BISYMMETRIC AND QUASITRIVIAL OPERATIONS: CHARACTERIZATIONS AND ENUMERATIONS

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Abstract. We investigate the class of bisymmetric and quasitrivial binary operations on a given set X and provide various characterizations of this class as well as the subclass of bisymmetric, quasitrivial, and order-preserving binary operations. We also determine explicitly the sizes of these classes when the set X is finite.

1. Introduction

Let X be an arbitrary nonempty set. We use the symbol X_n if X is finite of size $n \geq 1$, in which case we assume w.l.o.g. that $X_n = \{1, \ldots, n\}$.

Recall that a binary operation $F: X^2 \to X$ is *bisymmetric* if it satisfies the functional equation

$$F(F(x,y), F(u,v)) = F(F(x,u), F(y,v))$$

for all $x, y, u, v \in X$. The bisymmetry property for binary real operations has first been studied by Aczél [2, 3]. Since then, it has been investigated in the theory of functional equations, especially in characterizations of mean functions (see, e.g., [4, 11, 12]). This property has also been extensively investigated in algebra where it is called *mediality*. For instance, a groupoid (X, F) where F is a bisymmetric binary operation on X is called a *medial groupoid* (see, e.g., [13–17]).

In this paper, which is a continuation of [8], we investigate the class of binary operations $F: X^2 \to X$ that are bisymmetric and quasitrivial, where quasitriviality means that F always outputs one of its input values. It is known that any bisymmetric and quasitrivial operation is associative (see Kepka [17, Corollary 10.3]). This observation is of interest since it shows that the class of bisymmetric and quasitrivial operations is a subclass of the class of associative and quasitrivial operations. The latter class was characterized independently by several authors (see, e.g., Kepka [17, Corollary 1.6] and Länger [20, Theorem 1]) and a recent and elementary proof of this characterization is available in [8, Theorem 2.1]. We also investigate certain subclasses of bisymmetric and quasitrivial operations by adding properties such as order-preservation and existence of neutral and/or annihilator elements. In the finite case (i.e., $X = X_n$ for any integer $n \geq 1$), we enumerate the class of bisymmetric and quasitrivial operations as well as the latter subclasses.

The outline of this paper is as follows. After presenting some definitions and preliminary results (including the above-mentioned characterization of the class of

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associative and quasitrivial operations, see Theorem 2.14) in Section 2, we introduce in Section 3 the concept of "quasilinear weak ordering" and show how it can be used to characterize the class of bisymmetric and quasitrivial operations $F: X^2 \to X$ (see Theorem 3.6). We also provide an alternative characterization of the latter class when $X = X_n$ for some integer $n \geq 1$ (see Theorem 3.6). In particular, the latter characterizations give answers to open questions asked in [7, Section 5, Question (b)] and [8, Section 6]. We then recall the weak single-peakedness property (see Definition 3.7) as a generalization of single-peakedness to arbitrary weakly ordered sets and we use it to characterize the class of bisymmetric, quasitrivial, and order-preserving operations (see Theorem 3.14). In Section 4, we restrict ourselves to the finite case where $X = X_n$ and compute the size of the class of bisymmetric and quasitrivial operations as well as the sizes of some subclasses discussed in this paper. By doing so, we point out some known integer sequences and introduce new ones. In particular, the search for the number of bisymmetric and quasitrivial binary operations on X_n for any integer $n \geq 1$ (see Proposition 4.2) gives rise to a sequence that was previously unknown in the Sloane's On-Line Encyclopedia of Integer Sequences (OEIS, see [24]). All the (old and new) sequences that we consider are given in explicit forms and through their generating functions or exponential generating functions (see Propositions 4.1, 4.2, 4.4, and 4.5). Finally, in Section 5 we further investigate the quasilinearity property of the weak orderings on X that are weakly single-peaked w.r.t. a fixed linear ordering on X.

2. Preliminaries

In this section we recall and introduce some basic definitions and provide some preliminary results.

Recall that a binary relation R on X is said to be

- *total* if $\forall x, y$: xRy or yRx;
- *transitive* if $\forall x, y, z$: xRy and yRz implies xRz;
- *antisymmetric* if $\forall x, y$: xRy and yRx implies $x = y$.

A binary relation $\leq$ on X is said to be a *linear ordering on X* if it is total, transitive, and antisymmetric. In that case the pair $(X, \leq)$ is called a *linearly ordered set*. For any integer $n \geq 1$, we can assume w.l.o.g. that the pair $(X_n, \leq_n)$ represents the set $X_n = \{1, \ldots, n\}$ endowed with the linear ordering relation $\leq_n$ defined by $1 <_n \cdots <_n n$.

A binary relation $\lesssim$ on X is said to be a *weak ordering on X* if it is total and transitive. In that case the pair $(X, \lesssim)$ is called a *weakly ordered set*. We denote the symmetric and asymmetric parts of $\lesssim$ by $\sim$ and $<$, respectively. Recall that $\sim$ is an equivalence relation on X and that $<$ induces a linear ordering on the quotient set $X/\sim$. For any $u \in X$ we denote the equivalence class of u by $[u]_\sim$, i.e., $[u]_\sim = \{x \in X : x \sim u\}$.

For any linear ordering $\preceq$ and any weak ordering $\lesssim$ on X, we say that $\preceq$ is *subordinated to* $\lesssim$ if for any $x, y \in X$, we have that $x < y$ implies $x \prec y$.

For a weak ordering $\lesssim$ on X, an element $u \in X$ is said to be *maximal* (resp. *minimal*) *for* $\lesssim$ if $x \lesssim u$ (resp. $u \lesssim x$) for all $x \in X$. We denote the set of maximal (resp. minimal) elements of X for $\lesssim$ by $\max_\lesssim X$ (resp. $\min_\lesssim X$).

Definition 2.1. An operation $F: X^2 \to X$ is said to be

- *idempotent* if $F(x, x) = x$ for all $x \in X$.

- *quasitrivial* (or *conservative*) if $F(x,y) \in \{x,y\}$ for all $x,y \in X$.
- *commutative* if $F(x,y) = F(y,x)$ for all $x,y \in X$.
- *associative* if $F(F(x,y),z) = F(x,F(y,z))$ for all $x,y,z \in X$.
- *bisymmetric* if $F(F(x,y),F(u,v)) = F(F(x,u),F(y,v))$ for all $x,y,u,v \in X$.
- $\leq$-*preserving* for some linear ordering $\leq$ on X if for any $x,y,x',y' \in X$ such that $x \leq x'$ and $y \leq y'$ we have $F(x,y) \leq F(x',y')$.

Definition 2.2. Let $F\colon X^2 \to X$ be an operation.

- An element $e \in X$ is said to be a *neutral element of* F if $F(e,x) = F(x,e) = x$ for all $x \in X$. In this case we can easily show by contradiction that the neutral element is unique.
- An element $a \in X$ is said to be an *annihilator of* F if $F(x,a) = F(a,x) = a$ for all $x \in X$. In this case we can easily show by contradiction that the annihilator is unique.
- The points (x,y) and (u,v) of X^2 are said to be F-*connected* if $F(x,y) = F(u,v)$. The point (x,y) of X^2 is said to be F-*isolated* if it is not F-connected to another point of X^2.

When X_n is endowed with $\leq_n$, the operations $F\colon X_n^2 \to X_n$ can be easily visualized by showing their contour plots, where we connect by edges or paths all the points of X_n^2 having the same values by F. For instance, the operation $F\colon X_3^2 \to X_3$ defined by $F(2,2) = F(2,3) = 2$, $F(1,x) = 1$, and $F(3,x) = F(2,1) = 3$ for $x = 1,2,3$, is idempotent. Its contour plot is shown in Figure 1.

