HIGHER TOPOLOGICAL HOCHSCHILD HOMOLOGY OF PERIODIC COMPLEX K-THEORY

BRUNO STONEK

ABSTRACT. We describe the topological Hochschild homology of the periodic complex K-theory spectrum, $THH(KU)$, as a commutative KU-algebra: it is equivalent to $KU[K(\mathbb{Z},3)]$ and to $F(\Sigma KU_{\mathbb{Q}})$, where F is the free commutative KU-algebra functor on a KU-module. Moreover, $F(\Sigma KU_{\mathbb{Q}}) \simeq KU \vee \Sigma KU_{\mathbb{Q}}$, a square-zero extension. In order to prove these results, we first establish that topological Hochschild homology commutes, as an algebra, with localization at an element.

Then, we prove that $THH^n(KU)$, the n-fold iteration of $THH(KU)$, i.e. $T^n \otimes KU$, is equivalent to $KU[G]$ where G is a certain product of integral Eilenberg-Mac Lane spaces, and to a free commutative KU-algebra on a rational KU-module. We prove that $S^n \otimes KU$ is equivalent to $KU[K(\mathbb{Z}, n+2)]$ and to $F(\Sigma^n KU_{\mathbb{Q}})$. We describe the topological André-Quillen homology of KU.

1. INTRODUCTION

Topological Hochschild homology (THH) of structured ring spectra was introduced by Bökstedt [Bök85] and Breen [Bre78]; for an introduction to the subject, see [DGM13, Chapter 4], [EKMM97, Chapter IX] and [Shi00]. It is the generalization to structured ring spectra of classical Hochschild homology (HH) of rings.

One reason for its importance is its relation to algebraic K-theory. If R is a (discrete) ring, then the trace map $K(R) \to HH(R)$ factors through the topological Hochschild homology of the Eilenberg-Mac Lane ring spectrum of R. Moreover, the trace map $K(A) \to THH(A)$ exists for any ring spectrum A. Out of topological Hochschild homology one can build topological cyclic homology, which has close ties to algebraic K-theory: see [DGM13]. We might thus see THH as a more easily approachable stepping stone on the way to the more fundamental algebraic K-theory.

In this paper we are interested with the topological Hochschild homology of KU, the periodic complex topological K-theory spectrum. Previously, McClure and Staffeldt [MS93, Theorem 8.1] showed that $THH(L) \simeq L \vee \Sigma L_{\mathbb{Q}}$ as spectra, where L is the p-adic completion of the Adams summand of KU for a given odd prime p. More recently, a lot of effort was devoted to describe $THH(ku)$, where ku is connective complex K-theory [Aus05]: that case is markedly harder. It should also be noted that, rationally, $K(KU)$ was determined in [AR12, Theorem 3.6].

Our first expression for $THH(KU)$ as a commutative KU-algebra is obtained in **Theorem 5.5**:

$$THH(KU) \simeq KU[K(\mathbb{Z},3)],$$

where the underlying KU-module of $KU[K(\mathbb{Z},3)]$ is $KU \wedge K(\mathbb{Z},3)_+$. The second one is given in **Theorem 5.10**: there is a morphism of commutative KU-algebras

$$\tilde{f} : F(\Sigma KU_{\mathbb{Q}}) \to THH(KU)$$

which is a weak equivalence. Here $F(\Sigma KU_{\mathbb{Q}})$ is the free commutative KU-algebra on the KU-module $\Sigma KU_{\mathbb{Q}}$. Moreover, $F(\Sigma KU_{\mathbb{Q}})$ is weakly equivalent as a commutative KU-algebra to the split square-zero extension $KU \vee \Sigma KU_{\mathbb{Q}}$. We would like to note that the previous results for the topological Hochschild homology of L and KO (see Remark 5.13 for a more detailed account) do not deal with the multiplicative structure on THH.

Topological Hochschild homology of a commutative ring spectrum can be iterated. Using the tensoring of commutative ring spectra over spaces, $THH(A)$ can be expressed as $S^1 \otimes A$. Similarly, the n-fold iterated version, $THH^n(A)$, can be expressed as $T^n \otimes A$, where T^n is an n-torus. See [CDD11]: they propose $THH^n(A)$ as "a computationally tractable cousin of n-fold iterated algebraic K-theory".

