THE REPRESENTATION THEORY OF 2-SYLOW SUBGROUPS OF SYMMETRIC GROUPS

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Abstract. In this paper we develop a quick recursive algorithm to calculate the characters of the 2-Sylow subgroups of the S_n. The algorithm exploits the construction of these subgroups as automorphism groups of certain sets of binary trees. Such a construction reduces the problem to the case where n is a power of 2. The Sylow subgroups in this case are built recursively from lower powers of 2. This leads to a recursive characterisation and enumeration of the conjugacy classes and the irreducible representations, and thus allows us to compute character values from those of the lower subgroups. Finally we describe the Bratteli diagram of this family, and its one dimensional representations, which by McKay's correspondence are in bijection with odd-dimensional representations of the symmetric group.

1. Introduction

The 2-Sylow subgroups of a symmetric group arise as automorphism groups of binary trees with appropriate labellings on the external vertices.

Definition 1.1. *A complete binary tree is defined recursively [7] as either:*

- *A single vertex (called the trivial tree).*
- *A graph formed by taking two complete binary trees, adding a vertex, and adding an edge directed from the new vertex to the root of each binary tree.*

We differentiate the branches of a nontrivial binary tree by designating one the left and the other the right subtree. Vertices are internal if they have a nontrivial left and right subtree, and external otherwise. Given vertices x and y, $x \leq y$ if x occurs in one of the subtrees associated to y. This is a partial order, and the Hasse diagram of the poset is the familiar depiction of a binary tree as in Figure 1. The tree admits an obvious rank function(where the root has rank 0) and the rank of a vertex is the number of edges on the unique path connecting it to the root vertex. A complete binary tree is of height k if the rank of every external vertex is k.

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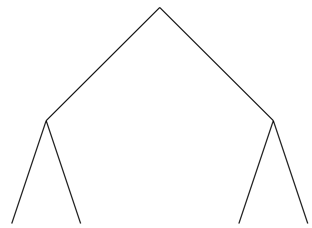


FIGURE 1. A complete binary tree of height 2.

Let H_k denote the automorphism group of the complete binary tree of height k. An automorphism of a binary tree acts on each internal vertex as automorphisms of its left and right subtree, followed by either fixing or exchanging their designations as right and left subtree. Given such an automorphism, a labelling, L is a bijection between external vertices and the set $\{1, 2, \ldots, 2^k\}$. L can be extended to internal vertices by concatenating the labels of its children, left to right if the automorphism fixes the subtrees under this vertex, and right to left otherwise. At the completion of this process, as in Figure 2, the label of the root node is a permutation of S_{2^k}. Fixing a particular labelling L of the external vertices, this process defines an embedding $\theta_L : H_k \to S_{2^k}$. Henceforth the labelling will be $1, 2, \ldots, 2^k$ from left to right, as in Figure 2.

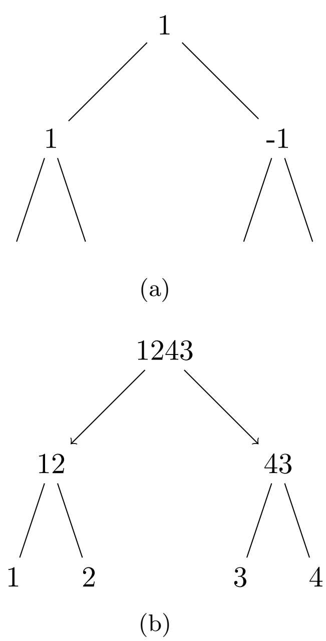


FIGURE 2. Automorphisms as (a) weights on a binary tree, and (b) permtuations.

Proposition 1.1. *H_k is the 2-Sylow subgroup of S_{2^k}.*

Proof. Let $v_2(n)$ be the highest exponent of 2 that divides n. Then by Legendre's formula, $v_2(n!) = \lfloor n/2 \rfloor + \lfloor n/4 \rfloor + \cdots + 1$. In particular:

$$\begin{aligned} v_2(2^k!) &= \left\lfloor 2^k/2 \right\rfloor + \left\lfloor 2^k/4 \right\rfloor + \cdots + 1 \\ &= 2^{k-1} + 2^{k-2} + \cdots + 1 \\ &= 2^k - 1 \end{aligned}$$

Each automorphism of a binary tree of height k can be represented as weights on the internal vertices, as in Figure 2 : -1 if the automorphism flips the subtrees under the vertex and 1 otherwise. This is immediate by induction and the preceding discussion on the action of an automorphism on internal vertices of the binary tree. There are $2^k - 1$ internal vertices. Thus, there are 2^{2^k-1} automorphisms of the binary tree of height k.

□

Definition 1.2. *The binary digits of an integer n is the set of distinct nonnegative integers $\{k_1, k_2, \ldots, k_l\}$ such that $n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_l}$.*

Let $\{k_1, \ldots, k_l\}$ be the binary digits of an n. Then $v_2(n!) = v_2(2^{k_1}!) + v_2(2^{k_2}!) + \cdots + v_2(2^{k_r}!)$. This is the cardinality of $H_{k_1} \times H_{k_2} \times \cdots \times H_{k_r}$. Thus, this Cartesian product must be a 2-Sylow subgroup of S_n, which embeds in $S_{2^{k_1}} \times S_{2^{k_2}} \times \cdots \times S_{2^{k_r}} \subset S_n$. We denote by P_n the 2-Sylow subgroup of S_n and note that $P_{2^k} = H_k$.

Proposition 1.2. *$H_k = H_{k-1} \wr C_2$.*

Proof. C_2 has an obvious action on the set $\{1, 2\}$. So we can form the group $H_{k-1} \wr C_2$. We prove that this group is H_k.

Consider $\psi \in H_k$. ψ is an automoprhism of the complete binary tree of height k. Let ϕ_1 and ϕ_2 be its restriction to the left and right subtrees. Further let ϵ be the weight on the root vertex(see Figure 2). Then $(\phi_1, \phi_2)^\epsilon \in H_{k-1} \wr C_2$. It is a routine calculation to verify that under this designation, the product in the wreath product corresponds to left multiplication in the group as a subgroup of S_{2^k}.

Now consider $(\phi_1, \phi_2)^\epsilon \in H_{k-1} \wr C_2$. The element $\phi_2 \in S_{2^k-1}$ acts on the labels $\{2^{k-1} + 1, \ldots, 2^k\}$, while ϕ_1 acts on $\{1, \ldots, 2^{k-1}\}$. By concatenating these strings, ϕ_1 first unless $\epsilon = -1$, we obtain a permutation of $\{1, \ldots, 2^k\}$. This operation again is consistent with multiplication in H_k.

□

An element of H_k is expressed as $(\sigma_1, \sigma_2)^\epsilon$, where $\epsilon = \pm 1$, with $\epsilon = -1$ representing the non-trivial action of C_2 and (σ_1, σ_2) is a tuple of elements from $H_{k-1} \times H_{k-1}$.

