

Signed counts of real simple rational functions

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We study the problem of counting real simple rational functions φ with prescribed ramification data (i.e. a particular class of oriented real Hurwitz numbers of genus 0). We introduce a signed count of such functions that is invariant under change of the branch locus, thus providing a lower bound for the actual count (which does depend on such change). We prove (non-)vanishing theorems for these signed counts and study their asymptotic growth when adding further simple branch points. The approach is based on [IZ16] which treats the polynomial case.

1 Introduction

A *simple rational function* of degree d is a function $\varphi : \mathbf{CP}^1 \rightarrow \mathbf{CP}^1$ which, in affine coordinates, has the form

$$\varphi(z) = \frac{\psi(z)}{z - p}$$

with $\psi \in \mathbf{C}[z]$, $\deg(\psi) = d$, $p \in \mathbf{C}$ and $\psi(p) \neq 0$. We call φ *real* if $\psi \in \mathbf{R}[z]$ and $p \in \mathbf{R}$, and *increasing* if the leading coefficient of ψ is positive. Two such functions are considered equivalent if they differ by linear coordinate change $z \mapsto \lambda z + \mu$, $\lambda, \mu \in \mathbf{R}$, $\lambda > 0$. A unique representative in each equivalence class of increasing functions is given by *normalized* functions, which are of the form $\varphi(z) = \psi(z)/z$, where $\psi(0) \neq 0$ and the leading coefficient of ψ is 1.

Let $b_1, \dots, b_k \in \mathbf{C}$ be the critical values (also called branch points) of φ and let $\Lambda_1, \dots, \Lambda_k$ be the corresponding ramification profiles. This means $\Lambda_j = (\Lambda_j^1 \geq \dots \geq \Lambda_j^{l_j})$ is the partition of d such that b_j has l_j preimages at which f locally takes the form $z \mapsto z^{\Lambda_j^i}$. We call $\{(b_j, \Lambda_j)\}_{j=1, \dots, k}$ the *ramification data* of φ .

Let us now fix k distinct *real* points $\mathcal{P} = (b_1, \dots, b_k)$, $b_j \in \mathbf{R}$ and k partitions of d $\Lambda = (\Lambda_1, \dots, \Lambda_k)$. We are interested in the *number of normalized real simple rational*

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functions f of degree d with ramification data $\{(b_j, \Lambda_j)\}_{j=1, \dots, k}$. We denote the set of such functions by $\mathcal{S}(\mathcal{P}, \mathbf{\Lambda})$. The number $|\mathcal{S}(\mathcal{P}, \mathbf{\Lambda})|$ does not depend on the position of the branch points, as long as we do not change their order of appearance on the real line. However, in general $|\mathcal{S}(\mathcal{P}, \mathbf{\Lambda})|$ is *not* invariant under a permutation of this order. To remedy the situation, we will define a sign $\varepsilon(\varphi) \in \{\pm 1\}$ for any $\varphi \in \mathcal{S}(\mathcal{P}, \mathbf{\Lambda})$ such that the following theorem holds.

Theorem 1.1 (Invariance theorem). *The number*

$$\sum_{\varphi \in \mathcal{S}(\mathcal{P}, \mathbf{\Lambda})} \varepsilon(\varphi)$$

does not depend on the position of the branch points $b_1, \dots, b_k \in \mathbf{R}$. We call it the S -number of $\mathbf{\Lambda}$ and denote it by $S(\mathbf{\Lambda})$.

Note that by definition $|S(\mathbf{\Lambda})|$ gives a lower bound for $|\mathcal{S}(\mathcal{P}, \mathbf{\Lambda})|$. It is therefore interesting to find criteria for $S(\mathbf{\Lambda}) \neq 0$ (equivalent to the existence of functions) or to prove statements about the asymptotic growth. To formulate our results, we need the following notation. For a partition $\Lambda = (\Lambda^1, \dots, \Lambda^t)$ of d , the *reduced* partition λ is the partition obtained from $(\Lambda^1 - 1, \dots, \Lambda^t - 1)$ after removing all zeros. A branch point is called *simple* if its reduced ramification profile is $\lambda = (1)$. Let us now fix k decreasing finite sequences $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k)$ and $m \in \mathbf{N}$. We set

$$d := m + \sum_{i,j} \lambda_j^i,$$

and let $\mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_{k+m})$ be the unique collection of partitions of d such that $\Lambda_1, \dots, \Lambda_k$ reduce to $\lambda_1, \dots, \lambda_k$ and $\Lambda_{k+1} = \dots = \Lambda_{k+m}$ reduce to (1) . Hence $S(\boldsymbol{\lambda}, m) := S(\mathbf{\Lambda})$ is the S -number of normalized simple rational functions with $k+m$ branch points, k of which have reduced ramification profiles $\lambda_1, \dots, \lambda_k$ while the remaining m branch points are simple. We collect these numbers in two generating series, separately for d odd and even:

$$F_{\boldsymbol{\lambda}}^{\text{odd}}(q) = \sum_{\substack{m=0 \\ d \text{ odd}}}^{\infty} S(\boldsymbol{\lambda}, m) \frac{q^m}{m!}, \quad F_{\boldsymbol{\lambda}}^{\text{even}}(q) = \sum_{\substack{m=0 \\ d \text{ even}}}^{\infty} S(\boldsymbol{\lambda}, m) \frac{q^m}{m!}.$$

Theorem 1.2 ((Non-)Vanishing theorem). *The generating series $F_{\boldsymbol{\lambda}}^{\text{odd}}(q)$ is not identically zero if and only if the following conditions hold:*

- *In each partition λ_j at most one odd number appears an odd number of times and at most one even number appears an odd number of times.*
- *There exists an even number of partitions λ_j having exactly one even number appearing an odd number of times.*

The generating series $F_{\boldsymbol{\lambda}}^{\text{even}}(q)$ is not identically zero if and only if the following conditions hold:

- *In each partition λ_j at most one odd number appears an odd number of times and at most one even number appears an odd number of times.*

Theorem 1.3 (Logarithmic growth). *Fix $\lambda = (\lambda_1, \dots, \lambda_k)$ and the parity of d such that the non-vanishing criteria in Theorem 1.2 are satisfied. Set $\pi \equiv d - \sum_{i,j} \lambda_j^i \pmod{2}$ the corresponding parity of m . Then the logarithmic growth of $S(\lambda, m)$ in m is given by*

$$\ln |S(\lambda, m)| \underset{m \equiv \pi \pmod{2}}{\overset{m \rightarrow \infty}{\sim}} m \ln(m).$$

These statements should be compared to [IZ16, Theorems 1,3,4,5] in the polynomial case. Since all the main theorems crucially depend on the definition of the signs $\varepsilon(\varphi)$, let us insert its definition here.

Definition 1.4. Let $\Sigma = (a_1, \dots, a_n)$ be a finite sequence of integers. A *disorder* of Σ is pair $i < j$ such that $a_i > a_j$. The number of disorders is denoted by $\text{dis}(\Sigma)$.

For any $a \in \mathbf{C}$ we denote by $R_\varphi(a) \in \mathbf{N}$ the ramification index of φ at a , i.e. the order of vanishing of $\varphi(z) - \varphi(a)$ for $a \neq p$, and $R_\varphi(p) = 1$.

Definition 1.5. Let φ be a real simple rational function with simple pole p and let b be a branch point of φ . We set Σ_b to be the sequence of ramification indices $R_\varphi(a)$ for $a \in (\varphi^{-1}(b) \cap \mathbf{R}) \sqcup \{p\}$, ordered according to their appearance on the real line. Let b_1, \dots, b_k denote the collection of branch points of φ . The *sign* of φ is

$$\varepsilon(\varphi) := (-1)^{\text{dis}(\varphi)} \quad \text{where } \text{dis}(\varphi) := \text{dis}(\Sigma_{b_1}) + \dots + \text{dis}(\Sigma_{b_k}).$$

Remark 1.6. The main difference with [IZ16] is that our definition considers the simple pole p as part of the preimage for any critical level. It will become clear in Section 3 that this is, under some assumptions, the only definition with a chance to satisfy to Theorem 1.1. We call the disorders involving p *pole disorders* and all other ones *level disorders*.

Let us give a brief outline of the paper. In Sections 2, 3 and 4 we closely follow the approach in [IZ16] to prove the Invariance Theorem 1.1 using dessins d'enfant. The main difference in these sections is that instead of trees we deal with graphs with a loop. Our presentation focuses on the difference to *loc. cit.* while being mostly self-contained. Sections 5, 7 and 8 deal with Theorems 1.2 and 1.3. Here, the differences to *loc. cit.* are more significant. In particular, in Section 6 we introduce the generating series of *broken alternations*, show that it obeys a certain differential equation and use this to express it in terms of the generating series of (ordinary) alternations. This is crucial for the extension of the ideas from *loc. cit.* to the case of simple rational functions.

The counts of simple rational functions under investigation here can be considered as *oriented* versions of *real Hurwitz numbers* as defined for example in [MR15; GMR16]. The approach to study these numbers via an invariant signed count is in the spirit of Welschinger invariants [Wel05; IKS04], even though the definition of Welschinger signs is of different flavor than Definition 1.5. Real Hurwitz numbers and their invariance/asymptotic properties have also been studied in the context of topological quantum field theories, matrix models, moduli spaces of real algebraic curves [AN06; GZ15; Or17], however, mostly in the context of *completely imaginary* configurations of branch points \mathcal{P} , in contrast to the *completely real* configurations $\mathcal{P} \subset \mathbf{RP}^1$ we consider here.

2 Dessins d'enfant for simple rational functions

Real simple rational functions can be described by dessins d'enfant of the following type. We fix d and a sequence $\mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_k)$ of partitions of d such that $d = \sum_{i,j} (\Lambda_j^i - 1)$. Note that this implies $k > 1$. For a partition Λ we denote by $l(\Lambda)$ the number of parts of Λ . For the vertex v of a graph G , its *degree* $\deg(v)$ is the number of adjacent halfedges.

Definition 2.1. A *real simple rational dessin* of degree d and type $\mathbf{\Lambda}$ is a graph $\Gamma \subset \mathbf{CP}^1$ whose vertices are labelled by elements of the set $\{1, \dots, k, \infty\}$ such that the following conditions hold.

- (a) The labelled graph Γ is invariant under complex conjugation conj .
- (b) The real circle \mathbf{RP}^1 is a union of edges of Γ .
- (c) Exactly two vertices of Γ are labelled by ∞ , namely $P := \infty \in \mathbf{CP}^1$ and $p \in \mathbf{R}$, with $\deg(P) = 2d - 2$ and $\deg(p) = 2$.
- (d) For each integer $1 \leq j \leq k$, the graph Γ has exactly $l(\Lambda_j)$ vertices labelled by j and their degrees are equal to the elements of Λ_j , multiplied by 2.
- (e) Each edge of Γ is one of the following $k + 1$ types: $\infty \rightarrow 1$, $1 \rightarrow 2$, $2 \rightarrow 3$, \dots , $k - 1 \rightarrow k$, $k \rightarrow \infty$; in particular, this induces an orientation on Γ .
- (f) For any connected component C of $\mathbf{CP}^1 \setminus \Gamma$, each type of edges appears exactly once in the boundary ∂C of C .
- (g) There exists an edge in \mathbf{RP}^1 of type $k \rightarrow \infty$ whose orientation agrees with that of \mathbf{RP}^1 .

Remark 2.2. Given a real dessin $\Gamma \subset \mathbf{CP}^1$, we call $\Gamma \setminus \{P\} \subset \mathbf{C}$ the *affine dessin* for Γ . It is a graph with $2d - 2$ unbounded ends of type $\infty \rightarrow 1$ or $k \rightarrow \infty$, and it is easy to adapt the above conditions into *affine* versions such that both contain exactly the same information. We will mostly use affine dessins in our figures. An example for $d = 21, k = 4$ is given in Figure 7. The vertex p is drawn in red.

It is easy to associate a real simple rational dessin Γ_φ to any increasing real simple rational function φ with branch points $b_1 < \dots < b_k \in \mathbf{R}$. As a set, we define $\Gamma_\varphi := \varphi^{-1}(\mathbf{RP}^1)$. We label the two preimages of $\infty \in \mathbf{CP}^1_{\text{target}}$ by ∞ and the preimages of b_j by j .

Definition 2.3. Two real simple rational dessins Γ, Γ' are *equivalent* if there exists a orientation preserving homeomorphism $\Phi : \mathbf{CP}^1 \rightarrow \mathbf{CP}^1$ such that Φ and conj commute, $\Phi(\Gamma) = \Gamma'$ and Φ preserves the labels. We denote the set of equivalence classes by $\text{Des}(\mathbf{\Lambda})$.

Theorem 2.4. Fix arbitrary real points $\mathcal{P} = \{b_1 < \dots < b_k\} \in \mathbf{RP}^1$. Then the map

$$\mathcal{S}(\mathcal{P}, \mathbf{\Lambda}) \rightarrow \text{Des}(\mathbf{\Lambda}), \quad \varphi \mapsto [\Gamma_\varphi],$$

is a bijection.

Proof. The proof is a straightforward application of dessins d'enfant and the Riemann existence theorem and can be easily adapted from [IZ16, Propositions 2.6 and 2.7]. ■

Remark 2.5. The sign of a simple rational function can be easily read of from its dessin Γ_φ . The sequence Σ_{b_j} of ramification indices for the branch point b_j can be described in terms of Γ_φ as the sequence of degrees of the vertices labelled by i and the pole p divided by 2 (cf. Definition 1.5).

