HOW THE KREWERAS TRIANGLE APPEARS IN THE UNIVERSAL sl_2 WEIGHT SYSTEM

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Abstract. The theory of finite order knot invariants applied to the Lie algebra sl_2 provides a weight system which maps the chord diagrams to polynomials in a single variable with integer coefficients. In this paper, we show that the Kreweras triangle, known to refine the normalized median Genocchi numbers, appears naturally in this weight system.

Notations

For all pair of integers $n < m$, the set $\{n, n+1, \ldots, m\}$ is denoted by $[n, m]$, and the set $[1, n]$ by $[n]$. The set of the permutations of $[n]$ is denoted by $\mathfrak{S}_n$. If two polynomials A and B of the ring $\mathbb{Z}[x]$ have the same congruence modulo $C \in \mathbb{Z}[x]$, then we write $A \overset{C}{\equiv} B$.

1. Introduction

1.1. **About the chord diagrams and the universal sl_2 weight system.** Let n be a positive integer. In the theory of finite order knot invariants (see [3, 14]), a chord diagram of order n, or n-chord diagram, is an oriented circle with $2n$ distinct points paired into n disjoint pairs named chords, considered up to orientation-preserving diffeomorphisms of the circle. It can be assimilated into a tuple $((p_i, p_i^*) : i \in [n])$ such that $\{p_1, p_1^*, p_2, p_2^*, \ldots\} = [2n]$ with $p_1 < p_2 < \ldots < p_n$ and $p_i < p_i^*$ for all i : for such a tuple, the corresponding chord diagram is obtained by labelling $2n$ points on a circle with the consecutive labels $1, 2, \ldots, 2n$ (following the counterclockwise direction), and pairing the points labelled with p_i and p_i^* for all $i \in [n]$. For example, the tuples $((1,3),(2,5),(4,6))$ and $((1,5),(2,4),(3,6))$ are two representations of the 3-chord diagram depicted in Figure 1.

A *weight system* is a function f on the chord diagrams that satisfies the 4-term relations depicted in Figure 2.

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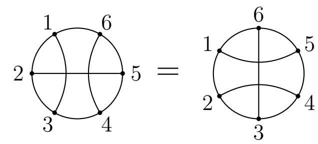


FIGURE 1. Two distinct labellings of a 3-chord diagram.

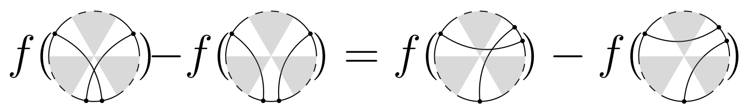


FIGURE 2. The 4-term relations.

The theory provides the construction of nontrivial weight systems from semisimple Lie algebras, among which the Lie algebra sl_2 of the 2×2 matrices whose trace is zero, which raises a weight system φ_{sl_2} mapping the n-chord diagrams to elements of $\mathbb{Z}[x]$ with degree n. With precisions, it gives birth to a family of weight systems $(\varphi_{\mathrm{sl}_2,\lambda})_{\lambda\in\mathbb{R}}$ related by the following equation for all $\lambda \in \mathbb{R}$ and for all n-chord diagram $\mathcal{D}$:

$$\lambda^n \varphi_{\mathrm{sl}_2,\lambda}(\mathcal{D})(x/\lambda) = \varphi_{\mathrm{sl}_2}(\mathcal{D})(x). \tag{1}$$

In the rest of this paper, we consider the weight system

$$\varphi = \varphi_{\mathrm{sl}_2,2}.$$

The following is a combinatorial definition of φ from [4].

Definition 1. Let $\mathcal{D}$ be an n-chord diagram. The weight $\varphi(\mathcal{D})$ is defined as x if $\mathcal{D}$ is the unique 1-chord diagram $\mathcal{D}_1 = ((1,2))$, otherwise $n \geq 2$ and $\varphi(\mathcal{D})$ is defined by the following inductive formula :

$$\varphi(\mathcal{D}) = (x-k)\varphi(\mathcal{D}_a) + \sum_{\{i,j\}\subset I_a} \Delta_{i,j}(\mathcal{D}_a) \tag{2}$$

where, if $\mathcal{D} = ((p_i, p_i^*) : i \in [n])$:

— a is any given chord (p_i, p_i^*) of D (it is then a nontrivial result that this definition does not depend on the choice of a);

— $\mathcal{D}_a$ is the $(n-1)$-chord diagram obtained from $\mathcal{D}$ by deleting the chord a;

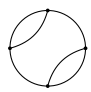
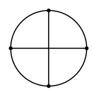
— $k = \#I_a$ where I_a is the set of the integers $i \in [n]$ such that the point p_i is located in the left half-plane defined by the support of a, and such that the chord (p_i, p_i^*) intersects a;

— for all $\{i,j\} \subset I_a$, $\Delta_{i,j}(\mathcal{D}_a) = \varphi(\mathcal{D}^1_{i,j}) - \varphi(\mathcal{D}^2_{i,j})$ where $\mathcal{D}^1_{i,j}$ (respectively $\mathcal{D}^2_{i,j}$) is the $(n-1)$-chord diagram obtained from

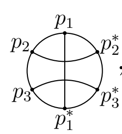
$\mathcal{D}_a$ by replacing the chords (p_i, p_i^*) and (p_j, p_j^*) by (p_i, p_j) and (p_i^*, p_j^*) (respectively by (p_i, p_j^*) and (p_j, p_i^*)).

