A NOTE ON FLIPS IN DIAGONAL RECTANGULATIONS

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ABSTRACT. Rectangulations are partitions of a square into axis-aligned rectangles. A number of results provide bijections between combinatorial equivalence classes of rectangulations and families of pattern-avoiding permutations. Other results deal with local changes involving a single edge of a rectangulation, referred to as flips, edge rotations, or edge pivoting. Such operations induce a graph on equivalence classes of rectangulations, related to so-called *flip graphs* on triangulations and other families of geometric partitions. In this note, we consider a family of flip operations on the equivalence classes of *diagonal rectangulations*, and their interpretation as transpositions in the associated *Baxter* permutations, avoiding the vincular patterns $\{3\underline{14}2, 2\underline{41}3\}$. This complements results from Law and Reading (JCTA, 2012) and provides a complete characterization of flip operations on diagonal rectangulations, in both geometric and combinatorial terms.

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1. INTRODUCTION

1.1. **Flip graphs.** The analysis of geometric partitions of space, such as triangle meshes, binary space partitions, and floorplans for integrated circuits plays a major role in discrete and computational geometry and its applications. In order to understand the underlying combinatorial structure of these partitions, it is often useful to define elementary operations that modify this structure locally. We can then connect distinct partitions using sequences of such operations.

In triangulations, such a notion is known under the term of *flip*. A flip in a triangulation is typically defined as the replacement of an edge shared by two triangles forming a convex quadrilateral by the other diagonal of the quadrilateral. This allows the definition of a *flip graph*, the vertices of which are triangulations, and in which two triangulations are adjacent whenever one can be obtained from the other by a single flip.

Flip graphs have applications in enumeration and random generation of geometric partitions, as well as optimization. The notion of flip graph has been studied for many distinct families of triangulations (maximal planar graphs and triangulations of a point set [7, 8], triangulations of a topological surface [18], see also [15] and references therein), and generalized to other families of geometric partitions, such as domino tilings [23], quadrangulations [17], and rectangulations, the topic of the present contribution. Flip graphs have been shown to have intimate links with many important structures in combinatorics, such as the Catalan objects [25], the Tamari lattice and the associahedra [16], cyclohedra [24], and partial cubes [11].

1.2. **Geometric partitions and pattern-avoiding permutations.** There exists a collection of results establishing bijections between families of geometric space partitions and pattern-avoiding permutations. We will use the *word notation* for permutations, in which a permutation σ is denoted by the word $\sigma(1)\sigma(2)\dots\sigma(n)$. A permutation σ is said to contain the *pattern* π, where π is another permutation, whenever there exists a subsequence of σ whose elements are in the same relative order as the elements of π. Hence for instance the permutation 45213 contains the pattern 213, but also the pattern 3412 in the form of the subsequence 4513. Pattern-avoiding permutations are families of permutations that do not contain any occurence of one or more given patterns.

It is well known, for instance, that triangulations of a convex polygon on $n+2$ vertices are in one-to-one correspondence with 312-avoiding permutations on n elements, and those are counted by the Catalan numbers (OEIS[1] A000108). Similarly, guillotine partitions of a square into n rectangles, obtained by recursive splitting with a horizontal or vertical cut, can easily be seen to be in one-to-one correspondence with $\{3142, 2413\}$-avoiding permutations, called *separable* permutations [6], which are counted by the Schröder numbers (OEIS A006318). This has recently been generalized to separable *d-permutations* and higher-dimensional guillotine partitions [5].

We will use the so-called *vincular* notation for more complex forbidden patterns in permutations. In this notation, an underlined pair of elements indicates that they need to occur consecutively in the permutation. For instance, forbidding the pattern $3\underline{14}2$ amounts to forbidding all occurences of the pattern 3142 with the added condition that 1 and 4 must occur consecutively.

The objects of interest in this paper are *rectangulations*, defined as partitions of a square into axis-aligned rectangles. Several combinatorial equivalence classes of such rectangulations have been defined, which are known to be in one-to-one correspondence with families of permutations avoiding certain vincular patterns. *Mosaic floorplans*, for instance, have been shown to be in correspondence with *Baxter permutations*, avoiding the patterns $3\underline{14}2$ and $2\underline{41}3$. This bijection seems to go back to the work of Dulucq and Guibert on Baxter permutations involving *twin binary trees* [10]. A complete description in terms of twin binary trees is given in Section 6 of Felsner et al. [12]. A simple description of a bijection for mosaic floorplans is given by Ackerman et al. [2]. Mosaic

[1] Online Encyclopedia of Integer Sequences: https://oeis.org/

Permutations	Rectangulations
Separable: $\{3142, 2413\}$-avoiding	Slicing floorplans, or guillotine partitions
Baxter: $\{3\underline{14}2, 2\underline{41}3\}$-avoiding Twisted Baxter: $\{3\underline{41}2, 2\underline{41}3\}$-avoiding $\{3\underline{14}2, 2\underline{14}3\}$-avoiding	Mosaic floorplans, or diagonal rectangulations, or R-equivalent rectangulations [26, 2, 14]
$\{3\underline{41}2, 2\underline{14}3\}$-avoiding	S-equivalent rectangulations [4]
2-clumped: $\{3\underline{51}24, 3\underline{51}42, 24\underline{51}3, 42\underline{51}3\}$-avoiding	generic rectangulations, or rectangular drawings [21]
Separable d-permutations	Guillotine partitions of 2^{d-1}-dimensional boxes [5]

TABLE 1. Known bijections between families of pattern-avoiding permutations and rectangulations.

floorplans are also in bijection with *twisted Baxter permutations* avoiding the patterns $3\underline{41}2$ and $2\underline{41}3$ [14]. These two families are therefore in one-to-one correspondence, together with the family of $\{3\underline{14}2, 2\underline{14}3\}$-avoiding permutations. They will be instrumental in what follows. Interestingly, the $\{3\underline{41}2, 2\underline{14}3\}$-avoiding permutations are *not* in one-to-one correspondence with (twisted) Baxter permutations. These were actually shown by Asinowski et al. to count other equivalence classes of rectangulations [4], namely those preserving the neighborhood relation between the *segments* of the rectangulation. Equivalence classes of *generic* rectangulations, also known as *rectangular drawings*, have been shown to be in one-to-one correspondence with *2-clumped* permutations, avoiding the vincular patterns $\{3\underline{51}24, 3\underline{51}42, 24\underline{51}3, 42\underline{51}3\}$ [21]. Table 1 lists the known bijections between families of pattern-avoiding permutations and rectangulations.

1.3. **Flips in rectangulations.** Different types of local operations can be defined on rectangulations, which have been given different names, such as flips, local move, edge rotations, or edge pivoting. In general, they all consist of replacing a horizontal edge of the rectangulation by a vertical one, or vice versa. In what follows, and with a slight abuse of terminology, we will refer to all those under the common name of *flip*.