3 Bipartite graphs

In this section, we introduce auxiliary combinatorial objects that are used in the proof of the invariance theorem. They are equivalent to simple rational dessins with $k = 2$ critical values.

Definition 3.1. A *black and white simple graph*, or short, *bw-graph* is a graph G embedded into \mathbf{C} whose vertices are colored in black and white in alternation, and it has only one cycle of edge length 2 (see Figure 1). A *bw-graph* is said to be *real* if it is invariant (including the colors) under complex conjugation. Two (real) *bw-graphs* are isomorphic if one can be transformed into the other by an (conj-equivariant) orientation preserving homeomorphism of \mathbf{C} .

For a real *bw-graph* G the *real part* is $\mathbf{R}G := G \cap \mathbf{R}$. The leftmost and rightmost vertices of $\mathbf{R}G$ are called the *border vertices*. The graph is called *white-sided* (respectively *black-sided*) if its rightmost border vertex is white (respectively, black). It is called *short* if the the cycle of G contains a border vertex. Otherwise, we call it *long*.

We denote by (Σ_w, Σ_b) the sequence of degrees of its real white and real black vertices, respectively, from left to right. A *level disorder* is a disorder in one of these sequences, while a *pole disorder* is real vertex of G to the left of the cycle and with degree larger than 1 (including the left cycle vertex). In this section, it is more convenient to keep track of level and pole disorders separately. We denote them by $\text{lev}(G)$ and $\text{pol}(G)$, respectively. The *sign* of G is

$$\varepsilon(G) = (-1)^{\text{lev}(G) + \text{pol}(G)}.$$

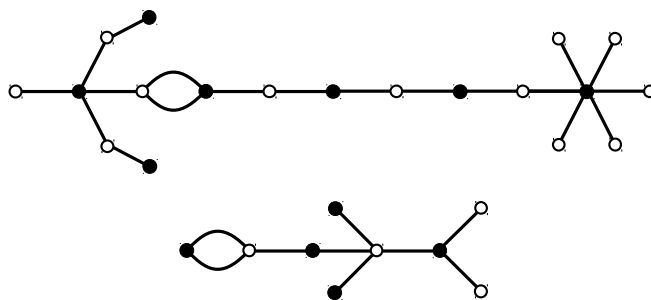


Figure 1 Two real *bw-graphs*.

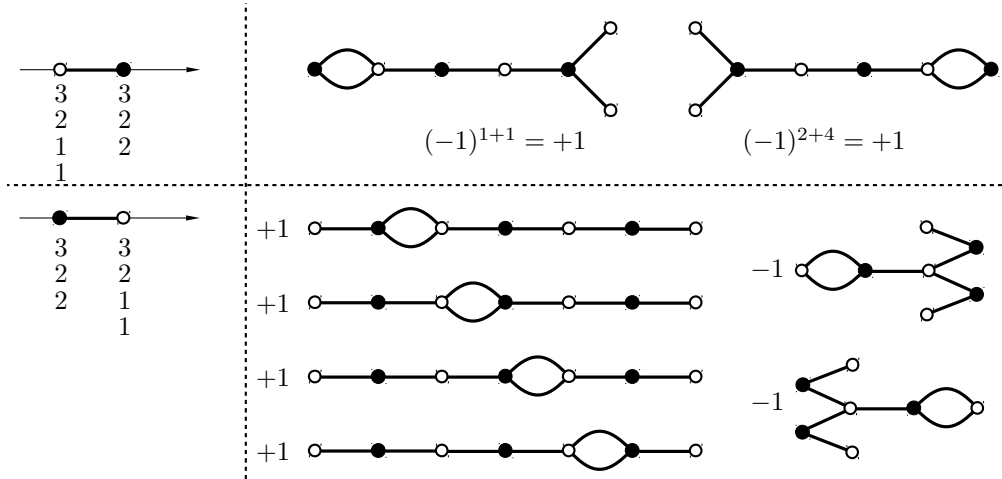


Figure 2 An example of $S(\mathbf{W}) = S(\mathbf{B})$ for $\Lambda_w = (3, 2, 1, 1)$ and $\Lambda_b = (3, 2, 2)$.

Example 3.2. The real bw -graph appearing at the top of Figure 1 is a white-sided long graph. It has $(\Sigma_w, \Sigma_b) = (132221, 43226)$ and its sign is equal to $(-1)^{12+2} = +1$. The real bw -graph G appearing at the bottom of Figure 1 is a black-sided short graph. It has $(\Sigma_w, \Sigma_b) = (34, 223)$ and its sign is equal to $(-1)^{0+1} = -1$.

Remark 3.3. A vertex of G is called *even* or *odd* if its degree $\deg(v)$ is even or odd, respectively. A real bw -graph G can have four, two or zero real odd vertices. Only the first case is equivalent to G being long. Note that in this case not all four vertices have the same color.

For the rest of this section, we fix a positive integer d , and two partitions Λ_w, Λ_b of d . We denote by \mathbf{G} the set of real bw -graphs such that the degrees of their white and black vertices give Λ_w and Λ_b , respectively. We split $\mathbf{G} = \mathbf{W} \sqcup \mathbf{B}$ into the subsets of white-sided and black-sided graphs, respectively.

Convention 3.4. Let M be a set together with a sign function $\varepsilon : M \rightarrow \{\pm 1\}$. We set

$$S(M) := \sum_{x \in M} \varepsilon(x).$$

Theorem 3.5. Fix a positive integer d and two partitions Λ_w and Λ_b of d . Then

$$S(\mathbf{W}) = S(\mathbf{B}).$$

Example 3.6. Figure 2 shows all graphs with $\Lambda_w = (3, 2, 1, 1)$ and $\Lambda_b = (3, 2, 2)$. The sum of signs is equal to 2 both for the black-sided graphs on top and for the white-sided graphs on the bottom.

3.1 Symmetrizing graphs

To prove Theorem 3.5, we subsequently identify subsets of graphs which cancel each other, hence reducing the problem to special classes of graphs. The first step is to symmetrize graphs in a certain sense.

Definition 3.7. Let v be a vertex of the graph G . A *pair of trees* T growing at v is a pair of complex conjugated connected components $T = (\Theta, \bar{\Theta})$ of $G \setminus \{v\}$ with $\Theta \neq \bar{\Theta}$. Let T_1, \dots, T_k be the sequence of all pairs of trees growing at v , ordered according to their appearance in $\{z : \Im(z) > 0\} \subset \mathbf{C}$ following the clock-wise direction (see Figure 3). Then the *forest* growing at v is the sequence $F(v) = (T_1, \dots, T_{k-1})$ if v is an odd border vertex and $F(v) = (T_1, \dots, T_k)$ otherwise.

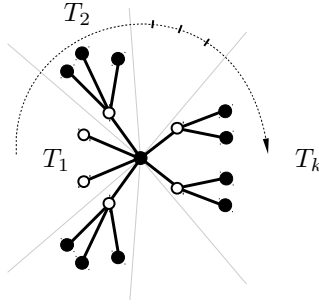


Figure 3 Ordering of the pairs of non-real trees growing at v .

Let k denote the number of pairs of trees growing at v . If v is even, then $\deg(v) = 2k + 2 = 2l(F(v)) + 2$. If v is odd, then $\deg(v) = 2k + 1$ or $\deg(v) = 2k + 3$ depending on whether v is a border vertex or not. Due to the shortening of $F(v)$ for border vertices we obtain $\deg(v) = 2l(F(v)) + 3$ for any odd vertex with $\deg(v) \neq 1$.

Definition 3.8. A collection of real vertices v_1, \dots, v_k of a given graph G are of the *same type* if they are of the same color, the same parity and $\deg(v_i) > 1$ for all i . Given a permutation σ of the v_i , we obtain a new graph $\sigma(G)$ by cutting off the forest $F(v_i)$ from v_i and replanting it at $v_{\sigma(v_i)}$, for all i . Hereby we keep the order of the individual pairs of trees in $F(v_i)$ and the position of the (fixed) right-most pair of trees in the case of a odd border vertex (see Figure 4). This determines $\sigma(G)$ uniquely up to homeomorphism. In the case of two vertices v_1, v_2 and σ the transposition, we call the result the *flipped graph* $\text{Flip}_{v_1}^{v_2}(G)$.

Note that in terms of the real sequences of G the operation $\text{Flip}_{v_1}^{v_2}$ acts like a transposition permuting the two entries corresponding to v_1 and v_2 .

Definition 3.9. Let $\Sigma = (e_1, \dots, e_{2m})$ be a integer sequence Σ of even length $2m$. Then Σ is *symmetric* if $e_{m-k} = e_{m+k+1}$ for all $k = 0, \dots, m-1$. If Σ is not symmetric, the *first non-symmetric pair* denotes the unique pair $(m-k, m+k+1)$ such that $e_{m-k} \neq e_{m+k+1}$ and $(e_{m-k+1}, \dots, e_{m+k})$ is symmetric.

A (general) integer sequence Σ is *nearly symmetric* if it is symmetric after applying the following two steps:

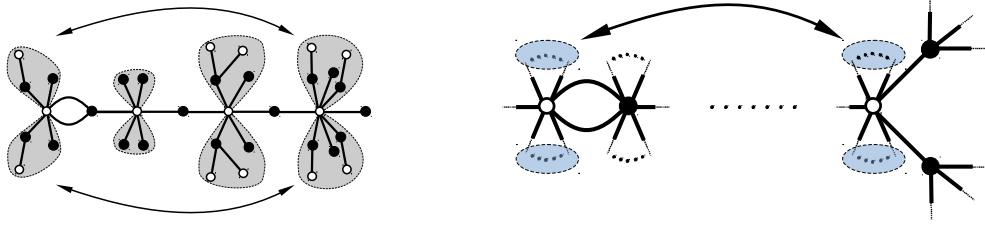


Figure 4 Exchanging forests for even and odd vertices, respectively.

- Remove all 1's.
- If the resulting sequence is of odd length, remove the first entry.

If Σ is not nearly symmetric then the *first non-symmetric pair* is the pair of indices which corresponds to the first non-symmetric pair after applying the same two steps. Given an integer sequence Σ , we denote by Σ^{even} and Σ^{odd} the subsequence of even and odd entries, respectively. A real *bw-graph* G is *nearly symmetric* if all the sequences $\Sigma_w^{\text{even}}, \Sigma_w^{\text{odd}}, \Sigma_b^{\text{even}}, \Sigma_b^{\text{odd}}$ are nearly symmetric separately. If G is not nearly symmetric, then the *first non-symmetric pair* is the pair of vertices corresponding to the first non-symmetric pair for the first sequence from before (in the given order) which is not nearly symmetric.

Remark 3.10. Let us unravel the previous definition in the given situation. Let Σ^{even} be a nearly symmetric sequence of only even entries, then either it is of the form $(e_1, \dots, e_m, e_m, \dots, e_1)$ or $(e_0, e_1, \dots, e_m, e_m, \dots, e_1)$. Let Σ^{odd} be a sequence of only odd entries and of length at most 3 and not of the form $(1, 1)$ or $(x, 1, y)$ (which cannot appear in practice). Then it is nearly symmetric if and only if it is of one of the following forms (where we assume $a \neq 1 \neq b$).

$$\emptyset, (a), (1), (a, a), (a, 1), (1, a), (1, a, 1), (a, a, 1), (1, a, a), (a, b, b)$$

If Σ^{odd} is not nearly symmetric, it is of the form $(\underline{c}, \underline{d}), (\underline{c}, \underline{d}, 1), (1, \underline{c}, \underline{d}), (a, \underline{c}, \underline{d})$ with $c \neq d$. The first non-symmetric pair is formed by the underlined entries.

Example 3.11. The top graph in Figure 1 is not nearly symmetric and its first non-symmetric pair is formed by the left- and right-most black vertex of degree 4 and 6, respectively. The bottom graph is nearly symmetric.

Lemma 3.12. *Let G be a graph which is not nearly symmetric and let $v_1 < v_2$ be the first non-symmetric pair of G . Then*

$$\varepsilon(\text{Flip}_{v_1}^{v_2}(G)) = -\varepsilon(G).$$

Proof. Let Σ be the sequence of degrees of some color and parity as v_1, v_2 . If $\deg v_1$ and $\deg v_2$ are consecutive entries of Σ , flipping the two obviously changes the number of disorders by 1. If not consecutive, they are separated by a stretch of the form

$(e_1, \dots, e_m, e_m, \dots, e_1)$. Such a stretch can be removed without changing the parity of $\text{dis}(\Sigma)$, since any disorder involving the stretch shows up an even number of times. Hence the previous case applies. The pole disorders are not affected by the flipping. ■

Proposition 3.13. *Consider the subsets $\mathbf{NW} \subset \mathbf{W}$ and $\mathbf{NB} \subset \mathbf{B}$ of nearly symmetric white-sided and black-sided graphs, respectively. Then we have*

$$S(\mathbf{W}) = S(\mathbf{NW}), \quad S(\mathbf{B}) = S(\mathbf{NB}). \quad (1)$$

Proof. We define the involution $\text{Sym} : \mathbf{W} \setminus \mathbf{NW} \rightarrow \mathbf{W} \setminus \mathbf{NW}$ by setting $\text{Sym}(G) = \text{Flip}_{v_1}^{v_2}(G)$, where $v_1 < v_2$ is the first non-symmetric pair of G . By Lemma 3.12 the pairs $G, \text{Sym}(G)$ cancel out in $S(\mathbf{W})$ and the statement follows. ■

3.2 Rotating and shifting vertices

The next step is based on rotating graphs by 180° . This operation reverses the real sequences. In order to stay in the class of nearly symmetric graphs, we need to make a slight adjustment.