Since P_n is the direct product of of the set of groups H_{k_i} where k_i ranges over the binary digits of n. This reduces the study of these groups to the case where n is a power of 2. Our aim in the next sections is to construct the character table of H_k. The character table of a group P_n is the tensored product, as matrices, of the character tables for the H_{k_i}.

2. Conjugacy classes of H_k

We begin building the character table of H_k by characterising the conjugacy classes of the group. Let $\sigma := (\sigma_1, \sigma_2)^\epsilon \in H_k$. Then σ is conjugate to $\tau := (\tau_1, \tau_2)^\kappa$ (denoted $\sigma \sim \tau$) if there exists a conjugating element $\phi = (\phi_1, \phi_2)^\delta$.This yields immediately that $\kappa = \epsilon$, and:

$$(\tau_1, \tau_2) = \begin{cases} (\phi_1^{-1}\sigma_1\phi_1, \phi_2^{-1}\sigma_2\phi_2) & \text{if } \delta = 1 \\ (\phi_1^{-1}\sigma_2\phi_1, \phi_2^{-1}\sigma_1\phi_2) & \text{if } \delta = -1 \end{cases} \tag{1}$$

when $\epsilon = 1$, and

$$(\tau_1, \tau_2) = \begin{cases} (\phi_1^{-1}\sigma_1\phi_2, \phi_2^{-1}\sigma_2\phi_1) & \text{if } \delta = 1 \\ (\phi_2^{-1}\sigma_2\phi_1, \phi_1^{-1}\sigma_1\phi_2) & \text{if } \delta = -1 \end{cases} \tag{2}$$

if $\epsilon = -1$

Equation (1) reveals two types of classes with $\epsilon = 1$:

- If σ_1 and σ_2 are not conjugate in H_{k-1}, the conjugacy class of $(\sigma_1, \sigma_2)^1$ comprises $(\tau_1, \tau_2)^1$ where either $\tau_1 \sim \sigma_1$ and $\tau_2 \sim \sigma_2$ or $\tau_1 \sim \sigma_2$ and $\tau_2 \sim \sigma_1$. This is the first type of conjugacy class, and there are $\binom{C_{k-1}}{2}$ such classes (where C_k is the number of conjugacy classes in H_k) for a choice of distinct classes in H_{k-1}.
- If $\sigma_1 \sim_{H_{k-1}} \sigma_2$, then $\sigma_1 \sim \tau_1 \sim \tau_2$. This is the second type of conjugacy class, and there are C_{k-1} of this type.

Evaluating equation (2) for $\sigma_1 = \mathrm{Id}$:

- For a choice of conjugacy class τ_1, $\tau_2 = \phi_1^{-1}\sigma$ where σ is any element conjugate to σ_2 in H_{k-1}. This gives the final type of conjugacy class, represented by $(\mathrm{Id}, \sigma_2)^{-1}$. There are C_{k-1} such classes.

We summarise these results in the table below.

	Representative	# classes	Size of class
1	$[(\sigma_1, \sigma_2)^1]$	$\binom{C_{k-1}}{2}$	$2c_{k-1}([\sigma_1])c_{k-1}([\sigma_2])$
2	$[(\sigma_1, \sigma_1)^1]$	C_{k-1}	$c_{k-1}([\sigma_1])^2$
3	$[(\mathrm{Id}, \sigma_1)^{-1}]$	C_{k-1}	$\lvert H_{k-1}\rvert c_{k-1}([\sigma_1])$

Table 1. Conjugacy classes of H_k

where $c_k([\sigma])$ is the size of the class $[\sigma]$.

There are $\binom{C_{k-1}}{2}$ classes of the first, and C_{k-1} classes each of the second and third type in H_k. Thus we have a recursion relation for C_k:

$$\begin{aligned} C_k &= \binom{C_{k-1}}{2} + 2C_{k-1} \\ C_0 &= 1 \end{aligned} \tag{3}$$

Theorem 2.1. *Let $b_m^{(k)}$ denote the number of conjugacy classes of H_k of cardinality 2^m and $f_k(t) = \sum b_m^{(k)} t^m$. Then $f_k(t)$ satisfies the recurrence relation:*

$$f_k(t) = \frac{t}{2}[f_{k-1}(t)^2 - f_{k-1}(t^2)] + f_{k-1}(t^2) + t^{2^{k-1}-1} f_{k-1}(t) \tag{4}$$

Proof. The proof uses operations on classes of combinatorial objects from the Symbolic method. A suitable reference is [6].

The class of conjugacy classes of H_k is denoted $Conj_k$. From Table 1 we know that $Conj_k$ is the disjoint union of three distinct types of conjugacy classes; we denote these $Conj_k^1$, $Conj_k^2$ and $Conj_k^3$. The generating function for $Conj_k$ is $\sum_\sigma t^{c_k([\sigma])} = f_k(t)$, where σ ranges over the representatives of conjugacy classes of H_k.

$Conj_k^2$ consists of classes of type $(\sigma, \sigma)^1$, which corresponds to the diagonal of the cartesian product $Conj_{k-1} \times Conj_{k-1}$, denoted $D(Conj_{k-1})$. The generating function for $Conj_k^2$ is thus $f_{k-1}(t^2)$.

$Conj_k^1$ consists of classes of type $(\sigma_1, \sigma_2)^1$, corresponding to unordered pairs of representations from H_{k-1}. The class $Conj_{k-1} \times Conj_{k-1} \backslash D(Conj_{k-1})$ is the set of ordered partitions. Its generating function is $f_{k-1}(t)^2 - f_{k-1}(t^2)$. The identification of a tuple (σ_1, σ_2) and (σ_2, σ_1) in this class gives the class of unordered pairs; its generating function is $\frac{1}{2}(f_{k-1}(t)^2 - f_{k-1}(t^2))$. Since the cardinality is twice the product of cardinalities of the classes σ_1 and σ_2, we multiply this expression by t in the final summation.

$Conj_k^3$ comprises classes of type $(\mathrm{Id}, \sigma)^{-1}$ and is isomorphic to $Conj_{k-1}$. The generating function for this class is thus $f_{k-1}(t)$. The constant $|H_{k-1}|$ multiplies the cardinality of all such classes, since our choice for $\sigma_1 = \mathrm{Id}$ in evaluating Equation 2 was arbitrary.

The terms in a distinct union add to give $f_k(t) = \frac{t}{2}(f_{k-1}(t)^2 - f_{k-1}(t^2)) + f_{k-1}(t^2) + t^{2^{k-1}-1} f_{k-1}(t)$.

□

3. Irreducible representations of H_k

This section enumerates and characterises the irreducible representations of H_k. We will find that these representations all occur as summands in representations induced from the normal subgroup $H_{k-1} \times H_{k-1}$. Irreducible representations of $H_{k-1} \times H_{k-1}$ are of the form $\phi_1 \times \phi_2$, for irreducible representations ϕ_1 and ϕ_2 of H_{k-1}. We adopt the notation $\gamma = \phi_1 \times \phi_2$ and Γ is the representation of H_k induced from γ.