Law and Reading [14] described a family of flips on rectangulations and provided an elegant combinatorial characterization. They showed that two rectangulations were connected by such a flip if and only if they were in the cover relation of a certain natural lattice structure, analogous to the Tamari lattice on triangulations (hence on 312-avoiding permutations), and part of the family of *Cambrian* lattices [20]. This lattice was also studied by Giraudo [13] under the name of *Baxter lattice*. Wide-reaching generalizations of these structures have been studied from the order-theoretic, algebraic, and polyhedral points of views by Reading [22], Chatel and Pilaud [9], and Pilaud and Santos [19], among others.

Ackerman, Barequet and Pinter [3] defined related flip operations on rectangulations of a point set. These rectangulations are defined on a given point set so that every point lies on a segment of the rectangulation, and vice versa. Ackerman et al. studied the flip graph induced by these operations [1]. The flips considered by Ackerman et al. are the same as the ones in Law and Reading whenever the point set lies on the diagonal. Their results include a linear upper bound on the diameter of this flip graph (see [1], Section 4).

1.4. **Contribution.** We first describe a known bijection from diagonal rectangulations to Baxter permutations. Then we consider flip operations on diagonal rectangulations, classify the different

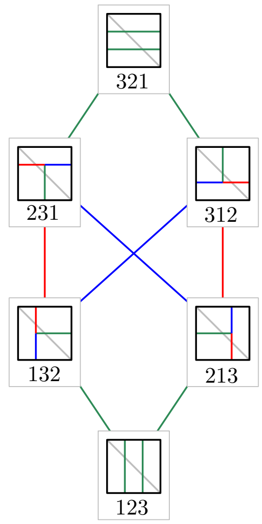


FIGURE 1. Flips in diagonal rectangulations with three rectangles, together with their associated Baxter permutation. The color code is explained in Section 4.

kinds of flips and give a combinatorial interpretation for each. Some of them, namely those involving edges that do not intersect the diagonal of the square, have already been characterized by Law and Reading [14]. We recall this characterization. For the others, we prove that the obtained flip graph is isomorphic to the graph on the corresponding Baxter permutations in which two Baxter permutations are adjacent whenever they differ by a single transposition of consecutive elements. We comment on the symmetry of the two interpretations. This provides a complete one-to-one correspondence not only between rectangulations and Baxter permutations, but also between these sets of natural operations on the geometric and combinatorial structures. Overall, this yields a complete characterization of flip operations in diagonal rectangulations. Illustrations of flip operations on rectangulations with three rectangles is given in Figure 1.

1.5. **Plan.** In Section 2 we provide some basic definitions and give a simple known bijection between diagonal rectangulations and Baxter permutations. In Section 3 we define and categorize a number of flip operations on diagonal rectangulations. Finally, in Section 4 we give combinatorial characterizations for all the described flip operations. We first summarize the Law-Reading characterization in terms of the lattice structure (4.1), then proceed with the characterization of other flips (4.2), which is our main new result.

2. DIAGONAL RECTANGULATIONS AND BAXTER PERMUTATIONS

In this section, we first define the combinatorial notion of diagonal rectangulation. Then we present maps from the set of diagonal rectangulations with n rectangles to the set of permutations on n elements. Those maps were described previously, and are known to be bijections between diagonal rectangulations and permutations avoiding some vincular patterns on four elements. They will be instrumental in the combinatorial interpretation of the flip graph on diagonal rectangulations. The material of this section is adapted from Ackerman et al. [2], and Law and Reading [14]. A description of an essentially equivalent map in terms of pairs of twin binary trees was given by Felsner et al. [12].

2.1. **Diagonal rectangulations.** A rectangulation is a partition of the unit square into axis-aligned rectangles. We define *vertices* as corners of the rectangles, and *edges* as line segment connecting

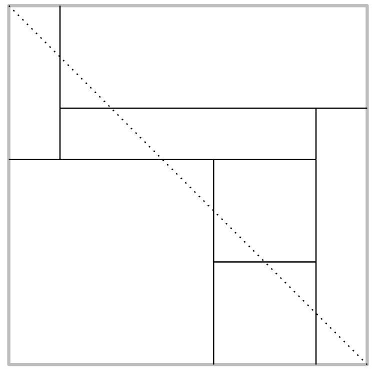


FIGURE 2. An example of diagonal rectangulation.

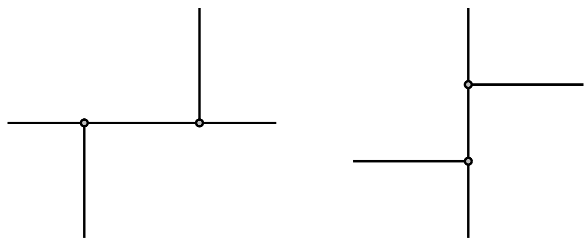


FIGURE 3. Forbidden configurations in a diagonal rectangulation.

two vertices, with no other vertex in between. The term *segment* is used to refer to inclusionwise maximal line segments of the rectangulation, possibly composed of several edges. We consider only rectangulations in which every vertex has exactly three incident edges, except the four vertices of the square, which have exactly two incident edges. We refer to the number of incident edges as the *degree* of the vertex, and classify the vertices into three self-explanatory classes depending on the orientation of their three incident edges and denoted by $\vdash$, $\dashv$, $\top$, and $\perp$.

We refer to the top-left to bottom-right diagonal of the square as the *main* diagonal, or simply the diagonal, when there is no amibiguity. A *diagonal* rectangulation is a rectangulation in which every rectangle intersects the main diagonal. However, since we deal with combinatorial structures, we actually define diagonal rectangulations as equivalence classes of such partitions of the square, with respect to moves that do not change the adjacency relation between the rectangles. Hence we allow changes in the positions of the vertices and edges, but we forbid moves that change the order of the vertices along a segment. An example of diagonal rectangulation is given in Figure 2. We have the following characterization of (the equivalence classes of) diagonal rectangulations.

Lemma 1. *A rectangulation is diagonal if and only if it does not contain one of the two forbidden configurations of Figure 3.*

We can also consider the equivalence classes of rectangulations for which we can change the relative position of vertices along a segment. Two rectangulations are said to be equivalent when one can be obtained from the other by performing so-called *wall slides*, as shown on Figure 4. Such moves may modify the adjacency relation among the rectangles. The equivalence relation is sometimes referred to as R*-equivalence* [4]. The R-equivalence classes are called *mosaic floorplans*.