Definition 3.14. Let $v_1 < \dots < v_k$ be a collection of ordered real vertices of the same type and let σ be the permutation which maps $v_i \mapsto v_{i+1}$, $v_k \mapsto v_1$. The *cyclic shift* of G based on $v_1 < \dots < v_k$ is the graph $\sigma(G)$ (see Definition 3.8 and Figure 5).

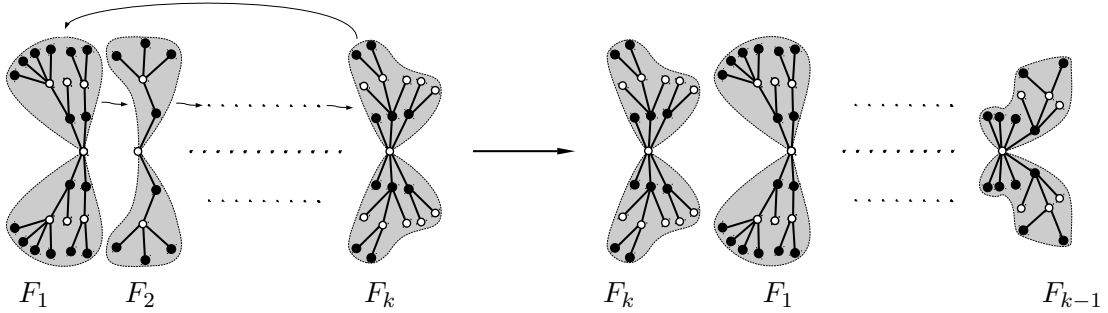


Figure 5 The cyclic shift operation.

Remark 3.15. Let Σ be a nearly symmetric sequence and let Σ' be the reversed sequence. Then either Σ' is nearly symmetric or can be made nearly symmetric by cyclically shifting all non-one entries of Σ' to the next non-one entry (keeping all 1's fixed). Obviously, if Σ' is part of the real sequence of a graph G , then the shifting, if necessary, can be performed by the cycled shifting operation based on the vertices corresponding to the non-one entries of Σ' .

Definition 3.16. We define the map $\text{Rot} : \mathbf{NG} \rightarrow \mathbf{NG}$, where $\text{Rot}(G)$ is the graph obtained from G by a rotation of 180° and, if necessary, performing cyclic shifts based on the higher-valent vertices of certain color and parity (see Remark 3.15).

Lemma 3.17. *Let G be a nearly symmetric graph and let Σ be its real sequence of white (or black) vertices. Let $\text{Rot}(\Sigma)$ denote the real sequence of white (or black) vertices of $\text{Rot}(G)$. Set $l := l(\Sigma^{\text{even}})$. Then we have*

$$\text{dis}(\text{Rot}(\Sigma)) - \text{dis}(\Sigma) \equiv \begin{cases} 0 & \Sigma^{\text{odd}} = \emptyset \text{ or } (a, a), \\ 1 & \Sigma^{\text{odd}} = (1, a) \text{ or } (a, 1), \\ l & \text{otherwise,} \end{cases} \pmod{2}.$$

Here we refer to the ten possible forms for Σ^{odd} introduced in Remark 3.10.

Proof. We set $\bar{\Sigma} = \text{Rot}(\Sigma)$. As in Lemma 3.12, throughout the proof we will use the fact that a consecutive pair of identical entries in Σ can be removed without changing the parity of $\text{dis}(\Sigma)$. We first focus on Σ^{odd} . Going through the ten types in Remark 3.10 we can easily check that $\text{dis}(\bar{\Sigma}^{\text{odd}}) - \text{dis}(\Sigma^{\text{odd}}) \equiv 1$ only if $\Sigma^{\text{odd}} = (1, a)$ or $(a, 1)$ and $\text{dis}(\bar{\Sigma}^{\text{odd}}) - \text{dis}(\Sigma^{\text{odd}}) \equiv 0$ otherwise.

We now reduce to this case by the following trick: Let k be the number of transposition necessary to transform Σ into $(\Sigma^{\text{odd}}, \Sigma^{\text{even}})$ (i.e. moving all odd entries to the beginning of the sequences). Since each of these transpositions flips an odd and even entry (i.e., two non-equal numbers), we get $\text{dis}(\Sigma) \equiv \text{dis}((\Sigma^{\text{odd}}, \Sigma^{\text{even}})) + k$. The pattern of odd and even entries in $\bar{\Sigma}$ is reversed to the pattern of Σ (the cyclic shifts only permute odd and even entries separately). Hence moving all odd entries in $\bar{\Sigma}$ to the right, we get $\text{dis}(\bar{\Sigma}) \equiv \text{dis}((\bar{\Sigma}^{\text{even}}, \bar{\Sigma}^{\text{odd}})) + k$. If we set $m := l(\Sigma^{\text{odd}})$ and move all the odd entries back to the left, we get $\text{dis}(\bar{\Sigma}) \equiv \text{dis}((\bar{\Sigma}^{\text{odd}}, \bar{\Sigma}^{\text{even}})) + k + lm$. Putting things together, we obtain

$$\begin{aligned} \text{dis}(\text{Rot}(\Sigma)) - \text{dis}(\Sigma) &\equiv \text{dis}((\bar{\Sigma}^{\text{odd}}, \bar{\Sigma}^{\text{even}})) - \text{dis}((\Sigma^{\text{odd}}, \Sigma^{\text{even}})) + lm \\ &\equiv \text{dis}(\bar{\Sigma}^{\text{odd}}) - \text{dis}(\Sigma^{\text{odd}}) + lm. \end{aligned} \quad (2)$$

Plugging in the possible types for Σ^{odd} , the claim follows. \blacksquare

Lemma 3.18. *Let $G \in \mathbf{NG}$ be a nearly symmetric graph. We have $\varepsilon(\text{Rot}(G)) = -\varepsilon(G)$ if there exists a sequence $\Sigma_c^{\text{odd}}(G)$ of G of some color c such that $\Sigma_c^{\text{odd}}(G) = (1, b, b)$, $(b, b, 1)$, or (1) , and $\varepsilon(\text{Rot}(G)) = \varepsilon(G)$ otherwise.*

Proof. Without loss of generality, we shall restrict to white-sided graphs. A white-sided graph with the left border vertex being white (resp. black) will be referred to as a white-white (resp. black-white) graph.

The total number of real vertices is odd for white-white graphs and even for black-white graphs. Let q be the number of one-valent real vertices, then

$$\text{pol}(\text{Rot}(G)) - \text{pol}(G) \equiv \begin{cases} q + 1 \pmod{2}, & \text{if } G \text{ is a white-white graph,} \\ q \pmod{2}, & \text{if } G \text{ is a black-white graph.} \end{cases} \quad (3)$$

This is true since each real vertex of G with $\deg(v) \neq 1$ contributes either to $\text{pol}(G)$ or, after rotation, to $\text{pol}(\text{Rot}(G))$.

For the case of white-white graphs, the sequences Σ_w^{odd} and Σ_b^{odd} have both odd length, and thus Lemma 3.17 implies $\text{lev}(\text{Rot}(G)) \equiv \text{lev}(G) + p \equiv \text{lev}(G) + 1$. Here p is the total number of even real vertices. Together with Equation (3), we get $\varepsilon(\text{Rot}(G)) = (-1)^q \varepsilon(G)$. Hence the sign changes if and only if Σ_w^{odd} is of type $(1, b, b)$, $(b, b, 1)$, or (1) .

In the case of black-white graphs, each of Σ_w^{odd} and Σ_b^{odd} has even length. Both, only one or none of them can be of type $(1, a)$ or $(a, 1)$, corresponding to $q = 2, 1$ or 0 . Lemma 3.17 implies $\text{lev}(\text{Rot}(G)) \equiv \text{lev}(G) + q$. Together with Equation (3) this gives $\varepsilon(\text{Rot}(G)) = (-1)^{2q} \varepsilon(G) = \varepsilon(G)$, which proves the claim. ■

Remark 3.19. A simple parity check shows the following: If d is even, all graphs in \mathbf{G} have border vertices of opposite color. If d is odd, all graphs in \mathbf{G} have border vertices of the same color.

Proof of Theorem 3.5 for d even. Rotation defines a bijection $\text{Rot} : \mathbf{NW} \rightarrow \mathbf{NB}$ with $\varepsilon(G) = \varepsilon(\text{Rot}(G))$ by Lemma 3.18. ■

Definition 3.20. A *reduced* graph G is a nearly symmetric graph such that Σ_w^{odd} and Σ_b^{odd} are of the form (a) , $(1, a, 1)$, or (a, b, b) for $a, b \neq 1$. We denote by $\mathbf{RW} \subset \mathbf{NW}$ the subset of reduced white-sided graphs.

Corollary 3.21. *If d is odd, we have $S(\mathbf{W}) = S(\mathbf{RW})$.*

Proof. Rotation defines an involution on $\mathbf{NW} \setminus \mathbf{RW}$ which satisfies $\varepsilon(G) = -\varepsilon(\text{Rot}(G))$ by Lemma 3.18. Using Proposition 3.13 we get $S(\mathbf{RW}) = S(\mathbf{NW}) = S(\mathbf{W})$. ■

3.3 Midline operation and rest of the proof

In this subsection we assume d odd. We want to construct a bijective correspondence

$$\text{Tr} : \{\text{short graphs of } \mathbf{RG}\} \longrightarrow \{\text{long graphs of } \mathbf{RG}\}$$

which does *not* necessarily respect the color of a graph.

Construction 3.22. Without loss of generality, consider a white-sided short graph $G = G_0 \in \mathbf{RW}$ and assume that the cycle lies on the left hand side of G (see bottom graph of Figure 1).

- (a) We use the same midline cutting method as in [IZ16, Proof of Lemma 3.13] (see Figure 6) in order to obtain an long graph G_1 .
- (b) The graph G_1 is in general not of type Λ_w, Λ_b , since the former border vertices v_1, v_2 changed their degree. We repair this by applying $G_2 := \text{Flip}_{v_1}^{v_2}(G_1)$.
- (c) The graph G_2 is in general not nearly symmetric (nor reduced). Instead, the sequences $\Sigma_b^{\text{even}}(G_2)$ and $\Sigma_w^{\text{even}}(G_2)$ are of the form $(\alpha, \Xi, \alpha^{-1})$, $(\alpha, \Xi, e_1, e_1, \alpha^{-1})$, $(\alpha, e_0, \Xi, \alpha^{-1})$, or $(\alpha, \Xi, e_0, \alpha^{-1})$. The first case is symmetric, while in the latter three cases we perform a cyclic shift on the subsequences (Ξ, e_1, e_1) , (α, e_0) , and (α, Ξ, e_0) , respectively, to make the sequences nearly symmetric. Similarly, if the

sequence $\Sigma_w^{\text{odd}}(G_2)$ is of type (b, a, b) , $a, b \neq 1$, we apply a flip on (b, a) . We obtain a reduced nearly symmetric long graph G_3 , which we also denote $\text{Tr}(G)$.

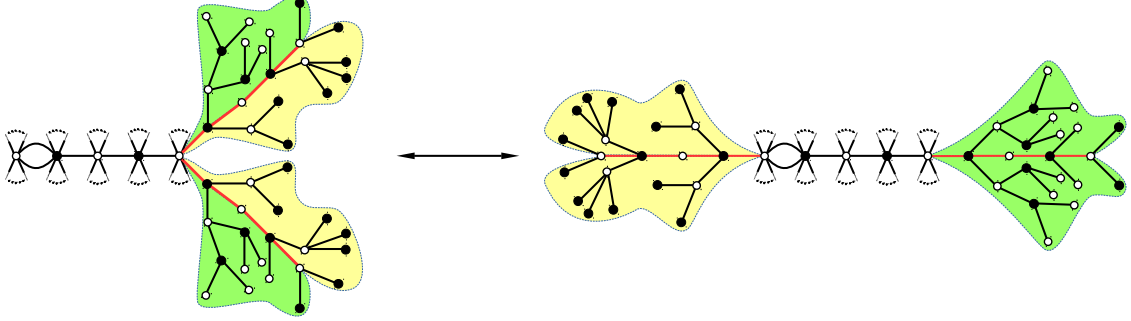


Figure 6 The midline cutting method.