We consider the iterated THH of KU. The first expression we gave above for $THH(KU)$ directly generalizes: one replaces $K(\mathbb{Z}, 3)$ by a suitable product of integral Eilenberg-Mac Lane spaces. See **Theorem 6.6**: there is a weak equivalence of commutative KU-algebras

$$THH^n(KU) \simeq KU\left[\prod_{i=1}^{n} K(\mathbb{Z}, i+2)^{\times\binom{n}{i}}\right].$$

The second expression for $THH(KU)$ also generalizes: this is **Theorem 6.10**, where we get the weak equivalence of commutative KU-algebras

$$F\left(\bigvee_{i=1}^{n}(S^i)^{\vee\binom{n}{i}} \wedge KU_{\mathbb{Q}}\right) \simeq T^n \otimes KU.$$

The expression $KU \vee \Sigma KU_{\mathbb{Q}}$ for $THH(KU)$ also generalizes to $THH^n(KU)$. In this case, the augmentation ideal $\overline{THH}^n(KU)$ is still rational, but it has a non-trivial non-unital commutative KU-algebra structure. We describe the non-unital commutative $\mathbb{Q}[t^{\pm 1}]$-algebra $\overline{THH}^n_*(KU)$ as iterated Hochschild homology. See **Theorem 6.19**.

We then shift our attention to $X \otimes KU$, where X is a pointed CW-complex which is a reduced suspension, e.g. a sphere S^n. In this case, the first description for $THH(KU)$ generalizes as a weak equivalence of commutative KU-algebras

$$S^n \otimes KU \simeq KU[K(\mathbb{Z}, n+2)]$$

and the second one generalizes as a weak equivalence of commutative KU-algebras

$$F(S^n \wedge KU_{\mathbb{Q}}) \to S^n \otimes KU.$$

This is a consequence of **Theorem 7.17**. Using these results, we establish a weak equivalence of KU-modules

$$TAQ(KU) \simeq KU_{\mathbb{Q}}$$

where $TAQ(KU)$ denotes the topological André-Quillen KU-module of KU.

We use an adaptation of the model for KU given by Snaith [Sna79], [Sna81], namely $\Sigma^\infty_+ K(\mathbb{Z}, 2)[x^{-1}]$, to the context of [EKMM97]. In Section 2, we review some model categorical aspects of [EKMM97], particularly those pertaining to commutative $\mathbb{S}$-algebras. In Section 3, we prove some elementary properties of localization of a commutative $\mathbb{S}$-algebra at an element. In Section 4, we review some needed aspects of topological Hochschild homology, and in **Corollary 4.12**, we prove that THH commutes with localization at an element. Section 5 contains the results pertaining to $THH(KU)$, and in Sections 6 and 7 we prove our results pertaining to $T^n \otimes KU$ and $S^n \otimes KU$. Finally, in Section 8, we determine the topological André-Quillen homology of KU.

1.1. **Conventions.** By *space* we will mean "compactly generated weakly Hausdorff space", and we will denote the cartesian closed category they form by **Top**. We will work with the categories of [EKMM97]: our main objects are $\mathbb{S}$-modules, commutative $\mathbb{S}$-algebras R, R-modules and commutative R-algebras A.

1.2. **Acknowledgments.** I would like to thank Christian Ausoni, my PhD supervisor, for suggesting this project, sharing his ideas and his support; Geoffroy Horel for his very useful suggestions; Christian Schlichtkrull for his careful reading and his suggestions; and Bjørn Dundas for so warmly and selflessly sharing so much of his time, ideas and insights at the University of Bergen.

Parts of the content of this article are part of the author's PhD dissertation at Université Paris 13. Research partially supported by the *ANR-16-CE40-0003 project ChroK* and the *Fondation Sciences Mathématiques de Paris* (FSMP).

2. Model structures

The category $\mathbb{S}$-**Mod** of $\mathbb{S}$-modules has a topological symmetric monoidal cofibrantly generated model structure [EKMM97, VII.4]. A commutative $\mathbb{S}$-algebra is, by definition, a commutative monoid in $\mathbb{S}$-**Mod**. The category they form, $\mathbb{S}$-**CAlg**, can also be described as the category of $\mathbb{P}$-algebras where $\mathbb{P}$ is the commutative monoid monad. The forgetful functor $U : \mathbb{S}\text{-}\mathbf{CAlg} \to \mathbb{S}\text{-}\mathbf{Mod}$ creates a model structure on $\mathbb{S}$-**CAlg**[1]. In particular, there is a Quillen adjunction $\mathbb{S}\text{-}\mathbf{Mod} \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{S}\text{-}\mathbf{CAlg}$. The category $\mathbb{S}$-**CAlg** has a topological symmetric monoidal cofibrantly generated model category strucutre.