Lemma 2. *Every mosaic floorplan, or R-equivalence class, has a unique representative as a diagonal rectangulation.*

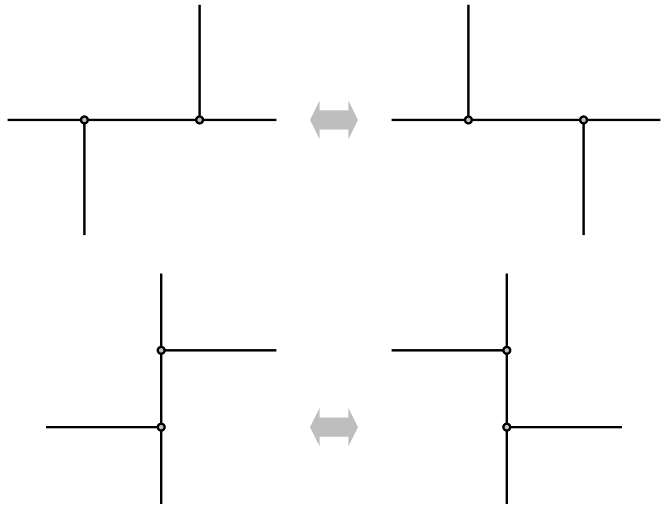


FIGURE 4. Wall slides.

2.2. **A map from permutations to diagonal rectangulations.** Before delving into the details of the bijections, we first describe a map ρ from any permutation to a diagonal rectangulation.

Given a permutation π on n elements, we consider the square to be dissected and divide its main diagonal into n intervals, that we label $1, 2, \ldots, n$, from left to right. We then proceed iteratively by growing rectangles intersecting the intervals $\pi_1, \pi_2, \ldots, \pi_n$. Before the ith step, we denote by T_{i-1} the union of the rectangles that have already been drawn, together with the left and bottom edges of the square. If $\pi_i = j$, we draw a rectangle intersecting the jth interval of the diagonal. We consider the left endpoint ℓ of this interval. If ℓ is on the boundary of T_{i-1}, then the upper left corner of the ith rectangle is the highest point above ℓ that belongs to T_{i-1}. Otherwise, the upper left corner is the rightmost point on the boundary of T_{i-1} that is directly left of ℓ. Similarly, consider the right endpoint r of the jth interval. If r is on the boundary of T_{i-1}, then the lower right corner of the new rectangle is the rightmost point of T_{i-1} that is directly right of r. Otherwise, it is the highest point on the boundary of T_{i-1} lying directly below r. The map is illustrated on Figure 5.

2.3. **Maps from diagonal rectangulations to permutations.** Given the algorithm above, one can realize that many distinct permutations can yield the same rectangulation. For instance, one can check on the example of Figure 5 that rectangles 6 and 5 can both be drawn before rectangle 1, hence that $\rho(4165372) = \rho(4651372)$. In general, given a diagonal rectangulation R, one can easily find a permutation π such that $\rho(\pi) = R$ by moving backwards in the order in which the rectangles are drawn. At each step, there always exists a rectangle in the rectangulation that is drawn correctly when the above procedure is applied. In order to define a map from rectangulations to permutations, we need a tie-breaking rule, which allows to decide univoquely which is the next rectangle to pick, and therefore to choose one well-defined element from each preimage $\rho^{-1}(R)$. There are two simple rules we can apply: the leftmost and rightmost rules. In those rules, the next rectangle we pick is the one that is leftmost (respectively rightmost) on the diagonal. In all cases, the rectangles of the given rectangulation are first labeled in what we call the $\boxbslash$-order, defined as the order in which they intersect the diagonal.

Let us first consider the leftmost rule, as illustrated on Figure 6. One can check that the permutation that is produced using the leftmost rule avoids the vincular patterns $3\underline{41}2$ and $2\underline{41}3$. Indeed, in both cases, the leftmost rule prescribes that rectangle 1 is chosen before rectangle 4. Permutations avoiding those two patterns are known as *twisted Baxter* permutations. In fact, only twisted

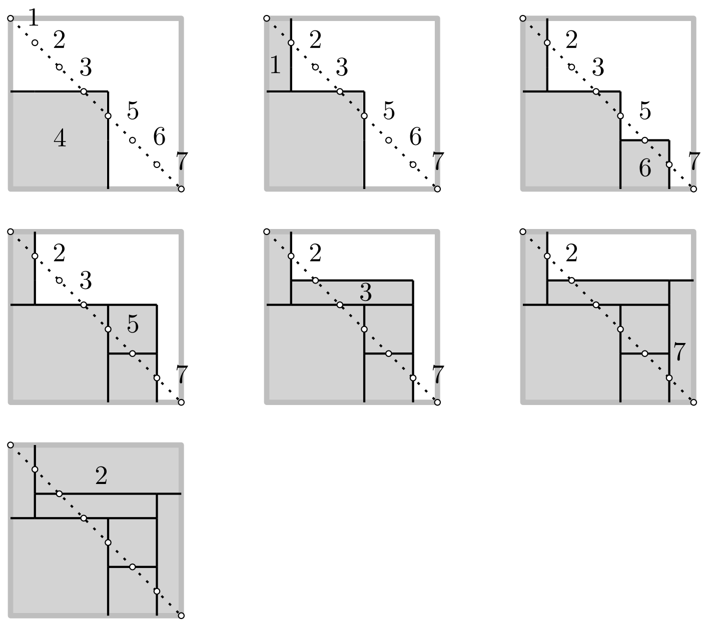


FIGURE 5. The map ρ from permutations to diagonal rectangulations. Construction of the diagonal rectangulation $\rho(4165372)$.

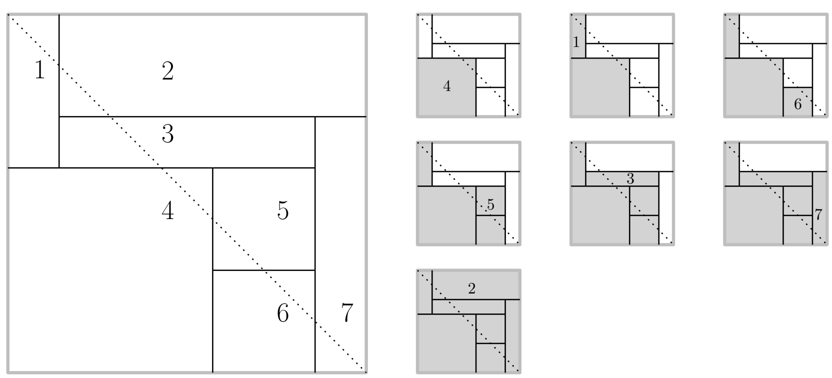


FIGURE 6. Illustration of the map given by the leftmost order on the rectangulation R of Figure 2: the labels of the rectangles are given by the ⧅-order (left), and listed in the leftmost order (right). The obtained twisted Baxter permutation is 4165372.

Baxter permutations can be produced, and the map is known to be a bijection between diagonal rectangulations and twisted Baxter permutations [14].

Similarly, the rightmost rule yields a bijection between diagonal rectangulations and $\{3\underline{14}2, 2\underline{14}3\}$-avoiding permutations.

