CANONICAL BASES FOR PERMUTOHEDRAL PLATES

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Abstract. We study three finite-dimensional quotient vector spaces constructed from the linear span of the set of characteristic functions of permutohedral cones by imposing two kinds of constraints: (1) neglect characteristic functions of higher codimension permutohedral cones, and (2) neglect characteristic functions of permutohedral cones which contain doubly infinite lines. We construct an ordered basis which is canonical, in the sense that it has subsets which map onto ordered bases for the quotients.

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1. Introduction

The purpose of this paper is two-fold: it contains general combinatorial and geometric results about generalized permutohedra, but from our perspective it is motivated by surprising connections to physics, in particular to the study of scattering amplitudes in quantum field theory and string theory.

This paper is devoted to the combinatorial analysis of the vector space of characteristic functions of generalized permutohedra, studied as *plates* by Ocneanu [14], and by the author in [7, 8]. We derive a certain *canonical* basis of the space which is spanned linearly by characteristic functions of permutohedral cones; these cones are *dual* to the faces of the arrangement of reflection hyperplanes, defined by equations $x_i - x_j = 0$.

The basis is called *canonical* because of its compatibility with quotient maps to three other spaces: subsets of the canonical plate basis descend to bases for the quotients. These maps are constructed from one or both of two geometrically-motivated conditions, according to which characteristic functions of (1) higher codimension faces, or (2) non-pointed cones, are sent to the zero element.

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Plates are *permutohedral* cones: the edges extend along the root directions $e_i - e_j$. Plates are labeled by ordered set partitions; when the blocks of an ordered partition are all singlets, then the corresponding plate is encoded by a directed tree of the form $\{(i_1, i_2), (i_2, i_3), \ldots, (i_{n-1}, i_n)\}$ and correspondingly have edge directions the roots $e_{i_j} - e_{i_{j+1}}$.

We begin in Section 2 by introducing notation and collecting basic results about polyhedral cones. In Section 3 we give linear relations which express the characteristic function of any product of mutually orthogonal plates as a signed sum of plates which have the ambient dimension $n-1$. We come to our main result, the construction of canonical plate basis, in Section 4 and give two graded dimension formulas. In Section 5 we give formulas which express the characteristic function of a permutohedral cone encoded by a directed tree as a signed sum of characteristic functions of plates. In Section 6 we present the formula which expands the characteristic function of any plate in the canonical basis.

For results more directly related to physics, readers may want to look toward the identities in Theorem 32 and Corollary 36; looking forward, it turns out that, using the theory of polyhedral geometry [3], specifically Theorem 3.1, these can be specialized to interpret the shuffle relations in [11], as well as the loop analogs of the Parke-Taylor factors [9, 10], using permutohedral cones.

2. Landscape: basic properties of cones and plates

Throughout this paper we shall assume $n \geq 1$.

The all-subset hyperplane arrangement lives inside the linear hyperplane $V_0^n \subset \mathbb{R}^n$ defined by $x_1 + \cdots + x_n = 0$, and consists of the *special* hyperplanes $\sum_{i \in I} x_i = 0$, as I runs through the proper nonempty subsets of $\{1, \ldots, n\}$. This paper deals with the subspace of functions supported on V_0^n lying in the span of characteristic functions $[U]$ of subsets U which are closed cones in the all-subset arrangement. Certain subsets U play a special role in this theory.

Definition 1. An *ordered set partition* of $\{1, \ldots, n\}$ is a sequence of *blocks* $(S_1, \ldots, S_k)$, where $\emptyset \neq S_i \subseteq \{1, \ldots, n\}$ with $S_i \cap S_j = \emptyset$ for $i \neq j$, and with $\cup_{i=1}^k S_i = \{1, \ldots, n\}$. For each such ordered set partition, the associated *plate* $\pi = [S_1, \ldots, S_k]$ is the cone defined by the system of inequalities

$$\begin{aligned} x_{S_1} &\geq 0 \\ x_{S_1 \cup S_2} &\geq 0 \\ &\vdots \\ x_{S_1 \cup \cdots \cup S_{k-1}} &\geq 0 \\ x_{S_1 \cup \cdots \cup S_k} &= 0, \end{aligned}$$

where $x_S = \sum_{i \in S} x_i$ for each $S \subseteq \{1, \ldots, n\}$. When the S_i's are singlets we write simply $\pi = [i_1, \ldots, i_n]$, where i_a stands for the unique element of S_a. We denote by $[\pi] = [[S_1, \ldots, S_k]]$ the characteristic function of the plate π.

Denote by $\mathrm{len}(\pi)$ the number of blocks in the ordered set partition which labels π.

This paper studies the four spaces $\hat{\mathcal{P}}^n, \mathcal{P}^n, \hat{\mathcal{P}}_1^n, \mathcal{P}_1^n$, as related by a diagram of linear surjections:

$$\begin{tikzcd} \hat{\mathcal{P}}^n \arrow[r] \arrow[d] & \hat{\mathcal{P}}_1^n \arrow[d] \\ \mathcal{P}^n \arrow[r] & \mathcal{P}_1^n \end{tikzcd} \tag{1}$$

The horizontal (respectively vertical) maps mod out by characteristic functions of cones which are not pointed (respectively not full-dimensional). In particular, the upper left space $\hat{\mathcal{P}}^n$ is the linear span of characteristic functions of all plates. The upper right space $\hat{\mathcal{P}}_1^n$ is the quotient of $\hat{\mathcal{P}}^n$ by the span of the characteristic functions of plates which are not pointed: they contain doubly infinite lines. The lower left space $\mathcal{P}^n$ requires somewhat more care to define: it is the quotient of $\hat{\mathcal{P}}^n$ by the span of those linear combinations of characteristic functions of plates which vanish outside a set of measure zero in V_0^n. The lower right space $\mathcal{P}_1^n$ is the quotient of $\hat{\mathcal{P}}^n$ by the span of both characteristic functions of non-pointed plates and linear combinations which vanish outside sets of measure zero in V_0^n.

The subscript 1 on $\hat{\mathcal{P}}_1^n$ and $\mathcal{P}_1^n$ is intended to remind that only characteristic functions of plates labeled by ordered set partitions having all blocks of size 1 are nonzero.

Our main result, Theorem 27, constructs a new basis which is unitriangularly related to the given basis of characteristic functions $[\pi]$, with the virtue that it contains subsets which will descend to bases for the other three spaces. In particular, this will show that they have dimensions given as follows:

$$\begin{array}{c|c} \dim\left(\hat{\mathcal{P}}^n\right) = \sum_{k=1}^n k!S(n,k) \quad \text{(Ordered Bell \#'s)} & \dim\left(\hat{\mathcal{P}}_1^n\right) \ = \ \sum_{k=1}^n s(n,k) = n! \\ \hline \dim\left(\mathcal{P}^n\right) = \sum_{k=1}^n (k-1)!S(n,k) \quad \text{(Cyclic Bell \#'s)} & \dim\left(\mathcal{P}_1^n\right) \ = \ s(n,1) = (n-1)! \end{array}.$$

Here $S(n,k)$ is the k^{th} Stirling number of the second kind, which counts the set partitions of $\{1,\ldots,n\}$ into k parts, and $s(n,k)$ is the (unsigned) Stirling number of the first kind which counts the number of permutations which have k cycles (including singlets) in their decompositions into disjoint cycles.

The canonical basis of $\hat{\mathcal{P}}^n$ is naturally graded, and the corresponding dimensions are given below. The first six rows are given below; note that the rows sum to the ordered Bell numbers $(1,3,13,75,541,4683)$.

1					
2	1				
6	6	1			
26	36	12	1		
150	250	120	20	1	
1082	2040	1230	300	30	1

In Corollary 30 we observe that the rows are given by the equation

$$T_{n,k} = \sum_{i=k}^{n} S(n,i)s(i,k),$$

where $S(n,i)$ is the Stirling number of the second kind which counts the number of set partitions of $\{1,\ldots,n\}$ into i blocks, and $s(i,k)$ is the Stirling number of the first kind which count the number of ways to order cyclically those blocks. This is given in O.E.I.S. A079641.

Further, as the canonical basis of $\hat{\mathcal{P}}^n$ passes to one for $\hat{\mathcal{P}}_1^n$, with graded dimensions the (unsigned) Stirling numbers of the first kind.

1					
1	1				
2	3	1			
6	11	6	1		
24	50	35	10	1	
120	274	225	85	15	1

2.1. **Notation and conventions.** Recall that a chain of inequalities $x_{i_1} \geq x_{i_2} \geq \cdots \geq x_{i_n}$ with $\sum x_i = 0$ cuts out a simplicial cone called a *Weyl chamber* in the arrangement of reflection hyperplanes, defined by equations $x_i - x_j = 0$, of type A_{n-1}. Weyl chambers are thus labeled by permutations $(i_1, \ldots, i_n)$ in one-line notation.

A *polyhedral cone* is an intersection of finitely many half spaces $\sum_{j=1}^n a_{i,j}x_j \geq 0$ in $\mathbb{R}^n$, for some integer constants $a_{i,j}$. For example, the chain of inequalities which define a Weyl chamber can be reorganized as

$$x_{i_1} - x_{i_2} \geq 0, \ldots, x_{i_{n-1}} - x_{i_n} \geq 0, \quad \text{with } \sum x_i = 0,$$

so it is a polyhedral cone. Letting $e_1, \ldots, e_n$ denote the standard basis for $\mathbb{R}^n$, if I is a proper subset of $\{1, \ldots, n\}$, set $e_I = \sum_{i\in I} e_i$ and let $\bar{e}_I = \frac{|I^c|}{n} e_I - \frac{|I|}{n} e_{I^c}$ be the projection of e_I onto V_0^n, along the vector $(1, \ldots, 1)$. A polyhedral cone is *pointed* if it does not contain any lines which extend to infinity in both directions.