Let $R \in \mathbb{S}\text{-}\mathbf{CAlg}$, and consider the category of R-modules, R-**Mod**. The forgetful functor $R\text{-}\mathbf{Mod} \to \mathbb{S}\text{-}\mathbf{Mod}$ creates a model structure on R-**Mod**, and R-**Mod** acquires a topological symmetric monoidal cofibrantly generated model category structure. The forgetful functor $U : R\text{-}\mathbf{CAlg} \to R\text{-}\mathbf{Mod}$ creates a model structure on R-**CAlg**, and thus R-**CAlg** has a topological symmetric monoidal cofibrantly generated model category structure. In these model categories, all objects are fibrant.

Cofibrancy is more delicate. The sphere $\mathbb{S}$-module $\mathbb{S}$ is not cofibrant as an $\mathbb{S}$-module, but it is cofibrant as a commutative $\mathbb{S}$-algebra. More generally, the underlying R-module of a cofibrant commutative R-algebra is generally not cofibrant as an R-module.

Let R be a commutative $\mathbb{S}$-algebra. We record the following useful properties:

(1) If M is a cofibrant R-module, then $M \wedge_R -$ preserves all weak equivalences of R-modules [EKMM97, III.3.8].
(2) Suppose R is cofibrant. Let A and B be cofibrant commutative R-algebras. Let $\gamma_A : \Gamma A \to A$ and $\gamma_B : \Gamma B \to B$ be cofibrant approximations of A and B in

[1] A functor $U : \mathcal{C} \to \mathcal{M}$ *creates a model structure* on $\mathcal{C}$ if $\mathcal{M}$ is a model category and $\mathcal{C}$ is a model category such that f is a fibration (resp. weak equivalence) in $\mathcal{C}$ if and only if Uf is a fibration (resp. weak equivalence) in $\mathcal{M}$. We say that U *strongly creates* the model structure of $\mathcal{C}$ if, in addition, f is a cofibration in $\mathcal{C}$ if and only if Uf is a cofibration in $\mathcal{M}$.

the category of R-modules. Then $\gamma_A \wedge_R \gamma_B : \Gamma A \wedge_R \Gamma B \to A \wedge_R B$ is a weak equivalence of R-modules [EKMM97, VII.6.5, VII.6.7].

(3) As in any model category, the coproduct of cofibrant objects is cofibrant. Hence, if A and B are cofibrant commutative R-algebras, then $A \wedge_R B$ is a cofibrant commutative R-algebra [EKMM97, VII.6.8].

(4) Let $\mathbb{S} \to A \to B$ be cofibrations of commutative $\mathbb{S}$-algebras. Then the functor $B \wedge_A - : A\text{-}\mathbf{CAlg} \to B\text{-}\mathbf{CAlg}$ preserves weak equivalences between commutative A-algebras which are cofibrant as commutative $\mathbb{S}$-algebras [EKMM97, VII.7.4].

(5) The category R-$\mathbf{CAlg}$ can also be described as the category of objects of $\mathbb{S}$-$\mathbf{CAlg}$ under R. The forgetful functor $R\text{-}\mathbf{CAlg} \to \mathbb{S}\text{-}\mathbf{CAlg}$ strongly creates a model structure on R-$\mathbf{CAlg}$ [MP12, Theorem 15.3.6]. This model structure coincides with the one described above [Hön17, Remark 2.4.1]. In conclusion, a map $f : A \to B$ is a cofibration in R-$\mathbf{CAlg}$ if and only if it is a cofibration in $\mathbb{S}$-$\mathbf{CAlg}$. In particular, if R is a cofibrant commutative $\mathbb{S}$-algebra and A is a cofibrant commutative R-algebra, then A is cofibrant as a commutative $\mathbb{S}$-algebra.

Note: in [EKMM97] they call *q-cofibration* what we call a cofibration. We will have no use for what they call a "cofibration".