We now describe the tie-breaking rule that allows to define a bijection B between diagonal rectangulations and *Baxter* permutations, which avoid the patterns $\{3\underline{14}2, 2\underline{41}3\}$. In order to define B, we define another linear order on the rectangles of a rectangulation: the ⧄-order. The ⧄-order is obtained by taking the representative ⧄R of R in the equivalence class of mosaic floorplans such that the bottom-left to top-right diagonal intersects every rectangle. From Lemma 2, this representative exists and is unique. The ⧄-order is then simply the order in which this diagonal intersects the rectangles. The Baxter permutation $\mathrm{B}(R)$ corresponding to a diagonal rectangulation

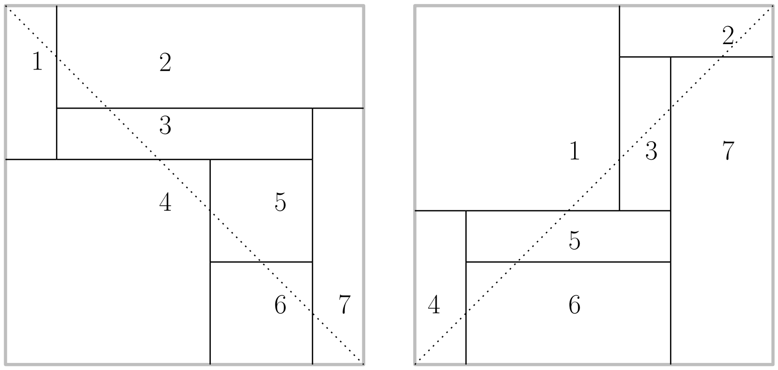


FIGURE 7. Illustration of the map B on the rectangulation R of Figure 2: the labels of the rectangles are given by the ⧅-order (left), and listed in the ⧄-order (right). The rectangulation on the right is in the same equivalence class of mosaic floorplans as the one on the left. The obtained Baxter permutation is $\mathrm{B}(R) = 4651372$.

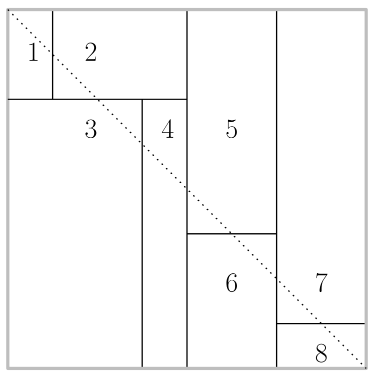


FIGURE 8. Example of diagonal rectangulation in which the leftmost, rightmost, and ⧄-orders are all distinct. The leftmost order, hence the twisted Baxter permutation is 31426587, the rightmost order is 34126857, and the ⧄-order, hence the Baxter permutation, is 34126587.

R is the order of the rectangle labels in the ⧄-order, see Figure 7. The map B can then be described concisely as follows:

(1) label the rectangles with respect to the ⧅-order,
(2) enumerate the labels of the rectangles in the ⧄-order.

The ⧄-order gives a tie-breaking rule that is distinct from both the leftmost and the rightmost order, and can be shown to avoid the Baxter patterns. An example illustrating this distinction is given on Figure 8. The following result is due to Ackerman et al. [2]. In their proof, the description of the ⧄-order involves *block deletion* operations, but can be seen to be equivalent to ours.

Theorem 1. *The map* B *is a bijection between diagonal rectangulations with* n *rectangles and Baxter permutations on* n *elements.*

2.4. **Inversion.** We now give a relation between rectangulations produced by a Baxter permutation π and its inverse π^{-1}. The map ρ from permutations of n elements to diagonal rectangulations with n rectangles is defined by iteratively drawing the rectangle given by the next element of the sequence on the main diagonal. We define a similar map ρ' that produces a rectangulation in which the rectangles intersect the other, bottom-left to top-right, diagonal. The map ρ' is simply defined as the composition of ρ with a reflection with respect to the horizontal axis. We observe that applying this map to the inverse permutation π^{-1} yields the alternative diagonal representation of $\rho(\pi)$.

Lemma 3. *Let $\pi = B(R)$ for a rectangulation R. Then $\rho'(\pi^{-1}) = \boxslash R$.*

Proof. Consider the map B' from rectangulations to permutations defined as follows:

(1) label the rectangles with respect to the $\boxslash$-order,
(2) enumerate the labels of the rectangles in the $\boxbslash$-order.

Note that this matches the description of the map B, except that we exchanged the roles of the two orders. Since the roles of the indices and the elements are now exchanged, we must have that $B'(R) = \pi^{-1}$. Now applying ρ' on the permutation π^{-1} amounts to inserting the rectangles in the order given by π^{-1} along the other, bottom-left to top-right diagonal. But since the rectangles were labeled by B' in the $\boxslash$-order, this must yield $\boxslash R$. $\square$

3. Flips

In this section, we present a geometric notion of flips in diagonal rectangulation. We consider only flipping edges that are not part of the boundary of the square. We say that an edge is *matched* at one of its endpoint whenever this endpoint is incident to another edge with the same (horizontal/vertical) orientation.

3.1. Simple flips.

Simple flips involve edges cutting a rectangle into two rectangles, which are precisely the edges that are unmatched at both endpoints. All such edges must intersect the diagonal. A simple flip consists in replacing such a horizontal edge by a vertical one, or vice versa. When replacing the edge, we can always do it in such a way that the resulting rectangulation remains diagonal. An example of simple flip in the rectangulation of Figure 2 is given in Figure 9a.

It is perhaps worth noting that flipping those edges is not sufficient to connect any pair of diagonal rectangulations. In other words, the simple flip graph is not connected. Two rectangulations that differ only by simple flips have been called *S-equivalent* by Asinowski et al. [4], and the corresponding equivalence classes are shown to be counted by the $\{3\underline{41}2, 2\underline{14}3$)-avoiding permutations.

3.2. Flips using rotations.

In some cases, an edge that is matched only at one of its endpoints can be *rotated* around this endpoint to yield another diagonal rectangulation. Examples of such flips are given in Figures 9b and 9c.

Note that when such a flip involves an edge that intersects the main diagonal, like in Figure 9c, merely replacing the rotated edge in a drawing of the rectangulation does not yield a drawing of the rectangulation that is diagonal, that is, some rectangles do not intersect the diagonal anymore. However, the rectangulation remains a proper diagonal rectangulation in the combinatorial sense, because no wall slide is needed to make all rectangles intersect the diagonal.

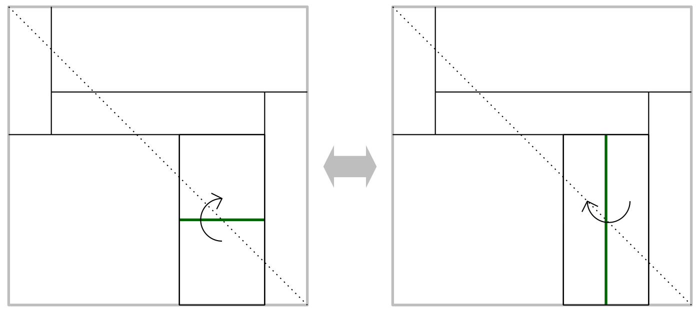
Together, all these flip operations define a flip graph on the set of diagonal rectangulations. However, not all edges can be flipped. An edge is said to be unflippable in two cases.