Remark 2. For all n-chord diagram $\mathcal{D}$, it is straightforward, by induction on n, that $\varphi(\mathcal{D})$ is a polynomial with degree n and integer coefficients, and is divisible by x.

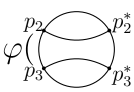
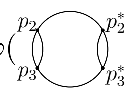
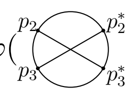
For example, there are two 2-chord diagrams :

 and ,

and their respective weights are x^2 and $(x-1)x$. To compute the weight of the 3-chord diagram $\mathcal{D}$ depicted hereafter

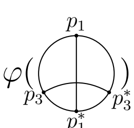


one can consider the chord $a = (p_1, p_1^*)$ to obtain

$\varphi(\mathcal{D}) = (x-2)\varphi($  $) + \varphi($  $) - \varphi($  $)$

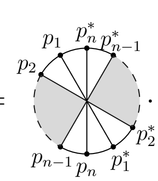
$= (x-2)x^2 + x^2 - (x-1)x = (x-1)^2x,$

though the choice of $a = (p_2, p_2^*)$ or $a = (p_3, p_3^*)$ provides a quicker computation :

$\varphi(\mathcal{D}) = (x-1)\varphi($  $)$

$= (x-1)^2x.$

Definition 3. For all $n \geq 1$, let $\mathcal{D}_n$ be the n-chord diagram where every chord intersects all the other chords, *i.e.*,

$\mathcal{D}_n = ((i, n+i) : i \in [n]) =$  .

We set $D_n = \varphi(\mathcal{D}_n) \in \mathbb{Z}[x]$.

The first elements of $(D_n)_{n\geq 1}$:

$$D_1 = x,$$

$$D_2 = (x-1)x \overset{x^2}{\equiv} -x,$$

$$D_3 = (x-2)(x-1)x \overset{x^2}{\equiv} 2x,$$

$$D_4 = (x-3)(x-2)(x-1)x + x^3 - (x-1)^2 x \overset{x^2}{\equiv} -7x.$$

Conjecture 4 (Lando,2016)**.** *For all $n \geq 1$,*

$$D_n \overset{x^2}{\equiv} (-1)^{n-1} h_{n-1} x$$

where $(h_n)_{n\geq 0} = (1, 1, 2, 7, 38, 295, \ldots)$ is the sequence of the normalized median Genocchi numbers [17]*, of which we give a reminder hereafter.*

1.2. **About the Genocchi numbers.** The Seidel triangle $(g_{i,j})_{1\leq j\leq i}$ [6] (see Figure 3) is defined by

$$g_{2p-1,j} = g_{2p-1,j-1} + g_{2p-2,j},$$
$$g_{2p,j} = g_{2p-1,j} + g_{2p,j+1},$$

with $g_{1,1} = 1$ and $g_{i,j} = 0$ if $i < j$.

⋮										⋰
5									155	…
4							17	17	155	…
3					3	3	17	34	138	…
2			1	1	3	6	14	48	104	…
1	1	1	1	2	2	8	8	56	56	…
j/i	1	2	3	4	5	6	7	8	9	…

FIGURE 3. The Seidel triangle.

The Genocchi numbers $(G_{2n})_{n\geq 1} = (1, 1, 3, 17, 155, 2073, \ldots)$ [15] and the median Genocchi numbers $(H_{2n+1})_{n\geq 0} = (1, 2, 8, 56, 608, \ldots)$ [16] can be defined as the positive integers $G_{2n} = g_{2n-1,n}$ and $H_{2n+1} = g_{2n+2,1}$ [6]. It is well known that H_{2n+1} is divisible by 2^n for all $n \geq 0$ [1]. The normalized median Genocchi numbers $(h_n)_{n\geq 0} = (1, 1, 2, 7, 38, 295, \ldots)$ are the positive integers defined by

$$h_n = H_{2n+1}/2^n.$$

Remark 5. In view of Formula (1) with $\lambda = 2$, Conjecture 4 is equivalent to

$$\varphi_{\mathrm{sl}_2}(\mathcal{D}_n) \overset{x^2}{\equiv} (-1)^{n-1} H_{2n-1} x$$

for all $n \geq 1$.