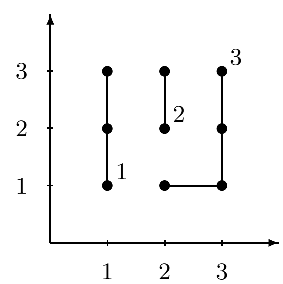

FIGURE 1. An idempotent operation on X_3 (contour plot)

Definition 2.3. Let $\leq$ be a linear ordering on X. We say that an operation $F\colon X^2 \to X$ has

- a $\leq$-*disconnected level set* if there exist $x,y,u,v,s,t \in X$, with $(x,y) < (u,v) < (s,t)$, such that $F(x,y) = F(s,t) \neq F(u,v)$.
- a *horizontal (resp. vertical)* $\leq$-*disconnected level set* if there exist $x,y,z,u \in X$, with $x < y < z$, such that $F(x,u) = F(z,u) \neq F(y,u)$ (resp. $F(u,x) = F(u,z) \neq F(u,y)$).

Fact 2.4. *Let $\leq$ be a linear ordering on X. If $F\colon X^2 \to X$ has a horizontal or vertical $\leq$-disconnected level set then it has a $\leq$-disconnected level set.*

Remark 1. We observe that, for any linear ordering $\leq$ on X, an operation $F\colon X^2 \to X$ having a $\leq$-disconnected level set need not have a horizontal or vertical $\leq$-disconnected level set. Indeed, the operation $F\colon X_3^2 \to X_3$ whose contour plot is depicted in Figure 2 has a $\leq_3$-disconnected level set since $F(1,1) = F(2,3) = 1 \neq 2 = F(2,2)$ but it has no horizontal or vertical $\leq_3$-disconnected level set.

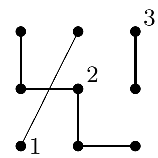

FIGURE 2. An idempotent operation on X_3

Lemma 2.5. *Let $\leq$ be a linear ordering on X. If $F\colon X^2 \to X$ is quasitrivial, then it has a $\leq$-disconnected level set iff it has a horizontal or vertical $\leq$-disconnected level set.*

Proof. (Necessity) Suppose that F has a $\leq$-disconnected level set and let us show that it has a horizontal or vertical $\leq$-disconnected level set. By assumption, there exist $x,y,u,v,s,t \in X$, with $(x,y) < (u,v) < (s,t)$, such that $F(x,y) = F(s,t) \neq F(u,v)$. Since F is quasitrivial, we have $F(x,y) \in \{x,y\}$. Suppose that $F(x,y) = x$ (the other case is similar). Also, since F is quasitrivial, we have $s = x$ or $t = x$. If $s = x$, then $u = x$ and thus F has a vertical $\leq$-disconnected level set. Otherwise, if $t = x$ and $s \neq x$, then $y \leq x$. If $y = x$, then $v = x$ and thus F has a horizontal $\leq$-disconnected level set. Otherwise, if $y < x$, then considering the point $(s,y) \in X^2$, we get $(x,y) < (s,y) < (s,x)$ and $F(x,y) = F(s,x) = x \neq F(s,y) \in \{s,y\} = \{F(s,s), F(y,y)\}$, which shows that F has either a horizontal or a vertical $\leq$-disconnected level set.

(Sufficiency) This follows from Fact 2.4. □

We define the *strict convex hull of $x,y \in X$ w.r.t.* $\leq$ by $\text{conv}_{\leq}(x,y) = \{z \in X \mid x < z < y\}$, if $x < y$, and $\text{conv}_{\leq}(x,y) = \{z \in X \mid y < z < x\}$, if $y < x$.

Recall that for any linear ordering $\leq$ on X, a subset C of X is said to be *convex w.r.t.* $\leq$ if for any $x,y,z \in X$ such that $y \in \text{conv}_{\leq}(x,z)$, we have that $x,z \in C$ implies $y \in C$.

Remark 2. (a) For any quasitrivial operation $F\colon X^2 \to X$ and any $x \in X$, consider the sets

$$L_x^h(F) = \{y \in X \mid F(y,x) = x\} \quad \text{and} \quad L_x^v(F) = \{y \in X \mid F(x,y) = x\}.$$

Clearly, for any linear ordering $\leq$ on X, a quasitrivial operation $F\colon X^2 \to X$ has no $\leq$-disconnected level set iff for any $x \in X$, the sets $L_x^h(F)$ and $L_x^v(F)$ are convex w.r.t. $\leq$. Indeed, if $L_x^h(F)$ or $L_x^v(F)$ is not convex w.r.t. $\leq$, then F has a horizontal or vertical $\leq$-disconnected level set and thus by Fact 2.4 it has a $\leq$-disconnected level set. Conversely, if F has a $\leq$-disconnected level set, then by Lemma 2.5 it has a horizontal or vertical $\leq$-disconnected level set. Suppose that it has a horizontal $\leq$-disconnected level set (the other case is similar). By assumption, there exist $x,y,z,u \in X$, with $x < y < z$, such that $F(x,u) = F(z,u) \neq F(y,u)$. Since F is quasitrivial, we have

$F(x,u) = F(z,u) = u$ and $F(y,u) = y$ which shows that the set $L_u^h(F)$ is not convex w.r.t. $\leq$.

(b) Recall that the *kernel of an operation* $F\colon X^2 \to X$ is the set

$$\ker(F) = \{((a,b),(c,d)) \in X^2 \times X^2 \mid F(a,b) = F(c,d)\}.$$

Clearly, $\ker(F)$ is an equivalence relation on X^2. It is not difficult to see that for any linear ordering $\leq$ on X, a quasitrivial operation $F\colon X^2 \to X$ has no $\leq$-disconnected level set iff for any $x \in X$, the class of (x,x) w.r.t. $\ker(F)$ is convex w.r.t. $\leq$.

Fact 2.6. *Let $\leq$ be a linear ordering on X. If $F\colon X^2 \to X$ is $\leq$-preserving then it has no $\leq$-disconnected level set.*

Proposition 2.7. *Let $\leq$ be a linear ordering on X. If $F\colon X^2 \to X$ is quasitrivial, then it is $\leq$-preserving iff it has no $\leq$-disconnected level set.*

Proof. (Necessity) This follows from Fact 2.6.

(Sufficiency) Suppose that F has no $\leq$-disconnected level set and let us show by contradiction that F is $\leq$-preserving. Suppose for instance that there exist $x,y,z \in X$, $y < z$, such that $F(x,y) > F(x,z)$. By quasitriviality we see that $x \notin \{y,z\}$. Suppose for instance that $x < y < z$ (the other cases are similar). By quasitriviality we have $F(x,y) = y$ and $F(x,z) = x = F(x,x)$, and hence by Lemma 2.5 F has a $\leq$-disconnected level set, a contradiction. □

Remark 3. We cannot relax quasitriviality into idempotency in Proposition 2.7. Indeed, the operation $F\colon X_3^2 \to X_3$ whose contour plot is depicted in Figure 1 is idempotent and has no $\leq_3$-disconnected level set. However it is not $\leq_3$-preserving.

Certain links between associativity and bisymmetry were investigated by several authors (see, e.g., [7, 17, 21, 23, 25]). We gather them in the following lemma.

Lemma 2.8 (see, [7, Lemma 22])**.** (i) *If $F\colon X^2 \to X$ is bisymmetric and has a neutral element, then it is associative and commutative.*

(ii) *If $F\colon X^2 \to X$ is associative and commutative, then it is bisymmetric.*

(iii) *If $F\colon X^2 \to X$ is quasitrivial and bisymmetric, then it is associative.*

Corollary 2.9. (i) *If $F\colon X^2 \to X$ is commutative and quasitrivial, then it is associative iff it is bisymmetric.*

(ii) *If $F\colon X^2 \to X$ has a neutral element, then it is associative and commutative iff it is bisymmetric.*

Remark 4. In [8, Theorem 3.3], the class of associative, commutative, and quasitrivial operations was characterized. In particular, it was shown that there are exactly $n!$ associative, commutative, and quasitrivial operations on X_n. Using Corollary 2.9(i) we can replace associativity with bisymmetry in [8, Theorem 3.3].