Given $\{v_1, \ldots, v_k\} \subset \mathbb{R}^n$, denote by $\langle v_1, \ldots, v_k\rangle_+ := \{c_1v_1 + \cdots + c_kv_k : c_i \geq 0\}$ their *conical hull*. For example, a Weyl chamber is a conical hull, since it can also be obtained as the set of all linear combinations of a set of vectors with nonnegative coefficients,

$$\left\langle \bar{e}_{i_1}, \bar{e}_{\{i_1,i_2\}}, \ldots, \bar{e}_{\{i_1,i_2,\ldots,i_{n-1}\}} \right\rangle_+ = \{c_1\bar{e}_{i_1} + c_2\bar{e}_{\{i_1,i_2\}} + \cdots + c_{n-1}\bar{e}_{\{i_1,\ldots,i_{n-1}\}} : c_i \geq 0\}.$$

Further, the conical hull of the *roots* of type A_{n-1}, $\langle e_{i_1} - e_{i_2}, \ldots, e_{i_{n-1}} - e_{i_n}\rangle_+$ is called a *root cone*, and it is easy to check that it coincides with the plate $[i_1, \ldots, i_n]$. We leave it as an exercise for the reader to check that, if $(S_1, \ldots, S_k)$ is an ordered set partition of $\{1, \ldots, n\}$, then we have

$$\begin{aligned} &[S_1, \ldots, S_k] \\ &= \langle e_a - e_b : \text{ either } a, b \in S_j \text{ for } j = 1, \ldots, k, \text{ or } (a, b) \in S_i \times S_{i+1} \text{ for } i = 1, \ldots, k-1\rangle_+ . \end{aligned} \tag{2}$$

Definition 2. The *Minkowski sum* of two polyhedral cones C_1, C_2 is given by $C_1 + C_2 = \{u + v : u \in C_1,\ v \in C_2\}$. Denote by $[C_1] \cdot [C_2] = [C_1 \cap C_2]$ the pointwise product of their characteristic functions, and denote by

$$[C_1] \bullet [C_2] = [C_1 + C_2] = [\{u + v : u \in C_1,\ v \in C_2\}]$$

their *convolution*, which is the characteristic function of the Minkowski sum $C_1 + C_2$.

Remark 3. See [3] for details on the constructions of $\cdot$ and $\bullet$ as bi-linear maps on the $\mathbb{Q}$-vector space of characteristic functions of cones. Note however that we extend the coefficient field to $\mathbb{C}$.

There is a notion of duality for polyhedral cones.

Definition 4. Let C be a polyhedral cone in V_0^n. The dual cone to C, denoted $C^\star$, is defined by the equation

$$C^\star = \{y \in V_0^n : y \cdot x \geq 0 \text{ for all } x \in C\}.$$

Remark 5. Dual cones are known to satisfy the following properties.

(1) The dual $C^\star$ of a cone C with nonempty interior is pointed.
(2) The dual $C^\star$ of a pointed cone C has nonempty interior.
(3) If a cone C is convex and topologically closed, then $(C^\star)^\star = C$.

See for example [5] for details.

Remark 6. We shall need the following results from [3], Theorem 2.7 and respectively Corollary 2.8, where we extend the field from $\mathbb{Q}$ to $\mathbb{C}$.

- Duality for cones respects linear relations among their characteristic functions: if $C_1, \ldots, C_k$ are cones and for some constants $c_1, \ldots, c_k \in \mathbb{C}$ we have
$$\sum_{i=1}^{k} c_i [C_i] = 0$$
then the same relation holds among the characteristic functions for the dual cones,
$$\sum_{i=1}^{k} c_i [C_i^\star] = 0,$$
interchanging the pointwise product and the convolution.
- There exists a linear map on the vector space spanned by characteristic functions of cones, which we also denote by $\star$, such that $[C]^\star = [C^\star]$.
- Duality for cones interchanges intersections and Minkowski sums: if C_1, C_2 are cones, then $[C_1 \cap C_2] = [C_1] \cdot [C_2]$, and moreover
$$([C_1] \cdot [C_2])^\star = [C_1^\star] \bullet [C_2^\star].$$

Example 7. Denote $\bar{e}_1 = (2, -1, -1)/3$ and $\bar{e}_2 = (-1, 2, -1)/3$, hence $\bar{e}_1 + \bar{e}_2 = (1, 1, -2)/3 = -\bar{e}_3$. Then the two cones, the simple root cone
$$\langle e_1 - e_2, e_2 - e_3 \rangle_+$$
and the Weyl chamber
$$\langle \bar{e}_1, \bar{e}_1 + \bar{e}_2 \rangle_+,$$
are dual to each other. See Figure 1.

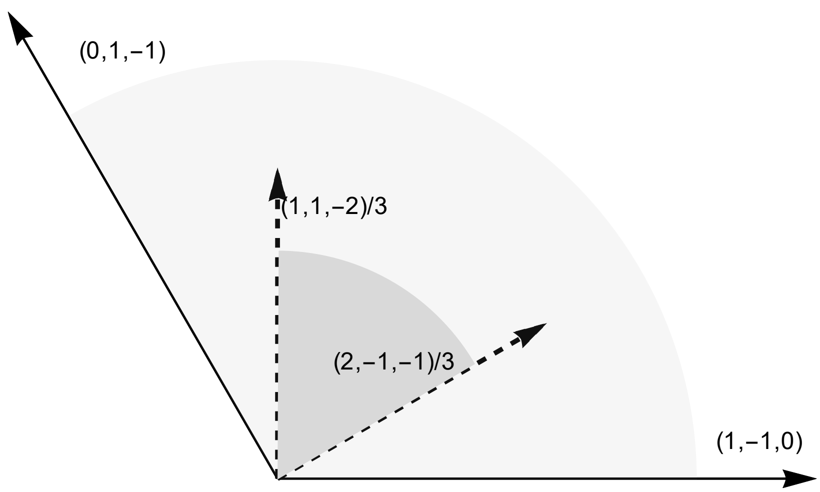

Figure 1. Dual Cones for Example 7

Definition 8. Denote by $[S_1, \ldots, S_k]^\star$ the face of the reflection arrangement labeled by the ordered set partition $(S_1, \ldots, S_k)$, given by the equations
$$(x_{s_{1,1}} = \cdots = x_{s_{1,l_1}}) \geq (x_{s_{2,1}} = \cdots = x_{s_{2,l_2}}) \geq \cdots \geq (x_{s_{k,1}} = \cdots = x_{s_{k,l_k}}), \quad x_1 + \cdots + x_n = 0$$
(where $s_{i,1}, \ldots, s_{i,l_i}$ are the elements of S_i), or for short
$$x_{(S_1)} \geq x_{(S_2)} \geq \cdots \geq x_{(S_k)}, \ x_1 + \cdots + x_n = 0,$$

where for compactness the symbol $x_{(S)}$ is defined to be the list of equations $(x_{i_1} = \cdots = x_{i_{|S|}})$, for a subset $S = \{i_1, \ldots, i_{|S|}\}$ of $\{1, \ldots, n\}$.

This can be given equivalently as the conical hull

$$\left\langle \bar{e}_{S_1}, \bar{e}_{S_1} + \bar{e}_{S_2}, \ldots, \bar{e}_{S_1} + \cdots + \bar{e}_{S_{k-1}} \right\rangle_+ .$$

Proposition 9. *If $(S_1, \ldots, S_k)$ is an ordered set partition of a subset $S \subseteq \{1, \ldots, n\}$, then the dual cone $[S_1, \ldots, S_k]^\star$ to*

$$[S_1, \ldots, S_k] = \left\{ \sum_{i \in S} x_i e_i \in V_0^n : \sum_{i \in S_1 \cup \cdots \cup S_j} x_i \geq 0, \text{ for each } j = 1, \ldots, k-1 \right\}$$

equals

$$\left\{ t_1 \sum_{i \in S_1} e_i + \cdots + t_k \sum_{i \in S_k} e_i \in V_0^n : t_1 \geq \cdots \geq t_k \right\}.$$

Proof. Suppose $y \cdot x \geq 0$ for all $x \in [S_1, \ldots, S_k]$. By Equation (2),

$$[S_1, \ldots, S_k] = \langle e_a - e_b : \text{ either } a, b \in S_i \text{ or } (a, b) \in S_i \times S_{i+1} \text{ for } i = 1, \ldots, k-1 \rangle_+ ,$$

it suffices to check that, for either $a, b \in S_i$ or $(a, b) \in S_i \times S_{i+1}$, we have

$$y_a - y_b = y \cdot (e_a - e_b) \geq 0 \text{ and } y_b - y_a = y \cdot (e_b - e_a) \geq 0.$$

Thus,

$$y = t_1 \sum_{i \in S_1} e_i + \cdots + t_k \sum_{i \in S_k} e_i + \sum_{i \in S^c} y_i e_i$$

for some $t_1, \ldots, t_k$ and y_i, such that $|S_1| t_1 + \cdots + |S_k| t_k + \sum_{i \in S^c} y_i = 0$. Now, for each $i = 1, \ldots, k-1$, for any $(a, b) \in S_i \times S_{i+1}$ we have $t_i - t_{i+1} = y \cdot (e_a - e_b) \geq 0$, or $t_i \geq t_{i+1}$. □

Corollary 10 follows from (2). For pedagogical reasons we include a different proof using duality.

Corollary 10. *If $\pi = [S_1, \ldots, S_k]$ is a plate, then we may factor its characteristic function using the pointwise product. Set theoretically we have*

$$\pi = [S_1, \ldots, S_k] = \bigcap_{i=1}^{k-1} [S_1 \cup \cdots \cup S_i, S_{i+1} \cup \cdots \cup S_k],$$

or in terms of characteristic functions,

$$[\pi] = [[S_1, S_2 \cup S_3 \cdots \cup S_k]] \cdot [[S_1 \cup S_2, S_3 \cup \cdots \cup S_k]] \cdots [[S_1 \cup S_2 \cdots \cup S_{k-1}, S_k]]$$

and using Minkowski sums as

$$\pi = \sum_{i=1}^{k-1} [S_i, S_{i+1}].$$

In terms of characteristic functions,

$$[\pi] = [[S_1, S_2]] \bullet [[S_2, S_3]] \bullet \cdots \bullet [[S_{k-1}, S_k]],$$

where

$$[S_i, S_{i+1}] = \left\{ \sum_{j \in S_i \cup S_{i+1}} x_j e_j \in V_0^n : \sum_{j \in S_i} x_j \geq 0, \sum_{j \in S_i \cup S_{i+1}} x_j = 0 \right\},$$

$e_1, \ldots, e_n$ being the standard basis for $\mathbb{R}^n$.