There exist many combinatorial models of the different kinds of Genocchi numbers. Here, for all $n \geq 0$, we consider :

— the set $PD2_n$ of the Dumont permutations of the second kind, that is, the permutations $\sigma \in \mathfrak{S}_{2n+2}$ such that $\sigma(2i-1) > 2i-1$ and $\sigma(2i) < 2i$ for all $i \in [n+1]$;

— the subset $PD2N_n \subset PD2_n$ of the normalized such permutations, defined as the $\sigma \in PD2_n$ such that $\sigma^{-1}(2i) < \sigma^{-1}(2i+1)$ for all $i \in [n]$.

It is known that $H_{2n+1} = \#PD2_n$ [5] and $h_n = \#PD2N_n$ [12, 8].

Kreweras [12] refined the integers h_n through the Kreweras triangle $(h_{n,k})_{n \geq 1, k \in [n]}$ (see Figure 4) defined by $h_{1,1} = 1$ and, for all $n \geq 2$ and $k \in [3, n]$,

$$\begin{aligned} h_{n,1} &= h_{n-1,1} + h_{n-1,2} + \ldots + h_{n-1,n-1}, \\ h_{n,2} &= 2h_{n,1} - h_{n-1,1}, \\ h_{n,k} &= 2h_{n,k-1} - h_{n,k-2} - h_{n-1,k-1} - h_{n-1,k-2}. \end{aligned} \tag{3}$$

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 2 & & 3 & & 2 & & & \\ & & 7 & & 12 & & 12 & & 7 & & \\ & 38 & & 69 & & 81 & & 69 & & 38 & \\ 295 & & 552 & & 702 & & 702 & & 552 & & 295 \end{array}$$

FIGURE 4. The six first lines of the Kreweras triangle.

It is easy to see that for all $n \geq 0$, the set $PD2N_n$ has the partition $\{PD2N_{n,k}\}_{k \in [n]}$ where $PD2N_{n,k}$ is the set of the $\sigma \in PD2N_n$ such that $\sigma(1) = 2k$. Kreweras and Barraud [13] proved that for all $n \geq 1$ and $k \in [n]$, the integer $h_{n,k}$ is the cardinality of $PD2N_{n,k}$. In particular, for all $n \geq 1$,

$$h_{n,1} = \sum_{i=1}^{n-1} h_{n-1,i} = h_{n-1}. \tag{4}$$

A visible property of the Kreweras triangle is the symmetry

$$h_{n,k} = h_{n,n-k+1} \tag{5}$$

for all $n \geq 1$ and $k \in [n]$. We can prove it combinatorially [13, 2], or directly from System (3), by first establishing the following easy result.

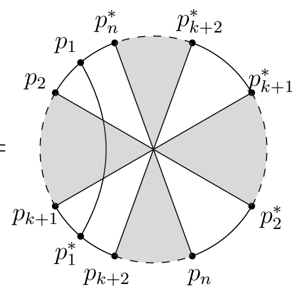
Proposition 6. *For all $n \geq 1$ and $k \in [n]$, we have*

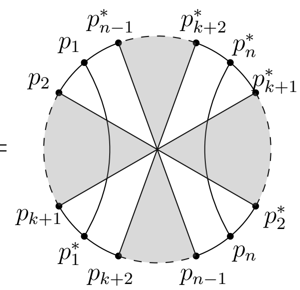
$$h_{n,k} - h_{n,k-1} = \sum_{i=k}^{n-1} h_{n-1,i} - \sum_{i=1}^{k-2} h_{n-1,i}$$

(where $h_{n,0}$ is defined as 0).

1.3. The Kreweras triangle in the universal $\mathfrak{sl}_2$ weight system.

Definition 7. Let $n \geq 1$ and $k \in [0, n-1]$. We define two n-chords diagrams $\mathcal{A}_{n,k}$ and $\mathcal{B}_{n,k}$ as follows.

$\mathcal{A}_{n,k} =$ 

$\mathcal{B}_{n,k} =$ 

We then define two polynomials $A_{n,k} = \varphi(\mathcal{A}_{n,k})$ and $B_{n,k} = \varphi(\mathcal{B}_{n,k})$. Note that :

- the chord (p_1, p_1^*) of $\mathcal{A}_{n,k}$ or $\mathcal{B}_{n,k}$ (and the chord (p_n, p_n^*) of $\mathcal{B}_{n,k}$) intersects exactly k chords;
- for all $n \geq 1$, $A_{n,0} = xD_{n-1}$ (where D_0 is defined as the polynomial 1), and $B_{n,0} = x^2 D_{n-2}$ for all $n \geq 2$, in particular their congruence modulo x^2 is 0 in view of Remark 2;
- $A_{n,n-1} = B_{n,n-1} = D_n$ for all $n \geq 1$.

We also set $A_{n,-1} = B_{n,-1} = 0$.