Proposition 3.1. *Γ is irreducible if γ is of type $\phi_1 \otimes \phi_2$, for inequivalent irreducible representations ϕ_1 and ϕ_2 of H_{k-1}, and is the sum of two irreducible representations otherwise.*

With γ and Γ as above, and using the formula for induced characters from [5, Chapter 5, pg 64] for the case $G = H_k$ and $H = H_{k-1} \times H_{k-1}$ and substituting values from table 1:

Table 2. Centralisers of conjugacy classes

Type of class	$\lvert C_{H_k}\rvert$	$\{x_i\}$	$\lvert C_{H_{k-1}}(x_i)\rvert$
$[(\sigma_1,\sigma_2)^1]$	$\lvert C_{H_{k-1}}(\sigma_1)\rvert\lvert C_{H_{k-1}}(\sigma_2)\rvert$	(σ_1,σ_2) (σ_2,σ_1)	$\lvert C_{H_{k-1}}(\sigma_1)\rvert\lvert C_{H_{k-1}}(\sigma_2)\rvert$ $\lvert C_{H_{k-1}}(\sigma_1)\rvert\lvert C_{H_{k-1}}(\sigma_2)\rvert$
$[(\sigma_1,\sigma_1)^1]$	$2\lvert C_{H_{k-1}}(\sigma_1)\rvert^2$	(σ,σ)	$\lvert C_{H_{k-1}}(\sigma_1)\rvert^2$

$$\Gamma((\sigma_1,\sigma_2)^\epsilon) = \begin{cases} \phi_1(\sigma_1)\phi_2(\sigma_2) + \phi_1(\sigma_2)\phi_2(\sigma_1) & \text{if } \epsilon = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

We are ready to present the proof of Proposition 3.1:

Proof. For an irreducible representation γ of $H_{k-1} \times H_{k-1}$, let γ^δ be the conjugate representation for $\delta = (\mathrm{Id}, \mathrm{Id})^{-1}$. The corollary to [8, Chapter 5, Proposition 23] posits that a necessary and sufficient condition for the irreducibility of Γ is that γ and γ^δ be non-isomorphic as $H_{k-1} \times H_{k-1}$ representations.

First let $\gamma = \phi_1 \times \phi_2$, so $\gamma^\delta = \phi_2 \times \phi_2$. These are nonisomorphic irreducible $H_{k-1} \times H_{k-1}$ representations since their characters are distinct. Thus the induced representation Γ is irreducible.

When $\gamma = \phi \times \phi$, $\gamma^\delta = \gamma = \phi \times \phi$. Thus the induced representation is reducible. The number of irreducible components of Γ is given by the inner product on characters, $\langle \Gamma, \Gamma \rangle$. From Frobenius reciprocity we have $\langle \Gamma, \Gamma \rangle = \langle \mathrm{Res}^{H_k}_{H_{k-1}\times H_{k-1}}(\Gamma), \gamma \rangle$. From Equation 5, $\langle \mathrm{Res}^{H_k}_{H_{k-1}\times H_{k-1}}(\Gamma), \gamma \rangle = 2$. Thus the induced representation of γ has two irreducible components.

□

Let $\mathrm{Irr}(H_k)$ denote the set of irreducible representations of H_k. If all the induced representations occuring in Proposition 3.1 were distinct, there would be $\binom{H_{k-1}}{2} + 2H_{k-1}$ of them. This, in addition to the condition $H_0 = 1$ follows equation 3 for the number of conjugacy classes. Thus all irreducible representations of H_k would be obtained by inducing irreducible representations from $H_{k-1} \times H_{k-1}$, if the induced representations were distinct. We use a theorem of Clifford on representations induced from normal subgroups to show that this is in fact the case.

Theorem 3.1. *[5, Theorem 1] Given an irreducible representation Γ of a group G, and a normal subgroup H of G over any field P, the representation $Res^G_H(\Gamma)$ is either itself irreducible or is fully reducible into irreducible components, all of the same degree. If γ is an irreducible component of $Res^G_H(\Gamma)$ then all the other irreducible components of $Res^G_H(\Gamma)$ are G-conjugates of γ, and all conjugate occur, and with the same multiplicity.*

Consider an irreducible representation Γ of H_k, occuring in the induction of an irreducible representation γ of $H_{k-1} \times H_{k-1}$. Let $\mathrm{Id} = (\mathrm{Id}, \mathrm{Id})^1$ and

$\delta = (\mathrm{Id}, \mathrm{Id})^{-1}$ be the coset representatives of $H_{k-1} \times H_{k-1}$ in H_k, and let γ^δ be the conjugate by δ of γ. Then by Theorem 3.1, $Res^{H_k}_{H_{k-1}\times H_{k-1}}(\Gamma) = \mathrm{m}(\gamma + \gamma^\delta)$, where m is the multiplicity. Now clearly,

$$\gamma^\delta = \begin{cases} \phi \otimes \phi & \text{if } \gamma = \phi \otimes \phi \\ \phi_2 \otimes \phi_1 & \text{if } \gamma = \phi_1 \otimes \phi_2 \end{cases} \tag{6}$$

So,

$$Res^{H_k}_{H_{k-1}\times H_{k-1}}(\Gamma) = \begin{cases} \mathrm{m}_1(\phi \otimes \phi) & \text{if } \gamma = \phi \otimes \phi \\ \mathrm{m}_2(\phi_1 \otimes \phi_2 + \phi_2 \otimes \phi_1) & \text{if } \gamma = \phi_1 \otimes \phi_2 \end{cases} \tag{7}$$

where, by Frobenius reciprocity, $\mathrm{m}_1 = \mathrm{m}_2 = 1$.

It is clear from this condition on the restrictions that the induced characters described in Proposition 3.1 are completely parametrised by the choice of two representations of H_{k-1}. Thus they are all distinct.

TABLE 3. Irreducible characters of H_k

Notation	Description	Action on $(\sigma_1, \sigma_2)^1$	Dimension
$\mathrm{Ind}(\phi_1, \phi_2)$	Induced from $\phi_1 \otimes \phi_2$	$\phi_1(\sigma_1)\phi_2(\sigma_2) + \phi_1(\sigma_2)\phi_2(\sigma_1)$	$2\dim(\phi_1)\dim(\phi_2)$
$\mathrm{Ext}^+(\phi)$	Positive extension of $\phi \otimes \phi$	$\phi(\sigma_1)\phi(\sigma_2)$	$\dim(\phi)\dim(\phi)$
$\mathrm{Ext}^-(\phi)$	Negative extension of $\phi \otimes \phi$	$\phi(\sigma_1)\phi(\sigma_2)$	$\dim(\phi)\dim(\phi)$

Now we restrict our attention to irreducible $H_{k-1} \times H_{k-1}$ characters of the type $\gamma = \phi \otimes \phi$. We have seen that the representation induced from γ consists of two irreducible representations of H_k. We call these characters $\mathrm{Ext}^+(\gamma)$ and $\mathrm{Ext}^-(\gamma)$ the positive and negative extensions of γ, for reasons explored below. From Equation 6 and Equation 7 we know that $Res^{H_k}_{H_{k-1}\times H_{k-1}}(\mathrm{Ext}^\pm(\gamma)) = \gamma$.