Lemma 3.23. *The map Tr is a bijection.*

Proof. Given a reduced nearly symmetric extended graph, the position of the cycle indicates the length of the two real segments on the left and right side of the graph that we want to reorganize. In particular, the length of α in step (3) can be recovered from G and hence, knowing the real sequences of G , step (3) can be reversed uniquely. Next, we can undo step (2) by flipping the two vertices which are at the inner ends of these two segments. Finally, there is unique way of cutting open the two segments and regluing them as additional tree at the right border vertex to undo step (1). ■

Lemma 3.24. *Let $G \in \mathbf{RW}$ be a short graph. Then*

$$\varepsilon(\text{Tr}(G)) = \begin{cases} -\varepsilon(G) & \text{if } \text{Tr}(G) \text{ is white-sided,} \\ +\varepsilon(G) & \text{if } \text{Tr}(G) \text{ is black-sided.} \end{cases} \quad (4)$$

Proof. Throughout the proof, Σ and $\bar{\Sigma}$ will refer to sequences for G and $\text{Tr}(G)$ respectively. By symmetry we may restrict our attention to graphs G with the cycle sitting on the left hand side. Fix a color c and let k_c be the number of transpositions needed to transform $\bar{\Sigma}_c$ to $(\bar{\Sigma}_c^{\text{odd}}, \bar{\Sigma}_c^{\text{even}})$, as in the proof of Lemma 3.17. Set $l_c = l(\bar{\Sigma}_c^{\text{even}})$ and let n_c be the number of even vertices of color c located *before* the cycle in $\text{Tr}(G)$. In other words, $n_c = l(\alpha)$ for the sequence α appearing in step (3). Note that the cyclic shifts applied in step (3) do not affect the number k_c and we can express it as

$$k_c = \begin{cases} n_c & \text{if } l(\bar{\Sigma}_c^{\text{odd}}) = 1, \\ n_c + l_c & \text{if } l(\bar{\Sigma}_c^{\text{odd}}) = 3. \end{cases} \quad (5)$$

Let δ_c denote the *change* of pole disorders from G to $\text{Tr}(G)$ caused by vertices of color c . We have $\delta_c = n_c$ for $\bar{\Sigma}_c^{\text{odd}} = (a)$ or $(1, a, 1)$ and $\delta_c = n_c + 1$ for $\bar{\Sigma}_c^{\text{odd}} = (a, b, b)$. Hence the

total contribution of the color c to the sign change can be summarized by the formula

$$\text{dis}(\overline{\Sigma}_c) - \text{dis}((\Sigma_c^{\text{odd}}, \Sigma_c^{\text{even}})) + \delta_c \equiv \begin{cases} 0 & \text{if } l(\overline{\Sigma}_c^{\text{odd}}) = 1, \\ l_c + 1 & \text{if } l(\overline{\Sigma}_c^{\text{odd}}) = 3. \end{cases} \quad (6)$$

Since by assumption G is a white-sided short graph with the cycle located to the left-hand side, we have $\Sigma_b = (\Sigma_b^{\text{odd}}, \Sigma_b^{\text{even}})$ and $\Sigma_w = (\Sigma_w^{\text{even}}, \Sigma_w^{\text{odd}})$. Hence for case $c = w$ we should take into account an extra contribution of l_w , and thus the sign change for black and white vertices is given in the following table.

	w	b
$l(\overline{\Sigma}_c^{\text{odd}}) = 1$	$(-1)^{l_w}$	$+1$
$l(\overline{\Sigma}_c^{\text{odd}}) = 3$	-1	$(-1)^{l_b+1}$

In the case where $\text{Tr}(G)$ is black-sided, we multiply the diagonal entries of the table. Since $l_w + l_b$ is odd, we obtain the total sign change $+1$. In the case where $\text{Tr}(G)$ is white-sided, we multiply the antidiagonal entries and obtain -1 . \blacksquare

Proof of Theorem 3.5 for d odd. Following Corollary 3.21, it is sufficient to prove that $S(\mathbf{RW}) = S(\mathbf{RB})$. The latter equality follows from Lemmata 3.23 and 3.24. \blacksquare

4 Invariance theorem

In this section we prove Theorem 1.1. We fix d and partitions $\mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_k)$ of d such that $d = \sum_{i,j} (\Lambda_j^i - 1)$. First note that the number of functions contained in $\mathcal{S}(\mathcal{P}, \mathbf{\Lambda})$ only depends on the ordering of the branch points in $\mathcal{P} = (b_1, \dots, b_k)$ on the real line, not on their exact position. This follows from Theorem 2.4. Hence it remains to prove that the signed count does not change under a permutation of the points in \mathcal{P} . Based on Section 2 it is slightly more convenient to take the symmetric point of view here: We fix the ordering of points by $b_1 < \dots < b_k$, and instead permute the partitions stored in the sequence $\mathbf{\Lambda}$. Of course, it suffices to prove invariance under the action the transposition exchanging Λ_j and Λ_{j+1} , $j = 1, \dots, k-1$.

Let us fix some $j \in \{1, \dots, k-1\}$. The relation to the bw -graphs from the previous section is as follows. We rename $\Lambda_j = \Lambda_b$ and $\Lambda_{j+1} = \Lambda_w$. Let Γ be a real simple rational dessin of degree d and type $\mathbf{\Lambda}$. We color the preimages of b_j and b_{j+1} in black and white, respectively. Let $E_{bw} := \varphi^{-1}([b_j, b_{j+1}]) \subset \Gamma$ denote the preimage of the closed segment joining b_j and b_{j+1} . We need to distinguish two cases. Let us recall from [IZ16, Definition 3.1] that a *black and white tree*, or *bw-tree*, is a graph as described in Definitions 3.1, except for requiring a tree graph, in exchange for our cycle condition.

- (1) E_{bw} is a union $T_1 \cup \dots \cup T_r$ of disjoint bw -trees, where some of them are real and others split into pairs of trees that are complex conjugate.
- (2) E_{bw} is a union $R_0 \cup T_1 \cup \dots \cup T_r$ of bw -trees T_1, \dots, T_r as before, together with exactly one real simple bw -graph R_0 (see Figure 7).

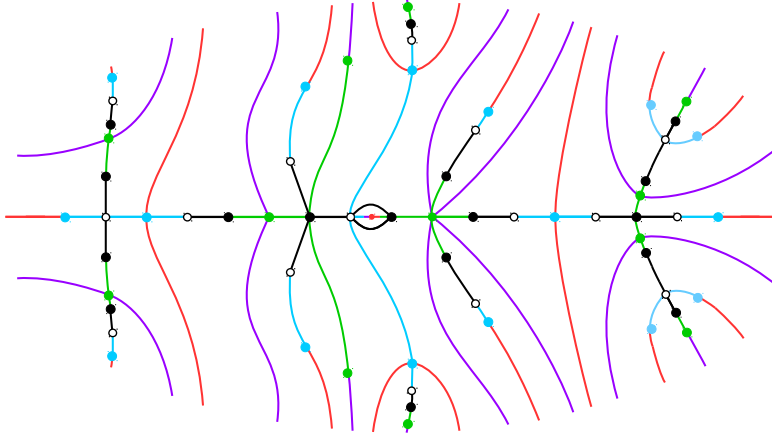


Figure 7 An example of a real dessin satisfying condition (2).

We focus on case (2). Case (1) is similar to the situation in [IZ16] and will be treated later. The following definition describes the type of graphs we obtain when we make b_j and b_{j+1} “collide” in \mathbf{R} , or equivalently, collapse the subgraph $E_{bw} \subset \Gamma$. A *real polynomial dessin* $\tilde{\Gamma} \subset \mathbf{CP}^1$ is a labelled graph satisfying all the conditions from Definition 2.1 except for condition (c), which we replace by

(c’) exactly one vertex is labelled by ∞ , namely $\infty \in \mathbf{CP}^1$.

Definition 4.1. A *bw-enhanced dessin* is a real polynomial dessin $\tilde{\Gamma}$ with (ordered) labels $\{1, \dots, j-1, bw, j+2, \dots, \infty\}$ together with the following data:

- A real *bw*-vertex V_0 , together with two partitions $\Pi_b(V_0), \Pi_w(V_0)$ of $\deg(V_0)/2 + 1$.
- Let $\mathcal{V}_{\mathbb{R}}$ be the set of real *bw*-vertices different from V_0 and let \mathcal{V}_+ be the set of *bw*-vertices with positive imaginary part. For each vertex $V \in \mathcal{V}_{\mathbb{R}} \cup \mathcal{V}_+$, fix two partitions $\Pi_b(V), \Pi_w(V)$ of $\deg(V)/2$.

From a simple rational dessin Γ of type (2) we can construct a *bw*-enhanced real dessin $\tilde{\Gamma}$ as follows. First, we obtain the graph $\tilde{\Gamma}$ by contracting the all edges $j \rightarrow j+1$ of Γ . This means that each component of $E_{bw} = R_0 \cup T_1 \cup \dots \cup T_r$ contracts to a point, and we label these points by *bw*. We set V_0 the vertex of $\tilde{\Gamma}$ obtained from contracting R_0 and define $\Pi_b(V_0)$ and $\Pi_w(V_0)$ as the partition of degrees of black and white vertices in R_0 , respectively. Similarly, for each vertex $V \in \mathcal{V}_{\mathbb{R}} \cup \mathcal{V}_+$, we define $\Pi_b(V)$ and $\Pi_w(V)$ as the partition of degrees of black and white vertices in T_i , where T_i is the *bw*-tree contracting to V . We call $\tilde{\Gamma}$ the *contraction* of Γ .

Definition 4.2. Let $\tilde{\Gamma}$ be a *bw*-enhanced dessin. For any label $j \neq bw, \infty$, we set Σ_j to the sequence of degrees of real vertices labelled by j , ordered from left to right. We denote by $\tilde{\Sigma}_w$ and $\tilde{\Sigma}_b$ the sequences of partitions $\Pi_w(V)$ and $\Pi_b(V)$, respectively, running through the real *bw*-vertices (from left to right), except for V_0 , for which we

use the partitions $(\Pi_w(V), 1)$ and $(\Pi_b(V), 1)$. A disorder of $\tilde{\Sigma}_w$ or $\tilde{\Sigma}_b$ is a pair of entries $\Pi_w^k(V) > \Pi_w^{k'}(V')$ or $\Pi_b^k(V) > \Pi_b^{k'}(V')$, respectively, with $V < V'$. Clearly, multiple entries of a partition are counted multiple times here. We set

$$\text{dis}(\tilde{\Gamma}) := \text{dis}(\tilde{\Sigma}_w) + \text{dis}(\tilde{\Sigma}_b) + \sum_{j \neq bw, \infty} \text{dis}(\Sigma_j), \quad \varepsilon(\tilde{\Gamma}) := (-1)^{\text{dis}(\tilde{\Gamma})}.$$

Recall from [IZ16] that the sign $\varepsilon(T)$ of a real bw -tree is $(-1)^{\text{dis}(\Sigma_w) + \text{dis}(\Sigma_b)}$.

Lemma 4.3. *If a bw -enhanced dessin $\tilde{\Gamma}$ is the contraction of a dessin Γ , then*

$$\varepsilon(\Gamma) = \varepsilon(\tilde{\Gamma}) \cdot \varepsilon(R_0) \cdot \prod_T \varepsilon(T),$$

where the product is taken over the real trees in E_{bw} .

Proof. The right hand side of the formula takes care of all disorders contributing to $\varepsilon(\Gamma)$ except for the ones involving a vertex in the interior of the (topological) edge E of Γ containing p . Note that for each label $i \in \{j+2, \dots, k, \infty, 1, \dots, j-1\}$ there is exactly one vertex labelled by i in the interior of E , and all these vertices have valance to 2. Hence such disorders appear in pairs: Any higher-degree real vertex to the left of p produces exactly one level and one pole disorder. This proves the formula. \blacksquare

Let $\tilde{\Gamma}$ be a bw -enhanced dessin. For any real bw -vertex V , we define the color $c(V)$ to be white if the real edge to the right of V is of type $bw \rightarrow j+2$. Otherwise we set $c(V)$ to be black. We denote by \mathcal{R}_0 the set of $c(V)$ -sided real simple bw -graphs of type $\Pi_b(V_0), \Pi_w(V_0)$. For any $V \in \mathcal{V}_{\mathbb{R}}$ we denote by \mathcal{T}_V the set of $c(V)$ -sided real bw -trees of type $\Pi_b(V), \Pi_w(V)$. For any $V \in \mathcal{V}_+$ we denote by \mathcal{T}_V the set of (non-real) bw -trees of type $\Pi_b(V), \Pi_w(V)$ with one marked half-edge emanating from a white vertex.

For any $V \in \mathcal{V}_+$, let us mark an adjacent edge v of type $bw \rightarrow j+2$. Let \mathcal{M} denote the set of increasing real simple dessins Γ that contract to $\tilde{\Gamma}$. Then we can define a map

$$F : \mathcal{M} \rightarrow \mathcal{R}_0 \times \prod_{V \in \mathcal{V}_{\mathbb{R}} \cup \mathcal{V}_+} \mathcal{T}_V, \\ \Gamma \mapsto (R_0, T_V : V \in \mathcal{V}_{\mathbb{R}} \cup \mathcal{V}_+),$$

where R_0 and T_V denote the components of E_{bw} contracting to V_0 and V , respectively. For $V \in \mathcal{V}_+$, we additionally mark the first half-edge in T_V touching e_V , counted in clockwise direction.

Lemma 4.4. *The map F is a bijection.*

Proof. We can easily describe the inverse map. Given $(R_0, T_V : V \in \mathcal{V}_{\mathbb{R}} \cup \mathcal{V}_+)$, we insert R_0 and T_V into a small disc removed around V_0 and $V \in \tilde{\Gamma}$, respectively, as described in [IZ16, Proof of Lemma 4.6]. Additionally, we fill the ‘‘hole’’ bounded by the cycle in R_0 as follows. Let $D \subset \mathbf{C}$ denote the unique affine real polynomial dessin of degree 1 labelled by $-\infty \rightarrow j+2 \rightarrow \dots \rightarrow k \rightarrow \infty \rightarrow 1 \rightarrow \dots \rightarrow j-1 \rightarrow +\infty$. We glue D into the hole such that the edge $-\infty \rightarrow j+2$ is attached to the white vertex in the boundary (see Figure 8). In this way we obtain real simple rational dessin Γ such that $F(\Gamma)$ is equal to the given data. \blacksquare

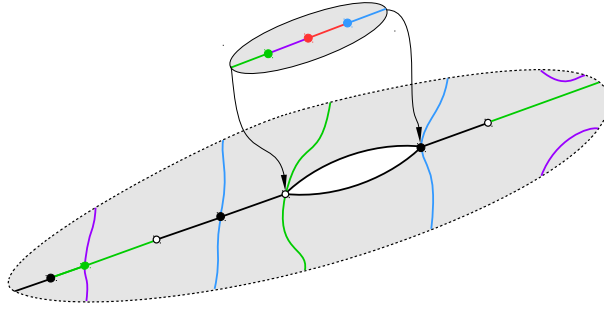


Figure 8 Filling the “hole” of R_0 .