Proof. The identity for the intersection products follows immediately from the defining inequalities for plates.

For the convolution product identity it is convenient first to dualize,

$$[[S_1, S_2]] \bullet [[S_2, S_3]] \bullet \cdots \bullet [[S_{k-1}, S_k]] \quad = \quad ([[S_1, S_2]]^\star \cdot [[S_2, S_3]]^\star \cdots [[S_{k-1}, S_k]]^\star)^\star \,,$$

where by Proposition 9 we have

$$[[S_{i-1}, S_i]]^\star = \left[\{y \in V_0^n : y_{(S_i)} \geq y_{(S_{i+1})}\}\right].$$

Thus,

$$\begin{aligned}
& ([[S_1, S_2]]^\star \cdot [[S_2, S_3]]^\star \cdots [[S_{k-1}, S_k]]^\star)^\star \\
= \;& \left(\left[\{x \in V_0^n : x_{(S_1)} \geq x_{(S_2)}\}\right] \cdots \left[\{x \in V_0^n : x_{(S_{k-1})} \geq x_{(S_k)}\}\right]\right)^\star \\
= \;& \left[\{x \in V_0^n : x_{(S_1)} \geq \cdots \geq x_{(S_k)}\}\right]^\star \\
= \;& \left[\left\{x \in V_0^n : x_{S_1} \geq 0, \ldots, \; x_{S_1 \cup \cdots \cup S_{k-1}} \geq 0\right\}\right] \\
= \;& [[S_1, \ldots, S_k]],
\end{aligned}$$

where as usual we use the shorthand notations $x_S = \sum_{i \in S} x_i$ and $x_{(S)} = (x_{s_1} = \cdots = x_{s_{|S|}})$, for S a subset of $\{1, \ldots, n\}$.

□

2.2. **Genericity.** The relationships between $\hat{\mathcal{P}}^n, \mathcal{P}^n, \hat{\mathcal{P}}_1^n$ and $\mathcal{P}_1^n$ reduce to variations on a single linear identity. If $S_1, S_2 \subsetneq \{1, \ldots, n\}$ are any two disjoint nonempty subsets, then the characteristic function of the set

$$\{x \in V_0^n : x_{S_1} = x_{S_2} = 0 \text{ and } x_j = 0 \text{ for all } j \notin S_1 \cup S_2\},$$

represented by the pointwise product $[[S_1, S_2]] \cdot [[S_2, S_1]]$, can be expressed via

$$[[S_1, S_2]] + [[S_2, S_1]] = [[S_1, S_2]] \cdot [[S_2, S_1]] + [[S_1 \cup S_2]].$$

The above identity holds in $\hat{\mathcal{P}}^n$. To obtain the identity which holds in $\mathcal{P}^n$ we specialize to

$$[[S_1, S_2]] + [[S_2, S_1]] = [[S_1 \cup S_2]],$$

where $[[S_1 \cup S_2]]$ is the characteristic function of the set

$$[S_1 \cup S_2] = \left\{\sum_{j \in S_1 \cup S_2} x_j e_j \in \mathbb{R}^n : \sum_{j \in S_1 \cup S_2} x_j = 0\right\} \subseteq V_0^n.$$

In $\hat{\mathcal{P}}_1^n$ and $\mathcal{P}_1^n$ we we assume that $S_1, S_2 \subset \{1, \ldots, n\}$ are singlets and specialize to respectively

$$[[S_1, S_2]] + [[S_2, S_1]] = [[S_1, S_2]] \cdot [[S_2, S_1]]$$

and

$$[[S, S^c]] + [[S^c, S]] = 0.$$

3. Plate homology: from plates to their faces

The main result of this section is Theorem 21, which expands the face of a plate as a linear combination characteristic functions of plates in $\hat{\mathcal{P}}^n$.

While the symmetric group does not play an essential role in this paper, let us point out some of the symmetry properties of plates which were implicit in [7]. The action of the symmetric group $\mathfrak{S}_n$ on plates is inherited from the coordinate permutation on $\mathbb{R}^n$. In the plate notation, $\sigma \in \mathfrak{S}_n$ acts on characteristic functions of plates $[[S_1, \ldots, S_k]]$ by permuting elements in the blocks S_i.

Remark 11. The permutation group $\mathfrak{S}_n$ preserves the following operations on characteristic functions of plates.

(1) Inclusion:
$$[[S_1, \ldots, S_k]] \mapsto [[T_1, \ldots, T_l]]$$
where $l \leq k$, each T_i is a union of some consecutive S_j's and we still have $\cup_{i=1}^l T_i = \{1, \ldots, n\}$. Note that from the defining inequalities we have the inclusion of cones
$$[S_1, \ldots, S_k] \subseteq [T_1, \ldots, T_l].$$
(2) Block permutation:
$$[[S_1, \ldots, S_k]] \mapsto [[S_{\tau_1}, \ldots, S_{\tau_k}]]$$
for a permutation $\tau \in \mathfrak{S}_k$.
(3) Restriction to a face:
$$[[S_1, \ldots, S_k]] = [[S_1, S_2]] \bullet \cdots \bullet [[S_{k-1}, S_k]] \mapsto [[S_{i_1}, S_{i_1+1}]] \bullet \cdots \bullet [[S_{i_l}, S_{i_l+1}]]$$
for any subset $\{i_1, \ldots, i_l\} \subseteq \{1, \ldots, k-1\}$.

Henceforth we shall not use the action of the symmetric group $\mathfrak{S}_n$.

Proposition 12. *The space $\hat{\mathcal{P}}^n$, the linear span of the characteristic functions of plates, is freely generated by characteristic functions of all plates π, and thus has linear dimension the ordered Bell number $\sum_{i=1}^n k! S_{n,k}$, where $S_{n,k}$ are the Stirling numbers of the second kind, which count the number of set partitions of $\{1, \ldots, n\}$ into k disjoint subsets.*

Proof. By Remark 6, the involution $\star$ preserves linear relations among characteristic functions; therefore it provides a natural isomorphism of vector spaces from $\hat{\mathcal{P}}^n$ onto the space spanned by the characteristic functions of faces of Weyl chambers. Further, for each dimension $k = 0, \ldots, n-1$, these faces have non-intersecting relative interiors and consequently their characteristic functions are linearly independent, and by the duality map $\star$, plates in $\hat{\mathcal{P}}^n$ are as well. Therefore, to extract the dimension formula it suffices to count the faces of the arrangement of reflection hyperplanes; but these are in bijection with the ordered set partitions of $\{1, \ldots, n\}$, which are counted by the ordered Bell numbers. □

Definition 13. Let $\mathbf{S} = (S_1, \ldots, S_k)$ be an ordered set partition of $\{1, \ldots, n\}$. Suppose $\mathbf{T} = (T_1, \ldots, T_l)$ is another ordered set partition of $\{1, \ldots, n\}$, such that each block of T_i is a union of blocks S_i of $\mathbf{S}$. Define $p_i \in \{1, \ldots, l\}$ by the condition that S_i is a subset of T_{p_i}. Thus if S_i is a subset of the first block of $\mathbf{T}$ then $p_1 = 1$. Then $\mathbf{T}$ has the orientation $p_i \leq p_j$ with respect to $\mathbf{S}$ whenever S_i appears to the left of S_j in $\mathbf{S}$, with equality if and only if $S_i \sqcup S_j \subseteq T_a$ for some $a \in \{1, \ldots, l\}$. In the case that $\mathbf{T}$ satisfies $p_i < p_j$, we shall say that T is *compatible* with the orientation $p_i < p_j$.

Definition 14. An ordered set partition $\mathbf{T} = (T_1, \ldots, T_m)$ is a *shuffle-lumping* of ordered set partitions

$$\mathbf{S}_1 = (S_1, S_2, \ldots, S_{k_1}), \mathbf{S}_2 = (S_{k_1+1}, S_{k_1+2}, \ldots, S_{k_1+k_2}), \ldots, \mathbf{S}_l = (S_{k_1+\cdots+k_{l-1}+1}, \ldots, S_{k_1+\cdots+k_l}),$$

provided that each ordered set partition $\mathbf{S}_i$ is properly oriented:

$$p_1 < p_2 < \cdots < p_{k_1}$$

$$p_{k_1+1} < p_{k_1+2} < \cdots < p_{k_1+k_2}$$

$$\vdots$$

$$p_{k_1+\cdots+k_{l-1}+1} < p_{k_1+\cdots+k_{l-1}+2} < \cdots < p_n.$$

Example 15. The plate $[S_1, \ldots, S_k]$ is uniquely characterized among its shuffle-lumpings by the set of orientations $p_1 < p_2 < \cdots < p_k$ on the blocks $S_1, \ldots, S_k$.

Example 16. If $(S_1, S_2, S_3, S_4) = (1, 4, 23, 5)$ and $(S_5, S_6) = (678, 9)$, then shuffle-lumped plates include for example

$$[1, 4678, 23, 59] \text{ and } [678, 1, 4, 23, 9, 5].$$

Then $[1, 4678, 23, 59]$ is a plate with the smallest possible number of blocks, while $[678, 1, 4, 23, 9, 5]$ is a plate with the largest possible number of blocks, in the shuffle-lumping of the set compositions (S_1, S_2, S_3, S_4) and (S_5, S_6).

Example 17. The shuffle-lumpings of $(S_1, S_2) = (\{1\}, \{2, 3\})$ and $(S_3, S_4) = (\{4\}, \{5\})$ are

$$\{(1, 23, 4, 5), (1, 234, 5), (1, 4, 23, 5), (14, 23, 5), (4, 1, 23, 5), (1, 4, 235), (14, 235), (4, 1, 235),$$

$$(1, 4, 5, 23), (14, 5, 23), (4, 1, 5, 23), (4, 15, 23), (4, 5, 1, 23)\}$$

In Lemma 18 we decompose the characteristic function of a union of closed Weyl chambers into an alternating sum of partially closed Weyl chambers in a canonical way that depends on descent positions, with respect to the natural order $(1, \ldots, n)$. See [18] for a systematic approach using so-called (P, ω)-partitions.