Lemma 2.10 (see [7, Proposition 4])**.** *Let $F\colon X^2 \to X$ be a quasitrivial operation and let $e \in X$. Then e is a neutral element of F iff (e,e) is F-isolated.*

For any integer $n \geq 1$, any $F\colon X_n^2 \to X_n$, and any $z \in X_n$, the *F-degree of z*, denoted $\deg_F(z)$, is the number of points $(x,y) \in X_n^2 \setminus \{(z,z)\}$ such that $F(x,y) = F(z,z)$. Also, the *degree sequence of F*, denoted $\deg_F$, is the nondecreasing n-element sequence of the degrees $\deg_F(x)$, $x \in X_n$. For instance, the degree sequence of the operation $F\colon X_3^2 \to X_3$ defined in Figure 1 is $\deg_F = (1,2,3)$.

Remark 5. If $F: X_n^2 \to X_n$ is a quasitrivial operation, then there exists at most one element $x \in X_n$ such that $\deg_F(x) = 0$. Indeed, otherwise by Lemma 2.10, F would have at least two distinct neutral elements, a contradiction.

Lemma 2.11. *If $F: X_n^2 \to X_n$ is quasitrivial, then for all $x \in X_n$, we have $\deg_F(x) \leq 2(n-1)$.*

Proof. Let $x \in X_n$. Since $F: X_n^2 \to X_n$ is quasitrivial, the point (x, x) can only be F-connected to points of the form (x, y) or (y, x) for some $y \in X_n \setminus \{x\}$. □

The following result was mentioned in [8] without proof.

Proposition 2.12. *Let $F: X_n^2 \to X_n$ be a quasitrivial operation and let $a \in X_n$. Then a is an annihilator of F iff $\deg_F(a) = 2(n-1)$.*

Proof. (Necessity) By definition of an annihilator, we have $F(x, a) = F(a, x) = a = F(a, a)$ for all $x \in X_n \setminus \{a\}$. Thus, we have $\deg_F(a) \geq 2(n-1)$ and hence by Lemma 2.11 we conclude that $\deg_F(a) = 2(n-1)$.

(Sufficiency) By quasitriviality, the point (a, a) cannot be F-connected to a point (u, v) where $u \neq a$ and $v \neq a$. Thus, since $\deg_F(a) = 2(n-1)$, we have $a = F(a, a) = F(x, a) = F(a, x)$ for all $x \in X_n \setminus \{a\}$. □

Remark 6. We observe that Proposition 2.12 no longer holds if we relax quasitriviality into idempotency. Indeed, the operation $F: X_3^2 \to X_3$ whose contour plot is depicted in Figure 3 is idempotent and the element $a = 1$ is the annihilator of F. However, $\deg_F(1) = 6 > 4$.

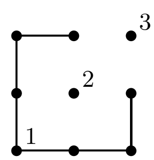

FIGURE 3. An idempotent operation with an annihilator on X_3

Corollary 2.13. *Let $F: X_n^2 \to X_n$ be a quasitrivial operation. Any element $a \in X$ such that $\deg_F(a) = 2(n-1)$ is unique and is of maximal F-degree.*

The *projection operations* $\pi_1: X^2 \to X$ and $\pi_2: X^2 \to X$ are respectively defined as $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ for all $x, y \in X$.

Given a weak ordering $\precsim$ on X, the *maximum* (resp. *minimum*) *operation on X w.r.t.* $\precsim$ is the commutative operation $\max_{\precsim}$ (resp. $\min_{\precsim}$) defined on $X^2 \setminus \{(x, y) \in X^2 : x \sim y,\ x \neq y\}$ as $\max_{\precsim}(x, y) = y$ (resp. $\min_{\precsim}(x, y) = x$) whenever $x \precsim y$. We also note that if $\precsim$ reduces to a linear ordering, then the operation $\max_{\precsim}$ (resp. $\min_{\precsim}$) is defined everywhere on X^2.

The following theorem provides a characterization of the class of associative and quasitrivial operations on X. In [1, Section 1.2], it was observed that this result is a simple consequence of two papers on idempotent semigroups (see [18] and [22]). It was also independently discovered by various authors (see, e.g., [17, Corollary 1.6] and [20, Theorem 1]). A short and elementary proof has also been given in [8, Theorem 2.1].

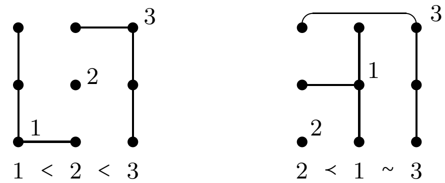

Figure 4. An associative and quasitrivial operation on X_3 that is not bisymmetric

Theorem 2.14. *An operation $F\colon X^2 \to X$ is associative and quasitrivial iff there exists a weak ordering $\precsim$ on X such that*

$$F|_{A\times B} = \begin{cases} \max_{\precsim}|_{A\times B}, & \text{if } A \neq B, \\ \pi_1|_{A\times B} \text{ or } \pi_2|_{A\times B}, & \text{if } A = B, \end{cases} \qquad \forall A,B \in X/\sim. \tag{1}$$

Remark 7. As observed in [8], for any associative and quasitrivial operation $F\colon X^2 \to X$, there is exactly one weak ordering $\precsim$ on X for which F is of the form (1). This weak ordering is defined by: $x \precsim y$ iff $F(x,y) = y$ or $F(y,x) = y$. Moreover, if $X = X_n$ for some integer $n \geq 1$, then $\precsim$ can be defined as follows: $x \precsim y$ iff $\deg_F(x) \leq \deg_F(y)$. The latter observation follows from [8, Proposition 2.2] which states that for any $x \in X_n$ we have

$$\deg_F(x) = 2 \times |\{z \in X_n : z \prec x\}| + |\{z \in X_n : z \sim x,\ z \neq x\}|. \tag{2}$$

Some associative and quasitrivial operations are bisymmetric. For instance, so are the projection operations π_1 and π_2. However, some other operations are not bisymmetric. Indeed, the operation $F\colon X_3^2 \to X_3$ whose contour plot is depicted in Figure 4 is associative and quasitrivial since it is of the form (1) for the weak ordering $\precsim$ on X_3 defined by $2 \prec 1 \sim 3$. However, this operation is not bisymmetric since $F(F(2,3),F(1,2)) = F(3,1) = 3 \neq 1 = F(1,3) = F(F(2,1),F(3,2))$. In the next section, we provide a characterization of the subclass of bisymmetric and quasitrivial operations (see Theorem 3.6).

3. Characterizations of bisymmetric and quasitrivial operations

In this section we provide characterizations of the class of bisymmetric and quasitrivial operations $F\colon X^2 \to X$ as well as the subclass of bisymmetric, quasitrivial, and $\leq$-preserving operations $F\colon X^2 \to X$ for some fixed linear ordering $\leq$ on X.

Definition 3.1. We say that a weak ordering $\precsim$ on X is *quasilinear* if there exist no pairwise distinct $a,b,c \in X$ such that $a \prec b \sim c$.

Hence, a weak ordering $\precsim$ on X is quasilinear iff for every $x,y \in X$, $x \neq y$, we have that $x \sim y$ implies $x,y \in \min_{\precsim} X$.

Remark 8. For any integer $n \geq 1$, the weak orderings $\precsim$ on X_n that are quasilinear are known in social choice theory as *top orders* (see, e.g., [10, Section 2]).