Corollary 4.5. *With the notation used in Lemma 4.4, we have*

$$S(\mathcal{M}) = \varepsilon(\tilde{\Gamma}) S(\mathcal{R}_0) \prod_{V \in \mathcal{V}_{\mathbb{R}}} S(\mathcal{T}_V) \prod_{V \in \mathcal{V}_+} |\mathcal{T}_V|.$$

Proof. Follows from Lemma 4.3 and Lemma 4.4. ■

Proof of Theorem 1.1. Since transpositions of the form $(j, j+1)$ generate the symmetric group, it is enough to prove $S(\mathcal{S}(\mathcal{P}, \Lambda)) = S(\mathcal{S}(\mathcal{P}, \Lambda'))$, where Λ' is the sequence of partitions with Λ_j and Λ_{j+1} swapped. Let $\mathcal{S}^{(1)}(\mathcal{P}, \Lambda)$ and $\mathcal{S}^{(2)}(\mathcal{P}, \Lambda)$ be the subsets of functions/dessins of type (1) and (2), respectively. Then $S(\mathcal{S}^{(1)}(\mathcal{P}, \Lambda)) = S(\mathcal{S}^{(1)}(\mathcal{P}, \Lambda'))$ follows, after some straightforward adaptation to our case, from [IZ16, Lemma 4.8]. It remains to show $S(\mathcal{S}^{(2)}(\mathcal{P}, \Lambda)) = S(\mathcal{S}^{(2)}(\mathcal{P}, \Lambda'))$. We can prove this for each contraction separately. More precisely, let $\tilde{\Gamma}, \tilde{\Gamma}'$ be a pair of *bw*-enhanced dessins obtained from each other by flipping the partitions $\Pi_b(V), \Pi_w(V)$ for any *bw*-vertex V . Let \mathcal{M} and \mathcal{M}' be the sets of dessins of type Λ and Λ' that contract to $\tilde{\Gamma}$ and $\tilde{\Gamma}'$, respectively. We show $S(\mathcal{M}) = S(\mathcal{M}')$ using Corollary 4.5. Indeed, the invariance of the various factors follows for $\varepsilon(\tilde{\Gamma})$ by definition, for $S(\mathcal{R}_0)$ by Theorem 3.5, for $S(\mathcal{T}_V)$ by [IZ16, Theorem 9], and for $|\mathcal{T}_V|$ by switching the colors of the vertices. Note that in the last case, the number of half-edges adjacent to a white and black vertex, respectively, is equal and hence the number of possible markings does not change. ■

Remark 4.6. Despite the combinatorial nature of the presented proof of Theorem 1.1 it is instructive to also get a geometric picture of the degeneration when b_j and b_{j+1} collide. While in our description the two discs adjacent to p just collapse to V_0 in $\tilde{\Gamma}$, the limit of this degeneration in $\overline{\mathcal{M}}_{0,0}(\mathbf{CP}^1, d)$ is a ramified cover with reducible source curve. The “bubbling” produces a source curve with two components (see Figure 9). The main component is described by $\tilde{\Gamma}$ and is mapped to \mathbf{CP}^1 by a polynomial map of degree $d-1$. The bubble corresponds to the two discs adjacent to p and is mapped isomorphically to \mathbf{CP}^1 . The vertex V_0 is the node connecting the two spheres. This phenomenon does not occur in the polynomial case of [IZ16]. What keeps the situation under control for simple rational functions is the fact that the degree 1 map on the bubble is essentially unique and hence can be discarded from the combinatorial data.

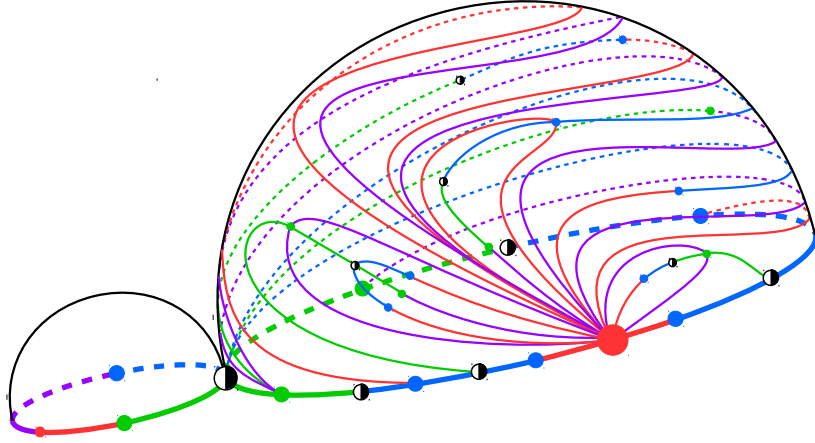


Figure 9 The bubbling of a dessin into a reducible dessin of degree $(d - 1, 1)$.

5 Vanishing Statements

This section is devoted to prove the vanishing statements contained in Theorem 1.2 or, in other words, the *only if* part. Let $\lambda := (\lambda_1, \dots, \lambda_k)$ be a sequence of *reduced* partitions as in the introduction.

Proposition 5.1 (First part of non-vanishing for all degrees). *If a partition λ_j has more than one odd number appearing an odd number of times or more than one even number appearing an odd number of times, then $F_\lambda^{odd}(q) = 0$ and $F_\lambda^{even}(q) = 0$.*

Proof. Fix $a \neq b$ of the same parity which both appear an odd number of times in λ_j . Then for any dessin Γ there is an odd number of real vertices of label i and degree $2a + 2$ and $2b + 2$, respectively. We reverse the order of appearance of these vertices using the flip operation analogous to $\text{Flip}_{v_1}^{v_2}(G)$ in Definition 3.8. To be more precise, there is only one adjustment needed: If the degrees are odd and we want to flip a vertex of type $j - 1 \rightarrow j \rightarrow j + 1$ with a vertex of type $j + 1 \rightarrow j \rightarrow j - 1$, we also flip the ordering of the pairs of trees in the forest of both vertices. It is shown in [IZ16, Page 37] that the involution constructed from this reversion operation flips the sign of φ and hence the statement follows. ■

Proposition 5.2 (non-vanishing for odd degrees). *Consider a sequence of partitions λ which does not satisfy the conditions of Proposition 5.1. If there is an odd number of partitions λ_j having an even element appearing an odd number of times, then*

$$F_\lambda^{odd}(q) = 0.$$

Proof. For d odd the operation $\varphi(z) \mapsto \tilde{\varphi}(z) = \varphi(-z)$ produces an increasing function again and hence defines an involution on $\mathcal{S}(\mathcal{P}, \mathbf{\Lambda})$. Let $\Sigma = \Sigma_{b_j}$ the ramification sequence of a critical value as in Definition 1.5. Then the corresponding sequence for g is given

by Σ^{-1} . One easily checks

$$\text{dis}(\Sigma) + \text{dis}(\Sigma^{-1}) \equiv_2 c(\Sigma),$$

where

$$c(\Sigma) := \#\{\text{pairs of numbers } m, n \text{ in } \Sigma \text{ such that } m \neq n\}.$$

Recall that Σ includes a 1 coming from the simple pole, so the sum of its entries is even. Now, if λ_i contains an even number a an odd number of times, then Σ contains two or three elements appearing an odd number of times ($1, a+1$, and possibly an even entry), and hence $c(\Sigma)$ is odd. If, on the other hand, λ_j does not contain an even element an odd number of times, then Σ contains zero or one elements appearing an odd number of times and hence $c(\Sigma)$ is even. The result then follows by the previous formula. \blacksquare

Remark 5.3. For d even, we could instead try to use the transformation $\varphi(z) \mapsto -\varphi(-z)$, which provides a bijection between $\mathcal{S}(\mathcal{P}, \mathbf{\Lambda})$ and $\mathcal{S}(-\mathcal{P}, \mathbf{\Lambda})$. The same argument as above shows that $F_{\lambda}^{\text{even}}(q) = 0$ if there is an odd number of partitions λ_j with an odd entry appearing an odd number of times. But note that such entries correspond to local extrema of φ . Hence the number is even.

6 Broken alternations

Definition 6.1. Let $a \in \text{Sym}(n)$ be a permutation on n elements, $i \mapsto a_i$. We call a an (*ordinary*) *alternation* if $a_n < a_{n-1} > a_{n-2} < a_{n-3} > \cdots \leq a_1$. The permutation a is said to be a *broken alternation* if it is an alternation except for exactly one index $i \in \{1, \dots, n-1\}$, where a_i, a_{i+1} is increasing if it should be decreasing, or the opposite. In this case, i is called the *break* of a , and we write $(a_1 \dots a_i | a_{i+1} \dots a_n)$. We denote by A_n, B_n and B_n^j the number of ordinary alternations, broken alternations, broken alternations with $a_j = n$, respectively.

It is clear that we have $A_1 = A_2 = B_2 = 1, B_1 = 0$, and $B_n = B_n^1 + \cdots + B_n^n$.

Proposition 6.2. *There is a one-to-one correspondence between broken alternations of n elements and real simple rational dessins of degree n and type $\mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$, $\Lambda_1 = \cdots = \Lambda_n = (21 \dots 1)$. Moreover, the sign of any such real simple rational dessin is equal to -1 to the power of $\lfloor n/2 \rfloor$.*

Remark 6.3. The exact same relation holds between ordinary alternations and real polynomial dessins, see [IZ16, Example 1.5]. We refer to Figure 10 for examples.

Proof of Proposition 6.2. Given a real simple rational dessin of type $\mathbf{\Lambda}$, the sequence of labels of the real vertices of degree greater than 2, ordered from left to right, produces a broken alternation of length n . The break corresponds to passing through the simple pole p . Vice versa, given a broken alternation a , we can construct a dessin as follows. We place 4-valent vertices on \mathbb{R} with labels as described in a . We put a 2-valent vertex with label ∞ on the segment between the vertices corresponding to the break of a . We

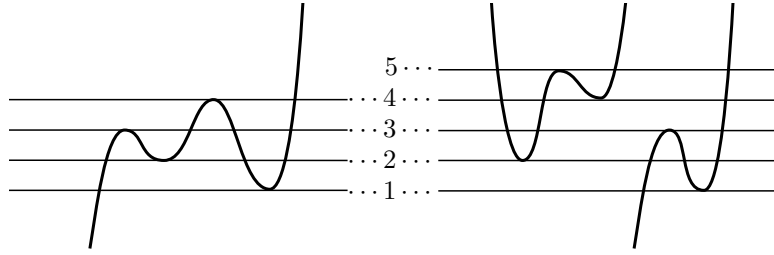


Figure 10 To the left, a polynomial representing the ordinary alternation $a = (3241)$. To the right, a simple rational function representing the broken alternation $a = (254|31)$.

glue the imaginary ends of the graph to infinity, except for the ends to the left and right of p , which we glue pairwise to obtain a pair of cycles. Finally we insert 2-valent vertices in the unique way to complete the graph to a dessin. The sign is deduced by noting that an real simple rational function of type Λ_n has exactly $\lfloor n/2 \rfloor$ local maxima, and each local maxima contributes by a factor of -1 . ■

We can easily derive recursive formulas for the numbers B_n, B_n^j similar to those for A_n . In accordance with Proposition 6.2, we set $B_0 = 0, A_0 = 1$.

Proposition 6.4. *For $n \geq 2$ we have*

$$B_n^j = \begin{cases} \binom{n-1}{j-1} \cdot (B_{j-1}A_{n-j} + A_{j-1}B_{n-j}) & \text{if } n+j \text{ odd,} \\ A_{n-1} & \text{if } j = n \text{ or } n \text{ odd, } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By our conventions, the pattern for ordinary alternations is $a_{j-1} < a_j > a_{j+1}$ for $n+j$ odd, and $a_{j-1} > a_j < a_{j+1}$ otherwise. In a broken alteration, we can flip at most one of these inequalities. Hence, if a is a broken alternation with $a_j = n$, we have the following cases. If $n+j$ is even, it is of the form $a = (n|\overline{a'})$ or $a = (\overline{a'}|n)$, where a' is an ordinary alternation of length $n-1$ and $\overline{a'}$ is defined by $\overline{a'_i} = n - a'_i$. The map $a \mapsto a'$ is obviously a bijection, which settles the second and third case. If $n+j$ is odd, the break of a is different from $j-1$ and j . Hence a is of the form $(a'na'')$, where either a' is ordinary and a'' is broken, or vice versa. To be precise, this is true after using the unique ordered relabeling of the values of a', a'' to $\{1, \dots, j-1\}$ and $\{1, \dots, n-j\}$, respectively. On other hand, a (reabeled) pair (a', a'') occurs exactly $\binom{n-1}{j-1}$ times in these constructions, since this is the number of ordered maps $\{1, \dots, j-1\} \rightarrow \{1, \dots, n-1\}$ (or, equivalently, $\{1, \dots, n-j\} \rightarrow \{1, \dots, n-1\}$). ■

The first few values of B_n are given in the following table.

n	1	2	3	4	5	6	7	8	9	10	11	12
B_n	0	1	2	7	26	117	594	3407	21682	151853	1160026	9600567

It seems that the sequence has not appeared in the literature before (cf. [Sta10]) and is not recorded on OEIS.