3.2.1. *Unflippable edges matched at both endpoints.*

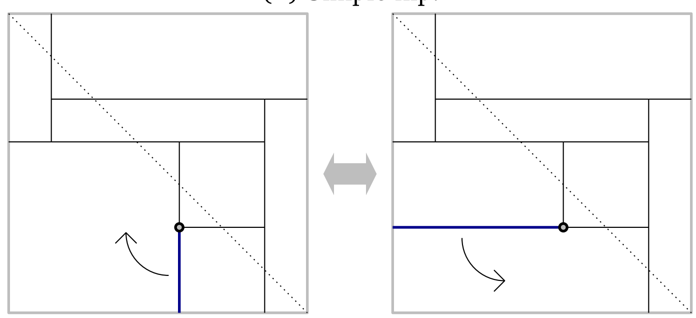
If the edge is matched at both endpoints, rotating this edge around any of the two endpoints yields a partition that is not a rectangulation. Examples are shown in Figure 10a.

3.2.2. *Unflippable edges matched at one endpoint.*

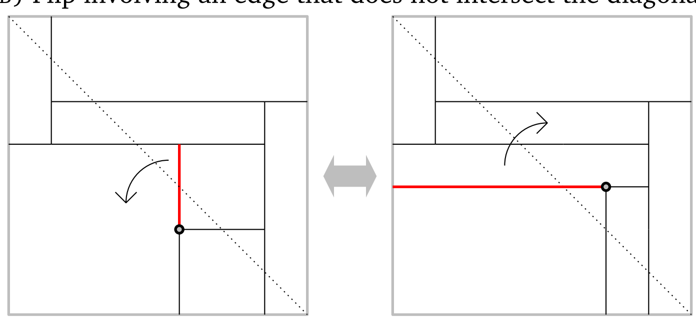
It can also be the case that an edge is matched at only one endpoint, and rotating it around this endpoint yields a rectangulation, but the obtained rectangulation is not diagonal. An illustration is given in Figure 10b. We have the following lemma characterizing such unflippable edges.



(A) Simple flip.

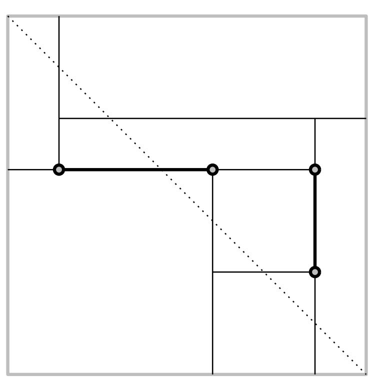


(B) Flip involving an edge that does not intersect the diagonal.

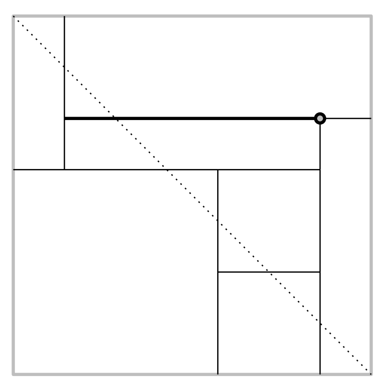


(C) Flip involving an edge intersecting the diagonal.

FIGURE 9. The three kinds of flips in a diagonal rectangulation.



(A) Unflippable edges matched at both endpoints.



(B) Unflippable edge matched at one endpoint.

FIGURE 10. Unflippable edges.

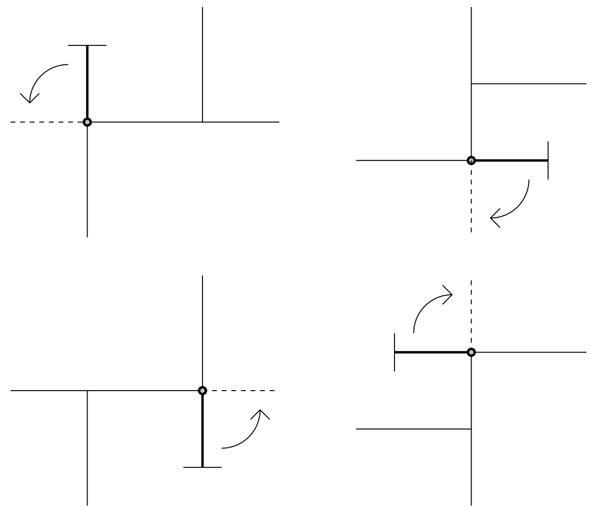


FIGURE 11. The four types of unflippable edges matched at only one endpoint.

Lemma 4. *Unflippable edges matched at only one endpoint must fall in one of the four types described in Figure 11. Furthermore, all of them must intersect the diagonal.*

Proof. By definition, flipping the edge must create one of the two configurations in Lemma 1, shown in Figure 3. Each of the two configurations can be forced to occur only after one of two edges have been rotated, hence can only happen in one of the four cases described. The second statement can be proved by contradiction, by considering the four types of unflippable edges. If, for one of them, the diagonal does not intersect the edge, then the rectangulation is not diagonal to start with. □

All edges that are neither simply flippable nor unflippable according to the previous definitions can be flipped using a rotation to get another diagonal rectangulation.

4. A COMPLETE COMBINATORIAL CHARACTERIZATION OF FLIPS

In this section, we give a combinatorial characterization of all edges in the flip graph of diagonal rectangulations. For this purpose, we will use some simple terminology on permutations. A *transposition* maps a permutation $\pi = \pi(1)\pi(2)\ldots\pi(j)\ldots\pi(k)\ldots\pi(n)$ to a permutation $\pi' = \pi(1)\pi(2)\ldots\pi(k)\ldots\pi(j)\ldots\pi(n)$. Furthermore, if the two values j and k satisfy $|\pi(j) - \pi(k)| = 1$, then the transposition is said to be a *transposition of consecutive elements*. If $k = j + 1$, then the transposition is said to be an *adjacent transposition*. Note that an adjacent transposition corresponds to a transposition of consecutive elements in the inverse permutation.

4.1. **Law-Reading flips.**

We first summarize a result of Law and Reading, characterizing some of the flip operations described above as a cover relation in a lattice, which can be found in Section 7 of [14]. In what follows, we will use the term *Law-Reading flips* to refer to those flips.

In the original description, Law-Reading flippable edges are constructed as follows: for every inner vertex, consider the two edges going towards (not necessarily intersecting) the diagonal. Consider the one that is matched and lock it. The remaining non-locked edges are Law-Reading flippable. It can be checked that all Law-Reading flippable edges are flippable with respect to the definitions of Section 3. The following lemma gives a simple alternative definition of Law-Reading flips.