Lemma 18. *We have the decomposition into disjoint sets*

$$V_0^n = \sqcup_{\sigma \in \mathfrak{S}_n} C_\sigma,$$

where each C_σ is the partially open Weyl chamber defined by

$$x_{\sigma_i} \geq x_{\sigma_{i+1}} \quad \text{if } \sigma_i < \sigma_{i+1}$$

and

$$x_{\sigma_i} > x_{\sigma_{i+1}} \quad \text{if } \sigma_i > \sigma_{i+1}.$$

The characteristic function of C_σ, with $\sigma = (\sigma_1, \ldots, \sigma_n)$ given, is

$$[C_\sigma] = \sum_\pi (-1)^{n-\mathrm{len}(\pi)} [\pi^\star],$$

where the sum is over the set of plates $\pi = [(S_1, \ldots, S_k)]$ which are labeled by ordered set partitions $(S_1, \ldots, S_k)$ which satisfy the following property: each block $S_i = \{i_1, \ldots, i_{|S_i|}\}$ is a maximal set of consecutive descending labels of σ, of the form $\sigma_{i_1} > \sigma_{i_1+1} > \cdots > \sigma_{i_{|S_i|}}$.

Proof. Any $x \in V_0^n$ not in *any* reflection hyperplane $x_i = x_j$ is in the interior of the Weyl chamber labeled by the order of its coordinate values, $x_{\sigma_1} > \cdots > x_{\sigma_n}$, say. Now if x is in the interior of a face labeled by an ordered set partition $(S_1, \ldots, S_k)$ of $\{1, \ldots, n\}$, having the form

$$x = t_1 e_{S_1} + \cdots + t_k e_{S_k},$$

for some $t_1 > t_2 > \cdots > t_k$ with $\sum_{i=1}^k |S_i| t_i = 0$, then we put $x \in C_\sigma$ where the permutation σ is obtained from $(S_1, \ldots, S_k)$ by placing the labels in each block S_i in increasing order and then concatenating the blocks.

The formula for $[C_\sigma]$ follows from the standard inclusion-exclusion expression for the characteristic function of the complement of the union of the codimension 1 faces $[\sigma_1, \ldots, \sigma_i\sigma_{i+1}, \ldots, \sigma_n]^\star$ corresponding to descents $\sigma_i > \sigma_{i+1}$ in σ:

$$[\sigma_1, \ldots, \sigma_n]^\star \setminus \left(\bigcup_{\sigma_i > \sigma_{i+1}} [\sigma_1, \ldots, \sigma_i\sigma_{i+1}, \ldots, \sigma_n]^\star \right),$$

that is

$$[C_\sigma] = [[\sigma_1, \ldots, \sigma_n]^\star] - \sum_{\pi'} (-1)^{n-1-\mathrm{len}(\pi')} [(\pi')^\star] = \sum_{\pi} (-1)^{n-\mathrm{len}(\pi)} [\pi^\star]$$

where the middle sum is over the lumpings π' of $[\sigma_1, \ldots, \sigma_n]$ at descents $\sigma_i > \sigma_{i+1}$ (excluding $[\sigma_1, \ldots, \sigma_n]$ itself), and the right sum now includes $[\sigma_1, \ldots, \sigma_n]$. ☐

Example 19. The characteristic functions of the partially open Weyl chambers respectively

$$\{x \in V_0^3 : x_1 \geq x_2 \geq x_3\}$$

$$\{x \in V_0^3 : x_2 > x_1 \geq x_3\}$$

$$\{x \in V_0^3 : x_3 > x_2 > x_1\}$$

can be obtained as linear combinations of characteristic functions of dual plates as

$$\begin{aligned} [C_{(1,2,3)}] &= [[1,2,3]]^\star \\ [C_{(2,1,3)}] &= [[2,1,3]]^\star - [[21,3]]^\star \\ [C_{(3,2,1)}] &= [[3,2,1]]^\star - [[32,1]]^\star - [[3,21]]^\star + [[321]]^\star \end{aligned}$$

More generally, in Lemma 18 we replace Weyl chambers, labeled by permutations, with higher codimension faces of the reflection arrangement which are labeled by ordered set partitions. Suppose

$$\mathbf{S}_1 = (S_1, \ldots, S_{k_1}), \ldots, \mathbf{S}_l = (S_{k_1+\cdots+k_{l-1}+1}, \ldots, S_{k_1+\cdots+k_l})$$

are ordered set partitions of respectively $\bigcup_{S \in \mathbf{S}_i} S$, $i = 1, \ldots, l$, and let $\sigma = (\sigma_1, \ldots, \sigma_l)$ be a permutation of $\{1, \ldots, l\}$. Define an embedding $\iota : V_0^l \hookrightarrow V_0^{k_1+\cdots+k_l}$ by

$$\sum_{i=1}^l t_i e_i \mapsto \sum_{i=1}^l \left(\frac{t_i}{|S_i|} \right) e_{S_i}.$$

Corollary 20. *We have*

$$[\iota(C_\sigma)] = \sum_{\pi} (-1)^{l-\mathrm{len}(\pi)} [\iota(\pi^\star)],$$

where the sum is the same as in Lemma 18.

In Theorem 21, we replace the natural order $(1,\dots,m)$ with an ordered set partition

$$(S_1, S_2, \dots, S_m)$$

of $\{1,\dots,n\}$, where $m = k_1 + \cdots + k_l$.

The proof in what follows of Theorem 21 illustrates the essential role of the duality isomorphism from Definition 4 and utilizes directly set-theoretic inclusion-exclusion arguments. Now, by way of Theorem 5.2 of [12], the same formula holds if we extend plates from V_0^n into the ambient space $\mathbb{R}^n$, in which case the last line of the plate equations becomes $x_1 + \cdots + x_n \geq 0$.

Theorem 21. *Given l ordered set partitions*

$$\mathbf{S}_1 = (S_1, S_2, \dots, S_{k_1}), \mathbf{S}_2 = (S_{k_1+1}, S_{k_1+2}, \dots, S_{k_1+k_2}), \dots, \mathbf{S}_l = (S_{k_1+\cdots+k_{l-1}+1}, \dots, S_{k_1+\cdots+k_l})$$

such that $\bigsqcup_{i=1}^{k_1+\cdots+k_l} S_i = \{1,\dots,n\}$, then we have the identity for characteristic functions of plates in $\hat{\mathcal{P}}^n$,

$$[[\mathbf{S}_1]] \bullet \cdots \bullet [[\mathbf{S}_l]] = \sum_{\pi} (-1)^{m-\mathrm{len}(\pi)} [\pi],$$

where $m = k_1 + \cdots + k_l$ and π runs over all shuffle-lumpings of $\mathbf{S}_1, \dots, \mathbf{S}_l$.

Proof. We shall work in the space of characteristic functions of faces of the reflection arrangement and then dualize to obtain the identity for characteristic functions of plates.

We have

$$[\mathbf{S}_i]^\star \quad = \quad \left\{ \sum_{i=1}^m t_i e_{S_i} \in V_0^n : t_{k_{i-1}+1} \geq \cdots \geq t_{k_i} \right\}$$

and thus

$$[\mathbf{S}_1]^\star \cap \cdots \cap [\mathbf{S}_l]^\star \quad = \quad \left\{ \sum_{i=1}^m t_i e_{S_i} : \begin{array}{c} t_1 \geq t_2 \geq \cdots \geq t_{k_1}, \\ t_{k_1+1} \geq \cdots \geq t_{k_1+k_2}, \\ \vdots \\ t_{k_1+\cdots+k_{l-1}+1} \geq \cdots \geq t_{k_1+\cdots+k_l} \end{array} \right\},$$

which lives in a copy of V_0^m embedded in V_0^n as $\iota(\sum_{i=1}^m t_i e_i) \mapsto \sum_{i=1}^m t_i (e_{S_i}/|S_i|)$.

Then $\iota^{-1}([\mathbf{S}_1]^\star \cap \cdots \cap [\mathbf{S}_l]^\star) \subseteq V_0^m$ is a union of Weyl chambers $\bigcup_\tau [\tau]^\star$ defined by $y_{\tau_1} \geq \cdots \geq y_{\tau_m}$ labeled by shuffles $(\tau_1, \dots, \tau_m)$ of

$$\sigma_1 = (1, 2, \dots, k_1), \dots, \sigma_l = (m - k_l + 1, \dots, m).$$

We replace each such (closed) Weyl chamber defined by $y_{\tau_1} \geq \cdots \geq y_{\tau_m}$ with the partially open Weyl chamber C_τ from Lemma 18 and obtain the disjoint union

$$\iota^{-1}([\mathbf{S}_1]^\star \cap \cdots \cap [\mathbf{S}_l]^\star) \quad \supseteq \quad \bigsqcup_\tau C_\tau,$$

where the disjoint union is dense in the (closed) left hand side. Thus, equality will follow once we establish that $\bigsqcup_\tau C_\tau$ is already topologically closed.

Supposing x is in a missing boundary face of some partially open Weyl chamber C_τ, then the coordinates of x satisfy an equality $x_d = x_{d+1}$ where $\tau_d > \tau_{d+1}$ is a descent of τ. But since $\sigma_1, \dots, \sigma_l$ are all increasing, this can happen only if τ_d and τ_{d+1} belong to two different permutations, say σ_i and respectively σ_j, for some $i \neq j$. This implies that the permutation τ' obtained from τ by switching τ_d and τ_{d+1} is also a shuffle of $\sigma_1, \dots, \sigma_l$, hence $x \in C_{\tau'}$, proving the equality.