It remains to find the value of $\mathrm{Ext}^\pm(\gamma)$ on classes of type $(\mathrm{Id}, \sigma)^{-1}$. We denote the representation corresponding to a character by the same name, and investigate the matrix of this representation with respect to an arbitrary basis of the representation space of γ. First note that $\mathrm{Ext}^\pm(\gamma)(\mathrm{Id}, \sigma_1)^{-1}) = \mathrm{Ext}^\pm(\gamma)((\mathrm{Id}, \sigma)^1)\mathrm{Ext}^\pm(\gamma)((\mathrm{Id}, \mathrm{Id})^{-1})$, and so we must find the matrix of the representation for the element $\delta = (\mathrm{Id}, \mathrm{Id})^{-1}$ to determine the character of the representation over all of H_k. Let $A_\pm = \mathrm{Ext}^\pm(\gamma)((\mathrm{Id}, \mathrm{Id})^{-1})$. Note that $A_\pm^2 = 1$.

(8)

$$\begin{aligned} A_\pm \gamma(\sigma_1, \sigma_2) A_\pm &= \mathrm{Ext}^\pm(\gamma)(((\mathrm{Id}, \mathrm{Id}), -1)\mathrm{Ext}^\pm(\gamma)((\sigma_1, \sigma_2), 1)\mathrm{Ext}^\pm(\gamma)((\mathrm{Id}, \mathrm{Id}), -1) \\ &= \mathrm{Ext}^\pm(\gamma)(((\sigma_2, \sigma_1), 1)) \\ &= \gamma(\sigma_2, \sigma_1) \end{aligned}$$

So

$$A_\pm \gamma = \overline{\gamma} A_\pm$$

where $\overline{\gamma}(\sigma_1, \sigma_2) = \gamma(\sigma_2, \sigma_1)$.

Thus $A_\pm$ is an intertwiner on the space $V \boxtimes V$, where V is the representation space of ϕ; by Schur's lemma, $A_\pm = \lambda T$ for some scalar λ and T be the intertwiner that takes the element $v \boxtimes w$ to $w \boxtimes v$. Since $I = A_\pm^2 = \lambda^2 T^2 = \lambda^2 I$, since $T^2 = I$, and so $\lambda = \pm 1$.

$$\begin{aligned} \mathrm{Ext}^\pm(\gamma)((1,\sigma)^{-1}) &= \mathrm{Ext}^\pm(\gamma)((\mathrm{Id},\sigma)^1)\mathrm{Ext}^\pm(\gamma)((\mathrm{Id},\mathrm{Id})^{-1}) \\ &= \phi(\mathrm{Id}) \otimes \phi(\sigma)(\pm \circ T) \\ &= \pm \mathrm{Id} \otimes \phi(\sigma) \circ T \end{aligned}$$

It now becomes clear why the extensions of $\phi \otimes \phi$ are called the positive and negative extensions. They can be distinguished by their value on the conjugacy class $(\mathrm{Id}, \mathrm{Id})^{-1}$. The extension whose value is positive on this conjugacy class is denoted $\mathrm{Ext}^+(\phi)$, and the other $\mathrm{Ext}^-(\phi)$ as in table 3.

Theorem 3.2. *Let $a_m^{(k)}$ denote the number of irreducible representations of H_k of dimension 2^m and $g_k(t) = \sum a_m^{(k)} t^k$. Then $g_k(t)$ satisfies the recurrence relation:*

$$g_k(t) = \frac{t}{2}[g_{k-1}(t)^2 - g_{k-1}(t^2)] + 2g_{k-1}(t^2). \tag{9}$$

Proof. Consider the class of irreducible representations G_k with generating function $\sum_{\Gamma \in \mathrm{Irr}(H_k)} t^{log_2(\dim(\Gamma))} = g_k(t)$. From Table 3 we know that G_k is the disjoint union of three types of representations. We denote these classes by G_k^{Ind}, G_k^+ and G_k^-.

G_k^- and G_k^+ are each $D(G_k)$- the diagonal of the Cartesian product $G_{k-1} \times G_{k-1}$. By the symbolic method, the generating function for each of G_k^+ and G_k^- is $g_{k-1}(t^2)$.

G_k^{Ind} is obtained by taking an unordered pair of distinct irreducible representations of H_{k-1}. The class $G_{k-1} \times G_{k-1} \setminus D(G_{k-1})$ is the set of ordered pairs of distinct representations. The corresponding generating function for ordered pairs of distinct representations is thus $(g_{k-1}(t)^2 - g_{k-1}(t^2))$. By identifying the tuples (ϕ_1, ϕ_2) and (ϕ_2, ϕ_1) in this class, one obtains the class of unordered pairs; its generating function is $\frac{1}{2}(g_{k-1}(t)^2 - g_{k-1}(t^2))$. Finally, in inducing the tensored representation so formed, the dimension is twice the product of representations, so we multiply the generating function by t.

Adding the generating functions for the three classes in the disjoint union gives $g_k(t) = \frac{t}{2}(g_{k-1}(t)^2 - g_{k-1}(t^2)) + 2g_{k-1}(t^2)$. $\square$

4. A recursive method for the character table

In this section we fill in the last gaps of the character table of H_k. It remains to find the character values for $\mathrm{Ext}^+(\phi)$ and $\mathrm{Ext}^-(\phi)$ on classes of the type $(\mathrm{Id}, \sigma)^{-1}$.

Recall that $\mathrm{Ext}^\pm(\phi)((\delta)) = \pm \mathrm{Id} \otimes \phi(\sigma) \circ T$, and so we must find the action of T on the matrix $\mathrm{Id} \otimes \phi(\sigma)$.

Theorem 4.1. $\mathrm{Tr}(\mathrm{Ext}^{\pm}(\phi)((\mathrm{Id},\sigma)^{-1}))) = \pm\mathrm{Tr}(\phi)$.

Proof. Consider a choice of basis $\{v_1, v_2, \dots, v_l\}$ for the representation space V of ϕ. Then the matrix for $\mathrm{Id} \otimes \phi(\sigma)$ has the structure:

$$\begin{pmatrix} \phi(\sigma) & 0 & \dots & \dots 0 \\ 0 & \phi(\sigma) & 0 & \dots 0 \\ 0 & 0 & \ddots & \dots 0 \\ 0 & 0 & \dots & \phi(\sigma) \end{pmatrix}$$

each of the blocks is an $l \times l$ size matrix. T acts on the space $V \boxtimes V$ by extending the action $T(v_i \boxtimes v_j) = v_j \boxtimes v_i$. Thus, the matrix for T is $(E_{ij})_{1\le i,j\le l}$, where E_{ij} is an $l \times l$ block that is 1 in the (j,i) position and 0 otherwise.