Fact 3.2. *Let $\precsim$ be a quasilinear weak ordering on X.*

- *If $|\min_{\precsim} X| = 1$ then it is a linear ordering.*
- *If $\max_{\precsim} X \neq \varnothing$, then $\max_{\precsim} X = X$ or $|\max_{\precsim} X| = 1$.*

Proposition 3.3. *Let $F: X^2 \to X$ be of the form (1) for some weak ordering $\precsim$ on X. Then, $\precsim$ is quasilinear iff for any linear ordering $\leq'$ on X subordinated to $\precsim$, F is $\leq'$-preserving.*

Proof. (Necessity) By the form (1) of F and by quasilinearity of $\precsim$, F is $\leq'$-preserving for any linear ordering $\leq'$ on X subordinated to $\precsim$.

(Sufficiency) We proceed by contradiction. Suppose that F is $\leq'$-preserving for any linear ordering $\leq'$ on X subordinated to $\precsim$. Suppose also that there exist pairwise distinct $a, b, c \in X$, such that $a \prec b \sim c$. Fix a linear ordering $\leq'$ on X subordinated to $\precsim$. Suppose that $a <' b <' c$ (the other case is similar). If $F|_{[b]^2_{\sim}} = \pi_1|_{[b]^2_{\sim}}$, then

$$F(b,c) = b <' c = F(a,c),$$

a contradiction. The case where $F|_{[b]^2_{\sim}} = \pi_2|_{[b]^2_{\sim}}$ is similar. □

Proposition 3.4. *Let $F: X^2 \to X$ be of the form (1) for some weak ordering $\precsim$ on X. Then, F is bisymmetric iff $\precsim$ is quasilinear.*

Proof. (Necessity) We proceed by contradiction. Suppose that F is bisymmetric and suppose also that there exist pairwise distinct $a, b, c \in X$ such that $a \prec b \sim c$. If $F|_{[b]^2_{\sim}} = \pi_1|_{[b]^2_{\sim}}$ then

$$F(F(a,c), F(b,a)) = F(c,b) = c \neq b = F(b,c) = F(F(a,b), F(c,a)),$$

a contradiction. The case where $F|_{[b]^2_{\sim}} = \pi_2|_{[b]^2_{\sim}}$ is similar.

(Sufficiency) Let $x, y, u, v \in X$, not all equal, and let us show that

$$F(F(x,y), F(u,v)) = F(F(x,u), F(y,v)).$$

By applying Fact 3.2 to the subset $\{x, y, u, v\}$ of X we have two cases to consider.

- If $x \sim y \sim u \sim v$ then $F|_{\{x,y,u,v\}} = \pi_i|_{\{x,y,u,v\}}$, $i \in \{1,2\}$, and hence $F|_{\{x,y,u,v\}}$ is bisymmetric.
- Otherwise, if $|\max_{\precsim}\{x, y, u, v\}| = 1$ then we can assume w.l.o.g. that $x \precsim y \precsim u \prec v$. By the form (1) of F it is clear that v is the annihilator of $F|_{\{x,y,u,v\}}$. Hence, we have

$$F(F(x,y), F(u,v)) = v = F(F(x,u), F(y,v)).$$

Therefore, F is bisymmetric. □

The following proposition provides an additional characterization of the class of bisymmetric and quasitrivial operations when $X = X_n$ for some integer $n \geq 1$.

Proposition 3.5. *Let $F: X_n^2 \to X_n$ be quasitrivial. Then, F is bisymmetric iff it is associative and satisfies*

$$\deg_F = (\underbrace{k-1, \ldots, k-1}_{k \text{ times}}, 2k, 2k+2, \ldots, 2n-2) \tag{3}$$

for some $k \in \{1, \ldots, n\}$.

Proof. (Necessity) By Lemma 2.8(iii), F is associative. By Proposition 3.4, there exists a quasilinear weak ordering $\precsim$ on X_n such that F is of the form (1). By Remark 7, we see that (2) holds for any $x \in X_n$. Therefore, we obtain the claimed form of the degree sequence of F.

(Sufficiency) By Theorem 2.14, there exists a weak ordering $\precsim$ on X_n such that F is of the form (1). By Remark 7, we have that $x \precsim y$ iff $\deg_F(x) \leq \deg_F(y)$.

Thus, according to our assumptions on the degree sequence of F, we have that $\precsim$ is quasilinear. Finally, using Proposition 3.4, F is bisymmetric. $\square$

The following theorem, which is an immediate consequence of Lemma 2.8(iii), Theorem 2.14, and Propositions 2.7, 3.3, 3.4, and 3.5, provides characterizations of the class of bisymmetric and quasitrivial operations.

Theorem 3.6. *Let $F\colon X^2 \to X$ be an operation. The following assertions are equivalent.*

(i) *F is bisymmetric and quasitrivial.*
(ii) *F is of the form (1) for some quasilinear weak ordering $\precsim$ on X.*
(iii) *F is of the form (1) for some weak ordering $\precsim$ on X and for any linear ordering $\leq'$ subordinated to $\precsim$, F is $\leq'$-preserving.*
(iv) *F is of the form (1) for some weak ordering $\precsim$ on X and for any linear ordering $\leq'$ subordinated to $\precsim$, F has no $\leq'$-disconnected level set.*

If $X = X_n$ for some integer $n \geq 1$, then any of the assertions (i)-(iv) is equivalent to the following one.

(v) *F is associative, quasitrivial, and satisfies (3) for some $k \in \{1,\ldots,n\}$.*

Remark 9. We observe that in [8, Theorem 3.3] the authors proved that an operation $F\colon X_n^2 \to X_n$ is associative, commutative, and quasitrivial iff it is quasitrivial and satisfies

$$\deg_F = (0,2,\ldots,2(n-1)).$$

Surprisingly, in Theorem 3.6(v), we have obtained a similar result by relaxing commutativity into bisymmetry. Moreover, it provides an easy test to check whether an associative and quasitrivial operation on X_n is bisymmetric. Indeed, the associative and quasitrivial operation $F\colon X_3^2 \to X_3$ whose contour plot is depicted in Figure 4 is not bisymmetric since $\deg_F = (0,3,3)$ is not of the form given in Theorem 3.6(v).

The rest of this section is devoted to the subclass of bisymmetric, quasitrivial, and $\leq$-preserving operations $F\colon X^2 \to X$ for a fixed linear ordering $\leq$ on X. In order to characterize this subclass, we first need to recall the concept of weak single-peakedness.

Definition 3.7. (see [8, Definition 4.3]) Let $\leq$ be a linear ordering on X and let $\precsim$ be a weak ordering on X. The weak ordering $\precsim$ is said to be *weakly single-peaked with respect to* $\leq$ if for all $x,y,z \in X$ such that $x < y < z$ we have $y \prec x$ or $y \prec z$ or $x \sim y \sim z$.

Remark 10. (a) If the weak ordering mentioned in Definition 3.7 is a linear ordering, then we simply say that it is *single-peaked with respect to* $\leq$ (see [9, Definition 3.8]). Note that single-peakedness was first introduced by Black [6] for linear orderings on finite sets. It is also easy to show by induction that there are exactly 2^{n-1} single-peaked linear orderings on X_n w.r.t. $\leq_n$ (see, e.g., [5]).

(b) In [8, Theorem 5.6] the authors proved that the weak orderings on X_n that are weakly single-peaked w.r.t. $\leq_n$ are exactly the weak orderings on X_n that are *single-plateaued w.r.t.* $\leq_n$ (see, [10, Definition 4 and Lemma 17]).