Lemma 5. *Law-Reading flips are exactly the flips that are either simple, or that involve the rotation of a flippable edge that does not intersect the diagonal, as illustrated in Figure 9b.*

Proof. Consider an edge that intersects the diagonal. If this edge is Law-Reading flippable, then it must be unmatched at both endpoints, since otherwise one of the endpoints would lock it. Hence it must be simply flippable. Conversely, suppose that this edge is flippable, but not simply flippable. Then it must be matched at exactly one endpoint. But for this endpoint, the edge is towards the diagonal and therefore must be locked. Hence Law-Reading flips of edges intersecting the diagonal are exactly the simple flips.

Consider now an edge that does not intersect the diagonal. We need to show that it is flippable if and only if it is Law-Reading flippable. Suppose it is flippable. Then it must be matched at exactly one endpoint. This endpoint must be the closest to the diagonal, for otherwise the rectangulation is not diagonal. But then it cannot be locked, and is Law-Reading flippable. On the other hand, suppose it is Law-Reading flippable. Then it cannot be locked, and can only be matched at the endpoint that is the closest from the diagonal. From Lemma 4, unflippable edges matched at one endpoint must intersect the diagonal. Therefore this edge must be flippable. □

We now give a combinatorial characterization of Law-Reading flips proved in [14] using the map from rectangulations to Baxter permutations. Before stating the result, we must define the lattice dRec_n of diagonal rectangulations with n rectangles.

4.1.1. *A lattice on diagonal rectangulations.* The *weak order* (also known as the weak *Bruhat* order) is a partial order on the set S_n of permutations of n elements in which a permutation π is smaller than another permutation π' whenever the set of inversions of π is a subset of the set of inversions of π'. The cover relation of the weak order is the set of pairs of permutations that differ by a single adjacent transposition. The weak order is a classical, well-studied order, and known to be a lattice.

The lattice dRec_n on diagonal rectangulations can be defined as the restriction of the weak order to the Baxter permutations corresponding to diagonal rectangulations with n rectangles. In fact, it can be shown that the preimages $\rho^{-1}(R)$ of ρ form a lattice congruence on the weak order. The lattice dRec_n is the quotient of the weak order with respect to this congruence. Therefore, dRec_n may as well be defined by restricting the weak order to any set of representatives of each congruence class. More concretely, we can pick for any rectangulation R any representative in $\rho^{-1}(R)$ and consider the order induced by those. For instance, the partial order induced by the weak order on the twisted Baxter permutations is isomorphic to dRec_n.

We can now state the connection between this order and Law-Reading flips in rectangulations. Recall that $B(R)$ is the Baxter permutation associated with the diagonal rectangulation R.

Theorem 2 (Law and Reading [14])**.** *Let R and R' be two diagonal rectangulations. Then R and R' are connected by a Law-Reading flip if and only if $\mathrm{B}(R)$ and $\mathrm{B}(R')$ are in a cover relation in dRec_n.*

This means that the two Baxter permutations corresponding to the pair of rectangulations are related by a monotone sequence of adjacent transpositions, and the intermediate permutations, if any, are not Baxter permutations. Note that the Law-Reading flips have a simple interpretation in the representation of a rectangulation by twin binary trees. A Law-Reading flip then corresponds to a rotation in one of the two binary tree (see Section 5.3 in Giraudo [13]).

4.2. **Barcelona flips.** We define Barcelona flips as those flips that involve a flippable edge intersecting the main diagonal. Barcelona flips are either simple flips, or flips involving the rotation of an edge intersecting the diagonal, as shown in Figure 9c.

4.2.1. *Barcelona flips in R are Law-Reading flips in $\boxslash R$.*

Lemma 6. *Let R and R' be two diagonal rectangulations that are connected by a Barcelona flip. Then $\boxslash R$ and $\boxslash R'$ are connected by a Law-Reading flip.*

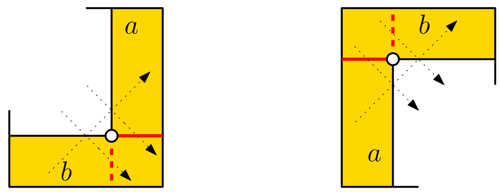


FIGURE 12. Illustration of the proof of Lemma 6. The $\boxbslash$ and $\boxslash$-orders of the two rectangles a and b are indicated by the dotted arrows.

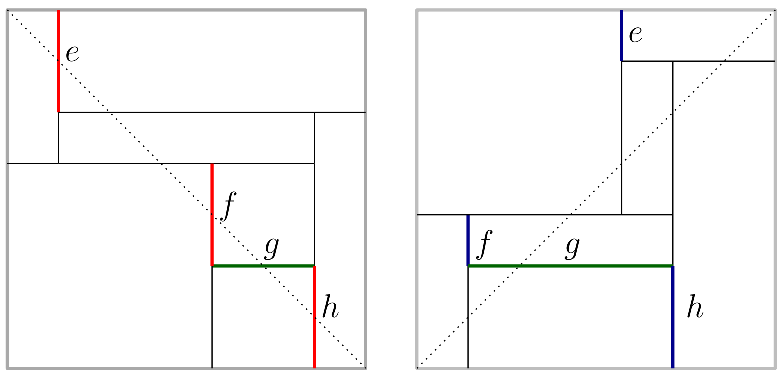


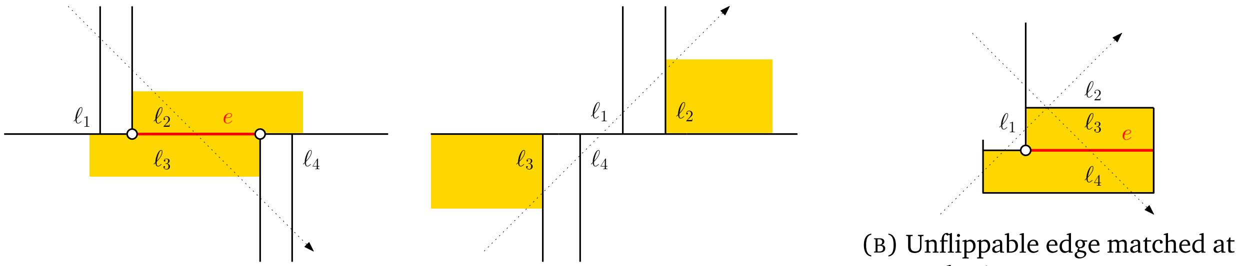
FIGURE 13. Illustration of Lemma 6: edges that can be flipped by a Barcelona flip in R (left) can be flipped by a Law-Reading flip in $\boxslash R$ (right).