Thus

$$\begin{aligned} \mathrm{Tr}(\mathrm{Ext}^{\pm}(\phi)((\mathrm{Id},\sigma)^{-1})) &= \pm \sum_{i=1}^{l} \mathrm{Tr}(\phi(\sigma)E_{ii}) \\ &= \pm\mathrm{Tr}(\phi(\sigma) \sum_{i=1}^{l} \mathrm{Tr}(E_{ii})) \\ &= \pm\mathrm{Tr}(\phi(\sigma)I) \\ &= \pm\mathrm{Tr}(\phi(\sigma)) \end{aligned}$$

□

This is a complete description of the character table of H_k.

The recursive method simplifies the calculation of character values of H_k. A lexicographic order on the conjugacy classes of H_{k-1} with the rule $1 < -1$ extends to a total order on H_k by identifying the class $(\sigma_1, \sigma_2)^{\epsilon}$ with the tuple $(\epsilon, \sigma_1, \sigma_2)$, for $\epsilon = \pm 1$. the columns arranged by this order resembles Table 4. A lexicographic ordering on irreducible representations of H_{k-1} along with the identifications $\mathrm{Ext}^{+}(\phi) \to (1, \phi, \phi)$, $\mathrm{Ext}^{-}(\phi) \to (2, \phi, \phi)$, $\mathrm{Ind}(\phi_1, \phi_2) \to (3, \phi_1, \phi_2)$ similarly gives a grouping on rows that resembles Table 4. Under such an arrangement, only the values marked with an asterisk in Table 4 need to be calculated anew.

We populate the tables for the values $k = 0, 1, 2, 3$, to demonstrate the recursive method.

TABLE 4. Template for the character table for H_k

	$(\sigma_1, \sigma_2)^1$	$(\mathrm{Id}, \sigma_1)^{-1}$
Positive Summands from $\phi \otimes \phi$	$\phi(\sigma_1)\phi(\sigma_2)^*$	character table for H_{k-1}
Negative Summands from $\phi \otimes \phi$	$\phi(\sigma_1)\phi(\sigma_2)$	-character table for H_{k-1}
Induced Irrep from $\phi_1 \otimes \phi_2$	$\phi_1(\sigma_1)\phi_2(\sigma_2) + \phi_1(\sigma_2)\phi_2(\sigma_1)^*$	0

Table 5. Character table for H_0:

	Id
Id	1

Table 6. Character table for H_1:

	$C_1 := (\mathrm{Id}, \mathrm{Id})^1$	$C_2 := (\mathrm{Id}, \mathrm{Id})^{-1}$
$\phi_1 := \mathrm{Id}^+$	1	1
$\phi_2 := \mathrm{Id}^-$	1	-1

Table 7. Character table for H_2:

	$(C_1, C_1)^1$	$(C_2, C_2)^1$	$(C_1, C_2)^1$	$(\mathrm{Id}, C_1)^{-1}$	$(\mathrm{Id}, C_2)^{-1}$
ϕ_1^+	1	1	1	1	1
ϕ_2^+	1	1	-1	1	-1
ϕ_1^-	1	1	1	-1	-1
ϕ_2^-	1	1	-1	-1	1
$\mathrm{Ind}(\phi_1, \phi_2)$	2	-2	0	0	0

Table 8. Character table for H_3:

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	−1	1	−1	−1	1	−1	−1	1	−1	1	1	−1	1	−1
1	1	1	1	1	1	1	−1	−1	1	−1	−1	−1	−1	1	1	1	1	−1	−1
1	1	1	1	1	1	−1	−1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1
4	4	4	0	0	−4	0	0	0	0	0	0	0	0	0	2	-2	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1
1	1	1	1	1	1	-1	1	-1	-1	1	-1	-1	1	-1	−1	−1	1	−1	1
1	1	1	1	1	1	1	-1	-1	1	-1	-1	-1	-1	1	−1	−1	−1	1	1
1	1	1	1	1	1	-1	-1	1	-1	-1	1	1	-1	-1	−1	−1	1	1	−1
4	4	4	0	0	-4	0	0	0	0	0	0	0	0	0	-2	2	0	0	0
2	2	-2	2	-2	2	0	2	0	0	2	0	0	-2	0	0	0	0	0	0
2	2	2	-2	-2	2	2	0	0	2	0	0	0	0	-2	0	0	0	0	0
2	2	-2	-2	2	2	0	0	2	0	0	2	-2	0	0	0	0	0	0	0
-4	-4	0	0	0	0	2	2	2	2	-2	-2	0	0	0	0	0	0	0	0
2	2	-2	-2	2	2	0	0	-2	0	0	-2	2	0	0	0	0	0	0	0
2	2	2	-2	-2	2	-2	0	0	-2	0	0	0	0	2	0	0	0	0	0
4	-4	0	0	0	0	-2	2	-2	2	-2	2	0	0	0	0	0	0	0	0
2	2	-2	-2	-2	2	0	-2	0	0	-2	0	0	2	0	0	0	0	0	0
4	-4	0	0	0	0	2	-2	-2	-2	2	2	0	0	0	0	0	0	0	0
4	-4	0	0	0	0	-2	-2	2	2	2	-2	0	0	0	0	0	0	0	0

5. The Bratteli diagram of P_n:

We now turn to the branching graph for the family $\{P_n\}$. In this section we introduce a bijection between irreducible representations of P_n and forests of binary trees. We use these objects to develop a description of the Bratteli diagram that is recursive and self-similar. Of particular interest are one-dimensional irreducible characters due to their bijective correspondence with odd dimensional representations of S_n. Bijections have been developed in [4]. Here we compare the subgraph of odd-dimensional representations of S_n to the subgraph of one-dimensional representations of P_n.

Theorem 5.1. *The restriction of an irreducible representation of H_k to P_{2^k-1} is multiplicity free.*

Proof. Consider $H_k \supset H_{k-1} \times H_{k-1} \supset H_{k-1} \times H_{k-2} \cdots \times H_0 \times H_0 \cong P_{2^k-1}$. Thus $\mathrm{Res}^{H_k}_{P_{2^k-1}} = \mathrm{Res}^{H_{k-1}\times H_{k-1}}_{P_{2^k-1}}(\mathrm{Res}^{H_k}_{H_{k-1}\times H_{k-1}})$. For an irreducible representation $\phi_1 \otimes \phi_2$ of $H_{k-1} \times H_{k-1}$, $\mathrm{Res}^{H_{k-1}\times H_{k-1}}_{P_{2^k-1}}(\phi_1 \otimes \phi_2) = \phi_1 \otimes \mathrm{Res}^{H_{k-1}}_{P_{2^{k-1}-1}}(\phi_2)$. This combined with

$$\mathrm{Res}^{H_k}_{H_{k-1}\times H_{k-1}}(\Gamma) = \begin{cases} \phi \otimes \phi & \Gamma = \mathrm{Ext}^{\pm}(\gamma) \\ \phi_1 \otimes \phi_2 + \phi_2 \otimes \phi_1 & \Gamma = \mathrm{Ind}(\phi_1, \phi_2) \end{cases}$$

proves that each summand occuring in the restriction is distinct. $\square$

Definition 5.1. *The up-set of an irreducible representation Γ of P_n, denoted Γ^+, is the set of irreducible representations of P_{n+1} that occur in the induction of Γ to P_{n+1}.*

Definition 5.2. *The down-set of an irreducible representation Γ of P_n, denoted Γ^-, is the set of irreducible representations of P_{n-1} that occur in the restriction of Γ to P_{n-1}.*

We now introduce 1-2 binary trees as the combinatorial object used to describe the Bratteli diagram. The motivation comes from [1], which uses the generating function from Equation 3 to count these objects.

