} \tag{5.6}$$

□

6. Entropy estimates for tree shifts determined by one-dimensional SFT's

Consider now a dyadic tree shift with vertices labeled from a finite alphabet $A = \{a_1, \dots, a_d\}$ with 1-step finite type restrictions given by a $0, 1$ matrix M indexed by the elements of A: adjacent nodes in the tree are allowed to have labels i for the first (closer to the root) and j for the second if and only if $M_{ij} = 1$. We wish to find, or at least estimate, the entropy h of the tree shift.

Proposition 6.1. *Suppose that the matrix M is irreducible, and for each $i = 1, \dots, d$ and $n \geq 0$ denote by $x_i(n)$ the number of labelings of Δ_n which are consistent with the adjacency matrix M and have the symbol a_i at the root. Then for each i and j,*

$$\lim_{n\to\infty} \frac{\log x_i(n)}{2^{n+1}} = \lim_{n\to\infty} \frac{\log x_j(n)}{2^{n+1}} = h. \tag{6.1}$$

Proof. By irreducibility, given $i, j \in \{1, \dots, d\}$, there is a word in the subshift Σ_M of some length k that begins with a_i and ends with a_j. By labeling each path from the root to a vertex at level k with this same word, we can find a labeling consistent with M that has a_i at the root and a_j at every vertex at level k. Now for any $n \geq 0$ the shifts to these vertices at level k of Δ_n can be labeled independently in $x_j(n)$ ways. This implies that

$$x_i(n+k) \geq x_j(n)^{2^k}, \tag{6.2}$$

from which the result follows. □

Theorem 6.2. *Let M be an irreducible d-dimensional $0, 1$ matrix, Σ_M the corresponding shift of finite type, and X_M the corresponding tree shift, labeled by elements of the alphabet A, with $|A| = d$, subject to the adjacency restrictions given by M. Then the topological entropy of Σ_M is less than or equal to the topological entropy of X_M: $h_{\text{top}}(\Sigma_M) \leq h$.*

Proof. Let v denote the positive Perron-Frobenius left eigenvector of M normalized so that $\sum v_i = 1$, and let $\lambda > 0$ denote the maximum eigenvalue of M. For each $n = 0, 1, \dots$ and $i = i, \dots, |A|$ denote by $x(n) = (x_i(n)), i = 1, \dots, |A|$, the vector that gives for each symbol $i \in A$ the number of trees of height n labeled according to the transitions allowed by M that have the symbol i at the root. Considering the symbols that can follow each symbol i in the last row of a labeling of Δ_n, and that they can be assigned in independent pairs to the nodes below, shows that these vectors satisfy the recurrence

$$x_i(0) = 1, \quad x_i(n+1) = (Mx(n))_i^2 \text{ for all } i = 1, \dots, d, \text{ all } n \geq 0. \tag{6.3}$$

(Cf. the nonlinear recurrences in [8].) Denote by 1 the vector $(1, 1, \dots, 1) \in \mathbb{R}^d$. We claim that

$$x(n) \cdot v \geq \lambda^{2^{n+1}-2} v \cdot 1 \quad \text{for all } n \geq 0. \tag{6.4}$$

Since M is irreducible and all entries of v arepositive, $x(n) \cdot v, x(n) \cdot 1$, and each $x_i(n)$ all grow at the same superexponential rate, so the result follows.

For $n = 0$ we have

$$x(0) \cdot v = \sum_i v_i = v \cdot 1. \tag{6.5}$$

Assuming that the inequality holds at stage n and using the inequality $\mathbb{E}(X^2) \geq [\mathbb{E}(X)]^2$ on the random variable $X_i = [Mx(n)]_i$ with discrete probabilities v_i, we have

$$\begin{aligned} &\sum_i x_i(n+1) v_i = \sum_i (Mx(n))_i^2 v_i \geq \left[\sum_i Mx(n)_i v_i\right]^2 \\ &= \left[\sum_i x(n)_i (vM)_i\right]^2 = \left[\sum_i x(n)_i \lambda v_i\right]^2 = [\lambda x(n) \cdot v]^2 \\ &\geq \left[\lambda^{2^{n+1}-2} \lambda v \cdot 1\right]^2 = \lambda^{2^{n+2}-2} v \cdot 1. \end{aligned} \tag{6.6}$$