Remark 8. For all $1 \leq i < j \leq n$, it is straightforward that

$$\Delta_{i,j}(\mathcal{D}_n) = B_{n,j-i-1} - B_{n,n-1-(j-i)}.$$

Theorem 9. *For all $n \geq 1$ and $k \in [0, n-1]$, we have*

$$(6_{n,k}) \qquad A_{n,k} \overset{x^2}{\equiv} (-1)^{n-1} \left(\sum_{i=1}^{k} h_{n-1,i} \right) x,$$

$$(7_{n,k}) \qquad B_{n,n-k-1} - B_{n,k-1} \overset{x^2}{\equiv} (-1)^{n-1} (h_{n,k+1} - h_{n,k}) x$$

where $h_{n,0}$ and $h_{0,1}$ are defined as 0.

In particular, either Formula $(6_{n,k})$ or Formula $(7_{n,k})$ proves Conjecture 4 in view of $D_n = A_{n,n-1} = B_{n,n-1} - B_{n,-1}$ and Equality (4).

Section 2 is dedicated to the proof of Theorem 9.

In Section 3, we discuss open problems related to it, among which a more general conjecture from Lando.

2. Proof of Theorem 9

We already know that Theorem 9 is true if $n = 1$ and $k = 0$. Assume that it is true for some $n \geq 1$ and for all $k \in [0, n-1]$.

Lemma 10. *For all* $k \in [n-1]$*,*

$$A_{n,k-1} + A_{n,n-k} \overset{x^2}{\equiv} A_{n,n-1} = D_n.$$

Proof. By hypothesis,

$$A_{n,k-1} + A_{n,n-k} \overset{x^2}{\equiv} (-1)^n \left(\sum_{i=1}^{k-1} h_{n-1,i} + \sum_{i=1}^{n-k} h_{n-1,i} \right) x,$$

$$A_{n,n-1} \overset{x^2}{\equiv} (-1)^n \left(\sum_{i=1}^{n-1} h_{n-1,i} \right) x,$$

so the lemma follows from Formula (5). □

Lemma 11. *For all* $k \in [0, n]$*,*

$$A_{n+1,k} - A_{n+1,k-1} \overset{x^2}{\equiv} \sum_{i=1}^{k} B_{n,i-2} - B_{n,n-i}.$$

Proof. For all $k \in [0, n]$, from Definition 1 (with $\mathcal{D} = \mathcal{A}_{n+1,k}$ and $a = (p_1, p_1^*)$) and Remark 8, we have the congruence

$$A_{n+1,k} \overset{x^2}{\equiv} -kD_n + \sum_{2 \leq i < j \leq k+1} B_{n,j-i-1} - B_{n,n-1-(j-i)}$$

from which the lemma follows in view of $-D_n = B_{n,-1} - B_{n,n-1}$. □

Lemma 11 and the assumption that Formula $(7_{n,k})$ is true for all $k \in [0, n-1]$ imply Formula $(6_{n+1,k})$ for all $k \in [0, n]$, and also Formula $(7_{n+1,0})$ in view of $B_{n+1,n} - B_{n+1,-1} = A_{n+1,n}$. It remains to prove Formula $(7_{n+1,k})$ for all $k \in [n]$.

Definition 12. For all n-chord diagram $\mathcal{D}$ and for all quadruplet of integers (a, b, c, d) such that $1 \leq a \leq b < c \leq d \leq n$, we define two polynomials

$$T_{a,b}(\mathcal{D}) = \sum_{a \leq s < t \leq b} \Delta_{s,t}(\mathcal{D}),$$

$$R_{a,b,c,d}(\mathcal{D}) = \sum_{s=a}^{b} \sum_{t=c}^{d} \Delta_{s,t}(\mathcal{D}).$$

They are related by the equality

$$T_{a,c}(\mathcal{D}) = T_{a,b}(\mathcal{D}) + T_{b+1,c}(\mathcal{D}) + R_{a,b,b+1,c}(\mathcal{D}). \tag{8}$$

Lemma 13. *For all $l \in [0, n-1]$,*

$$B_{n+1,l} \overset{x^2}{\equiv} -lA_{n,l} + T_{2,l+1}(\mathcal{A}_{n,l}), \tag{9}$$

$$A_{n+1,l} \overset{x^2}{\equiv} -(n-1)A_{n,l} + T_{2,n}(\mathcal{A}_{n,l}), \tag{10}$$

$$A_{n+1,l} \overset{x^2}{\equiv} -lD_n + \sum_{2\le i<j\le l+1} B_{n,j-i-1} - B_{n,n-1-(j-i)}. \tag{11}$$