This, together with the expansion from Lemma 18 for the characteristic function $[C_\tau]$ implies the identity for characteristic functions

$$[\iota^{-1}([\mathbf{S}_1]^\star \cap \cdots \cap [\mathbf{S}_l]^\star)] = \sum_\tau [C_\tau] = \sum_\tau \left(\sum_{\pi_\tau} (-1)^{m-\mathrm{len}(\pi_\tau)} [\iota^{-1}(\pi_\tau^\star)] \right),$$

where the inner sum is over all lumpings of the plate π_τ which can occur at the descents of τ.

It follows from Theorem 2.3 of [3] that the ι induces a unique linear map on the space of characteristic functions, and we obtain

$$[[\mathbf{S}_1]^\star \cap \cdots \cap [\mathbf{S}_l]^\star] = \sum_\tau [\iota(C_\tau)] = \sum_\tau \left(\sum_{\pi_\tau} (-1)^{m-\mathrm{len}(\pi_\tau)} [\pi_\tau^\star] \right).$$

We finally dualize again to obtain the sum over all shuffle-lumpings

$$[[\mathbf{S}_1]] \bullet \cdots \bullet [[\mathbf{S}_l]] = \sum_\pi (-1)^{m-\mathrm{len}(\pi)} [\pi].$$

□

It is worthwhile to point out that in the convolution $[[\mathbf{S}_1]] \bullet \cdots \bullet [[\mathbf{S}_l]]$, the plates $[\mathbf{S}_i]$ live in mutually orthogonal subspaces in V_0^n.

Example 22. Let $S_1 = \{1\}$ and $(S_2, S_3) = (\{2\}, \{3\})$. Then Theorem 21 says that

$$[[1]] \bullet [[2,3]] = [[1,2,3]] + [[2,1,3]] + [[2,3,1]] - ([[12,3]] + [[2,13]]).$$

See Figure 2.

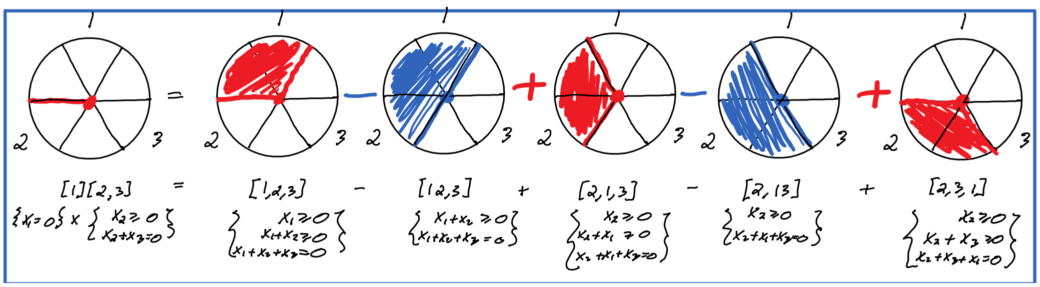

Figure 2. The characteristic function $[[1]] \bullet [[2,3]]$

4. Constructing the canonical plate basis

We generalize the notion of the cycle decomposition, from permutations to ordered set partitions. The geometric motivation is to establish a graded basis for $\hat{\mathcal{P}}^n$ such that the d^{th} graded piece is spanned by characteristic functions of faces of dimension d of the all-subset hyperplane arrangement.

In what follows, we fix once and for all the *standard* ordered set partition $(I_1, I_2, \ldots, I_n)$ of $\{1, \ldots, n\}$, where $I_j = \{j\}$. For compactness, we shall abuse notation and write j instead of I_j. However, it should not be forgotten that this obscures an action of the product group $\mathfrak{S}_n \times \mathfrak{S}_n$, where one factor permutes the order of the blocks and the other permutes their contents.

Definition 23. A *composite set partition* of $\{1, \ldots, n\}$ is a set $\{\mathbf{S}_1, \ldots, \mathbf{S}_l\}$ where each $\mathbf{S}_i$ is an ordered set partition of a subset $J_i \subset \{1, \ldots, n\}$, such that $\{J_1, \ldots, J_l\}$ is an (unordered) set partition of $\{1, \ldots, n\}$. If each $\mathbf{S}_i$ has the property that its first label contains the minimal label in J_i, then the composite set partition is called *standard.* Denote by COSP^n the set of composite ordered set partitions of $\{1, \ldots, n\}$.

In Lemma 24 we give a bijection between ordered set partitions $(S_1, \ldots, S_k)$ of the set $\{1, \ldots, n\}$ and standard composite set partitions. From this we shall derive a basis for $\hat{\mathcal{P}}^n$ which is compatible with all quotients $\hat{\mathcal{P}}^n_1, \mathcal{P}^n, \mathcal{P}^n_1$.

The image of the unitriangular map in $\hat{\mathcal{P}}^n$ appears quite analogous to the canonical decomposition of the homogeneous component of the free Lie algebra, see the discussion around Lemma 8.22 in [17]. Also note the similarity to Foata's transform. It would be very interesting to look into these further.

Lemma 24. *There exists a bijection $\mathcal{U}$ between ordered set partitions of $\{1, \ldots, n\}$ and standard composite set partitions.*

Proof. For an ordered set partition $(S_1, \ldots, S_k)$ of $\{1, \ldots, n\}$, with minimal labels respectively $(s_1, \ldots, s_k)$, let $s_{i_1} > s_{i_2} > \cdots > s_{i_l}$ be the maximal decreasing sequence of the minimal labels which satisfies $s_{i_1} = s_1$ and $s_{i_l} = 1$.

We construct from $(S_1, \ldots, S_k)$ a standard composite set partition such that we always have 1 in the first block of $\mathbf{S}_1$:

$$\begin{aligned}
\mathbf{S}_1 &= (S_{i_l}, \ldots, S_k) \\
\mathbf{S}_2 &= (S_{i_{l-1}}, \ldots, S_{i_l - 1}) \\
&\vdots \\
\mathbf{S}_{l-1} &= (S_{i_2}, \ldots, S_{i_3 - 1}) \\
\mathbf{S}_l &= (S_1, S_2, \ldots, S_{i_2 - 1}).
\end{aligned}$$

For the inverse, if $\{\mathbf{S}_1, \ldots, \mathbf{S}_l\}$ is a standard composite ordered set partition with minimal elements respectively $s_1, \ldots, s_l$, let $(s_{i_1}, \ldots, s_{i_l})$ be the permutation of $(s_1, \ldots, s_l)$ such that $s_{i_1} > \cdots > s_{i_l}$. We then reconstruct the ordered set partition as the concatenation $(\mathbf{S}_{i_l}, \mathbf{S}_{i_{l-1}}, \ldots \mathbf{S}_{i_1})$. □

It may be interesting to compare Theorem 27 in what follows with Theorem 5.1 of [17] on the construction of the Hall basis of the free Lie algebra, using the set of Lyndon words, ordered alphabetically, for the Hall set.

We define a map $\mathcal{C}$ from the set of ordered set partitions to the set of packed words in $\{0, \ldots, n-1\}$, that is sequences in $\{0, \ldots, n-1\}^n$ satisfying the conditions

(1) $0 \in \{p_1, \ldots, p_n\}$ and
(2) Successive values increase in steps of 1.

Proposition 25. *There exists a bijection*

$$\mathcal{C} : \{\textit{Ordered set partitions of } \{1, \ldots, n\}\} \to \textit{packed words in } \{1, \ldots, n\}.$$

Proof. Let $\mathbf{T} = (T_1, \ldots, T_k)$ be an ordered set partition of $\{1, \ldots, n\}$. We define a sequence $(c_1, \ldots, c_n)$ by $c_i = j - 1$, if $i \in T_j$. By construction $0 = c_i$ for some i and the sequence is contiguous: if some $c \geq 1$ satisfies $c \in \{c_1, \ldots, c_n\}$ then $c - 1$ satisfies the same. Define $\mathcal{C}(\mathbf{T}) = (c_1, \ldots, c_n)$. Conversely, if $(c_1, \ldots, c_n)$ is a contiguous sequence containing 0, define an ordered set partition $\mathbf{T} = (T_1, \ldots, T_k)$ by letting $T_i = \{m \in \{1, \ldots, n\} : c_m = i\}$. □

In what follows, we induce a partial order on the ordered set partitions from the lexicographic order on such sequences.

Definition 26. Given two plates $\pi_1 = [(T_1, \ldots, T_k)],\ \pi_2 = [(T'_1, \ldots, T'_l)]$, say that $\pi_1 \prec \pi_2$ if

$$\mathcal{C}(T_1, \ldots, T_k) < \mathcal{C}(T'_1, \ldots, T'_l)$$

in the lexicographic order. This induces a total order on the set of plates, and thus on the basis of their characteristic functions, in $\hat{\mathcal{P}}^n$. Clearly the first element of ordered basis is the characteristic function of V_0^n itself, since it is labeled by the trivial ordered set partition, hence $\mathcal{C}([12\cdots n]) = (0,\ldots,0)$. Similarly, the last element is labeled by the permutation $(n, n-1, \ldots, 2, 1)$, where we have $\mathcal{C}([n, n-1, \ldots, 2, 1]) = (n-1, n-2\ldots, 1, 0)$.

Theorem 27. *Let $\mathcal{B}^n$ be the basis of characteristic functions of plates for the space $\hat{\mathcal{P}}^n$, labeled by ordered set partitions of $\{1,\ldots,n\}$. For each basis element $[\pi] = [[S_1,\ldots,S_k]] \in \mathcal{B}^n$, plug the set partition $\mathbf{S}_1,\ldots,\mathbf{S}_l$ provided by Lemma 24 into Theorem 21:*

$$[\pi] \mapsto \mathcal{U}([\pi]) := \sum_{\pi'} (-1)^{m-\mathrm{len}(\pi')}[\pi'],$$

where the sum is over all plates π' which are labeled by shuffle-lumpings of the standard composite set partition $\mathbf{S}_1,\ldots,\mathbf{S}_l$.

Then, the matrix of the linear transformation $\mathcal{U}$ induced on $\hat{\mathcal{P}}^n$ is upper unitriangular with respect to the lexicographically ordered basis $\mathcal{B}^n$. In particular, $\mathcal{U}$ is invertible.