□

Proposition 6.3. *Let M be an irreducible d-dimensional $0, 1$ matrix, Σ_M the corresponding shift of finite type, and X_M the corresponding tree shift, labeled by elements of the alphabet A, with $|A| = d$, subject to the adjacency restrictions given by M. Let r be a positive right eigenvector of M, λ the maximum eigenvalue of M, $r_{\max} = \max\{r_i\}, r_{\min} = \min\{r_i\}$, and $c = r_{\max}/r_{\min}$. Then the entropy h of the tree shift X_M satisfies the estimate*

$$h \leq U = \frac{1}{2} \log c + \log \lambda. \tag{6.7}$$

Proof. We use the following estimate for the number $N_n(a)$ of n-blocks that can follow any symbol $a \in A$ in the SFT Σ_M (see [15, p. 107]):

$$N_n(a) \leq c\lambda^n. \tag{6.8}$$

To label the nodes of Δ_n with symbols from the alphabet A consistent with the adjacency restrictions prescribed by M, we first label the nodes down the left edge $LL \dots L$; there are no more than $dc\lambda^n$ ways to do this. Then we label the paths below, taking into account nodes above that have already been labeled. Thus there are less than or equal to $dc\lambda^2 c\lambda = dc^2\lambda^3$ ways to label Δ_1, no more than $dc\lambda^3 c\lambda c\lambda^2 c\lambda = dc^4\lambda^7$ to label Δ_2, etc. Using induction, we estimate that there are no more than

$$dc^{2^n} \lambda^{2^{n+1}-2} \tag{6.9}$$

ways to label Δ_n that observe the adjacency restrictions, and this gives

$$h \leq U = \frac{1}{2} \log c + \log \lambda. \tag{6.10}$$

□

7. Recurrence and numerics for the golden mean

We compute both entropies numerically for the golden mean tree shift of finite type, for which no two adjacent nodes are allowed to have the same label 1, by means of a recurrence equation in just one variable. This approach shows explicitly how the characteristic polynomial of the adjacency matrix of the one-dimensional SFT enters and may be helpful in the analysis of other examples.

Direct observation gives $p(0) = 2$ and $p(1) = 5$. With some more effort we find that $p(2) = 41$ and $p(3) = 2306$. Consulting the Online Encyclopedia of Integer Sequences [16] finds Sequence A076725, the number of independent sets (no two nodes in the set are adjacent) in a dyadic tree with $2^n - 1$ nodes.

Referencing Jonathan S. Braunhut, OEIS states that the sequence satisfies the recursion

$$p(n+1) = p(n)^2 + p(n-1)^4. \tag{7.1}$$

This may be verified as follows. A labeled shift of Δ_{n+1} with no 11 either begins with 0 at its root, in which case it can be followed by two independently labeled shifts of Δ_n's with no 11 below, or else it begins with 1^0_0 at the root followed by four independently labeled shifts of Δ_{n-1}'s below.

OEIS gives numerical evidence that

$$p(n) \asymp bc^{2^{n+2}}, \qquad \text{with } c \approx 1.28975 \quad \text{and } b \approx 0.6823278. \tag{7.2}$$

Then we find that

$$h = 2\log c \approx 0.509 \qquad \text{and} \qquad h_2 = \log 2 \approx 0.693. \tag{7.3}$$

We verify that the limit for h exists, by using the recurrence equation $p(n+1) = p(n)^2 + p(n-1)^4$.

Proposition 7.1. *Suppose that $p(0) = 2, p(1) = 5$, and for $n \geq 2$ we have $p(n+1) = p(n)^2 + p(n-1)^4$. Let $q(n) = p(n)/[p(n-1)]^2$ for $n \geq 2$. Then $\lim_{n\to\infty} q(n)$ exists (it is the real root of $x = 1 + 1/x^2$, which is approximately 1.46557), and therefore $\lim \log p(n)/2^{n+1}$ exists (it is approximately 0.509).*