We can turn the recursive relations into differential equations for generating series. Since it is more convenient to do this for odd and even indices separately, we define for the ordinary alternations

$$f(q) := \sum_{k=0}^{\infty} (-1)^k \frac{A_{2k+1}}{(2k+1)!} q^{2k+1}, \quad g(q) := \sum_{k=0}^{\infty} (-1)^k \frac{A_{2k}}{(2k)!} q^{2k},$$

and for the broken alternations

$$u(q) := \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k+1}}{(2k+1)!} q^{2k+1}, \quad v(q) := \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k)!} q^{2k}.$$

Lemma 6.5. *The power series f, g, u, v satisfy the following differential equations.*

$$u' = 2(g - fu - 1), \tag{7}$$

$$v' = -(f + fv + gu). \tag{8}$$

Proof. From Proposition 6.4, for $k \geq 1$ we have

$$B_{2k+1} = 2A_{2k} + \sum_{j=1}^k \binom{2k}{2j-1} \cdot (B_{2j-1}A_{2k+1-2j} + A_{2j-1}B_{2k+1-2j}) \tag{9}$$

$$= 2 \left(A_{2k} + \sum_{j=1}^k \binom{2k}{2j-1} \cdot B_{2j-1}A_{2k-(2j-1)} \right). \tag{10}$$

which proves the first equation for all non-constant terms. For the constant term, we note that by our conventions $B_1 = 2(A_0 - 0 - 1) = 0$. Again by Proposition 6.4, for $k \geq 1$ we have

$$B_{2k} = A_{2k-1} + \sum_{j=0}^{k-1} \binom{2k-1}{2j} \cdot (B_{2j}A_{2k-2j-1} + A_{2j}B_{2k-2j-1}), \tag{11}$$

$$= A_{2k-1} + \sum_{j=0}^{2k-1} \binom{2k-1}{j} \cdot B_j A_{2k-1-j}, \tag{12}$$

The sum in (12) corresponds to the anti-symmetric part of $(f + g)(u + v)$ which is $fv + gu$. Moreover, because of the antisymmetry of the second equation, no constant term appears, which completes the proof. \blacksquare

We can solve these differential equations explicitly. Recall the 19th century result [And81] (see also [Sta10]) that

$$f(q) = \tanh(q) \quad \text{and} \quad g(q) = \operatorname{sech}(q).$$

Corollary 6.6. *The generating series u and v can be expressed as*

$$u(q) = -f - q + qf^2 + 2fg, \tag{13}$$

$$v(q) = 1 - 2f^2 - g + qfg. \tag{14}$$

Proof. The given descriptions for u and v are the unique solutions to the equations (7) and (8) with initial conditions $u(0) = v(0) = 0$. \blacksquare

7 Polynomiality

Keeping the notations from the introduction, we want to prove the following result.

Theorem 7.1. *For any sequence of reduced partitions λ , the generating series*

$$F_\lambda^{odd}(q) \quad \text{and} \quad F_\lambda^{even}(q)$$

are polynomials in q , $f(q)$ and $g(q)$ with rational coefficients.

Based on the invariance theorem 1.1, we fix points $b_1 < b_2 < \dots < b_{k+m}$ and consider real simple rational functions with reduced ramification profile $\lambda_1, \dots, \lambda_k$ at the branch points b_{m+1}, \dots, b_{k+m} and simple branch points b_1, \dots, b_m . We choose an additional point $b_k < \alpha < b_{k+1}$.

Let φ be such a function and Γ be the associated affine dessin of φ . We set B' to be the union of connected components of $\varphi^{-1}([\alpha, \infty))$ which contain a critical point of φ (see Figure 11). We also include the components of $\varphi^{-1}([\alpha, \infty))$ to the left and right which contain unbounded pieces of \mathbb{R} . Since the left one only exists if d is odd, this is just a convenient way of storing the parity of d in B' .

Definition 7.2. A connected component C of $\mathbf{R} \setminus B'$ is a *chain* of Γ . Each point in the inverse image of α is labelled by \blacksquare , and is called an α -vertex. We order the chains from left to right as C_1, \dots, C_l . The *special chain* $C_s, s \in \{1, \dots, l\}$ is chain containing the simple pole p . The tuple $B_\Gamma := (B', s)$ is called the *base* of Γ .

Let q_1, q_2 denote the closest critical points before and after the simple pole p . We need to distinguish three cases (see Figures 11, 13, and 14 for examples):

- (A) $\varphi(q_1), \varphi(q_2) > \alpha$,
- (B) $\varphi(q_1) > \alpha > \varphi(q_2)$ or $\varphi(q_2) > \alpha > \varphi(q_1)$,
- (C) $\varphi(q_1), \varphi(q_2) < \alpha$.

The three cases can be distinguished in terms of the base as follows.

- (A) The base B_Γ contains a cycle.
- (B) The base B_Γ contains no cycle, but a bounded open end (with the simple pole p as endpoint).
- (C) Otherwise.

Remark 7.3. To motivate the following definition, let us make a few simple observations. Let C_1, \dots, C_l be the sequence of chains, the order of which is according to the orientation of \mathbb{R} . Let r_i denote the number of critical points occurring in C_i . Then the parity of r_i is given by

$$r_i \equiv_2 \begin{cases} d & i = 1, \\ 1 & i > 1, \end{cases} \quad (15)$$

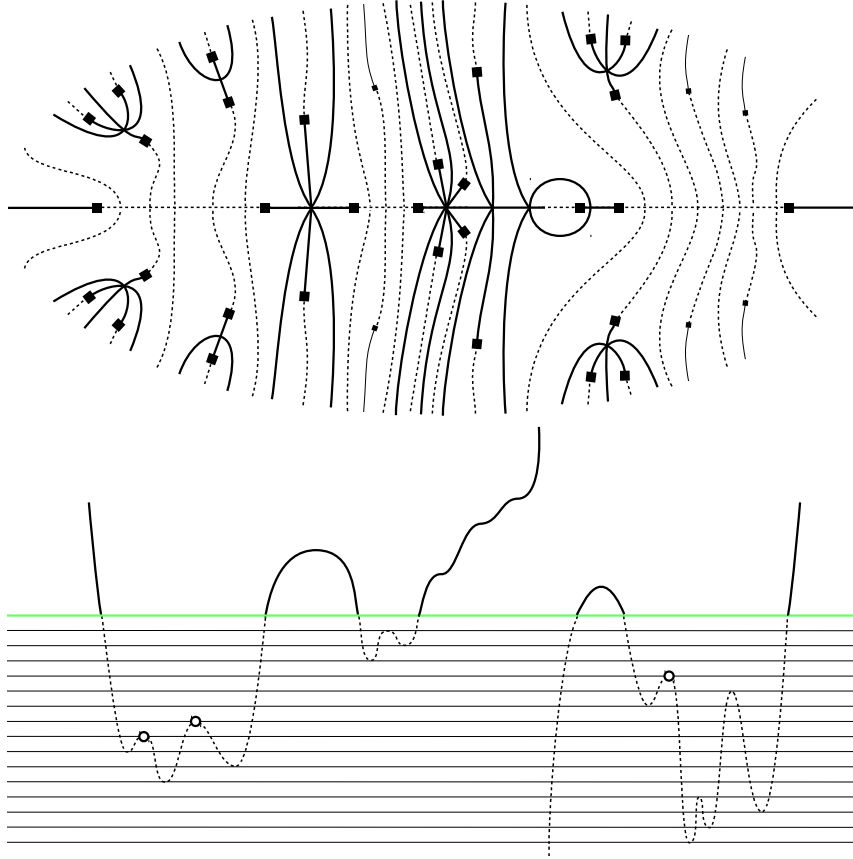


Figure 11 A base B of type (A), represented as a union of thick lines. Here, $l = 4$, $s = 3$ and $(c_1, c_2, c_3, c_4) = (2, 0, 0, 1)$. The black squares are the α -vertices. The dotted lines represent the inverse image of $(-\infty, \alpha)$ for a dessin Γ with $B_\Gamma = B$.

except for the following cases:

$$r_s \equiv_2 \begin{cases} d - 1 & s = 1, \text{ and either type (A) or (B),} \\ 0 & s > 1, \text{ and either type (A) or (B).} \end{cases} \quad (16)$$

Moreover, we have

$$r_s = 0 \quad \text{if } B \text{ is of type (A).} \quad (17)$$

In the following, we fix a base $B = (B', s)$ of type (A), (B) or (C) and chains C_1, \dots, C_l . Note that for type (A) and (B), the special index s is uniquely determined by B' , while for type (C) any $s \in \{1, \dots, l\}$ can occur. Let Z_+ be a connected component of B which is contained in $\{z : \Im(z) > 0\} \subset \mathbf{C}$. It has a unique α -vertex which can be connected in $\mathbf{C} \setminus B$ to a unique chain C_i . We say Z_+ is *adjacent* to C_i and denote by c_i the number of such Z_+ (see Figure 11).

Definition 7.4. A *chain data set* for a base B consists of a tuple $(I_i, a_i, M_i)_{i \in \{1, \dots, l\}}$ with the following properties.

- (a) The sets I_i form a disjoint union $I_1 \sqcup \dots \sqcup I_l = \{1, \dots, m\}$ and the numbers $r_i := |I_i|$ satisfy equations (15), (16) and (17).
- (b) For any $i \in \{1, \dots, \hat{s}, \dots, l\}$, a_i is an ordinary alternation a_i of length r_i . Additionally, a_s is an ordinary or broken alternation of length r_s if type is (B) or (C). For type (A), we set $a_s = \emptyset$.
- (c) We set $m_i := \lfloor r_i/2 \rfloor$ for $i \neq s$. We set $m_s := \lfloor r_s/2 \rfloor - 1$, except for the case type (B), d even, and $s = 1$, in which we set $m_s := \lfloor r_s/2 \rfloor$. Then the last ingredient is a choice of subsets $M_i \subset \{1, \dots, m_i\}$ of size c_i .

Proposition 7.5. *Fix a base B as above. Then the set of dessins Γ with $B = B_\Gamma$ is in one-to-one correspondence with the set of chain data sets for B .*

Proof. We first associate a chain data set to each Γ . Fix $i \in \{1, \dots, l\}$. We define I_i such that $j \in I_i$ if and only if the simple critical point with value b_j is contained in C_i . After reparametrizing I_i by $\{1, \dots, r_i\}$ keeping the ordering, the order of appearance of the critical points in C_i defines a permutation of $\{1, \dots, r_i\}$. If the type is (B), $i = s$, and the simple pole is located on the right hand side of C_s , we additionally compose with the permutation with $x \mapsto r_s + 1 - x$. The result is used as a_i . Note that m_i appearing in Definition 7.4 is equal to the number of maxima in C_i for $i \neq s$, and equal to the number of maxima minus one if $i = s$ and the type is (B) or (C). In this case, there is a unique maximum in C_s which is closest to the simple pole p (see Figure 12). After removing this maximum, the remaining maxima can be ordered increasingly and labelled by $\{1, \dots, m_i\}$. Let Z_+ be a connected component of B , as before, adjacent to C_i . Since Γ is connected, there is path connecting Z_+ and \mathbf{R} . It is easy to see that this path is unique, its starting point is a α -vertex of Z_+ and its endpoint is a maximum of C_i . Varying Z_+ , this construction distinguishes c_i maxima of C_i , whose labels we collect in $M_i \subset \{1, \dots, m_i\}$. By the properties of Γ is clear that we constructed a well-defined chain data set for B .

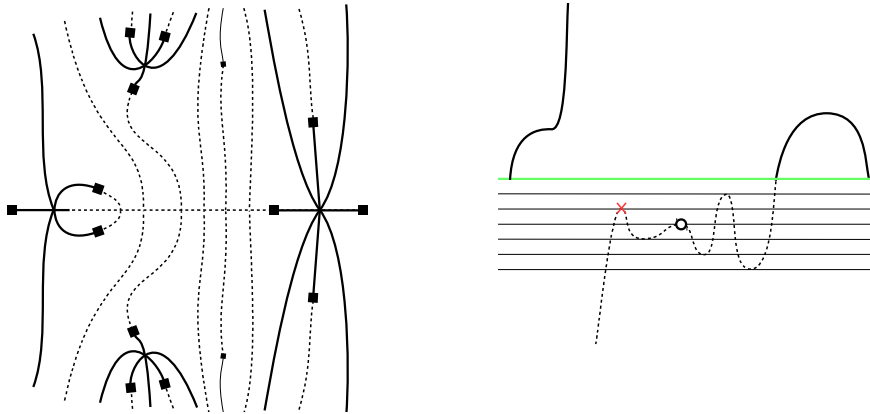


Figure 12 The special chain of a base of type (B). The ticked maximum cannot be marked in M_s .