3. Inversion of an element

In this section, we recall the procedure of inverting a homotopy element in a commutative $\mathbb{S}$-algebra following [EKMM97] and prove some properties which will be needed below.

Theorem 3.1. [EKMM97, VIII.2.2, VIII.4.2] *Let R be a cofibrant commutative $\mathbb{S}$-algebra and $x \in \pi_* R$. There exists a cofibrant commutative R-algebra $R[x^{-1}]$ with unit $j : R \to R[x^{-1}]$ satisfying that $\pi_*(R[x^{-1}]) = \pi_*(R)[x^{-1}]$, and if $f : R \to T$ is a map in $\mathbb{S}$-$\mathbf{CAlg}$ such that $(\pi_* f)(x) \in \pi_* T$ is invertible, then there exists a map $\tilde{f} : R[x^{-1}] \to T$ in $\mathbb{S}$-$\mathbf{CAlg}$ making the following diagram commute:*

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "j"'] & T \\ R[x^{-1}] \arrow[ur, dotted, "\tilde{f}"'] & \end{tikzcd},$$

and $\tilde{f}$ is unique up to homotopy through maps of commutative $\mathbb{S}$-algebras. Moreover, if the map $\pi_*(R)[x^{-1}] \to \pi_* T$ coming from the universal property for localizations of commutative $\pi_*(R)$-algebras is an isomorphism, then $\tilde{f}$ is a weak equivalence.

The previous theorem is valid, *mutatis mutandis*, if $\mathbb{S}$ is replaced by some cofibrant commutative $\mathbb{S}$-algebra.

Lemma 3.2. *The multiplication map $\mu : R[x^{-1}] \wedge_R R[x^{-1}] \to R[x^{-1}]$ is a weak equivalence of commutative $R[x^{-1}]$-algebras.*

Proof. The Tor spectral sequence [EKMM97, IV.4.1] here takes the form

$$E^2_{*,*} = \mathrm{Tor}^{\pi_* R}_{*,*}(\pi_* R[x^{-1}], \pi_* R[x^{-1}]) \Rightarrow \pi_*(R[x^{-1}] \wedge_R R[x^{-1}]).$$

Since the localization morphism $\pi_*R \to \pi_*R[x^{-1}]$ is flat, the spectral sequence is concentrated in the 0-th column and thus the edge homomorphism

$$\nabla : \pi_*R[x^{-1}] \otimes_{\pi_*R} \pi_*R[x^{-1}] \to \pi_*(R[x^{-1}] \wedge_R R[x^{-1}]) \tag{3.3}$$

is an isomorphism. Since $\wedge_R$ is the coproduct in the category of commutative R-algebras, we can consider the canonical maps $i_1, i_2 : R[x^{-1}] \to R[x^{-1}] \wedge_R R[x^{-1}]$. The edge homomorphism ∇ coincides with the map $(\pi_* i_1, \pi_* i_2)$ defined via the universal property of the coproduct of commutative π_*R-algebras. We have the following commutative diagram of commutative π_*R-algebras:

$$\begin{tikzcd}
\pi_*R[x^{-1}] \arrow[r, "\iota_1"] \arrow[dr, "\pi_* i_1"'] \arrow[ddr, bend right, "\mathrm{id}"'] & \pi_*R[x^{-1}] \otimes_{\pi_*R} \pi_*R[x^{-1}] \arrow[d, "\nabla"] & \pi_*R[x^{-1}] \arrow[l, "\iota_2"'] \arrow[dl, "\pi_* i_2"] \arrow[ddl, bend left, "\mathrm{id}"] \\
& \pi_*(R[x^{-1}] \wedge_R R[x^{-1}]) \arrow[d, "\pi_*\mu"] & \\
& \pi_*R[x^{-1}] &
\end{tikzcd}$$

where ι_1, ι_2 are the canonical inclusions into a coproduct of commutative π_*R-algebras. Again, by the universal property of the coproduct of commutative π_*R-algebras, there is a unique arrow $\pi_*R[x^{-1}] \otimes_{\pi_*R} \pi_*R[x^{-1}] \to \pi_*R[x^{-1}]$ making the outer diagram commute. One such arrow is the canonical isomorphism that one has for any such algebraic localization, i.e. $h : S^{-1}A \otimes_A S^{-1}A \xrightarrow{\cong} S^{-1}A$ for any commutative ring A and multiplicative subset $S \subset A$. Another such arrow is $\pi_*\mu \circ \nabla$. Therefore, $h = \pi_*\mu \circ \nabla$. Since ∇ and h are isomorphisms, so is $\pi_*\mu$. $\square$