Proof. By applying Definition 1 on $\mathcal{D} = \mathcal{B}_{n+1,k}$ with $a = (p_1, p_1^*)$ or $a = (p_{n+1}, p_{n+1}^*)$, we obtain Formula (9). By applying it on $\mathcal{D} = \mathcal{A}_{n+1,k}$ with $a = (p_{n+1}, p_{n+1}^*)$ (respectively $a = (p_1, p_1^*)$), we obtain Formula (10) (respectively Formula (11) in view of Remark 8). □

Lemma 14. *For all $k \in [n]$,*

$$A_{n+1,n-k} - B_{n+1,n-k} \overset{x^2}{\equiv} -(k-1)A_{n,n-k} + \sum_{j=n-k+2}^{n} \sum_{i=2}^{j-1} \Delta_{i,j}(\mathcal{A}_{n,n-k}).$$

Proof. It is an application of Formula (9) and Formula (10) with $l = n-k$. □

Lemma 15. *For all $k \in [n]$,*

$$\begin{aligned} A_{n+1,k-1} - B_{n+1,k-1} \overset{x^2}{\equiv} & -(k-1)A_{n,n-k} \\ & + \sum_{2\le i<j\le k} B_{n,j-i-1} - B_{n,n-1-(j-i)} - \Delta_{i,j}(\mathcal{A}_{n,k-1}). \end{aligned}$$

Proof. It is an application of Formula (9) and Formula (11) with $l = k-1$, in view of Lemma 10. □

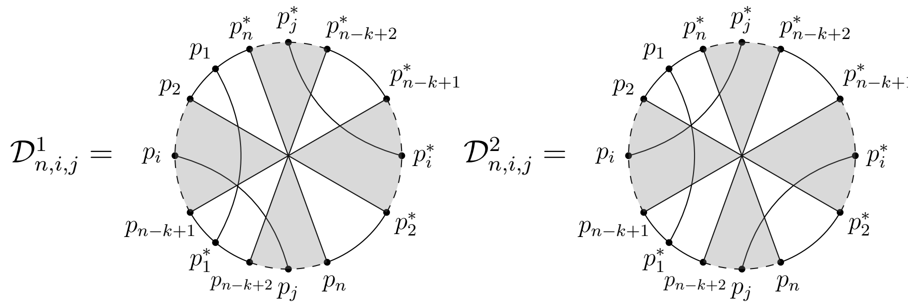
As we will see at the end of this section, the rest of the proof is to show that the polynomials in Lemma 14 and Lemma 15 are congruent modulo x^2, in other words, to obtain

$$\sum_{j=n-k+2}^{n} \sum_{i=2}^{j-1} \Delta_{i,j}(\mathcal{A}_{n,n-k}) \overset{x^2}{\equiv} \sum_{2\le i<j\le k} B_{n,j-i-1} - B_{n,n-1-(j-i)} - \Delta_{i,j}(\mathcal{A}_{n,k-1}). \tag{12}$$

Lemma 16. *For all $k \in [n]$, $j \in [n-k+2, n]$ and $i \in [2, n-k+1]$,*

$$\Delta_{i,j}(\mathcal{A}_{n,n-k}) + \Delta_{n-k+3-i,2n-k+2-j}(\mathcal{A}_{n,n-k}) = 0.$$

Proof. $\Delta_{i,j}(\mathcal{A}_{n,n-k}) = \varphi(\mathcal{D}^1_{n,i,j}) - \varphi(\mathcal{D}^2_{n,i,j})$ where



Now, if Σ is the axial symmetry that maps p_1 to p_1^*, it is easy to check that

$$\Sigma(\mathcal{D}^1_{n,i,j}) = \mathcal{D}^2_{n,n-k+3-i,2n-k+2-j}.$$

Moreover, from Definition 1, it is straightforward by induction on the order n of any chord diagram $\mathcal{D}$ that $\varphi(\Sigma(\mathcal{D})) = \varphi(\mathcal{D})$, thence the lemma. ☐

In view of Lemma 16, Formula (12) that we need to prove becomes (13)

$$\sum_{n-k+2\le i<j\le n} \Delta_{i,j}(\mathcal{A}_{n,n-k}) \overset{x^2}{\equiv} \sum_{2\le i<j\le k} B_{n,j-i-1} - B_{n,n-1-(j-i)} - \Delta_{i,j}(\mathcal{A}_{n,k-1}).$$

Lemma 17. *For all $2 \le i < j \le k \le n$,*

$$\begin{aligned} B_{n,j-i-1} \overset{x^2}{\equiv} & -(n-3)B_{n-1,j-i-1} \\ & + T_{2,n-2}(\mathcal{B}_{n-1,j-i-1}) \\ & - 2R_{2,k-i,k-i+1,n-2}(\mathcal{B}_{n-1,j-i-1}) \\ B_{n,n-1-(j-i)} \overset{x^2}{\equiv} & -(n-1)B_{n-1,n-2-(j-i)} \\ & + T_{1,n-1}(\mathcal{B}_{n-1,n-2-(j-i)}) \\ & - 2R_{1,k-j+1,k-j+2,n-1}(\mathcal{B}_{n-1,n-2-(j-i)}). \end{aligned}$$

Incidentally, the families of polynomials $R_{2,k-i,k-i+1,n-2}(\mathcal{B}_{n-1,j-i-1})$ and $R_{1,k-j+1,k-j+2,n-1}(\mathcal{B}_{n-1,n-2-(j-i)})$ do not depend on k.