Proof. We first consider the case where the Barcelona flip is a simple flip. Then the edge is unmatched a both endpoints, and remain so in $\boxslash R$. Hence $\boxslash R$ and $\boxslash R'$ are connected by a simple flip as well.

In the case where the Barcelona flip is not simple, it must involve two rectangles with labels a and b that can be in two possible distinct relative positions, as depicted in Figure 12.

We first remark that in the configuration on the left of Figure 12 in R, the top left corners of a and b must respectively be $\top$ and $\vdash$ vertices. This is because otherwise the rectangulation after or before the flip contains one of the two forbidden configurations of Figure 3 and cannot be diagonal. Similarly in the configuration on the right, the bottom right corners of a and b must respectively be $\perp$ and $\dashv$ vertices. Hence the relative position of the two rectangles a and b cannot be changed by wall slides, and remains the same in $\boxslash R$. This in turn implies that the other diagonal in $\boxslash R$ does not intersect the flipped edge, and that this edge is still matched at only one endpoint. Since it does not intersect the diagonal, Lemma 4 implies that the edge is flippable in $\boxslash R$, and the flip is a Law-Reading flip. Applying the same reasoning starting with R', we conclude that flipping this edge in $\boxslash R$ yields the rectangulation $\boxslash R'$. $\square$

The Lemma is illustrated in Figure 13. Combining the above lemma with Lemma 3 on the way to obtain $\boxslash R$ from $B(R)$, and the characterization of Law-Reading flips in Theorem 2, one can already conclude that a Barcelona flip in a rectangulation R corresponds to a sequence of adjacent transpositions in the inverse permutation $B(R)^{-1}$. It is perhaps tempting to conjecture at this stage that Barcelona and Law-Reading flips are exactly dual to each other, in the sense that the set of Barcelona flips in R is in bijection with the set of Law-Reading flips in $\boxslash R$. This is not the case. In what follows, we show that the Barcelona flips are in correspondence with the Law-Reading flips in $\boxslash R$ that involve a *single* adjacent transposition in $B(R)^{-1}$, that is, a single transposition of consecutive elements in $B(R)$.



(A) Unflippable edge matched at both endpoints.

(B) Unflippable edge matched at one endpoint.

FIGURE 14. Illustrations for the proof of Theorem 3.

4.2.2. *A characterization of Barcelona flips.*

Theorem 3. *Let R and R' be two diagonal rectangulations. Then R and R' are connected by a Barcelona flip if and only if $\mathrm{B}(R)$ and $\mathrm{B}(R')$ differ by a single transposition of consecutive elements.*

Proof. ($\Rightarrow$) First suppose that R and R' are connected by such a flip. If this is a simple flip, it is not difficult to verify, by referring to the descriptions of the map B, that the permutations indeed differ by a single transposition of consecutive elements.

Now suppose it is not a simple flip. From Lemma 6, we have that ⧄R and ⧄R' are connected by a nonsimple Law-Reading flip. But those involve precisely the edges that do not intersect the diagonal, hence the ⧄-order labels of the rectangles in R and R' are the same. It remains to observe that since flipping the edge does not create any obstruction to the rectangulation being diagonal, the ⧅-order of the rectangles a and b (refer to Figure 12) is simply reversed. We conclude that the flip corresponds to the single transposition of the two elements a and b in $\mathrm{B}(R)$.

($\Leftarrow$) We now suppose that the two Baxter permutations $\pi = \mathrm{B}(R)$ and $\pi' = \mathrm{B}(R')$ differ by a single transposition of two consecutive elements a and b. By definition of B the adjacent transposition must correspond to an edge e in R that intersects the diagonal. We need to show that e is flippable. We proceed by contradiction, and suppose that e is unflippable. Unflippable edges come in two flavors, and we consider the two cases separately.

In the first case, e is unflippable because it is matched at both endpoints. Suppose first that e is horizontal and let $\ell_2 = a$ and $\ell_3 = b$ be the labels of the two rectangles above and below the diagonal, corresponding to the two labels involved in the transposition. The two endpoints of e must have their other incident edges oriented as shown on Figure 14a.

Now consider the leftmost rectangle having his lower horizontal edge on the same segment as e, and denote its label by ℓ_1 (this may be the rectangle that is just on the left of ℓ_2, or another rectangle further left). The diagonal must intersect rectangle ℓ_1 before rectangle ℓ_2. Similarly, consider the rightmost rectangle having it upper horizontal edge on the same segment as e, and denote its label by ℓ_4. We must have $\ell_1 < \ell_2 < \ell_3 < \ell_4$.

In the ⧄-order, the configuration of the rectangles is obtained by sliding the edges orthogonal to e so that all the edges above e are on the right of the edges below e. By considering the four rectangles in the ⧄-order, as illustrated on the right of Figure 14a, we can check that π contains the pattern $\ell_3 \ldots \ell_4 \ell_1 \ldots \ell_2$. Therefore, π' contains the subsequence $\ell_2 \ldots \ell_4 \ell_1 \ldots \ell_3$, an occurence of the forbidden pattern $2\underline{41}3$, and π' cannot be a Baxter permutation, a contradiction. A similar, symmetric, reasoning can be done when the unflippable edge is vertical, and then the forbidden pattern is $3\underline{14}2$.

In the second case, e is matched at one endpoint only, but still unflippable because rotating it around its matched endpoint would yield a non-diagonal rectangulation. Again, we have four symmetric cases, illustrated in Figure 11. We detail the case where e is horizontal, and is matched

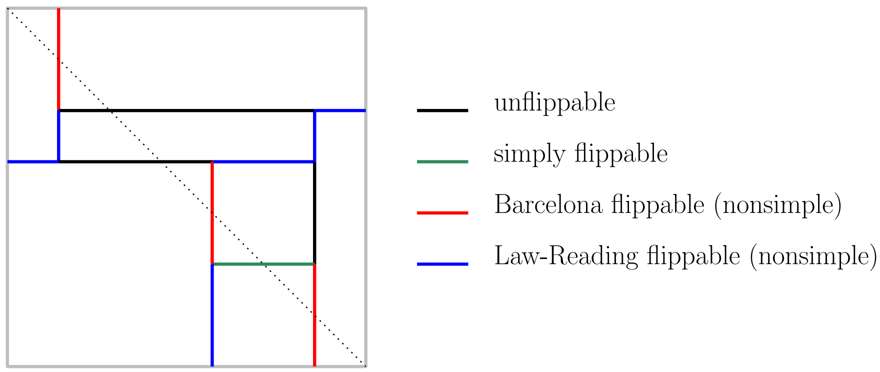


FIGURE 15. The various types of flippable and unflippable edges.

at its left endpoint (top right case in the figure). Figure 14b illustrates what happens in this case. Let $a = \ell_3$ and $b = \ell_4$ be the labels of the two rectangles above and below e, respectively.