If $f : R \to T$ is a morphism between cofibrant commutative $\mathbb{S}$-algebras and $x \in \pi_*R$, then Theorem 3.1 gives us a map of cofibrant commutative $\mathbb{S}$-algebras

$$\begin{tikzcd}
R \arrow[r, "f"] \arrow[d, "j_R"'] & T \arrow[d, "j_T"] \\
R[x^{-1}] \arrow[r, dotted, "{f[x^{-1}]}"'] & T[(\pi_*f)(x)^{-1}].
\end{tikzcd}$$

Note that $f[x^{-1}]$ turns $T[(\pi_*f)(x)^{-1}]$ into a commutative $R[x^{-1}]$-algebra.

The previous square induces an arrow from the pushout $R[x^{-1}] \wedge_R T$ in R-**CAlg**. The following theorem tells us that it is a weak equivalence. Compare with [EKMM97, V.1.15] which handles the case where T is replaced by an R-module.

Proposition 3.4 (Base change for localization)**.** *Let $f : R \to T$ be a morphism of cofibrant commutative $\mathbb{S}$-algebras and $x \in \pi_*R$. The morphism of commutative R-algebras*

$$(f[x^{-1}], j_T) : R[x^{-1}] \wedge_R T \to T[(\pi_*f)(x)^{-1}] \tag{3.5}$$

is a weak equivalence.

Note that (3.5) is also an equivalence in $R[x^{-1}]$-**CAlg** and in T-**CAlg**.

Proof. Denote the morphism $(f[x^{-1}], j_T)$ by h, for simplicity. Like in the proof of Lemma 3.2, the Tor spectral sequence that computes the homotopy groups of $R[x^{-1}] \wedge_R T$ from those of $R[x^{-1}]$ and T collapses, since $\pi_*R \to \pi_*R[x^{-1}] = (\pi_*R)[x^{-1}]$ is flat. Therefore, the map π_*h, fitting in a commutative diagram

$$\begin{tikzcd} \pi_*(R[x^{-1}] \wedge_R T) \arrow[r, "\pi_*h"] & (\pi_*T)[(\pi_*f)(x)^{-1}] \\ (\pi_*R)[x^{-1}] \otimes_{\pi_*R} \pi_*T, \arrow[u, "\cong"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

is an isomorphism, since the diagonal map is an isomorphism. Indeed, this is the map appearing in the analogous statement in commutative algebra of the theorem we are proving, applied to $\pi_*f : \pi_*R \to \pi_*T$. But this statement of commutative algebra is not hard to prove: it follows from the universal properties and the extension-restriction of scalars adjunction. $\square$

Proposition 3.6. *Let R and T be cofibrant commutative $\mathbb{S}$-algebras, $x \in \pi_n R$ and $y \in \pi_m T$. Denote by $x \wedge y$ the image of $x \otimes y$ under the morphism*

$$\pi_*R \otimes_{\pi_*\mathbb{S}} \pi_*T \longrightarrow \pi_*(R \wedge T) .$$

There is a weak equivalence of commutative $\mathbb{S}$-algebras

$$R[x^{-1}] \wedge T[y^{-1}] \to (R \wedge T)[(x \wedge y)^{-1}].$$

Note that this is is also a map of commutative $R[x^{-1}]$ and $T[y^{-1}]$-algebras.

Proof. Let $i_1 : R \to R \wedge T$, $i_2 : T \to R \wedge T$ be the canonical maps into the coproduct. There exists a map f making the following diagram commute.

$$\begin{tikzcd} R \arrow[r, "i_1"] \arrow[d, "j_R"'] & R \wedge T \arrow[r, "j_{R\wedge T}"] & (R \wedge T)[(x \wedge y)^{-1}] \\ R[x^{-1}] \arrow[urr, dotted, "f"'] & & \end{tikzcd}$$