Proof. First note that as $\mathcal{U}([\pi])$ contains $[\pi]$ itself as a summand, it suffices to prove that $\mathcal{U}$ is order non-increasing with respect to the lexicographic ordering from Definition 26 on ordered set partitions.

Let $\{\mathbf{S}_1,\ldots,\mathbf{S}_l\}$ be the resulting standard composite set partition coming from Lemma 24 applied to a plate $\pi = [S_1,\ldots,S_k]$, and consider an arbitrary (signed) summand of $\mathcal{U}(\pi)$. Such a summand is labeled by a shuffle-lumping $\mathbf{T} = (T_1,\ldots,T_m)$ of the ordered set partitions $\mathbf{S}_1,\ldots,\mathbf{S}_l$, with respect to the ordered set partition $(\{1\},\ldots,\{n\})$. By construction of the ordered set partitions $\mathbf{S}_1,\ldots,\mathbf{S}_l$ from π in Lemma 24, among all of their shuffle-lumpings π occurs and is maximal in the lexicographic order $\prec$. It follows that the matrix for $\mathcal{U}$ is upper triangular; but we noted at the beginning of the proof that the matrix for $\mathcal{U}$ with respect to the basis $\mathcal{B}^n$ has 1's on the diagonal. It follows that $\mathcal{U}$ is invertible. ☐

Intuitively, a convolution of characteristic functions of plates $[\pi_1]\bullet\cdots\bullet[\pi_k]$ is an element of the canonical basis of $\hat{\mathcal{P}}^n$ if and only if (1) $\pi_1,\ldots,\pi_k$ live in mutually orthogonal subspaces of V_0^n and (2) each plate π_i is standard, with its minimal element in the first block. Moreover, according to Theorem 27 the map $\mathcal{U}$ sends $[\pi]$ to the characteristic function of the unique face of π which is in the canonical basis and which is maximal with respect to inclusions of sets.

Example 28. With respect to the lexicographic order

$$(0,0,0),(0,0,1),(0,1,0),(0,1,1),(0,1,2),(0,2,1),(1,0,0),(1,0,1),$$

$$(1,0,2),(1,1,0),(1,2,0),(2,0,1),(2,1,0),$$

via the bijection $\mathcal{C}$, on the plate basis we have respectively

$$[123],[12,3],[13,2],[1,23],[1,2,3],[1,3,2],[23,1],[2,13],$$

$$[2,1,3],[3,12],[3,1,2],[2,3,1],[3,2,1]$$

and the map $\mathcal{U}$ takes the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where columns 7 through 13 label linear combinations of characteristic functions of plates which vanish outside a higher codimension subset. It is informative to verify (for example, graphically, using inclusion-exclusion, as in Figure 2) that the rightmost column, an alternating sum over all 13 plates, encodes the characteristic function of the point $(0,0,0)$, and that column 7 expresses

$$[[23,1]] \mapsto [[23,1]] + [[1,23]] - [[123]] = [[1]] \bullet [[23]].$$

Finally, column 12 encodes

$$[[2,3,1]] \mapsto [[2,3,1]] + [[2,1,3]] - [[2,13]] + [[1,2,3]] - [[12,3]] = [[1]] \bullet [[2,3]],$$

or, in an order in which it is perhaps easier to see the shuffle-lumping,

$$[[1]] \bullet [[2,3]] = [[2,3,1]] - [[2,13]] + [[2,1,3]] - [[12,3]] + [[1,2,3]].$$

The following more involved example will serve to illustrate the upper-triangularity of Theorem 27.

Example 29. We have

$$\mathcal{U}([[4\ 11, 10, 3, 5\ 7, 6\ 8, 1\ 9, 2]]) = [[1\ 9, 2]] \bullet [[3, 5\ 7, 6\ 8]] \bullet [[4\ 11, 10]],$$

where we omit the (rather long) alternating sum over all shuffle-lumpings of the ordered set partitions

$$(1\ 9, 2), (3, 5\ 7, 6\ 8), (4\ 11, 10).$$

It is a useful exercise to apply Lemma 24 to check that

$$\mathcal{C}(\{4,11\},\{10\},\{3\},\{5,7\},\{6,8\},\{1,9\},\{2\}) = (5,6,2,0,3,4,3,4,5,1,0)$$

and verify that $(\{4,11\},\{10\},\{3\},\{5,7\},\{6,8\},\{1,9\},\{2\})$ is the shuffle-lumping that is maximal with respect to the lexicographic ordering: switching or merging any two blocks which are not in the same ordered set partition results in a lexicographically smaller ordered set partition. For example, the shuffle-lumping obtained by merging $\{3\}$ and $\{10\}$, which are in distinct ordered set partitions, obviously decreases the lexicographic order:

$$\mathcal{C}(\{4,11\},\{3,10\},\{5,7\},\{6,8\},\{1,9\},\{2\}) = (4,5,1,0,2,3,2,3,4,1,0),$$

while switching $\{3\}$ and $\{10\}$ (slightly less obviously) does as well:

$$\mathcal{C}(\{4,11\},\{3\},\{10\},\{5,7\},\{6,8\},\{1,9\},\{2\}) = (5,6,1,0,3,4,3,4,5,2,0).$$

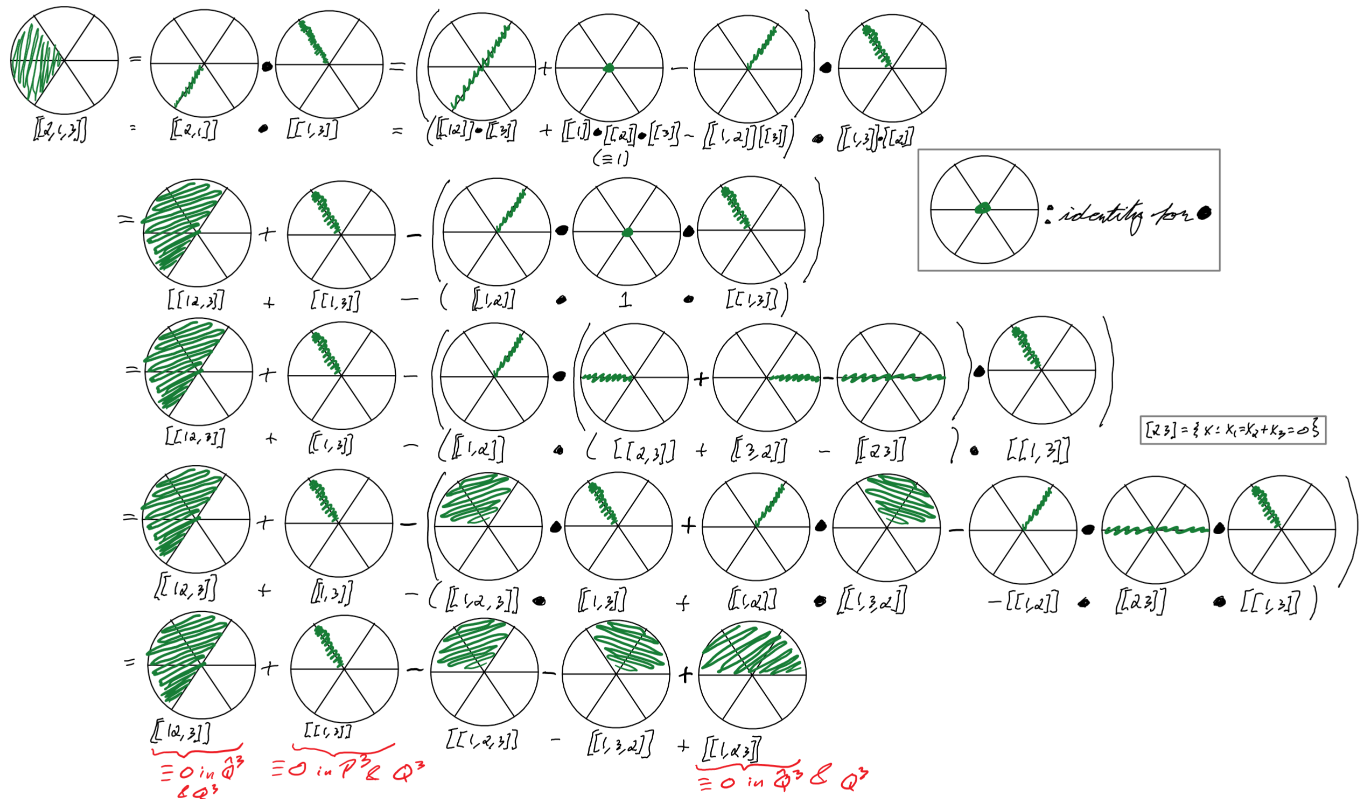

FIGURE 3. Some computations in the convolution algebra of permutohedral cones: expansion in the canonical plate basis

Figure 3 expands the plate $[[2, 1, 3]]$ in the canonical plate basis.

As a consequence of Theorem 27 we have Corollary 30.

Corollary 30. *The linear dimension of the degree k component $(\hat{\mathcal{P}}^n)_k$ of the space $\hat{\mathcal{P}}^n$, consisting of linear combinations of characteristic functions of total dimension k Minkowski sums of plates, is equal to the number of standard composite set partitions $\{\mathbf{S}_1, \ldots, \mathbf{S}_k\}$ of $\{1, \ldots, n\}$. Namely,*

$$\dim((\hat{\mathcal{P}}^n)_k) = \sum_{i=k}^{n} S(n, i)s(i, k).$$

Likewise, the linear dimension of the degree k component of the space $\hat{\mathcal{P}}^n_1$, consisting of linear combinations of characteristic functions of pointed, total dimension k Minkowski sums of plates, is equal to the kth Stirling number of the first kind,

$$\dim((\hat{\mathcal{P}}^n_1)_k) = S(n, n)s(n, k) = s(n, k).$$

Here $S(n, i)$ is the Stirling number of the second kind, which counts the number of set partitions of $\{1, \ldots, n\}$ into i blocks, and $s(i, k)$ is the Stirling number of the first kind, which counts the number of permutations of $\{1, \ldots, i\}$ which decompose as a product of k disjoint cycles.