Proof. By applying Definition 1 on $\mathcal{D} = \mathcal{B}_{n,j-i-1}$ and $a = (p_{k-i+1}, p^*_{k-i+1})$ (respectively on $\mathcal{D} = \mathcal{B}_{n,n-1-(j-i)}$ and $a = (p_{k-j+2}, p^*_{k-j+2})$), we obtain the two respective formulas

$$\begin{aligned}
B_{n,j-i-1} &\overset{x^2}{\equiv} -(n-3)B_{n-1,j-i-1} \\
&\quad + T_{2,k-i}(\mathcal{B}_{n-1,j-i-1}) + T_{k-i+1,n-2}(\mathcal{B}_{n-1,j-i-1}) \\
&\quad - R_{2,k-i,n-2}(\mathcal{B}_{n-1,j-i-1}), \\
B_{n,n-1-(j-i)} &\overset{x^2}{\equiv} -(n-1)B_{n-1,n-2-(j-i)} \\
&\quad + T_{1,k-j+1}(\mathcal{B}_{n-1,n-2-(j-i)}) + T_{k-j+2,n-1}(\mathcal{B}_{n-1,n-2-(j-i)}) \\
&\quad - R_{1,k-j+1,n-1}(\mathcal{B}_{n-1,n-2-(j-i)}),
\end{aligned}$$

and the equations of the lemma then follow from Formula (8). □

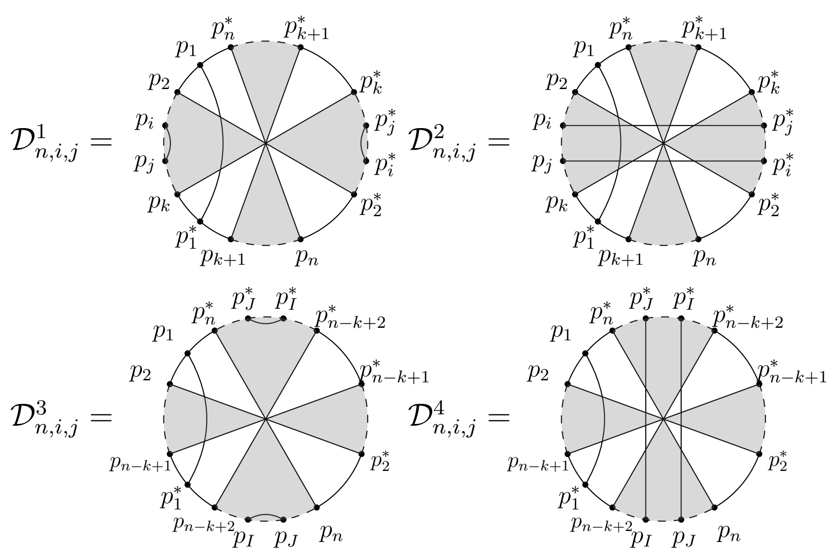
Lemma 18. *For all* $k \in [n]$ *and* $2 \leq i < j \leq k$,

$$\Delta_{n-k+i,n-k+j}(\mathcal{A}_{n,n-k}) = B_{n,j-i-1} - B_{n,n-1-(j-i)} - \Delta_{i,j}(\mathcal{A}_{n,k-1}).$$

Proof. Let $(I, J) = (n-k+i, n-k+j)$. We have

$$\begin{aligned}
\Delta_{i,j}(\mathcal{A}_{n,k-1}) &= \varphi(\mathcal{D}^1_{n,i,j}) - \varphi(\mathcal{D}^2_{n,i,j}), \\
\Delta_{I,J}(\mathcal{A}_{n,n-k}) &= \varphi(\mathcal{D}^3_{n,I,J}) - \varphi(\mathcal{D}^4_{n,I,J})
\end{aligned}$$