Indeed, applying π_* to the horizontal composition, we get the map

$$\pi_*(j_{R\wedge T} \circ i_1) : \pi_*R \to \pi_*(R \wedge T)[(x \wedge y)^{-1}]$$

which maps x to $x \wedge 1$. This is an invertible element with inverse $(1 \wedge y)(x \wedge y)^{-1}$, since the map $(\pi_*i_1, \pi_*i_2) : \pi_*R \otimes_{\pi_*\mathbb{S}} \pi_*T \to \pi_*(R \wedge T)$ is multiplicative. Therefore, the property of Theorem 3.1 provides us with the arrow f in $\mathbb{S}$-**CAlg**. Similarly, we get a map $g : T[y^{-1}] \to (R \wedge T)[(x \wedge y)^{-1}]$. We assemble f and g into the coproduct map in $\mathbb{S}$-**CAlg**

$$(f, g) : R[x^{-1}] \wedge T[y^{-1}] \to (R \wedge T)[(x \wedge y)^{-1}].$$

Now recall from [EKMM97, Section V.1] that $R[x^{-1}]$ is weakly equivalent, in R-**Mod**, to the homotopy colimit of the tower

$$R \xrightarrow{x} \Sigma^{-n}R \xrightarrow{x} \Sigma^{-2n}R \xrightarrow{x} \dots .$$

The T-module $T[y^{-1}]$ is described similarly. The $R \wedge T$-module $(R \wedge T)[(x \wedge y)^{-1}]$ is weakly equivalent to the homotopy colimit of the tower

$$R \wedge T \xrightarrow{x\wedge y} \Sigma^{-n-m}R \wedge T \xrightarrow{x\wedge y} \Sigma^{-2n-2m}R \wedge T \xrightarrow{x\wedge y} \dots .$$

Smashing the homotopy colimit computing $R[x^{-1}]$ with the one computing $T[y^{-1}]$ we obtain the homotopy colimit computing $(R \wedge T)[(x \wedge y)^{-1}]$, since the diagonal map $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is homotopy cofinal. The map (f, g) is compatible with these identifications, hence it is a weak equivalence. $\square$

4. Topological Hochschild homology

4.1. **Symmetric monoidal categories.** Let $(\mathcal{V}, \otimes, \mathbb{1})$ be a symmetric monoidal category. Denote by $\mathbf{Mon}(\mathcal{V})$ and $\mathbf{CMon}(\mathcal{V})$ the corresponding symmetric monoidal categories of monoids and commutative monoids in $\mathcal{V}$, respectively. Denote by $s\mathcal{V}$ the category of simplicial objects in $\mathcal{V}$; it is a symmetric monoidal category with levelwise tensor product.

Suppose $\mathcal{V}$ is closed and cocomplete. Then if $A \in \mathbf{CMon}(\mathcal{V})$, the category A-$\mathbf{Mod}$ of A-modules gets a relative tensor product $\otimes_A$ such that $(A\text{-}\mathbf{Mod}, \otimes_A, A)$ is a symmetric monoidal category. One can thus speak of commutative A-algebras, which are the commutative monoids in A-$\mathbf{Mod}$. We denote by A-$\mathbf{CAlg}$ the category they form.

Let $F : \mathcal{V} \to \mathcal{W}$ be a strong symmetric monoidal functor between cocomplete closed symmetric monoidal categories, and suppose F preserves colimits. Then F induces a functor on commutative monoids, on modules over commutative monoids, and on commutative algebras. More specifically, there is an induced functor $F : A\text{-}\mathbf{CAlg} \to F(A)\text{-}\mathbf{CAlg}$.

We will need the following

Lemma 4.1. *Let $F : \mathcal{V} \to \mathcal{W}$ be a functor as above, and let $A \in \mathbf{CMon}(\mathcal{V})$. Then there is a natural isomorphism*

$$\begin{tikzcd} \mathbf{CMon}(\mathcal{V}) \arrow[r, "A\otimes -"] \arrow[d, "F"'] \arrow[dr, phantom, "\Uparrow"] & A\text{-}\mathbf{CAlg} \arrow[d, "F"] \\ \mathbf{CMon}(\mathcal{W}) \arrow[r, "F(A)\otimes -"'] & F(A)\text{-}\mathbf{CAlg}. \end{tikzcd}$$

Proof. First, note that there is a strong symmetric monoidal functor $A \otimes - : \mathcal{V} \to A\text{-}\mathbf{Mod}$, which therefore induces the functor at the top of the diagram, and similarly for the one in the bottom. The isomorphism

$$\nabla : F(A) \otimes F(B) \to F(A \otimes B)$$

natural in $B \in \mathbf{CMon}(\mathcal{V})$ is given by the structure isomorphism of F. The only thing one needs to check is that ∇ is a map of $F(A)$-commutative algebras, but this is a straightforward verification. $\square$

4.2. **Simplicial cyclic bar construction in general.** The results in this section are similar to the ones in [Sto18, Section 1] which are done for the simplicial *reduced* bar construction.