Proof. In the formula for $\dim((\hat{\mathcal{P}}^n)_k)$, the contribution $S(n, i)s(i, k)$ is the product of the number of set partitions of n with i blocks, times the number of permutations of $\{1, \ldots, i\}$ which decompose into k disjoint cycles. This is exactly the enumeration of the standard composite set partitions $\{\mathbf{S}_1, \ldots, \mathbf{S}_k\}$: each $\mathbf{S}_a$ is a standard ordered set partition and

$$\bigcup_{S \in \mathbf{S}_1} S \cup \cdots \cup \bigcup_{S \in \mathbf{S}_k} S = \{1, \ldots, n\}.$$

The formula for the dimension of the k^{th} graded component $(\hat{\mathcal{P}}_1^n)_k$ follows by taking the unique ordered set partition $\{\{1\},\ldots,\{n\}\}$ of $\{1,\ldots,n\}$ and counting the number of permutations of $\{1,\ldots,n\}$ which decompose into k disjoint cycles. □

Example 31. The formula in Corollary 30 is given in O.E.I.S. A079641, the matrix product of the Stirling numbers of the second kind with the unsigned Stirling numbers of the first kind. The first six rows are given below; note that the rows sum to the ordered Bell numbers $(1, 3, 13, 75, 541, 4683)$.

$$\begin{array}{cccccc}
1 \\
2 & 1 \\
6 & 6 & 1 \\
26 & 36 & 12 & 1 \\
150 & 250 & 120 & 20 & 1 \\
1082 & 2040 & 1230 & 300 & 30 & 1
\end{array}$$

The canonical basis for $\hat{\mathcal{P}}_1^n$ has graded dimension given by the Stirling numbers of the first kind,

$$\begin{array}{cccccc}
1 \\
1 & 1 \\
2 & 3 & 1 \\
6 & 11 & 6 & 1 \\
24 & 50 & 35 & 10 & 1 \\
120 & 274 & 225 & 85 & 15 & 1.
\end{array}$$

5. Plates and trees

Let $\{(i_1, j_1), \ldots, (i_{n-1}, j_{n-1})\}$ be the set of oriented edges of a directed tree $\mathcal{T}$ on the vertex set $\{1,\ldots,n\}$. This data encodes a certain permutohedral cone which is also simplicial, given explicitly as the conical hull

$$\pi_{\mathcal{T}} = \langle e_{i_1} - e_{j_1}, \ldots, e_{i_{n-1}} - e_{j_{n-1}} \rangle_+.$$

In Theorem 32 we present a combinatorial formula which expands the characteristic function of the permutohedral cone assigned to any oriented tree on n vertices as a signed sum of characteristic functions of plates.

The proof of Theorem 32 follows closely that of Theorem 21. The idea is to decompose a union of overlapping closed Weyl chambers into a disjoint union of partially open Weyl chambers; then the characteristic function of the disjoint union expands using the formula in Lemma 18. Then we dualize to get the permutohedral cone $\pi_{\mathcal{T}}$.

Theorem 32. *Let $\mathcal{T} = \{(i_1, j_1), \ldots (i_{n-1}, j_{n-1})\}$ be a directed tree. We have, in the space $\hat{\mathcal{P}}^n$, the identity of characteristic functions*

$$[\langle e_{i_1} - e_{j_1}, \ldots, e_{i_{n-1}} - e_{j_{n-1}} \rangle_+] = \sum_{\pi : p_{i_a} < p_{j_a}} (-1)^{n - \mathrm{len}(\pi)} [\pi],$$

where we recall that $p_{i_a} < p_{j_a}$ if and only if in the ordered set partition which labels π, the label i_a is in a block strictly to the left of the block containing j_a.

Proof. The dual cone is defined by the equations

$$\langle e_{i_1} - e_{j_1}, \ldots, e_{i_{n-1}} - e_{j_{n-1}} \rangle_+^\star = \left\{ y \in V_0^n : y_{i_1} \geq y_{j_1}, \ldots, y_{j_{n-1}} \geq y_{j_{n-1}} \right\}.$$

We first claim that this is a union of those Weyl chambers $\cup_\tau [\tau]^\star$ defined by $y_{\tau_1} \geq \cdots \geq y_{\tau_m}$ which satisfy the $n-1$ conditions $\tau_{i_1} > \tau_{j_1}, \ \ldots, \ \tau_{i_{n-1}} > \tau_{j_{n-1}}$. To see this, let $\bar{e}_{I_1}, \ldots, \bar{e}_{I_{n-1}}$ be the basis which is orthogonally dual to $e_{i_1}-e_{j_1},\ldots,e_{i_{n-1}}-e_{j_{n-1}}$, so that $\bar{e}_{I_a}\cdot(e_{i_b}-e_{j_b})=\delta_{a,b}$. Then we have from the corresponding vector space isomorphism a bijection of cone points

$$\langle e_{i_1}-e_{j_1},\ldots,e_{i_{n-1}}-e_{j_{n-1}}\rangle_+ \to \langle e_{i_1}-e_{j_1},\ldots,e_{i_{n-1}}-e_{j_{n-1}}\rangle_+^\star = \langle \bar{e}_{I_1},\ldots,\bar{e}_{I_{n-1}}\rangle_+$$

defined by $e_{i_a}-e_{j_a}\mapsto \bar{e}_{I_a}$, that is

$$\sum_{a=1}^{n-1} t_a(e_{i_a}-e_{j_a}) \mapsto \sum_{a=1}^{n-1} t_a \bar{e}_{I_a}.$$

Let $[\alpha_1]^\star,\ldots,[\alpha_m]^\star$ be the minimal set of Weyl chambers such that

$$\langle e_{i_1}-e_{j_1},\ldots,e_{i_{n-1}}-e_{j_{n-1}}\rangle_+^\star \subseteq \cup_{i=1}^m [\alpha_i]^\star,$$

which means that the permutations $\alpha_1,\ldots,\alpha_m$ are all compatible with the orders $(i_1,j_1),\ldots,(i_{n-1},i_{n-1})$. We show that this is an equality: for each $y\in\cup_{i=1}^m[\alpha_i]^\star$, since $\bar{e}_{I_1},\ldots,\bar{e}_{I_{n-1}}$ is a basis for V_0^n we have $y=\sum_{a=1}^{n-1}t_a\bar{e}_{I_a}$ for some $t_a\in\mathbb{R}$, for equality it suffices to show that $y_{i_a}-y_{j_a}=y\cdot(e_{i_a}-e_{j_a})=t_a\geq 0$ for all $a=1,\ldots,n-1$. But having $y_{i_a}-y_{j_a}<0$ for some a would imply that α_i is not compatible with the order $(i_1,j_1),\ldots,(i_{n-1},i_{n-1})$.

As in Theorem 21, we replace the Weyl chambers $[\tau]^\star$ with the (mutually disjoint) partially open Weyl chambers C_τ from Lemma 18. By construction these all satisfy the inequalities defining the dual cone, and we correspondingly have, for characteristic functions,

$$[\langle e_{i_1}-e_{j_1},\ldots,e_{i_{n-1}}-e_{j_{n-1}}\rangle_+^\star]=\sum_\tau [C_\tau],$$

where the sum is over all permutations $\tau=(\tau_1,\ldots,\tau_n)$ satisfying the $n-1$ conditions $\tau_{i_1}>\tau_{j_1},\ \ldots,\ \tau_{i_{n-1}}>\tau_{j_{n-1}}$.

But from Lemma 18, for each such τ we have the further decomposition

$$[C_\tau]=\sum_\pi (-1)^{n-\mathrm{len}(\pi)}[\pi^\star],$$

where the sum is over all plates $\pi=[S_1,\ldots,S_k]$ which are labeled by ordered set partitions $(S_1,\ldots,S_k)$ such that each block is labeled by a permutation which has the set of consecutive descents of τ, of the form $\tau_{i_1}>\tau_{i_2}>\cdots>\tau_{i_{|S_i|}}$. Summing over all such τ we obtain

$$[\langle e_{i_1}-e_{j_1},\ldots,e_{i_{n-1}}-e_{j_{n-1}}\rangle_+]=\sum_{\pi:p_{i_a}<p_{j_a}}(-1)^{n-\mathrm{len}(\pi)}[\pi],$$

which completes the proof.

□

Example 33. Let $\mathcal{T}=\langle e_1-e_2,e_1-e_3\rangle_+$. Then

$$[\langle e_1-e_2,e_1-e_3\rangle_+]=[[1,2,3]]+[[1,3,2]]-[[1,23]].$$

Corollary 34. *Let $\mathcal{T}=\{(i_1,j_1),\ldots,(i_{k-1},j_{k-1})\}$ be a directed tree, where $i_a,j_a\in\{1,\ldots,k\}$ with $i_a\neq j_a$. Let $(S_1,\ldots,S_k)$ be an ordered set partition of $\{1,\ldots,n\}$. We have*

$$[[S_{i_1},S_{j_1}]]\bullet\cdots\bullet[[S_{i_{k-1}},S_{j_{k-1}}]]=\sum_{\pi:p_{i_a}<p_{j_a}}(-1)^{k-\mathrm{len}(\pi)}[\pi].$$

Proof. The proof of Theorem 32 generalizes with minimal adjustment to the present case, when $(1,2,\ldots,n)$ is replaced by any ordered set partition $(S_1,\ldots,S_k)$. □