Fact 3.8. *Let $\leq$ be a linear ordering on X and let $\precsim$ be a quasilinear weak ordering on X that is weakly single-peaked w.r.t. $\leq$. If $|\min_{\precsim} X| = 1$, then $\precsim$ is a linear ordering that is single-peaked w.r.t. $\leq$.*

The next proposition provides a characterization of the class of associative, quasitrivial, and $\leq$-preserving operations $F\colon X^2 \to X$ for a fixed linear ordering $\leq$ on X.

Proposition 3.9. *(see [8, Theorem 4.5]) Let $\leq$ be a linear ordering on X. An operation $F\colon X^2 \to X$ is associative, quasitrivial, and $\leq$-preserving iff F is of the form* (1) *for some weak ordering $\precsim$ on X that is weakly single-peaked w.r.t. $\leq$.*

Remark 11. In [8, Theorem 3.7], the class of associative, commutative, quasitrivial, and order-preserving operations has been characterized. In particular, it was shown that there are exactly 2^{n-1} associative, commutative, quasitrivial, and $\leq_n$-preserving operations on X_n. Using Corollary 2.9(i) we can replace associativity with bisymmetry in [8, Theorem 3.7].

Using Lemma 2.8(iii), Theorem 3.6, and Proposition 3.9 we can easily derive the following result.

Proposition 3.10. *Let $\leq$ be a linear ordering on X. An operation $F\colon X^2 \to X$ is bisymmetric, quasitrivial, and $\leq$-preserving iff F is of the form* (1) *for some quasilinear weak ordering $\precsim$ on X that is weakly single-peaked with respect to $\leq$.*

When $X = X_n$ for some integer $n \geq 1$, we provide in Proposition 3.13 an additional characterization of the class of bisymmetric, quasitrivial, and $\leq_n$-preserving operations. We first consider two preliminary results.

Proposition 3.11. *An operation $F\colon X_n^2 \to X_n$ is quasitrivial, $\leq_n$-preserving, and satisfies $\deg_F = (n-1,\ldots,n-1)$ iff $F = \pi_1$ or $F = \pi_2$.*

Proof. (Necessity) Since F is quasitrivial we know that $F(1,n) \in \{1,n\}$. Suppose that $F(1,n) = n = F(n,n)$ (the other case is similar). Since F is $\leq_n$-preserving, we have $F(x,n) = n$ for all $x \in X_n$. Since $\deg_F(n) = n-1$, it follows that $F(n,y) = y$ for all $y \in X_n$. In particular, we have $F(n,1) = 1 = F(1,1)$, and by $\leq_n$-preservation we obtain $F(x,1) = 1$ for all $x \in X_n$. Finally, since $\deg_F(1) = n-1$, it follows that $F(1,y) = y$ for all $y \in X_n$. Thus, since F is $\leq_n$-preserving, we have

$$y = F(1,y) \leq_n F(x,y) \leq_n F(n,y) = y, \quad x,y \in X_n$$

which shows that $F = \pi_2$.

(Sufficiency) Obvious. □

Remark 12. We observe that Proposition 3.11 no longer holds if quasitriviality is relaxed into idempotency. Indeed, the operation $F\colon X_3^2 \to X_3$ whose contour plot is depicted in Figure 5 is idempotent, $\leq_3$-preserving, and satisfies $\deg_F = (2,2,2)$ but it is neither π_1 nor π_2. We also observe that Proposition 3.11 no longer holds if we omit $\leq_n$-preservation. Indeed, the operation $F\colon X_3^2 \to X_3$ whose contour plot is depicted in Figure 6 is quasitrivial and satisfies $\deg_F = (2,2,2)$ but it is neither π_1 nor π_2.

Lemma 3.12. *Let $F\colon X_n^2 \to X_n$ be a quasitrivial and $\leq_n$-preserving operation and let $a \in X_n$. If a is an annihilator of F, then $a \in \{1,n\}$.*

Proof. We proceed by contradiction. Suppose that $a \in X_n \setminus \{1,n\}$. Since F is quasitrivial, we have $F(1,n) \in \{1,n\}$. Suppose that $F(1,n) = 1 = F(1,1)$ (the other case is similar). Then

$$1 = F(1,1) \leq_n F(1,a) \leq_n F(1,n) = 1,$$

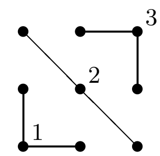

Figure 5. An idempotent and $\leq_3$-preserving operation

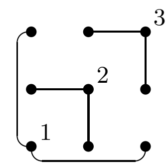

Figure 6. A quasitrivial operation that is not $\leq_3$-preserving

and hence $F(1,a) = 1$ a contradiction. □

Proposition 3.13. *Let $F\colon X_n^2 \to X_n$ be quasitrivial and $\leq_n$-preserving. Then F is bisymmetric iff it satisfies* (3) *for some $k \in \{1,\ldots,n\}$.*

Proof. (Necessity) This follows from Proposition 3.5.

(Sufficiency) We proceed by induction on n. The result clearly holds for $n = 1$. Suppose that it holds for some $n \geq 1$ and let us show that it still holds for $n+1$. Assume that $F\colon X_{n+1}^2 \to X_{n+1}$ is quasitrivial, $\leq_{n+1}$-preserving, and satisfies

$$\deg_F = (\underbrace{k-1,\ldots,k-1}_{k \text{ times}}, 2k, 2k+2, \ldots, 2n)$$

for some $k \in \{1,\ldots,n+1\}$. If $k = n+1$ then by Proposition 3.11 we have that $F = \pi_1$ or $F = \pi_2$ and hence F is clearly bisymmetric. Otherwise, if $k \in \{1,\ldots,n\}$ then, by the form of the degree sequence of F, there exists an element $a \in X_{n+1}$ such that $\deg_F(a) = 2n$. Using Proposition 2.12 we have that a is an annihilator of F. Moreover, by Lemma 3.12, we have $a \in \{1, n+1\}$. Suppose that $a = n+1$ (the other case is similar). Then, $F' = F|_{X_n^2}$ is clearly quasitrivial, $\leq_n$-preserving, and satisfies (3). Thus, by induction hypothesis, F' is bisymmetric. Since $a = n+1$ is the annihilator of F, we necessarily have that F is bisymmetric. □

The following theorem, which is an immediate consequence of Theorem 3.6 and Propositions 3.10 and 3.13, provides characterizations of the class of bisymmetric, quasitrivial, and order-preserving operations.

Theorem 3.14. *Let $\leq$ be a linear ordering on X and let $F\colon X^2 \to X$ be an operation. The following assertions are equivalent.*

(i) *F is bisymmetric, quasitrivial, and $\leq$-preserving.*
(ii) *F is of the form* (1) *for some quasilinear weak ordering $\precsim$ on X that is weakly single-peaked w.r.t. $\leq$.*
(iii) *F is of the form* (1) *for some weak ordering $\precsim$ on X that is weakly single-peaked w.r.t. $\leq$ and for any linear ordering $\leq'$ subordinated to $\precsim$, F is $\leq'$-preserving.*
(iv) *F is of the form* (1) *for some weak ordering $\precsim$ on X that is weakly single-peaked w.r.t. $\leq$ and for any linear ordering $\leq'$ subordinated to $\precsim$, F has no $\leq'$-disconnected level set.*

If $(X,\leq) = (X_n,\leq_n)$ for some integer $n \geq 1$, then any of the assertions (i)-(iv) is equivalent to the following one.