Definition 5.3. *A 1-2 binary tree is defined recursively as either the trivial tree- $\emptyset$, or a tuple (r, S), where r is called the root vertex, and S is a multiset of either one or two 1-2 binary trees of the same height.*

The definition of the subtrees as a set is to indicate that we 'forget' their positions as left or right subtrees. Since S comprises trees of the same height, we may define the height of the tree recursively as one more than the common height of trees in S(the trivial tree is said to have height 0). This notion of height is equivalent to that defined for complete binary trees in Section 1. Set S may comprise two distinct trees, two identical trees or a single tree. We define a bijection that maps these to representations of the type $\mathrm{Ind}(\phi_1, \phi_2)$, $\mathrm{Ext}^+(\phi)$ and $\mathrm{Ext}^-(\phi)$ respectively.

$$\theta_k(\Gamma) = \begin{cases} (r, \{\theta_{k-1}(\phi_1), \theta_{k-1}(\phi_2)\}), & \Gamma = \mathrm{Ind}(\phi_1, \phi_2) \\ (r, \{\theta_{k-1}(\phi), \theta_{k-1}(\phi)\}), & \Gamma = \mathrm{Ext}^+(\phi) \\ (r, \{\theta_{k-1}(\phi)\}), & \Gamma = \mathrm{Ext}^-(\phi) \end{cases} \tag{10}$$

In fact we have defined a recursive family of bijections. Trees inherit the notion ot type from the representation they denote. Hence we may extend the notation Ind, Ext^+ and Ext^- from irreducible characters to trees Since irreducible characters of P_n are tensored products of representations of certain H_{k_i}, the bijection is naturally extended to such representations by introducing tuples of binary trees, which we call forests.

Definition 5.4. *A forest of n is a tuple of r 1-2 binary trees of heights $(k_r, \dots, k_1)$, where $k_r > \cdots > k_1$ are the binary digits of n.*

The bijection is now extended componentwise, as under:

With $\Gamma = \phi_1 \times \phi_2 \times \dots \phi_l$, where ϕ_j is an irreducible representation of H_{k_j}:

$$\theta_{[n]}(\Gamma) = (\theta_{k_l}(\phi_l), \theta_{k_{l-1}}(\phi_{l-1}), \dots, \theta_{k_0}(\phi_0)) \tag{11}$$

Given a tree T or a forest F, the up-set T^+(or F^+) and down-set T^-(or F^-) are defined as the images of the up-sets and down-sets(Definition 5.1 and Definition 5.2) of the representations corresponding to these trees(or forests).

Proposition 5.1. *With $T = (r, S)$ of height k:*

$$T^- = \begin{cases} T_1 \times (T_2)^- \sqcup T_2 \times (T_1)^- & S = \{T_1, T_2\} \\ T \times T^- & S = \{T, T\} \\ T \times T^- & S = \{T\} \end{cases}$$

Proof. We describe T^- through the effect on trees of the restriction defined in the proof of Theorem 5.1. In restricting a representation Γ of H_k to the subgroup $H_{k-1} \times H_{k-1}$ we obtain the set $S \times S$, where each tuple (T_1, T_2) is understood to denote the representation $\theta_{k-1}^{-1}(T_1) \otimes \theta_{k-1}^{-1}(T_2)$. This is the set of images of the summands occuring in the restriction of Γ.

The restriction to $P_{2^k - 1}$ then proceeds on each summand by restricting to $P_{2^{k-1}-1}$ along the second component of the tensor. It acts on a tuple of trees (T_1, T_2) by leaving unchanged the first component and replacing T_2 by a tuple in its downset. □

Definitions 5.1 and 5.2 are equivalent, in that if we have representations Γ and γ of H_k and $P_{2^k - 1}$ respectively, $\gamma \in \Gamma^-$ iff $\Gamma \in \gamma^+$. Thus we have:

Corollary 5.1.1. *With $F = T_1 \times \overline{F}$ a forest of $2^k - 1$:*

$$F^+ = \begin{cases} \{(r, \{T_1, T_2\}) | T_2 \in \overline{F}^+\} \cup \{(r, \{T_1\})\} & \text{if } T_1 \in (\overline{F})^+ \\ \{(r, \{T_1, T_2\}) | T \in \overline{F}^+\} & \text{otherwise} \end{cases}$$

We now have a complete characterisation of the branching between the levels 2^k and $2^k - 1$, recursively from the levels 2^{k-1} and 2^{k-1}. We now prove that the tree is self-similar at every level, and so that Proposition 5.1 and Corollary 5.1.1 provide sufficient information to construct the entire Bratteli diagram.

Proposition 5.2. *With k and n integers such that $n < 2^k$. Given a representation $F = T_1 \times \overline{F}$ of $2^k + n$, where T_1 is a tree of height k, and $\overline{F}$ is a forest of n. Then $F^- = T_1 \otimes \overline{F}^-$.*

Proof. The restriction $P_n \to P_{n-1}$ induces the restriction $H_k \times P_n \to H_k \times P_{n-1}$, such that its restriction to H_k is the identity map. A forest F is the image of an irreducible representation of P_{2^k+n}, where T_1 is the image of an irreducible H_k and $\overline{F}$ is the image of an irreducible representation of P_n. Under the specified restriction, the H_k representation is fixed while the representation of P_n is restricted to P_{n-1}. Thus the image set $F^- = \{(T_1, \overline{f}) | \overline{f} \in \overline{F}^-\}$ and the result follows.

□

The analogous condition for the up-set follows:

Corollary 5.1.2. *With k and n integers such that $n < 2^k$. Given a representation $f = T_1 \times \overline{f}$ of $2^k + n - 1$, where T_1 is a tree of height k, and $\overline{f}$ is a forest of $n - 1$. Then $f^+ = T_1 \otimes \overline{f}^+$.*

Proposition 5.2 and Corollary 5.1.2 may be applied repeatedly on an integer n to strip away powers of 2 from the binary digits of n to reduce the branching to that between H_{k_0} to $P_{2^{k_0}-1}$, where k_0 is the least significant binary digit of n.

For example consider the branching between P_{12} and P_{11}. The binary digits of n are $3, 2$. Thus $P_{12} = H_3 \times H_2$ and $P_{11} = H_3 \times H_1 \times H_0$. Applying Proposition 5.1 for $k = 3$, $n = 4$ gives that the branching between P_{12} and P_{11} looks 'locally' like the branching between H_2 and P_3. What is meant by locally is that for a given tree T of H_3, if all trees with T are collected at the levels 12 and 11, then the branching between these forests is a copy of the subgraph consisting of the 4th and 3rd levels of the diagram.