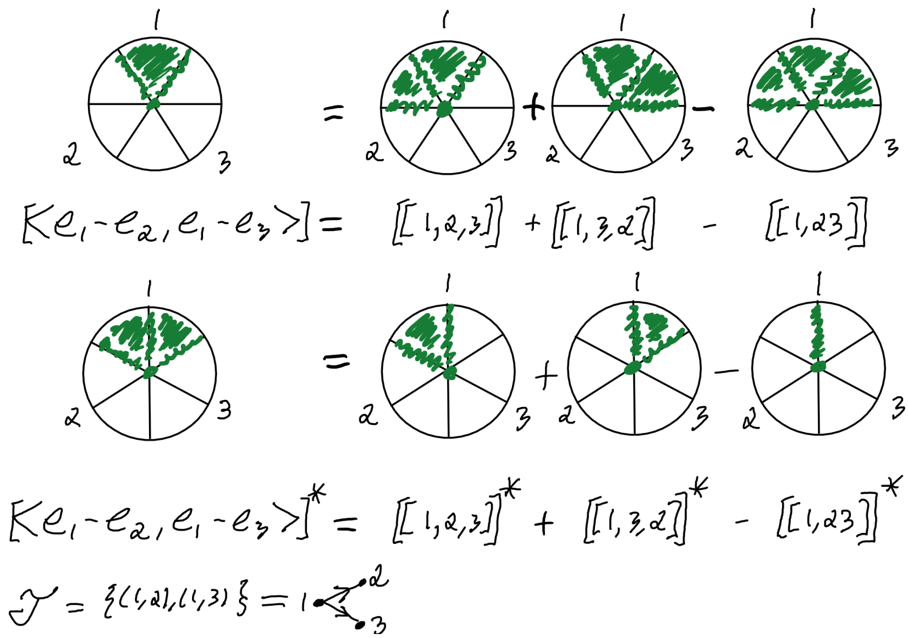

FIGURE 4. The relation for Example 33 and its dual

6. Straightening plates to the canonical basis

We derive in Theorem 35 the general expression for the expansion of a plate in the canonical basis for $\hat{\mathcal{P}}^n$. This implies the result of Ocneanu's original computation of the so-called plate relations in which he worked in a vector space generated formally by rooted binary trees. However, in practice one usually works in one of the quotient spaces $\mathcal{P}^n$, $\hat{\mathcal{P}}_1^n$ or $\mathcal{P}_1^n$, see Corollary 36.

Theorem 35. *Let* $[\pi] = [[S_l, S_{l-1}, \ldots, S_1, S_{l+1}, \ldots, S_k]] \in \hat{\mathcal{P}}^n$. *Denote* $[\pi_a] = [[S_a, S_{a+1}]]$ *and* $[\pi_a^\cup] = [[S_a \cup S_{a+1}]]$.

Then

$$[\pi] \;=\; \sum_{J\subseteq\{1,\ldots,l-1\}} \sum_{M\subseteq J} \sum_{\pi'} (-1)^{|M|}(-1)^{(c_1+1+k-l)-len(\pi)} \left(\left(\prod_{m\in M\cap C^c} [\pi_m] \right) \bullet \left(\prod_{m'\in (J\setminus M)\cap C^c} [\pi_{m'}^\cup] \right) \bullet [\pi'] \right),$$

where the sum $\sum_{\pi'}$ *is over all shuffle-lumpings of* $[S_1 \cup \cdots \cup S_{c_0}, T_1, \ldots, T_{c_1}]$ *satisfying* $p_1 \le p_2 \le \cdots \le p_{c-1} \le p_c$ *and* $p_1 < p_{l+1} < \cdots < p_k$, *where* $C = \{1, 2, \ldots, c-1, c\} \subseteq J$ *is the connected component of* J *containing* 1, *so* $c+1 \notin J$. *Here the set* M *determines* c_0 *as well as the* c_1 *blocks* T_i, *each of which is a union of consecutive* S_i, *such that* $S_1 \cup \cdots \cup S_{c_0} \cup T_1 \cup \cdots \cup T_{c_1} = \cup_{c=1}^{l} S_i$.

Proof. In the decomposition

$$[\pi] = [[S_l, S_{l-1}, \ldots, S_1]] \bullet [[S_1, S_{l+1}, \ldots, S_k]],$$

the left factor becomes

$$\begin{aligned}[[S_l, S_{l-1}, \dots, S_1]] &= \sum_{J\subseteq\{1,\dots,l-1\}} (([[S_{j_1}\cup S_{j_1+1}]] - [[S_{j_1}, S_{j_1+1}]]) + 1) \bullet \cdots \\ &\cdots \bullet \left(([[S_{j_{|J|}}\cup S_{j_{|J|+1}}]] - [[S_{j_{|J|}}, S_{j_{|J|+1}}]]) + 1\right) \\ &= \sum_{J\subseteq\{1,\dots,l-1\}} ([\pi^{\cup}_{j_1}] - [\pi_{j_1}]) \bullet \cdots \bullet ([\pi^{\cup}_{j_{|J|}}] - [\pi_{j_{|J|}}]) \\ &= \sum_{J\subseteq\{1,\dots,l-1\}} \sum_{M\subseteq J} (-1)^{|M|} \left(\prod_{m\in M} [\pi_m]\right) \bullet \left(\prod_{m'\in (J\setminus M)} [\pi^{\cup}_{m'}]\right).\end{aligned}$$

Consider the case when $1 \notin J$; then all summands in the expression

$$\sum_{M\subseteq J} (-1)^{|M|} \left(\prod_{m\in M} [\pi_m]\right) \bullet \left(\prod_{m'\in (J\setminus M)} [\pi^{\cup}_{m'}]\right) \bullet [[S_1, S_{l+1}, \dots, S_k]]$$

are already convolution products of characteristic functions of mutually orthogonal standard plates and are thus already in the canonical basis.

Therefore let us consider the case when $1 \in J$, and let $c \in \{1, \dots, l\}$ be such that $C = \{1, \dots, c\} \subseteq J$ is the connected component of J containing 1. Denote $S_{[a,b]} = \cup_{i=a}^b S_i$. For each $M \subseteq J$ we decompose $J = C \cup C^c$ and obtain

$$(-1)^{|M|} \left(\prod_{m\in M} [\pi_m]\right) \bullet \left(\prod_{m'\in (J\setminus M)} [\pi^{\cup}_{m'}]\right) \bullet [[S_1, S_{l+1}, \dots, S_k]]$$

$$= (-1)^{|M|} \left(\prod_{m\in M\cap C^c} [\pi_m]\right) \bullet \left(\prod_{m'\in (J\setminus M)\cap C^c} [\pi^{\cup}_{m'}]\right) \bullet [[S_{[1,c_0]}, T_1, \dots, T_{c_1}]] \bullet [[S_1, S_{l+1}, \dots, S_k]].$$

We shall use Corollary 34 to expand in the canonical basis the factor

$$[[S_{[1,c_0]}, T_1, \dots, T_{c_1}]] \bullet [[S_1, S_{l+1}, \dots, S_k]]$$

for some $c_0 \leq c$. Here each T_i is a union of consecutive S_j's and $(\cup_{i=1}^{c_0} S_i) \bigcup (\cup_{j=1}^{c_1} T_j) = \cup_{i=1}^{c} S_i$.

We have

$$\begin{aligned}&[[S_{[1,c_0]}, T_1, \dots, T_{c_1}]] \bullet [[S_1, S_{l+1}, \dots, S_k]] \\ =\ & [[S_{[1,c_0]}, T_1, \dots, T_{c_1}]] \bullet [[S_{[1,c_0]}, S_{l+1}, \dots, S_k]] \\ =\ & \sum_{\pi} (-1)^{(c_1+1+k-l)-\mathrm{len}(\pi)} [\pi],\end{aligned}$$

by the formula of Corollary 34. Here the exponent of the sign $(-1)^{(c_1+1+k-l)-\mathrm{len}(\pi)}$ counts the decrease in the number of blocks, from $c_1 + l + (k-l)$ for $[[S_{[1,c_0]}, T_1, \dots, T_{c_1}]] \bullet [[S_1, S_{l+1}, \dots, S_k]]$ down to $\mathrm{len}(\pi)$ for the summand $[\pi]$.

The sum is over all ordered set partitions $\{S_1, S_2, \dots, S_c, S_{l+1}, \dots, S_k\}$ satisfying $p_1 \leq p_2 \leq \cdots \leq p_{c-1} \leq p_c$ and $p_1 < p_{l+1} < \cdots < p_k$, where $\{1, 2, \dots, c-1, c\} \subseteq J$ is the connected component of J containing 1, so $c+1 \notin J$. This completes the proof.

□

Corollary 36. *Let $[[S_j, S_{j-1}, \dots, S_1, S_{j+1}, \dots, S_k]] \in \hat{\mathcal{P}}^n$ for an ordered set partition $\{S_1, \dots, S_k\}$ of $\{1, \dots, n\}$. Then, passing to the quotient $\mathcal{P}^n$ we have*

$$[[S_j, S_{j-1}, \dots, S_1, S_{j+1}, \dots, S_k]] = (-1)^{j-1} \sum_{p_j \geq p_{j-1} \geq \cdots \geq p_1 < p_{j+1} < \cdots < p_k} (-1)^{k-\mathrm{len}(\pi)} [\pi].$$

Now let $[[S_j, S_{j-1}, \ldots, S_1, S_{j+1}, \ldots, S_n]]$ *be an ordered set partition of* $\{1, \ldots, n\}$ *where all* S_i *are singlets.*

Then in $\hat{\mathcal{P}}_1^n$ *we have*

$$[[S_j, S_{j-1}, \ldots, S_1, S_{j+1}, \ldots, S_n]] \;\; = \;\; \sum_{J \subseteq \{1,\ldots,j-1\}} \left((-1)^{|J|} \sum_{\{p_{a+1} > p_a : a \in J\} \cup \{p_1 < p_{j+1} < \cdots < p_k\}} [\pi] \right),$$

where the inner sum is over all plates π *whose blocks are shuffle-lumpings of blocks of the ordered set partition* $\{S_j : j \in J\}$*.*

In $\mathcal{P}_1^n$ *we have*

$$[[S_j, S_{j-1}, \ldots, S_1, S_{j+1}, \ldots, S_n]] \;\; = \;\; (-1)^{j-1} \sum_{p_j > \cdots > p_2 > p_1 < p_{j+1} < \cdots < p_k\}} [\pi],$$

Proof. Reduce the formula of Theorem 35, as follows. In the quotient $\mathcal{P}^n$ of $\hat{\mathcal{P}}^n$, all summands become zero except those where $J = \{1, \ldots, j-1\}$. For the quotient $\hat{\mathcal{P}}_1^n$, only those summands with $M = J$ survive. For $\mathcal{P}_1^n$, only those terms with $J = \{1, \ldots, j-1\}$ and $M = J$ survive. $\square$

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