From Lemma 4, the top horizontal edge of rectangle ℓ_3 must be part of the obstruction to the rectangulation being diagonal. Let us denote by ℓ_2 the label of the predecessor of ℓ_3 in the $\boxbslash$-order. Consider the rectangle labeled ℓ_1 to the left of the left vertical edge of ℓ_3. We clearly must have $\ell_1 < \ell_2 < \ell_3 < \ell_4$.

Now remark that the top left corner of rectangle ℓ_4 must be of type $\vdash$, since otherwise we would have a forbidden configuration for the diagonal representation. Therefore, in the $\boxslash$-order, no wall slide can be involved, and the rectangles ℓ_1, ℓ_3, and ℓ_4 have the same relative positions. By considering the four rectangles in the $\boxslash$-order, we can check that π must contain the pattern $\ell_4 \ldots \ell_1 \ell_3 \ldots \ell_2$ (see Figure 14b). By definition, π' must contain the pattern $\ell_3 \ldots \ell_1 \ell_4 \ldots \ell_2$, which is an instance of the forbidden pattern $3\underline{14}2$. This is again a contradiction to the fact that π' is a Baxter permutation. The same reasoning can be done on the remaining three types of unflippable edges matched at one endpoint shown on Figure 11. In all cases, we identify one of the two forbidden patterns in π'.

Therefore, the edge corresponding to the adjacent transposition in π must be flippable, and the transposition indeed corresponds to a flip operation, as claimed. $\square$

Hence this characterization is very similar to the previous one, except that the transpositions involve *consecutive* elements instead of *adjacent* elements, and that only a *single* transposition is needed.

4.3. **Characterization.**

We can summarize our combinatorial characterization of flips in diagonal rectangulations as follows.

Theorem 4. *Two diagonal rectangulations R and R' are connected by a flip if and only if one of these two conditions hold:*

- *$\mathrm{B}(R)$ and $\mathrm{B}(R')$ differ by a single transposition of consecutive elements,*
- *$\mathrm{B}(R)$ and $\mathrm{B}(R')$ are in a cover relation in dRec_n.*

Furthermore, R and R' are connected by a simple flip if and only if both conditions hold.

Figure 15 shows all types of flippable and unflippable edges on our running example. The flip graph on diagonal rectangulations with four rectangles is given in Figure 16.

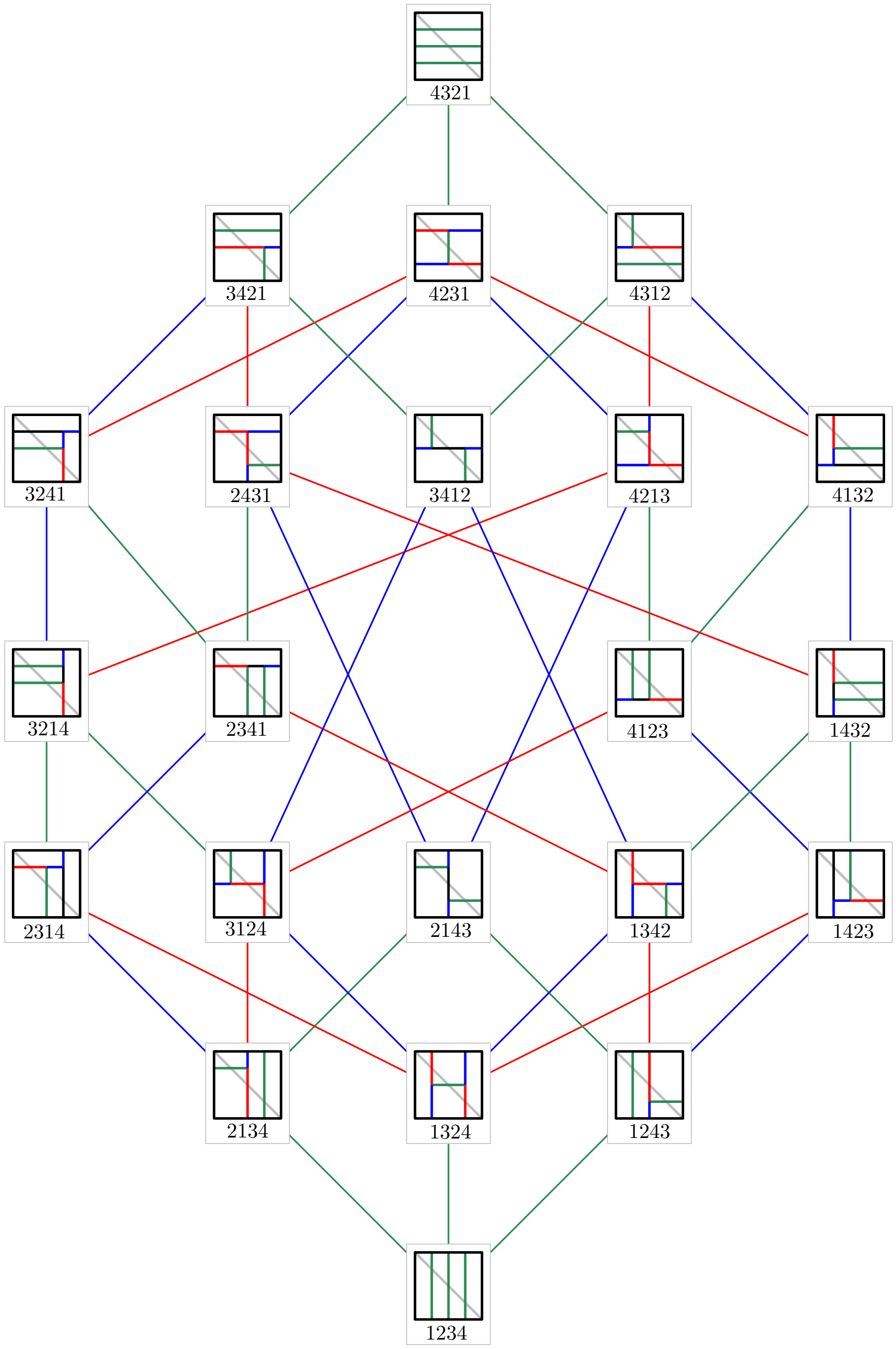


FIGURE 16. The flip graph on diagonal rectangulations made of four rectangles. In each rectangulation, the green edges are simply-flippable, and the blue and red edges are respectively Law-Reading and Barcelona-flippable, but not simply flippable. The links between the rectangulations are color-coded similarly. The Baxter permutation corresponding to each rectangulation is given.

Acknowledgments

This work was initiated while the first author was visiting the UPC Research Group on Discrete, Combinatorial and Computational Geometry in March-April 2017. The authors wish to thank Andrei Asinowski, Stefan Felsner, and Vincent Pilaud for useful discussions and comments on a preliminary version of this manuscript. V.S. and R.I.S. were partially supported by projects Gen. Cat. DGR 2014SGR46 and MTM2015-63791-R (MINECO/FEDER).

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