Proof. We have $q(n) = 1 + 1/q(n-1)^2, q(2) = 5/4 = 1.25, q(3) = 41/25 \approx 1.64, q(4) = 625/1681 \approx 1.3718$. Using the recurrence,

$$q_n - q_{n-1} = \frac{q_{n-2}^2 - q_{n-1}^2}{q_{n-1}^2 q_{n-2}^2}, \tag{7.4}$$

so that $q_n - q_{n-1}$ is alternating in sign. Moreover,

$$|q_n - q_{n-1}| = |q_{n-2} - q_{n-1}| \frac{q_{n-2} + q_{n-1}}{q_{n-1}^2 q_{n-2}^2}, \tag{7.5}$$

and the factor $q_{n-2} + q_{n-1}/q_{n-1}^2 q_{n-2}^2$ is bounded by $0.891657 < 1$, since, viewing the first few values of the sequence $q(n)$ and using the fact that it is alternately increasing and decreasing we have both of $q_{n-2}, q_{n-1} \geq 1.25$ and at least one of them no less than 1.3718. Numerical evidence from OEIS shows that

$$q_n = \frac{p_n}{p_{n-1}^2} \to 1.46557\ldots. \tag{7.6}$$

We show now that this implies the existence of the limit $\lim \log p_n/2^{n+1}$. From above, since $q_n \to 1.46557\ldots$, we have $\log p_n - 2\log p_{n-1} \to \log 1.46557\cdots = a = 0.382244\ldots$. Thus

$$|\log p_n - 2\log p_{n-1} - a| = \epsilon_n \to 0, \tag{7.7}$$

and hence

$$\left|\frac{\log p_n}{2^{n+1}} - \frac{\log p_{n-1}}{2^n} - \frac{a}{2^{n+1}}\right| = \left|\left(\frac{\log p_n}{2^{n+1}} + \frac{a}{2^{n+1}}\right) - \left(\frac{\log p_{n-1}}{2^n} + \frac{a}{2^n}\right)\right| < \frac{\epsilon_n}{2^{n+1}}. \tag{7.8}$$

Therefore the sequence

$$\left|\left(\frac{\log p_n}{2^{n+1}} + \frac{1.46557\ldots}{2^{n+1}}\right)\right| \tag{7.9}$$

converges, and hence so does the sequence $\log p_n/2^{n+1}$. □

Proposition 7.2. *For each $n \geq 0$ denote by A_n the number of dyadic trees labeled by $A = \{0,1\}$ in such a way that no two adjacent nodes both have the label 1. (Thus $A_{n+1} = (A_n + A_{n-1}^2)^2$ for $n \geq 1$ and $(A_n) = (1, 4, 25, 1681, 5317636, \ldots)$.) Denote by γ the golden mean $(1+\sqrt{5})/2$, so that $\gamma + 1 = \gamma^2$ and the one-dimensional golden mean shift of finite type Σ_M has entropy $\log\gamma$. Then*

$$A_n \geq \gamma^{2^{n+1}-1} \quad \text{for } n \geq 4, \tag{7.10}$$

so that

$$h = \lim_{n\to\infty} \frac{\log A_n}{2^{n+1}-1} \geq \log\gamma = h_{\text{top}}(\Sigma_M). \tag{7.11}$$

Proof. Direct calculation verifies the inequality for $n = 4, 5$. Using induction,

$$\begin{aligned} A_{n+1} &= (A_n + A_{n-1}^2)^2 \geq [\gamma^{2^{n+1}-1} + (\gamma^{2^n-1})^2]^2 \\ &= (\gamma^{2^{n+1}-1} + \gamma^{2^{n+1}-2})^2 = \gamma^{2^{n+2}-2} + 2\gamma^{2^{n+2}-3} + \gamma^{2^{n+2}-4} \\ &= \gamma^{2^{n+2}-4}(\gamma^2 + 2\gamma + 1) = \gamma^{2^{n+2}-4}(\gamma+1)^2 = \gamma^{2^{n+2}-4}\gamma^4 \\ &= \gamma^{2^{n+2}} \geq \gamma^{2^{n+2}-1}. \end{aligned} \tag{7.12}$$

□

8. Examples and questions

The following table shows *Mathematica* computations for one 2×2 and fourteen 3×3 one-dimensional shifts of finite type, with entropies h_{top} determining labelings of the dyadic tree. The entropy h of the tree is estimated by recurrence up to $n = 15$. If the transition matrix M has all row sums equal to s, then $h = h_{\text{top}}(\Sigma_M) = U = \log s$. For an upper estimate we use $U = (\log c)/2 + \log\lambda$ from Proposition 6.3.