We proceed to show the existence of the inverse map. Assume we are given a chain data set for B . Fix some $i \in \{1, \dots, l\}$. Using the bijections in Proposition 6.2 and [IZ16, Example 1.5], we can associate to a_i a unique (affine) real dessin D_i with labels $1 \rightarrow \dots \rightarrow r_i$. Reversing the relabeling from above, we can rename the labels by elements of I_i . We complete this to a labeling of type $-\infty \rightarrow 1 \rightarrow \dots \rightarrow m \rightarrow +\infty$ by adding two-valent vertices as necessary. The next step is to glue the D_i into \mathbf{C} such that their real parts go to C_i keeping the orientation. We do this such that \mathbf{R} is fully covered by B and the D_i (in type (A) and (B), we have to throw in by hand the point p which closes up the open bounded end of B). The next step is to connect some non-real ends of D_i to B . Of course, it suffices to describe the construction in the upper half plane, and then copy it symmetrically. We start with type (B) and $i = s$. In this case there is a unique maximum in D_s closest to p , and an end of D_s emanating from it. There is a unique α -vertex contained in the connected component of B adjacent to p and to which this end of D_i can be glued. We connect them. Among the m_i maxima of D_i which are not “adjacent” to p , we select the subset of size c_i corresponding to M_i . We consider the c_i ends of D_i which emanate from the these maxima. There is a unique way (up to homeomorphism) to glue these ends to α -vertices of the connected components of B adjacent to C_i without producing extra intersections. Again, we connect them like this. We now complete the dessins as follows. Each end of D_i of type $m \rightarrow \infty$ not yet glued to a α -vertex is replaced by a sequence of edges $m \rightarrow m+1 \rightarrow \dots \rightarrow m+k \rightarrow +\infty$. This includes, in type (C), the end adjacent to the simple pole in D_s . Finally, all non-real ends $-\infty \rightarrow 1$ and $m+k \rightarrow +\infty$ are extended to infinity to obtain an honest affine dessin Γ . It is clear that this is indeed the inverse map we are looking for. \blacksquare

Using this Proposition, we can explicitly describe the generating series of S -numbers for a given base B . We use the same conventions and point configuration \mathcal{P} as at the beginning of this section.

Definition 7.6. Given a base $B = (B', s)$, we denote by $\mathcal{S}_B(\mathcal{P}, \mathbf{\Lambda})$ the set of real simple rational dessins Γ with $B_\Gamma = B$ (up to real orientation preserving homeomorphism). We set $S_B(\boldsymbol{\lambda}, m) := S(\mathcal{S}_B(\mathcal{P}, \mathbf{\Lambda}))$ and

$$F_B := \sum_{m \geq 0} S_B(\boldsymbol{\lambda}, m) \frac{q^m}{m!}.$$

Remark 7.7. Clearly, after fixing the parity of d and $\boldsymbol{\lambda}$, there is only a finite collection of bases that occur as B_Γ . Hence we can express F_λ^{odd} and F_λ^{even} as finite sums

$$F_\lambda^{\text{odd}} = \sum_B F_B \quad \text{and} \quad F_\lambda^{\text{even}} = \sum_B F_B.$$

Definition 7.8. Let B be a base and let Σ_j be the sequence of degrees of real vertices of B contributing to λ_j (i.e., the vertices with label $m+j$), with an extra 1 inserted at the position of C_s *only* if B is of type (A) or (B). The *sign* of a base B is

$$\varepsilon(B) := (-1)^{\text{dis } \Sigma_1 + \dots + \text{dis } \Sigma_k}.$$

Remark 7.9. In case (C), two consecutive 1's should be inserted, one for the simple pole p , and one for the 2-valent vertex on the segment in $\varphi^{-1}([\alpha, \infty))$ adjacent to p , which is not part of B . In the definition, equivalently, we do not insert anything.

We denote the differential operator $D = q \frac{d}{dq}$. We introduce the following functions

$$\begin{aligned} f_c &= \frac{1}{2^c c!} (D-1)(D-3) \cdots (D-2c+1)f, \\ g_c &= \frac{1}{2^c c!} D(D-2) \cdots (D-2c+2)g, \\ \tilde{g}_c &= \frac{1}{2^c c!} (D-2)(D-4) \cdots (D-2c)g, \\ u_c &= \frac{1}{2^c c!} (D-3)(D-5) \cdots (D-2c-1)u, \\ v_c &= \frac{1}{2^c c!} (D-2)(D-4) \cdots (D-2c)v. \end{aligned}$$

Theorem 7.10. *Fix a base B and set $\varepsilon = \varepsilon(B)$. Then, depending on the different cases, the generating series F_B can be written as in following table.*

F_B	type (A)		type (B)	type (C)
d odd	$\varepsilon \prod_{\substack{i=1 \\ i \neq s}}^l f_{c_i}$		$\varepsilon \tilde{g}_{c_s} \prod_{\substack{i=1 \\ i \neq s}}^l f_{c_i}$	$\varepsilon u_{c_s} \prod_{\substack{i=1 \\ i \neq s}}^l f_{c_i}$
d even	$\varepsilon g_{c_1} \prod_{\substack{i=1 \\ i \neq s}}^l f_{c_i}$	$s = 1$	$-\varepsilon f_{c_1} \prod_{i=2}^l f_{c_i}$	$\varepsilon v_{c_1} \prod_{i=2}^l f_{c_i}$
		$s \neq 1$	$\varepsilon g_{c_1} \tilde{g}_{c_s} \prod_{\substack{i=2 \\ i \neq s}}^l f_{c_i}$	$\varepsilon g_{c_1} u_{c_s} \prod_{\substack{i=2 \\ i \neq s}}^l f_{c_i}$

Proof. Up to signs, the statement follows directly from Proposition 7.5. Indeed, let us compare to the ingredients of a chain data set from Definition 7.4. Ingredient (b) corresponds to the coefficients in f, g, u, v by Proposition 6.2 and Remark 6.3, ingredient (c) corresponds to the extra factors in the coefficients of $f_c, g_c, \tilde{g}_c, u_c, v_c$, compared to the non-indexed counterparts, and finally ingredient (a) corresponds to taking the product of the generating series. For the sign computations, first note that all disorders $\text{dis } \Sigma_{b_j}, j = m+1, \dots, m+k$ are accounted for in $\varepsilon(B)$ (see also Remark 7.9). For any simple branch point $b_j, j = 1, \dots, m$, the parity of $\text{dis } \Sigma_{b_j}$ only depends on whether the corresponding critical point is a local maximum or minimum, which is compatible with the signs in f, g, u, v by Proposition 6.2. There is one exception, namely if the special chain for type (B) is of the form $[a, b)$ (pole on the right), in which case an extra reflection was necessary. There are two subcases: For $s \neq 1$ the number of maxima and minima in D_s is equal, hence the the sign is not affected by the reflection. For $s = 1$, however, the number of maxima and minima differs by one, which is fixed by the extra minus sign in this case. \blacksquare

Remark 7.11. Recall from [IZ16, Lemma 5.3] that

$$Dq = q, \quad Df = q(1 - f^2), \quad \text{and} \quad Dg = -qfg. \quad (18)$$

In particular, if $h \in \mathbf{Q}[q, f]$, then $Dh \in \mathbf{Q}[q, f]$, and if $h \in \mathbf{Q}[q, f]g$, then $Dh \in \mathbf{Q}[q, f]g$.

Proof of Theorem 7.1. By Corollary 6.6 and Remark 7.11, we have $f_c, g_c, \tilde{g}_c, u_c, v_c \in \mathbf{Q}[q, f, g]$. Moreover, by Theorem 7.10, we have that $F_B \in \mathbf{Q}[q, f, g]$. Therefore, the result follows from Remark 7.7. \blacksquare

8 Non-vanishing statements for generating series

The functions f and g satisfy the relation $f^2 + g^2 = 1$, which implies

$$\mathbf{Q}[q, f, g] = \mathbf{Q}[q, f] + \mathbf{Q}[q, f]g.$$

The corresponding decomposition of an element in $\mathbf{Q}[q, f, g]$, removing higher powers of g , is called *g-reduced*. The following lemma asserts uniqueness of this representation.

Proposition 8.1. *The family of power series $q^i f^j, q^i f^j g, i, j \in \mathbf{N} \cup \{0\}$ is linearly independent over \mathbf{C} .*

Proof. Regarded as meromorphic functions, $f = \tanh$ and $g = \operatorname{sech}$ have the same set of poles $p_k := (1/2 + k)i\pi, k \in \mathbf{Z}$, all of them simple. Moreover, a simple calculation shows $\operatorname{Res}_{p_k}(f) = i$ and $\operatorname{Res}_{p_k}(g) = (-1)^{k-1}i$. We consider an equality of power series or, equivalently, meromorphic functions

$$0 = a + \sum_{j=1}^{\delta} (a_j f + b_j g) f^{j-1},$$

with $a, a_j, b_j \in \mathbf{C}[q]$. Looking at the highest order term, we can observe that all poles in $a_\delta f + b_\delta g$ need to cancel out, since otherwise the expression on the right hand side has a pole of order δ . But this implies

$$0 = \operatorname{Res}_{p_k}(a_\delta f + b_\delta g) = i(a_\delta + (-1)^{k-1}b_\delta)(p_k)$$

for all $k \in \mathbf{Z}$. This implies that both polynomials $a_\delta + b_\delta$ and $a_\delta - b_\delta$ have infinitely many zeros, and hence $a_\delta = b_\delta = 0$. Proceeding recursively, we deduce that all the polynomials a_j, b_j and finally a are zero. \blacksquare

From now on, we make constant use of the proposition and always replace elements in $\mathbf{Q}[q, f, g]$ by their unique *g-reduced* representation.

Set $\deg(q^i f^j) = (i, j)$ and extend to a degree function on $\mathbf{Q}[q, f]$ using the lexicographic order, i.e. $(i, j) < (i', j')$ if and only if $i < i'$ or $i = i'$ and $j < j'$. For an element $F = F_1 + F_2 g \in \mathbf{Q}[q, f] \oplus \mathbf{Q}[q, f]g$, we define the *f-degree* $\deg_f(F) := \deg(F_1)$ and the *g-degree* $\deg_g(F) := \deg(F_2) + (0, 1)$ if $F_2 \neq 0$, otherwise $\deg_g(F) = 0$. Note that for $F, F' \in \mathbf{Q}[q, f]g$, we have $\deg_f(F \cdot F') = \deg_g(F) + \deg_g(F')$. The derivation rules in Equation (18) imply the following statement.

Corollary 8.2. For any $F \in \mathbf{Q}[q, f]f \oplus \mathbf{Q}[q, f]g$ we have

$$\deg_f(DF) = \deg_f(F) + (1, 1), \quad \deg_g(DF) = \deg_g(F) + (1, 1).$$

Moreover, the leading coefficient of DF with respect to f -degree and g -degree is $-j$ times the corresponding leading coefficient of F , where $\deg_f(F) = (i, j)$, respectively, $\deg_g(F) = (i, j)$.

Proposition 8.3. Let B be a base with l chains and let $c = \sum_{i=1}^l c_i$ be the total number of connected components of B contained in $\{\mathfrak{S}(z) > 0\}$. Then $\deg(F_B)$ takes value as described in the following table.

		type (A)	type (B)	type (C)
d odd	$\deg_f(F_B)$	$(c, c + l - 1)$	$(0, 0)$	$(c + 1, c + l + 1)$
	$\deg_g(F_B)$	$(0, 0)$	$(c, c + l)$	$(c, c + l + 1)$ [1]
d even	$\deg_f(F_B)$	$(0, 0)$	$(c, c + l)$ [2]	$(c, c + l + 1)$ [3]
	$\deg_g(F_B)$	$(c, c + l - 1)$	$(0, 0)$	$(c + 1, c + l + 1)$

Moreover, the coefficients of the leading terms are $\varepsilon(B)(-1)^c/2^c$, except for the cases marked by [1], [2], [3], which have leading coefficients

$$[1] \ \varepsilon(B)(-1)^c/2^{c-1} \quad [2] \ \varepsilon(B)(-1)^{c-1}/2^c \quad [3] \ \varepsilon(B)(-1)^{c-1}/2^{c-1}.$$

Proof. Applying Corollary 8.2, we can compute the degrees of the modified generating functions as described in the following table.

	f_n	g_n	\tilde{g}_n	u_n	v_n
\deg_f	$(n, n + 1)$	$(0, 0)$	$(0, 0)$	$(n + 1, n + 2)$	$(n, n + 2)$
\deg_g	$(0, 0)$	$(n, n + 1)$	$(n, n + 1)$	$(n, n + 2)$	$(n + 1, n + 2)$

Then the statement about degrees follows from Theorem 7.10. Similarly, by Corollary 8.2 the various leading coefficients of these functions are $(-1)^n/2^n$, except for the following special cases.

- [1] The term u_{c_s} carries an extra factor of 2 by Corollary 6.6.
- [2] The terms $-f_{c_1}$ and $g_{c_1}\tilde{g}_{c_s}$ produce an extra -1 .
- [3] The terms v_{c_1} and $g_{c_1}u_{c_s}$ carry an extra factor of -2 by Corollary 6.6.

Hence the statement about leading coefficients follows. ■

Notation 8.4. For a fixed set of reduced partitions $\boldsymbol{\lambda} := (\lambda_1, \dots, \lambda_k)$, denote by \mathfrak{c} the number of pairs in each partition of $\boldsymbol{\lambda}$, and by \mathfrak{o} (resp. \mathfrak{e}) the number of partitions in $\boldsymbol{\lambda}$ with an odd (resp. even) entry appearing an odd number of times.

From now on, we fix the parity of d and a sequence of reduced ramification profiles $\boldsymbol{\lambda}$ such that the non-vanishing criteria in Theorem 1.2 are satisfied. We denote by \mathfrak{o} (resp. \mathfrak{e}) the number of partitions in $\boldsymbol{\lambda}$ with an odd (resp. even) entry appearing an odd number of times. We denote by \mathfrak{c} the number of pairs of equal entries in each partition of $\boldsymbol{\lambda}$, such that $2\mathfrak{c} + \mathfrak{o} + \mathfrak{e} = \sum_j l(\lambda_j)$.