where



By applying Definition 1 with $a = (p_1, p_1^*)$, we obtain

$$\begin{aligned}
\varphi(\mathcal{D}^1_{n,i,j}) \overset{x^2}{\equiv} &- (k-3)B_{n-1,j-i-1} \\
&+ T_{2,k-i}(\mathcal{B}_{n-1,j-i-1}) + T_{n-i+1,n-2}(\mathcal{B}_{n-1,j-i-1}) \\
&- R_{2,k-i,n-i+1,n-2}(\mathcal{B}_{n-1,j-i-1}), \\
\varphi(\mathcal{D}^2_{n,i,j}) \overset{x^2}{\equiv} &- (k-1)B_{n-1,n-2-(j-i)} \\
&+ T_{1,k-j+1}(\mathcal{B}_{n-1,n-2-(j-i)}) + T_{n-j+2,n-1}(\mathcal{B}_{n-1,n-2-(j-i)}) \\
&- R_{1,k-j+1,n-j+2,n-1}(\mathcal{B}_{n-1,n-2-(j-i)}), \\
\varphi(\mathcal{D}^3_{n,i,j}) \overset{x^2}{\equiv} &- (n-k)B_{n-1,j-i-1} + T_{k-i+1,n-i}(\mathcal{B}_{n-1,j-i-1}), \\
\varphi(\mathcal{D}^4_{n,i,j}) \overset{x^2}{\equiv} &- (n-k)B_{n-1,n-2-(j-i)} + T_{k-j+2,n-j+1}(\mathcal{B}_{n-1,n-2-(j-i)}).
\end{aligned}$$

It is then a consquence of Formula (8) and Lemma 17 that

$$\begin{aligned}
\varphi(\mathcal{D}^1_{n,i,j}) + \varphi(\mathcal{D}^3_{n,i,j}) &= B_{n,j-i-1}, \\
\varphi(\mathcal{D}^2_{n,i,j}) + \varphi(\mathcal{D}^4_{n,i,j}) &= B_{n,n-1-(j-i)},
\end{aligned}$$

in view of

$$\begin{aligned}
R_{2,k-i,k-i+1,n-i} + R_{2,k-i,n-i+1,n-2} &= R_{2,k-i,k-i+1,n-2}, \\
R_{1,k-j+1,k-j+2,n-j+1} + R_{1,k-j+1,n-j+2,n-1} &= R_{1,k-j+1,k-j+2,n-1}.
\end{aligned}$$

This proves the lemma. □

Lemma 18 proves Formula 13. In other words, the results from Lemma 14 to Lemma 18 imply that

$$B_{n+1,n-k} - B_{n+1,k-1} \overset{x^2}{\equiv} A_{n+1,n-k} - A_{n+1,k-1} \tag{14}$$

for all $k \in [n]$. Now, at this step we know that Formula $(6_{n+1,k})$ is true, so Formula 14 gives

$$B_{n+1,n-k} - B_{n+1,k-1} \overset{x^2}{\equiv} (-1)^n \left(\sum_{i=1}^{n-k} h_{n,i} - \sum_{i=1}^{k-1} h_{n,i} \right) x,$$

which, in view of Formula 5 and Proposition 6, proves Formula $(7_{n+1,k})$ for all $k \in [n]$, and ends the proof of Theorem 9.

3. Open problems

Conjecture 4 proved by Theorem 9 is a particular case of the following conjecture, as we explain afterwards.

Conjecture 19 (Lando,2016)**.** *The generating function* $\sum_{t\geq 0} D_n(x)t^n$ *has the continued fraction expansion*

$$\cfrac{1}{1-b_0(x)t-\cfrac{\lambda_1(x)t^2}{1-b_1(x)t-\cfrac{\lambda_2(x)t^2}{\ddots}}}$$

where $b_k(x) = x - k(k+1)$ *and* $\lambda_k(x) = -k^2x + \binom{k}{2}\binom{k+1}{2}$*.*

Following Flajolet's theory of continued fractions [10], recall that a Motzkin path of length $n \geq 0$ is a tuple $(p_0, \ldots, p_n) \in ([0,n] \times [0,n])^n$ such that $p_0 = (0,0)$, $p_n = (n,0)$ and $\overrightarrow{p_{i-1}p_i}$ equals either $(1,1)$ (we then say it is an *up step*), or $(1,0)$ (an *horizontal step*), or $(1,-1)$ (a *down step*), for all $i \in [n]$. Conjecture (19) is equivalent to

$$D_n(x) = \sum_{\gamma \in M_n} \omega_{b_\bullet(x),\lambda_\bullet(x)}(\gamma)$$

for all $n \geq 0$, where $\omega_{b_\bullet(x),\lambda_\bullet(x)}(\gamma)$ is the product of the weigths of the steps of $\gamma \in M_n$, where an up step is weighted by 1, an horizontal step from (x,y) to $(x+1,y)$ by $b_y(x)$, and a down step from (x,y) to $(x+1,y-1)$ by $\lambda_y(x)$.