Name	Matrix= M	$h_{\text{top}}(\Sigma_M)$	h (est)	U
Γ	$11, 10$	.481	.509	.721
X_0	$010, 101, 101$	.481	.509	.722
X_1	$110, 001, 110$	.481	.509	.722
X_2	$011, 101, 100$	.481	.509	.722
X_3	$011, 111, 101$	.81	.846	1.104
X_4	$111, 110, 100$	.81	.846	1.214
X_5	$110, 011, 101$	.693	.693	.693
X_6	$011, 101, 110$	.693	.693	.693
X_7	$110, 001, 111$	.693	.768	1.04
X_8	$110, 011, 110$	.693	.693	.693
X_9	$011, 101, 101$	.693	.693	.693
X_{10}	$011, 111, 100$	.693	.774	1.242
X_{11}	$111, 100, 100$	.693	.763	1.04
A_1	$110, 101, 001$	.481	.611	∞
A_2	$110, 011, 010$	.481	.575	.962

Table 1. Estimates of some tree shift entropies

Example 8.1. For the matrix M whose rows are $010, 001, 110$, we find numerically that the maximum eigenvalue is $\lambda \approx 0.2812$, the right and left eigenvectors are $r \approx (0.57, 0.75, 1)$ and $v \approx (0.75, 1.32, 1)$, so that the constant in Proposition 6.3 is $c \approx 1.75$, and the (rigorous) upper estimate for h is $U \approx 0.56$. Thus for this example we have

$$0 < h_{\text{top}}(\Sigma_M) = \log \lambda \approx 0.28 \leq h \leq U \approx 0.56 < \log 2. \tag{8.1}$$

(Numerically, using the recurrence, we find $h \approx 0.36$.)

Questions:

1. What is the *maximal* possible complexity function of a dyadic tree all of whose infinite paths starting at the root are labeled by sequences from a fixed Sturmian system? (Does this concept even make sense?)

2. Suppose that in a dyadic tree we label the infinite paths from the root with sequences from a fixed Sturmian system in the following lexicographic manner. Down the leftmost path put the lexicographically minimal sequence. Starting at the left edge, so long as there is no choice for the next symbol (the block completed is not right special), just copy the entry on the left edge at both nodes below. If the block arrived at allows two successor symbols, put a 0 on the node below and to the left, a 1 on the node below and to the right. Continue to apply this rule along all infinite paths from the root. We should arrive with the tree completely filled in with *all* sequences in the Sturmian system written along the uncountably many infinite paths from the root, in lexicographic order left to right, with the minimal sequence down the left edge and the maximal sequence down the right edge. The labeling is accomplished by following paths along the Hofbauer-Buzzi Markovian diagram of the Sturmian system (see [11]). What is the complexity function $p_\tau(n)$ of this labeled tree?

3. What if we follow the preceding scheme to label the dyadic tree, except that whenever we have a choice we put 0 and 1 on the two nodes below randomly and independently of all other choices? What complexity function do we get with probability 1? Could the labeled tree have positive entropy?

4. Is there an example for which the limit defining h (see Theorem 5.1) is not equal to the infimum?

5. For each $n \geq 0$ denote by Φ_n the set of all allowable labelings of Δ_n. For $n \geq 0, \phi \in \Phi_n$, and $a \in A$, denote by $s_\phi(a)$ the number of the 2^n terminal nodes of Δ_n that have the label a. For each $a \in A$ denote by t_a the a'th row sum of M, that is, the number of outgoing arrows from the vertex a in the graph of the shift of finite type defined by M. Then

$$|\Phi_{n+1}| = \sum_{\phi \in \Phi_n} \prod_{a \in A} (t_a^2)^{s_\phi(a)}. \tag{8.2}$$

Switching the order of summation might be a starting point for obtaining upper and lower estimates for h.

6. Let us imagine that for most ϕ the distribution of the $s_\phi(a)$ is approximately given by the measure μ of maximal entropy for the subshift defined by M, so that $s_\phi(a) \sim 2^n \mu[a]$. Then

$$|\Phi_{n+1}| \sim |\Phi_n| \prod_{a \in A} (t_a^2)^{2^n \mu[a]} \sim |\Phi_0| \prod_{a \in A} t_a^{(2^{n+1}-1)\mu[A]}, \tag{8.3}$$

so we might suppose that

$$\lim \frac{\log |\Phi_{n+1}|}{2^{n+2}} \sim U_m := \log \prod_{a \in A} t_a^{\mu[a]} = \sum_{a \in A} \mu[a] \log t_a. \tag{8.4}$$

The latter expression is the average of the row sums of the transition matrix M using weights for the symbols given by the measure of maximal entropy for the one-dimensional shift of finite type. This may often be a good first approximation to h, usually from below.

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