Definition 4.2. The *simplicial cyclic bar construction* is a functor

$$B^{\mathrm{cy}}_{\bullet} : \mathbf{Mon}(\mathcal{V}) \to s\mathcal{V}$$

defined as follows. If $A \in \mathbf{Mon}(\mathcal{V})$ with multiplication $\mu : A \otimes A \to A$ and unit $\eta : \mathbb{1} \to A$, then $B^{\mathrm{cy}}_n(A) = A^{\otimes n+1}$. The faces $d_i : A^{\otimes n+1} \to A^{\otimes n}$, $i = 0, \ldots, n$ are defined as

$$d_i = \mathrm{id}^{\otimes i} \otimes \mu \otimes \mathrm{id}^{\otimes n-i-1} \quad \text{if } i = 0, \ldots, n-1, \text{ and}$$

$$d_n = (\mu \otimes \mathrm{id}^{\otimes(n-1)}) \circ \sigma_{n+1}$$

where $\sigma_{n+1} : A^{\otimes n+1} \to A^{\otimes n+1}$ is the isomorphism that puts the last A term at the beginning. The degeneracies $s_i : A^{\otimes n+1} \to A^{\otimes n+2}$ are

$$s_i = \mathrm{id}^{\otimes i+1} \otimes \eta \otimes \mathrm{id}^{\otimes n-i} \qquad \text{for all } i = 0, \ldots, n.$$

This is a strong symmetric monoidal functor, thus by the Eckmann-Hilton argument it induces a functor

$$B^{\mathrm{cy}}_\bullet : \mathbf{CMon}(\mathcal{V}) \to s\mathbf{CMon}(\mathcal{V}).$$

For a commutative monoid A in $\mathcal{V}$, we have that $B^{\mathrm{cy}}_\bullet(A) \in sA\text{-}\mathbf{CAlg}$. Indeed, there is a morphism $cA \to B^{\mathrm{cy}}_\bullet A$ in $s\mathbf{CMon}(\mathcal{V})$, where cA denotes the constant simplicial object at A. In simplicial level n, it is $\mathrm{id} \otimes \eta^{\otimes n} : A \to A^{\otimes n+1}$.

We could specify the simplicial commutative A-algebra structure of $B^{\mathrm{cy}}_\bullet(A)$ more explicitely: the A-module structure on $A^{\otimes n+1} \cong A \otimes A^{\otimes n}$ is given by acting on the first factor, and the multiplication over A is given by

$$A^{\otimes n+1} \otimes_A A^{\otimes n+1} \cong A \otimes (A^{\otimes n} \otimes A^{\otimes n}) \xrightarrow{\mathrm{id}\otimes\mu} A \otimes A^{\otimes n}$$

where μ denotes the product of $A^{\otimes n} \in \mathbf{CMon}(\mathcal{V})$.

There is a relative version of this construction: if M is an A-bimodule, then one can define $B^{\mathrm{cy}}_\bullet(A, M)$ which has $B^{\mathrm{cy}}_n(A, M) = M \otimes A^{\otimes n}$ with similar faces and degeneracies.

Let $F : \mathcal{V} \to \mathcal{W}$ be a strong symmetric monoidal functor between cocomplete closed symmetric monoidal categories which preserves colimits. Since it takes commutative A-algebras to commutative $F(A)$-algebras, then $F(B^{\mathrm{cy}}_\bullet A)$ is a simplicial commutative $F(A)$-algebra. The structure morphisms of F provide a natural isomorphism

$$B^{\mathrm{cy}}_\bullet(FA) \xrightarrow{\cong} F(B^{\mathrm{cy}}_\bullet A) \tag{4.3}$$

of simplicial commutative $F(A)$-algebras.