(v) *F is quasitrivial, $\leq_n$-preserving, and satisfies* (3) *for some $k \in \{1,\ldots,n\}$.*

Remark 13. We observe that in [8, Theorem 3.3] the authors proved that an operation $F\colon X_n^2 \to X_n$ is associative, quasitrivial, commutative, and $\leq_n$-preserving iff it is quasitrivial, $\leq_n$-preserving, and satisfies

$$\deg_F = (0, 2, \ldots, 2(n-1)).$$

Surprisingly, in Theorem 3.14(v), we have obtained a similar result by relaxing commutativity into bisymmetry. Moreover, it provides an easy test to check whether a quasitrivial and $\leq_n$-preserving operation on X_n is bisymmetric. Indeed, the quasitrivial and $\leq_3$-preserving operation $F\colon X_3^2 \to X_3$ whose contour plot is depicted in Figure 4 (left) is not bisymmetric since $\deg_F = (0,3,3)$ is not of the form given in Theorem 3.14(v). It is important to note that in this test, associativity of the given operation need not be checked.

4. Enumerations of bisymmetric and quasitrivial operations

In [8, Section 4], the authors computed the size of the class of associative and quasitrivial operations $F\colon X_n^2 \to X_n$ as well as the sizes of some subclasses obtained by considering commutativity and order-preservation. Some of these computations gave rise to previously unknown integer sequences, which were then posted in OEIS (for instance A292932 and A293005). In the same spirit, this section is devoted to the enumeration of the class of bisymmetric and quasitrivial operations $F\colon X_n^2 \to X_n$ as well as some of its subclasses. The integer sequences that emerge from our investigation are also now posted in OEIS (see, e.g., Axxxxxx).

We also consider either the (ordinary) generating function (GF) or the exponential generating function (EGF) of a given sequence $(s_n)_{n\geq 0}$. Recall that when these functions exist, they are respectively defined by the power series

$$S(z) \;=\; \sum_{n\geq 0} s_n\, z^n \quad\text{and}\quad \hat{S}(z) \;=\; \sum_{n\geq 0} s_n\, \frac{z^n}{n!}\,.$$

For any integer $n \geq 0$ we denote by $p(n)$ the number of weak orderings on X_n that are quasilinear. We also denote by $p_e(n)$ (resp. $p_a(n)$) the number of weak orderings $\precsim$ on X_n that are quasilinear and for which X_n has exactly one minimal element (resp. one maximal element) for $\precsim$. Clearly, $p_e(n)$ is the number of linear orderings on X_n, namely $p_e(n) = n!$ for $n \geq 1$. Proposition 4.1 below provides explicit formulas for $p(n)$ and $p_a(n)$. The first few values of these sequences are shown in Table 1.[1]

Proposition 4.1. *The sequence $(p(n))_{n\geq 0}$ satisfies the linear recurrence equation*

$$p(n+1) - (n+1)p(n) = 1, \qquad n \geq 0,$$

with $p(0) = 0$, and we have the closed-form expression

$$p(n) = n! \sum_{k=1}^{n} \frac{1}{k!}, \qquad n \geq 1.$$

Moreover, its EGF is given by $\hat{P}(z) = (e^z - 1)/(1-z)$. Furthermore, for any integer $n \geq 2$ we have $p_a(n) = p(n) - 1$, with $p_a(0) = 0$ and $p_a(1) = 1$.

[1] Note that the sequence A000142 differs from $(p_e(n))_{n\geq 0}$ only at $n = 0$.

Proof. We clearly have

$$p(n) = \sum_{k=1}^{n} \binom{n}{k,1,\ldots,1}, \qquad n \geq 1,$$

where the multinomial coefficient $\binom{n}{k,1,\ldots,1}$ provides the number of ways to put the elements $1,\ldots,n$ into $(n-k+1)$ classes of sizes $k,1,\ldots,1$. The claimed linear recurrence equation and the EGF of $(p(n))_{n\geq 1}$ follow straightforwardly. Regarding the sequence $(p_a(n))_{n\geq 0}$ we observe that $\max_{\precsim} X_n \neq X_n$ whenever $n \geq 2$ (see Fact 3.2). □

n	$p(n)$	$p_e(n)$	$p_a(n)$
0	0	0	0
1	1	1	1
2	3	2	2
3	10	6	9
4	41	24	40
5	206	120	205
6	1237	720	1236
OEIS	A002627	A000142	Axxxxxx

TABLE 1. First few values of $p(n)$, $p_e(n)$, and $p_a(n)$

For any integer $n \geq 0$ we denote by $q(n)$ the number of bisymmetric and quasitrivial operations $F\colon X_n^2 \to X_n$. We also denote by $q_e(n)$ (resp. $q_a(n)$) the number of bisymmetric and quasitrivial operations $F\colon X_n^2 \to X_n$ that have neutral elements (resp. annihilator elements). Proposition 4.2 provides explicit formulas for these sequences. The first few values of these sequences are shown in Table 2.[2]

Proposition 4.2. *The sequence $(q(n))_{n\geq 0}$ satisfies the linear recurrence equation*

$$q(n+1) - (n+1)q(n) = 2, \qquad n \geq 1,$$

with $q(0) = 0$ and $q(1) = 1$, and we have the closed-form expression

$$q(n) = 2p(n) - n! = n!\left(2\sum_{i=1}^{n} \frac{1}{i!} - 1\right), \qquad n \geq 1.$$

Moreover, its EGF is given by $\hat{Q}(z) = (2e^z - 3)/(1-z)$. Furthermore, for any integer $n \geq 1$ we have $q_e(n) = n!$, with $q_e(0) = 0$. Also, for any integer $n \geq 2$ we have $q_a(n) = q(n) - 2$, with $q_a(0) = 0$ and $q_a(1) = 1$.

Proof. It is not difficult to see that the number of bisymmetric and quasitrivial operations on X_n is given by

$$q(n) = 2p(n) - n! = n!\left(2\sum_{i=1}^{n} \frac{1}{i!} - 1\right), \qquad n \geq 1.$$

Indeed, since $F\colon X_n^2 \to X_n$ is bisymmetric and quasitrivial, we have by Theorem 3.6 that F is of the form (1) for some quasilinear weak ordering $\precsim$ on X_n. Since $F|_{\min_{\precsim} X_n} = \pi_1$ or π_2, we have to count twice the number of k-element subsets of X_n,

[2]Note that the sequence A000142 differs from $(q_e(n))_{n\geq 0}$ only at $n = 0$.

for every $k \in \{1, \ldots, n\}$. However, the number of linear orderings on X_n should be counted only once (indeed, by Remark 4 there is a one-to-one correspondence between linear orderings and bisymmetric, commutative, and quasitrivial operations on X_n). Hence, $q(n) = 2p(n) - n!$. The claimed linear recurrence equation and the EGF of $(q(n))_{n\geq 1}$ follow straightforwardly. Using Corollary 2.9(ii) and Remark 4, we observe that the sequence $(q_e(n))_{n\geq 0}$, with $q_e(0) = 0$, gives the number of linear orderings on X_n. Finally, regarding the sequence $(q_a(n))_{n\geq 0}$, we observe that $\max_{\lesssim} X_n \neq X_n$ whenever $n \geq 2$ (see Fact 3.2). □

n	$q(n)$	$q_e(n)$	$q_a(n)$
0	0	0	0
1	1	1	1
2	4	2	2
3	14	6	12
4	58	24	56
5	292	120	290
6	1754	720	1752
OEIS	Axxxxxx	A000142	Axxxxxx

TABLE 2. First few values of $q(n)$, $q_e(n)$, and $q_a(n)$

We are now interested in computing the size of the class of bisymmetric, quasitrivial, and $\leq_n$-preserving operations $F\colon X_n^2 \to X_n$. To this extent, we first consider a preliminary result.