McKay's conjecture, proved true for $p = 2$ for symmetric groups in [3], says that the one-dimensional representations of P_n are equinumerous with odd dimensional representations of S_n. The subgraph of odd-dimensional representations is called the MacDonald tree in [2]. We describe the subgraph of one dimensional representations in the Bratteli diagram.

From 3, the dimension of the representation induced from $\phi_1 \otimes \phi_2$ is $2\dim(\phi_1)\dim(\phi_2)$; the dimension of the positive and negative extension of $\phi \otimes \phi$ is $\dim(\phi)^2$. Clearly the one dimensional irreducible representations of H_k are the positive and negative extensions of the one dimensional irreducible representations of H_{k-1}. The dimension of a forest of trees is the product of dimensions of the trees in this forest. Thus one dimensional forests are tuples of one dimensional trees.

As a consequence of this, there are 2^k one-dimensional representations of H_k, and $2^{k_0+\cdots+k_l}$ one-dimensional representations of P_n, where $\{k_l, \ldots, k_0\}$ are the binary digits of n. This indicates a way to represent such trees by strings of binary digits. First denote the unique one-dimensional representation of H_0 by the digit 1. Now given a one-dimensional representation of H_{k-1} represented by a binary string b of length $k - 1$, the binary strings for the positive and negative extensions are obtained by appending 1 and 0 respectively to the start of b. Henceforth the binary string for T is used interchangeably with the tree T.

With B_k being the set of binary strings of length k, let $L : B_k \to B_{k-1}$ truncate the leftmost bit from a binary string. Then the binary string of the unique subtree of a one-dimensional tree with binary string B is given by $L(B)$. Given such a tree T, $T^- = (L(B), L^2(B), \ldots, 1)$ from 5.1. Thus each one-dimensional tree has a unique ancestor. Applying Corollary 5.1.1 to a forest $F = T_1 \times \overline{F}$ of $2^k - 1$, an extension is possible if and only if $\overline{F} = (L(B), L^2(B), \ldots, 1)$, where B is the binary string for T_1. In this case there are two extensions, $0B$ and $1B$.

This discussion leads to a recursive description of the subgraph. Let O_k represent the subgraph of one-dimensional representations upto the level $2^k - 1$. At the highest level, each node is a one-dimensional forest of $2^k - 1$. For a one-dimensional tree T of height $k-1$, there is a unique forest of $2^k - 1$ with T as its first element that yields successors at the 2^k level, and such a node yields two successors. For each of these trees, its branching to the $2^k - 1$ level is a copy of O_k, by Proposition 5.2. Thus, to obtain O_{k+1} from O_k,

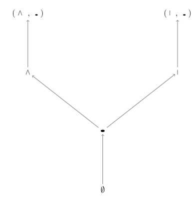
- Begin with a single copy of O_k.
- Choose 2^{k-1} nodes at the level $2^k - 1$ level of O_k, such that the unique path from these nodes downwards pairwise do not intersect at the 2^{k-1} level. This corresponds to a choice of distinct trees of height $k - 1$.
- To each such node attach two copies of O_k.

An example of this is provided in Figure 3. Here, O_3 is constructed from O_2.

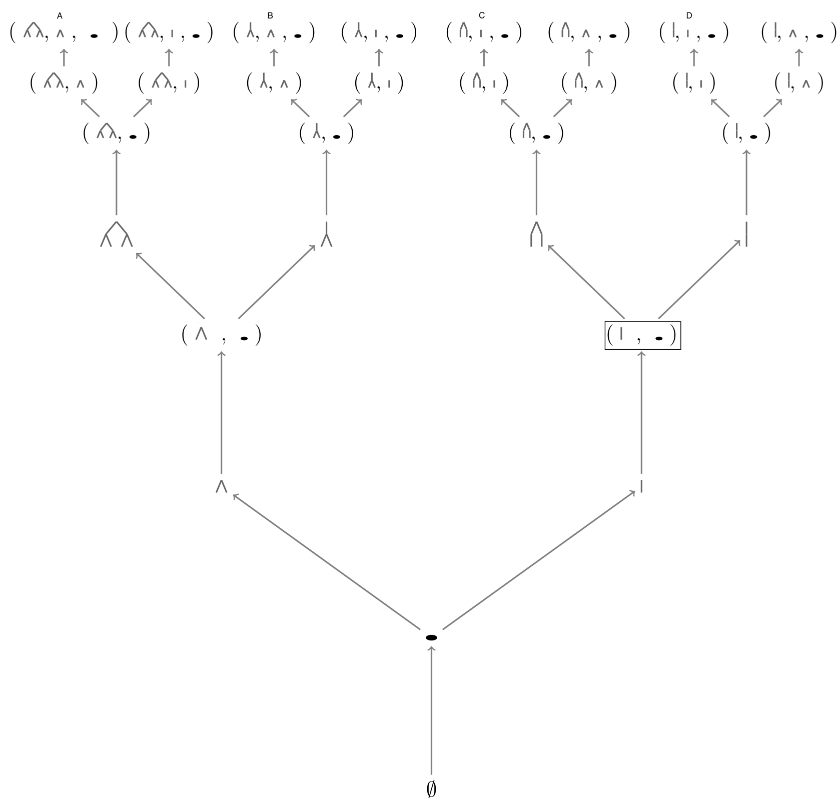
From this description it is clear that this tree is non-isomorphic to the MacDonald tree in [2]. For example, the subgraph considered here has infinitely many infinite rays, since each representation can be extended repeatedly in a canonical way. This means in particular that there cannot exist a bijection between odd-dimensional representations of S_n and one-dimensional representations of P_n that preserves the subgraph structure.

We refer to [2] the recursive description of the MacDonald tree. A portion of this tree has been reproduced in Figure 4.

A Sage implementation of the bijection between forests and irreducible representations(also conjugacy classes) can be found at https://github.com/sridharpn/2-Sylow.