Theorem 8.5. *The degrees of F_λ^{odd} and F_λ^{even} can be bounded by*

$$\deg_f F_\lambda^{odd} \leq (\mathfrak{c} + 1, \mathfrak{c} + \mathfrak{o} + 2), \quad \deg_g F_\lambda^{odd} \leq (\mathfrak{c}, \mathfrak{c} + \mathfrak{o} + 2), \quad (19)$$

$$\deg_f F_\lambda^{even} \leq (\mathfrak{c}, \mathfrak{c} + \mathfrak{o} + 2), \quad \deg_g F_\lambda^{even} \leq (\mathfrak{c} + 1, \mathfrak{c} + \mathfrak{o} + 2). \quad (20)$$

Proof. Let B be a base of given type. We note that $c \leq \mathfrak{c}$, since any connected component contained in $\{\Im(z) > 0\}$ accounts for at least one pair of complex conjugated critical points. Assume that B is a base with $c = \mathfrak{c}$. Any connected component $B \cap \mathbf{R}$ of the form $[a, b]$ (a bounded closed interval) contains at least one vertex of degree $4n$, hence the number of such components is bounded by \mathfrak{o} . It follows that the number of chains l is bounded by $l \leq \mathfrak{o} + 2$ for type (A) and (B) and by $l \leq \mathfrak{o} + 1$ for type (C). Then the claim follows from Proposition 8.3. \blacksquare

Our goal in the remaining subsections is to prove sharpness of these estimates (in same cases) by proving that the corresponding leading coefficient is non-zero.

Definition 8.6. A *simple base* B is a base of type (B) or (C) such that

- the number of connected components contained in $\{\Im(z) > 0\}$ is \mathfrak{c} ,
- the number of bounded closed connected components in $B \cap \mathbf{R}$ is \mathfrak{o} ,

Remark 8.7. Simple bases are exactly the maximal bases appearing in the proof of Theorem 8.5. Each connected component Z_+ contained in $\{\Im(z) > 0\}$ contains exactly one vertex of degree greater than 2. Each bounded closed connected component of $B \cap \mathbf{R}$ is \mathfrak{o} contains exactly one vertex of degree $4n$ and no two of these vertices carry the same label. They correspond to local maxima and are called the *maxima* of B . There are \mathfrak{e} vertices of degree $4n + 2$, $n > 0$, which we call the *crossings* of B . Again, the labels of crossings are pairwise distinct. In type (B), there exists exactly one finite half-closed connected component of $B \cap \mathbf{R}$, and it contains no maximum, but at least one crossing. In particular, if $\mathfrak{e} = 0$, simple bases of type (B) do not exist. However, simple bases of

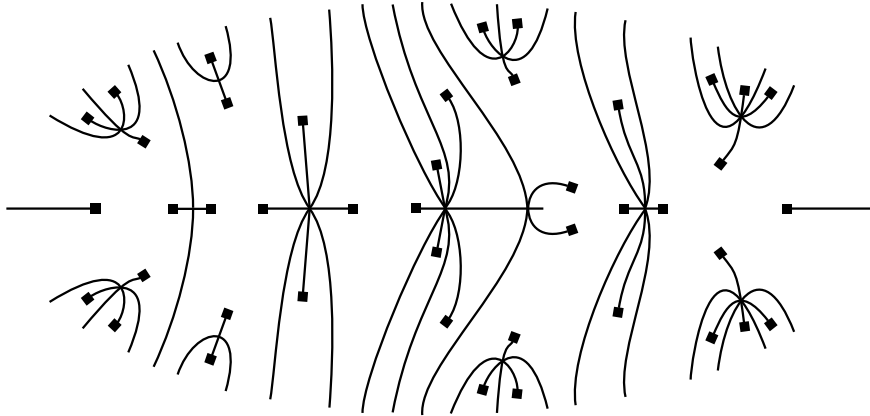


Figure 13 A simple base of type (B) with $\mathfrak{c} = 4$, $\mathfrak{o} = 3$ and $\mathfrak{e} = 2$.

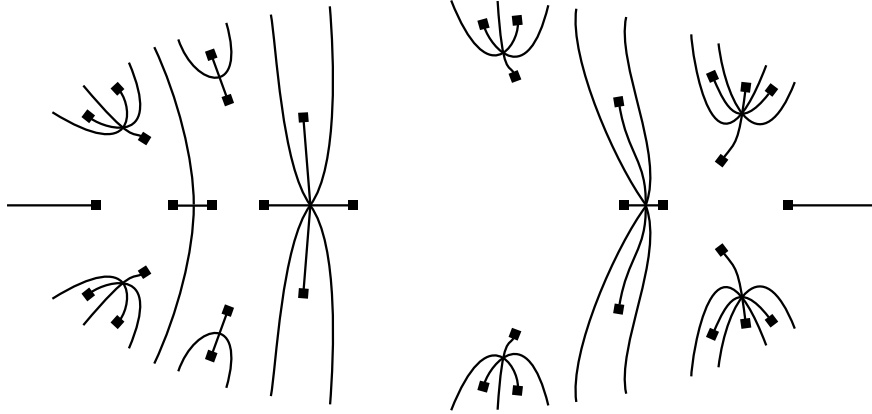


Figure 14 A simple base of type (C) with $c = 4$, $o = 3$ and $\epsilon = 0$.

type (B) do exist if $\epsilon > 0$, as well as simple bases of type (C) for any value of ϵ (cf. their polynomial counterparts defined in [IZ16, Section 5]). Examples are given in Figures 13 and 14.

It is now easy to describe the coefficients in front of the monomials corresponding to the upper bounds in Proposition 8.3. Let $\mathbf{SB}_{(B)}$ and $\mathbf{SB}_{(C)}$ denote the set of simple bases of type (B) and type (C), respectively. We defined signs on these sets in Definition 7.8 and can take the signed counts $S(\mathbf{SB}_{(B)})$ and $S(\mathbf{SB}_{(C)})$.

Corollary 8.8. *The coefficients of the monomials corresponding to f -degree $(c + 1, c + o + 2)$, g -degree $(c, c + o + 2)$, for d odd, and f -degree $(c, c + o + 2)$, g -degree $(c + 1, c + o + 2)$, for d even, are equal, in the same order, to $\frac{(-1)^c}{2^c}$ times*

$$S(\mathbf{SB}_{(C)}), \quad S(\mathbf{SB}_{(B)}) + 2S(\mathbf{SB}_{(C)}), \quad -S(\mathbf{SB}_{(B)}) - 2S(\mathbf{SB}_{(C)}), \quad S(\mathbf{SB}_{(C)}).$$

Proof. The statement follows by Proposition 8.3, Theorem 8.5, and Remark 8.7. ■

8.1 The case $\epsilon = 0$

Lemma 8.9. *If $\epsilon = 0$, no simple bases of type (B) exist, and all simple bases of type (C) have the same sign $(-1)^o$.*

Proof. The non-existence of bases of type (B) is explained in Remark 8.7. Assume B is a simple base of type (C). Since for each label j there exists at most one real vertex with label j , and all of them are maxima (i.e., the non-one entries in Σ_{b_j} are succeeded by an odd number of 1's), each maximum contributes -1 to $\varepsilon(B)$, and the claim follows. ■

8.2 The case $\epsilon > 0$

Definition 8.10. Given a simple bases B , a crossings v and a 2-valent vertex w of the same label, we denote by $\text{Flip}_w^v(B)$ the base obtained from B by exchanging the two

vertices together with adjacent edges of v in $\mathbf{C} \setminus \mathbf{R}$. If we flip an “increasing” and a “decreasing” vertex, we also reverse the order of these edges. We denote by $[B]$ the equivalence class of simple bases up to changing the position of the crossings via the flip operation.

Proposition 8.11. *If ϵ is an even positive integer, then*

$$S([B]) = \begin{cases} 0 & d \text{ odd, type (C) ,} \\ (-1)^{\sigma+\mathfrak{b}} & d \text{ odd, type (B), or } d \text{ even, type (C).} \end{cases}$$

Here, \mathfrak{b} denotes the number of partitions λ_i with both an odd and even entry appearing an odd number of times, and the even entry is bigger than the odd entry.

Note that we neglect the case d even, type (B), which is not needed here.

Proof. The connected components of $B \cap \mathbf{R}$ of form $(-\infty, y]$, $[x, +\infty)$, $[x, y]$ and $[x, y)$ or $(x, y]$ are called left end, right end, ordinary segment, special segment in the following. For each crossing, there are zero or two possible positions on an ordinary segment and one position on any other segment. Let v be a crossing (labelled by j) sitting on the left end (only if d is odd) or on an ordinary segment. Let w be the 2-valent vertex of the same label on the right end or on the same ordinary segment. Then $\text{Flip}_w^v(B)$ has opposite sign. This is clear in the second case and easy to verify in the first case, since the sequence Σ_j we are manipulating is of the form

$$(a, \underbrace{1, \dots, 1}_{\text{even number}}, 1) \quad \text{or} \quad (a, \underbrace{1, \dots, 1, b, 1 \dots 1}_{\text{odd number}}, 1),$$

with $1 \neq a \neq b \neq 1$. The second case occurs if and only if there is a maximum with same label as v . We can easily turn this into an involution on $[B]$, reducing the computation of $S([B])$ to those bases for which all crossings are located on the special segment, if B is of type (B), and all crossings located on the right end, if B is of type (C) and d is even.

It remains to show that the sign of such bases is $(-1)^{\sigma+\mathfrak{b}}$. To see this, we first note that a crossing labelled with j is succeeded by an odd or even number of 1's in Σ_j , depending on whether the special segment is of the form $(x, y]$ or $[x, y)$. Since ϵ is even, the total contribution of the corresponding disorders is always even. It remains to count the disorders involving a maximum. A maximum labelled by j is succeeded by an odd number l_j of entries in Σ_j . If no crossing of label j and higher degree than the maximum exists, the number of disorders involving the isaximum is l_j . If a crossing of higher degree than the maximum exists, then the number of disorders involving the maximum is $l_j + 1$ or $l_j - 1$, depending on whether the crossing lies before or after the maximum. Hence the critical levels to be counted are exactly those which contain a maximum, but no crossing of higher degree. The number of such levels is $\sigma - \mathfrak{b}$. ■

8.3 Proofs of main theorems

Proof of Theorem 1.2. The vanishing statements are contained in Propositions 5.1 and 5.2, and it remains to prove non-vanishing under the given conditions. Fix λ and the parity of d such that the conditions in Theorem 1.2 are satisfied. Recall that Theorem 7.1 asserts that the series $F_\lambda^{odd}(q)$ and $F_\lambda^{even}(q)$ are polynomials in q , f and g . Therefore, by Proposition 8.1, it suffices to prove that in the g -reduced representations of $F_\lambda^{odd}(q)$ and $F_\lambda^{even}(q)$ at least one of the coefficients is non-zero.

d odd, $\epsilon = 0$: By Corollary 8.8 it suffices to show $S(\mathbf{SB}_{(C)}) \neq 0$. This follows by Lemma 8.9, since all elements in $\mathbf{SB}_{(C)}$ have the same sign.

d odd, $\epsilon > 0$: By Corollary 8.8 it suffices to show $S(\mathbf{SB}_{(B)}) + 2S(\mathbf{SB}_{(C)}) \neq 0$. This follows by Proposition 8.11, since $S(\mathbf{SB}_{(C)}) = 0$ and $S(\mathbf{SB}_{(B)}) = (-1)^{\mathfrak{o}+\mathfrak{b}}E$, where E denotes the number of equivalence classes in $\mathbf{SB}_{(B)}$.

d even: By Corollary 8.8 it suffices to show $S(\mathbf{SB}_{(C)}) \neq 0$. This follows by Propositions 8.9 and 8.11, since $S(\mathbf{SB}_{(C)}) = (-1)^{\mathfrak{o}+\mathfrak{b}}E$, where E denotes the number of equivalence classes in $\mathbf{SB}_{(C)}$. This completes the proof. \blacksquare

Remark 8.12. To compute the numbers $S(\mathbf{SB}_{(B)})$ and $S(\mathbf{SB}_{(C)})$ appearing in the previous proof, we note that simple bases belonging to different equivalence classes differ by the position of s , and the ordering of the pairs of complex conjugate critical points and real critical maxima. Moreover, we need to avoid overcounting bases having two critical points with same multiplicity and label in $\{\Im(z) > 0\}$. Denote by N_j^i the number of times an element i appears in λ_j and set

$$A(\lambda) = \prod_{j=1}^k \prod_i [N_j^i/2],$$

where the second product runs through all i with $N_j^i \geq 2$. Then for all d we have

$$\mathbf{SB}_{(C)} = \frac{(\mathfrak{o} + 1) \cdot (\mathfrak{o} + \mathfrak{c})!}{A(\lambda)},$$

and for d odd and $\epsilon > 0$ we have

$$\mathbf{SB}_{(B)} = \frac{2 \cdot (\mathfrak{o} + \mathfrak{c} + 1)!}{A(\lambda)}.$$

Proof of Theorem 1.3. Since we assume that the non-vanishing criteria from Theorem 1.2 are satisfied, we have $F := F_\lambda^{odd} \neq 0$ or $F := F_\lambda^{even} \neq 0$, respectively. By the proof of Proposition 8.1 we conclude that F has bounded convergence radius $\pi/2$, with poles $\pm i\pi/2$ on the boundary circle. Note that F is either an even or an odd function, depending on the parity of $\sum_{i,j} \lambda_j^i$. Following [IZ16, Proof of Theorem 5], let G denote the function obtained by dividing by q , if F is odd, and performing the variable change $Q = q^2$. Then G has a unique singularity at radius $\pi^2/4$ and hence $\ln |b_k| \sim -k \ln(\pi^2/4)$, where b_k denote the coefficients in G . Since $b_k = S(\lambda, m)/m!$ for $m = 2k$ or $m = 2k + 1$, respectively, the claim follows. \blacksquare

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