Lemma 4.3. *Let $\lesssim$ be a quasilinear weak ordering on X_n that is weakly single-peaked w.r.t. $\leq_n$. If $\max_{\lesssim} X_n \neq X_n$, then $\max_{\lesssim} X \subseteq \{1, n\}$ and $|\max_{\lesssim} X| = 1$.*

Proof. We proceed by contradiction. By Fact 3.2, the set $\max_{\lesssim} X_n$ contains exactly one element. Suppose that $\max_{\lesssim} X_n = \{x\}$, where, $x \in X \setminus \{1, n\}$. Then the triplet $(1, x, n)$ violates weak single-peakedness of $\lesssim$. □

For any integer $n \geq 0$ we denote by $u(n)$ the number of quasilinear weak orderings $\lesssim$ on X_n that are weakly single-peaked w.r.t. $\leq_n$. We also denote by $u_e(n)$ (resp. $u_a(n)$) the number of quasilinear weak orderings $\lesssim$ on X_n that are weakly single-peaked w.r.t. $\leq_n$ and for which X_n has exactly one minimal element (resp. one maximal element) for $\lesssim$. Proposition 4.4 provides explicit formulas for these sequences. The first few values of these sequences are shown in Table 3.[3]

Proposition 4.4. *The sequence $(u(n))_{n\geq 0}$ satisfies the linear recurrence equation*

$$u(n+1) = 2u(n) + 1, \qquad n \geq 0,$$

with $u(0) = 0$, and we have the closed-form expression

$$u(n) = 2^n - 1, \qquad n \geq 0.$$

Moreover, its GF is given by $U(z) = z/(2z^2 - 3z + 1)$. Furthermore, for any integer $n \geq 1$ we have $u_e(n) = 2^{n-1}$ with $u_e(0) = 0$. Also, for any integer $n \geq 2$ we have $u_a(n) = u(n) - 1$ with $u_a(0) = 0$ and $u_a(1) = 1$.

[3]Note that the sequence A131577 differs from $(u_e(n))_{n\geq 0}$ only at $n = 0$.

Proof. We clearly have $u(0)=0$ and $u(1)=1$. So let us assume that $n\geq 2$. If $\precsim$ is a quasilinear weak ordering on X_n that is weakly single-peaked w.r.t. $\leq_n$, then by Lemma 4.3, either $\max_{\precsim} X_n = X_n$ or $\max_{\precsim} X_n = \{1\}$ or $\max_{\precsim} X_n = \{n\}$. In the two latter cases, it is clear that the restriction of $\precsim$ to $X_n \setminus \max_{\precsim} X_n$ is quasilinear and weakly single-peaked w.r.t. the restriction of $\leq_n$ to $X_n \setminus \max_{\precsim} X_n$. It follows that the number $u(n)$ of quasilinear weak orderings on X_n that are weakly single-peaked w.r.t. $\leq_n$ satisfies the first order linear equation

$$u(n) = 1 + u(n-1) + u(n-1), \qquad n\geq 2.$$

The stated expression of $u(n)$ and the GF of $(u(n))_{n\geq 2}$ follow straightforwardly. Using Fact 3.8 and Remark 10(a), we observe that the sequence $(u_e(n))_{n\geq 0}$, with $u_e(0)=0$, gives the number of linear orderings on X_n that are single-peaked w.r.t. $\leq_n$. Finally, regarding the sequence $(u_a(n))_{n\geq 0}$, we observe that $\max_{\precsim} X_n \neq X_n$ whenever $n\geq 2$ (see Fact 3.2). □

n	$u(n)$	$u_e(n)$	$u_a(n)$
0	0	0	0
1	1	1	1
2	3	2	2
3	7	4	6
4	15	8	14
5	31	16	30
6	63	32	62
OEIS	A000225	A131577	Axxxxxx

TABLE 3. First few values of $u(n)$, $u_e(n)$, and $u_a(n)$

For any integer $n\geq 0$ we denote by $v(n)$ the number of bisymmetric, quasitrivial, and $\leq_n$-preserving operations $F\colon X_n^2\to X_n$. We also denote by $v_e(n)$ (resp. $v_a(n)$) the number of bisymmetric, quasitrivial, and $\leq_n$-preserving operations $F\colon X_n^2\to X_n$ that have neutral elements (resp. annihilator elements). Proposition 4.5 provides explicit formulas for these sequences. The first few values of these sequences are shown in Table 4.[4]

Proposition 4.5. *The sequence $(v(n))_{n\geq 2}$ satisfies the linear recurrence equation*

$$v(n+1) = 2v(n) + 2, \qquad n\geq 1,$$

with $v(0)=0$ and $v(1)=1$, and we have the closed-form expression

$$v(n) = 3\cdot 2^{n-1} - 2, \qquad n\geq 1.$$

Moreover, its GF is given by $V(z) = (5z-1)/(4z^2-6z+2)$. Furthermore, for any integer $n\geq 1$ we have $v_e(n) = 2^{n-1}$ with $v_e(0)=0$. Also, for any integer $n\geq 2$ we have $v_a(n) = v(n)-2$ with $v_a(0)=0$ and $v_a(1)=1$.

Proof. We clearly have $v(0)=0$ and $v(1)=1$. So let us assume that $n\geq 2$. If $F\colon X_n^2\to X_n$ is a bisymmetric, quasitrivial, and $\leq_n$-preserving operation, then by Theorem 3.14 it is of the form (1) for some quasilinear weak ordering $\precsim$ on X_n that is

[4]Note that the sequence A131577 differs from $(v_e(n))_{n\geq 0}$ only at $n=0$.

weakly single-peaked w.r.t. $\leq_n$. By Lemma 4.3, either $\max_{\precsim} X_n = X_n$ or $\max_{\precsim} X_n = \{1\}$ or $\max_{\precsim} X_n = \{n\}$. In the first case we have to consider the two projections $F = \pi_1$ and $F = \pi_2$. In the two latter cases, it is clear that the restriction of F to $(X_n \setminus \max_{\precsim} X_n)^2$ is still bisymmetric, quasitrivial, and $\leq_n$-preserving. It follows that the number $v(n)$ of quasitrivial, bisymmetric, and $\leq_n$-preserving operations $F\colon X_n^2 \to X_n$ satisfies the first order linear equation

$$v(n) = 2 + v(n-1) + v(n-1), \qquad n \geq 2.$$

The stated expression of $v(n)$ and the GF of $(v(n))_{n\geq 2}$ follow straightforwardly. Using Corollary 2.9(ii) and Remark 11, we observe that the sequence $(v_e(n))_{n\geq 0}$, with $v_e(0) = 0$, gives the number of linear orderings on X_n that are single-peaked w.r.t. $\leq_n$. Finally, regarding the sequence $(v_a(n))_{n\geq 0}$, we observe that $\max_{\precsim} X_n \neq X_n$ whenever $n \geq 2$ (see Fact 3.2). □

Remark 14. We observe that an alternative characterization of bisymmetric, quasitrivial, and $\leq_n$-preserving operations $F\colon X_n^2 \to X_n$ was obtained in [19, Section 5]. Also, the explicit expression of $v(n)$ as stated in Proposition 4.5 was independently obtained in [19, Section 6] by means of a totally different approach.

n	$v(n)$	$v_e(n)$	$v_a(n)$
0	0	0	0
1	1	1	1
2	4	2	2
3	10	4	8
4	22	8	20
5	46	16	44
6	94	32	92
OEIS	Axxxxxx	A131577	Axxxxxx

TABLE 4. First few values of $v(n)$, $v_e(n)$, and $v_a(n)$

Example 4.6. We show in Figure 7 the $q(3) = 14$ bisymmetric and quasitrivial operations on X_3. Among these operations, $q_e(3) = 6$ have neutral elements, $q_a(3) = 12$ have annihilator elements, and $v(3) = 10$ are $\leq_3$-preserving.

5. Quasilinearity and weak single-peakedness

In this section we investigate some properties of the quasilinear weak orderings on X that are weakly single-peaked w.r.t. a fixed linear ordering $\leq$ on X.

The following lemma provides a characterization of quasilinearity under weak single-peakedness.