O_2



O_3

FIGURE 3. O_3 is built recursively by attaching two copies of O_2 to appropriate nodes on the maximal level of O_2. Nodes on the highest level of O_3 that further propagate are labelled A-D

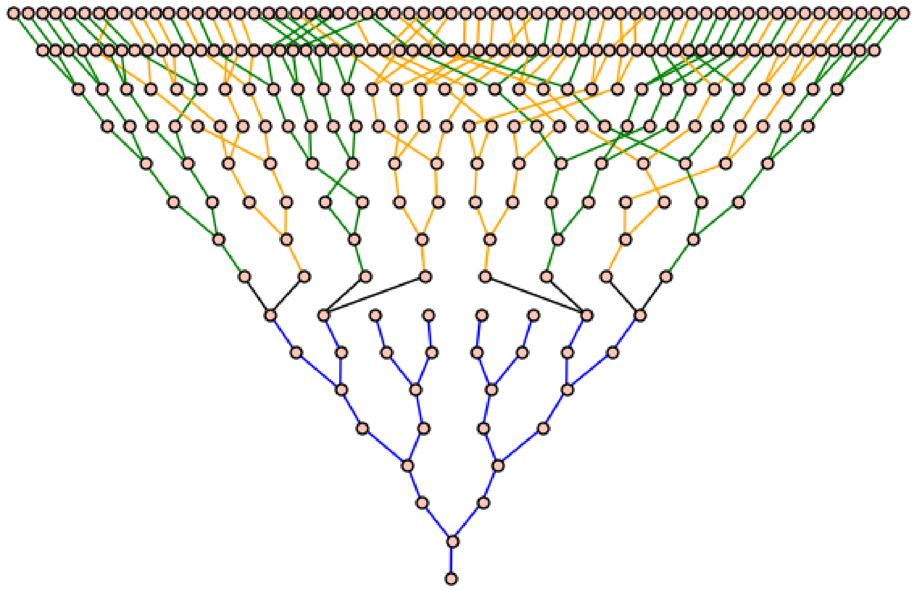


FIGURE 4. The MacDonald tree till $n = 15$

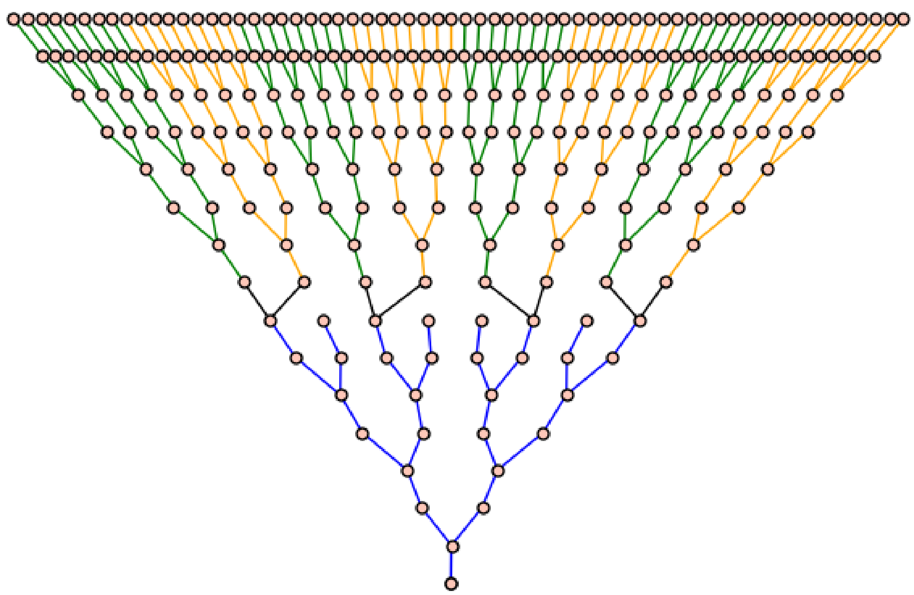


FIGURE 5. The subgraph of one-dimensional representations of P_n till n=15.

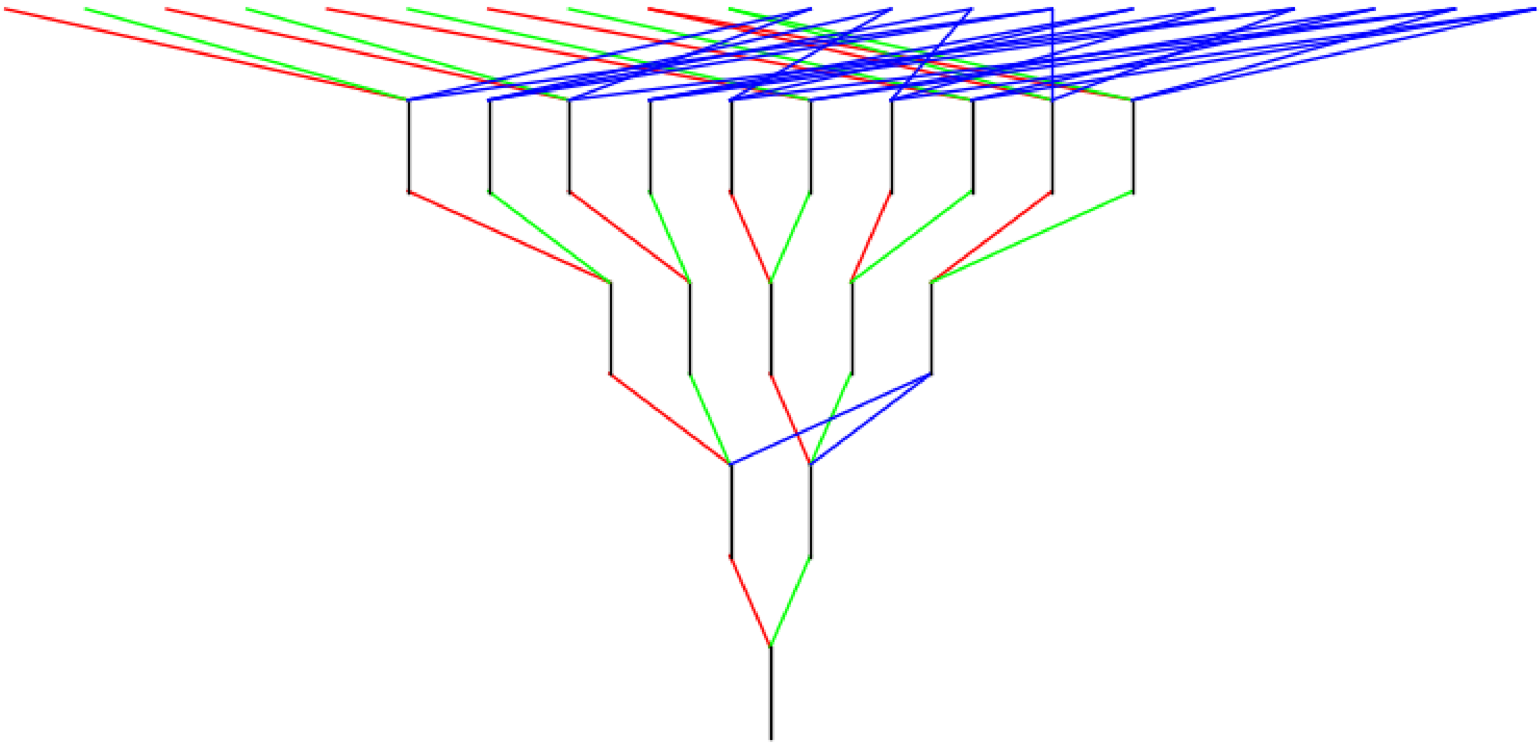


FIGURE 6. Branching of irreducible representations till $n = 8$. Green edges represent Ext^+ type representations, with red edges for Ext^- and blue for Ind type representations.

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