Now, for all $n \geq 2$, if M'_n is the subset of the paths $\gamma = (p_0, \ldots, p_n) \in M_n$ whose only points $p_i = (x_i, y_i)$ such that $y_i > 0$ are p_0 and p_n, then it is clear that

$$\begin{aligned}\sum_{\gamma \in M_n} \omega_{b_\bullet(x),\lambda_\bullet(x)}(\gamma) &\overset{x}{\equiv} \sum_{\gamma \in M'_n} \omega_{b_\bullet(x),\lambda_\bullet(x)}(\gamma),\\ &= -x \sum_{\gamma \in M_{n-2}} \omega_{b'_\bullet(x),\lambda'_\bullet(x)}(\gamma),\\ &\overset{x}{\equiv} -x \sum_{\gamma \in M_{n-2}} \omega_{\beta_\bullet,\Lambda_\bullet}(\gamma)\end{aligned}$$

where

$$\begin{aligned}b'_k(x) &= b_{k+1}(x) \overset{x}{\equiv} \beta_k = -(k+1)(k+2),\\ \lambda'_k(x) &= \lambda_{k+1}(x) \overset{x}{\equiv} \Lambda_k = \binom{k+1}{2}\binom{k+2}{2}.\end{aligned}$$

Conjecture 4 is then a particular case of Conjecture 19 in that

$$\sum_{n\geq 0}(-1)^n h_{n+1}t^n = \cfrac{1}{1-\beta_0 t-\cfrac{\Lambda_1 t^2}{1-\beta_1 t-\cfrac{\Lambda_2 t^2}{\ddots}}},$$

which we can obtain by applying Lemma 20 hereafter on the following formula (see [11, 9]) :

$$\sum_{n\geq 0}(-1)^n h_n t^n = \cfrac{1}{1-\cfrac{-\binom{2}{2}t}{1-\cfrac{-\binom{2}{2}t}{1-\cfrac{-\binom{3}{2}t}{1-\cfrac{-\binom{3}{2}t}{1-\cfrac{-\binom{4}{2}t}{\ddots}}}}}}.$$

Lemma 20 (Dumont and Zeng [7])**.** *Let $(c_n)_{n\geq 0}$ be a sequence of complex numbers, then*

$$\cfrac{c_0}{1-\cfrac{c_1 t}{1-\cfrac{c_2 t}{\ddots}}} = c_0 + \cfrac{c_0 c_1 t}{1-(c_1+c_2)t-\cfrac{c_2 c_3 t^2}{1-(c_3+c_4)t-\cfrac{c_4 c_5 t^2}{\ddots}}}.$$

Another ambitious problem would be to extend the combinatorial interpretations provided by Theorem 9 to any chord diagram.

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References

[1] D. Barsky and D. Dumont. “Congruences pour les nombres de Genocchi de 2e espèce (French)”. In: *Study Group on Ultrametric Analysis* 34 (7th–8th years: 1979–1981), pp. 112–129.

[2] A. Bigeni. *Combinatorial interpretations of the Kreweras triangle in terms of subset tuples*. 2017. eprint: arXiv:1712.01929.

[3] S. Chmutov, S. Duzhin and J. Mostovoy. *Introduction to Vassiliev knot invariants*. Cambridge University Press, 2012.

[4] S. Chmutov and A. Varchenko. “Remarks on the Vassiliev knot invariants coming from $\mathfrak{sl}_2$”. Topology 36 (1997), no. 1, 153–178.

[5] D. Dumont. “Interprétations combinatoires des nombres de Genocchi”. In: *Duke Math.* J. 41 (1974), pp. 305–318.

[6] D. Dumont and G. Viennot. “A combinatorial interpretation of the Seidel generation of Genocchi numbers”. In: *Ann. Discrete Math.* 6 (1980), pp. 77–87.

[7] D. Dumont and J. Zeng. “Further results on the Euler and Genocchi numbers”. In: *Aequationes Math.* 47.1 (1994), pp. 31–42.

[8] E. Feigin. "Degenerate flag varieties and the median Genocchi numbers". In: *Math. Res. Lett.* 18.6 (2011), pp. 1163–1178.

[9] E. Feigin. "The median Genocchi numbers, q-analogues and continued fractions". In: *European J. Combin.* 33.8 (2012), pp. 1913–1918.

[10] P. Flajolet. "Combinatorial aspects of continued fractions". In: *Discrete Math.* 32.2 (1980), pp. 125–161.

[11] G.-N. Han and J. Zeng. "On a q-sequence that generalizes the median Genocchi numbers". In: *Ann. Sci. Math. Québec* 23.1 (1999), pp. 63–72.

[12] G. Kreweras. "Sur les permutations comptées par les nombres de Genocchi de première et deuxième espèce (French)". In: *European J. Combin.* 18.1 (1997), pp. 49–58.

[13] G. Kreweras and J. Barraud. "Anagrammes alternés". In: *European J. Combin.* 18.8 (1997), pp. 887–891.

[14] S. K. Lando and A. K. Zvonkin. *Graphs on surfaces and their applications.* Springer, 2004. Chap. 6.

[15] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/A110501` (2011).

[16] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/A005439` (2011).

[17] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/A000366` (2011).