Lemma 5.1. *Let $\leq$ be a linear ordering on X and let $\precsim$ be a weak ordering on X that is weakly single-peaked w.r.t. $\leq$. Then $\precsim$ is quasilinear iff there exist no $a,b,c \in X$, with $a < b < c$, such that $b \prec a \sim c$.*

Proof. (Necessity) Obvious.

(Sufficiency) We proceed by contradiction. Suppose that there exist pairwise distinct $a,b,c \in X$, such that $a \prec b \sim c$. By weak single-peakedness we must have $b < a < c$ or $c < a < b$, a contradiction. □

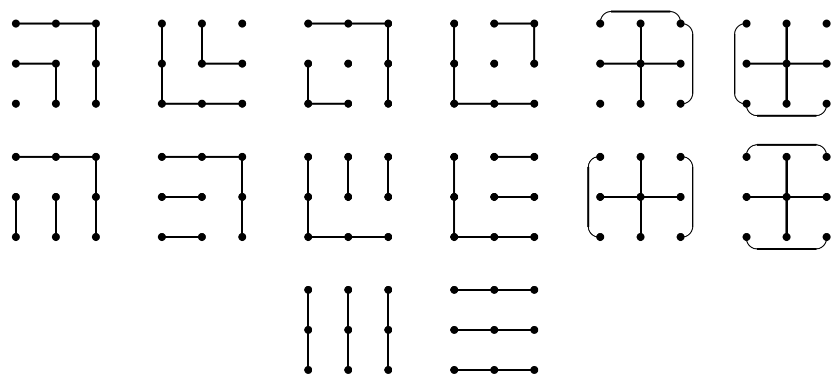

FIGURE 7. The 14 bisymmetric and quasitrivial operations on X_3

The following lemma provides a characterization of weak single-peakedness under quasilinearity.

Lemma 5.2. *Let $\leq$ be a linear ordering on X and let $\precsim$ be a weak ordering on X that is quasilinear. Then $\precsim$ is weakly single-peaked w.r.t. $\leq$ iff for any $a,b,c \in X$ such that $a < b < c$ we have $b \precsim a$ or $b \precsim c$.*

Proof. (Necessity) Obvious.

(Sufficiency) We proceed by contradiction. Suppose that there exist $a,b,c \in X$ satisfying $a < b < c$ such that $b \succ a$ and $b \succsim c$ (the case $b \succsim a$ and $b \succ c$ is similar). We only have two cases to consider. If $b \sim c$, then quasilinearity is violated. Otherwise, if $b \succ c$, we also arrive at a contradiction. □

Remark 15. In Theorems 3.6 and 3.14 we can replace quasilinearity with its equivalent condition stated in Lemma 5.1. Similarly, in Theorems 3.6 and 3.14 we can replace weak single-peakedness with its equivalent condition stated in Lemma 5.2.

From Lemmas 5.1 and 5.2 we obtain the following characterization.

Proposition 5.3. *Let $\leq$ be a linear ordering on X and let $\precsim$ be a weak ordering on X. Let us also consider the following assertions*

(i) *$\precsim$ is weakly single-peaked w.r.t. $\leq$.*
(ii) *There exist no $a,b,c \in X$, with $a < b < c$, such that $b \prec a \sim c$.*
(iii) *For all $a,b,c \in X$ such that $a < b < c$ we have $b \precsim a$ or $b \precsim c$.*
(iv) *$\precsim$ is quasilinear.*

Then the conjunction of assertions (i) and (ii) holds iff the conjunction of assertions (iii) and (iv) holds.

The following result can be easily derived from the previous Proposition 5.3.

Corollary 5.4. *Let $\leq$ be a linear ordering on X and let $\precsim$ be a weak ordering on X that satisfies assertions (i) and (ii) of Proposition 5.3. If $|\min_{\precsim} X| = 1$, then $\precsim$ is a linear ordering on X that is single-peaked w.r.t. $\leq$.*

Let us recall the following notion which was first introduced in [8, Definition 5.4].

Definition 5.5. Let $\leq$ be a linear ordering on X and let $\precsim$ be a weak ordering on X. A subset $P \subseteq X$, $|P| \geq 2$, is called a *plateau with respect to* $(\leq, \precsim)$ if P is convex with respect to $\leq$ and if there exists $x \in X$ such that $P \subseteq [x]_{\sim}$. Moreover, the plateau P is said to be $\precsim$*-minimal* if for all $a \in X$ verifying $a \precsim P$ there exists $z \in P$ such that $a \sim z$.

Proposition 5.6. *Let $\leq$ be a linear ordering on X and let $\precsim$ be a weak ordering on X. Consider the assertions (iii) and (iv) of Proposition 5.3 as well as the following one.*

(iv') *If there exist $a, b \in X$, $a \neq b$, such that $a \sim b$ then $\mathrm{conv}_{\leq}(a, b)$ is a plateau with respect to $(\leq, \precsim)$ and it is $\precsim$-minimal.*

Then we have ((iii) and (iv)) ⇒ (iv'), and (iv') ⇒ (iv).

Proof. $((iii)$ and $(iv)) \Rightarrow (iv')$. We proceed by contradiction. Let $a, b \in X$, $a \neq b$, such that $a \sim b$ and suppose that $\mathrm{conv}_{\leq}(a, b)$ is not a plateau. But then there exists $u \in \mathrm{conv}_{\leq}(a, b)$ such that either $u \prec a \sim b$, which contradicts (iv), or $u \succ a \sim b$ which contradicts (iii). Thus, $\mathrm{conv}_{\leq}(a, b)$ is a plateau and it is $\precsim$-minimal by (iv).

$(iv') \Rightarrow (iv)$. We proceed by contradiction. Suppose that there exist pairwise distinct $a, b, c \in X$ such that $a \prec b \sim c$. But then $\mathrm{conv}_{\leq}(b, c)$ is a plateau which is not $\precsim$-minimal, a contradiction to (iv'). □

6. Conclusion

This paper is based on two known results : (1) a characterization of the class of associative and quasitrivial operations on X (see Theorem 2.14) and (2) the fact that the class of bisymmetric and quasitrivial operations on X is a subclass of the latter one (see Lemma 2.8(iii)). By introducing the concept of quasilinearity for weak orderings on X (see Definition 3.1) we provided a characterization of the class of bisymmetric and quasitrivial operations on X (see Theorem 3.6). To characterize those operations that are $\leq$-preserving (see Theorem 3.14), we considered the concepts of weak single-peakedness (see Definition 3.7) and quasilinearity. Surprisingly, when $X = X_n$, we also provided a characterization of the latter classes (see Theorems 3.6 and 3.14) in terms of the degree sequences, which provides an easy test to check whether a quasitrivial operation is bisymmetric. The latter characterizations give an answer to an open question asked in [7, Section 5, Question (b)]. Moreover, we computed the size of the class of bisymmetric and quasitrivial operations. All the new sequences that arose from our results were posted in OEIS.

In view of these results, some questions arose and we list some of them below.

- We gave a partial answer to an open question asked in [8]. Namely, we were able to generalize [8, Theorems 3.3 and 3.7] by relaxing commutativity into bisymmetry (see Theorems 3.6 and 3.14). It would be interesting to search for a generalization of Theorems 3.6 and 3.14 by removing bisymmetry in assertion (i).
- Generalize Theorems 3.6 and 3.14 by relaxing quasitriviality into idempotency.
- Generalize Theorems 3.6 and 3.14 for n-variable operations, $n \geq 3$.
- The integer sequences A000142, A002627, A000225, and A131577 were previously known in OEIS to solve enumeration issues neither related to quasilinearity nor weak single-peakedness. It would be interesting to search for possible one-to-one correspondences between those problems and ours.

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