4.3. **Geometric realization.** Consider $F = \Sigma^\infty_+ : \mathbf{Top} \to \mathbb{S}\text{-}\mathbf{Mod}$: it is a strong symmetric monoidal functor [EKMM97, II.1.2]. If G is a topological commutative monoid, we denote by $\mathbb{S}[G]$ the $\mathbb{S}$-module $\Sigma^\infty_+ G$ together with the commutative $\mathbb{S}$-algebra structure induced by the monoid structure of G.

Endow the cartesian category $\mathbf{Top}$ with the standard cosimplicial space $\Delta^\bullet_{\mathrm{top}}$ and the symmetric monoidal category $\mathbb{S}$-$\mathbf{Mod}$ with the cosimplicial spectrum $\Sigma^\infty_+ \Delta^\bullet_{\mathrm{top}}$. By Theorem 2.9 of [Sto18], these beget strong symmetric monoidal functors of geometric realization

$$|-| : s\mathbf{Top} \to \mathbf{Top} \qquad \text{and} \qquad |-| : s\mathbb{S}\text{-}\mathbf{Mod} \to \mathbb{S}\text{-}\mathbf{Mod}.$$

If A is a topological commutative monoid or a commutative $\mathbb{S}$-algebra, define

$$B^{\mathrm{cy}}(A) := |B^{\mathrm{cy}}_\bullet(A)|.$$

It is a commutative A-algebra. In the $\mathbb{S}$-module case, this object defines the topological Hochschild homology of A, denoted $THH(A)$ (which has good homotopical behavior when A is a cofibrant commutative $\mathbb{S}$-algebra [EKMM97, IX.2.7]), and if M is an A-bimodule then $|B^{\mathrm{cy}}_\bullet(A, M)|$ defines $THH(A, M)$. Note that if $f : A \to B$ is a weak equivalence of cofibrant commutative $\mathbb{S}$-algebras, then $THH(A) \to THH(B)$ is a weak equivalence.

First note that $B^{\mathrm{cy}}_\bullet(A) \to B^{\mathrm{cy}}_\bullet(B)$ is a weak equivalence in each level. Indeed, $f^{\wedge p+1}$ is a weak equivalence since f is a weak equivalence and A and B are cofibrant. Then, since the simplicial cyclic bar construction gives a proper simplicial $\mathbb{S}$-module [EKMM97, IX.2.8], we can apply [EKMM97, X.2.4] to get the conclusion.

Since the functors $|-|$ are strong symmetric monoidal, by realizing the isomorphism (4.3) we obtain the following

Proposition 4.4. *Let G be a topological commutative monoid. There is an isomorphism of commutative $\mathbb{S}[G]$-algebras*

$$THH(\mathbb{S}[G]) \xrightarrow{\cong} \mathbb{S}[B^{\mathrm{cy}}(G)].$$

Versions of the previous proposition have appeared in [SVW00, Remark 4.4], in [HM97, Theorem 7.1] in the setting of functors with smash product, and in [DGM13, Example 4.2.2.7] in the setting of Γ-spaces. Note that we take care to note that this isomorphism respects the commutative $\mathbb{S}[G]$-algebra structures.

We will also need the following proposition, obtained by applying Lemma 4.1 to the strong symmetric monoidal functor $\Sigma^\infty_+ : \mathbf{Top} \to \mathbb{S}\text{-}\mathbf{Mod}$.

Proposition 4.5. *Consider $G, H \in \mathbf{CMon}(\mathbf{Top})$. There is an isomorphism of commutative $\mathbb{S}[G]$-algebras*

$$\mathbb{S}[G] \wedge \mathbb{S}[H] \xrightarrow{\cong} \mathbb{S}[G \times H]$$

natural in H.

4.4. **Cyclic bar construction of a topological abelian group.** Let G be a topological abelian group with unit 0. Denote by BG the model for the classifying space of G which is given as the geometric realization of the reduced bar construction $B_\bullet(0, G, 0)$ of G. Therefore, BG is a topological abelian group. Moreover, if G is a CW-complex and addition is a cellular map, then the same can be said of BG. All of this is due to Milgram [Mil67].

The space $G \times BG$ gets the structure of a commutative G-algebra, via the inclusion of the first factor $G \to G \times BG$, which is a morphism of topological abelian groups.

Proposition 4.6. *Let G be a topological abelian group. There is a homeomorphism of commutative G-algebras*

$$B^{\mathrm{cy}}G \cong G \times BG.$$

Proof. Let $G_\bullet$ denote the constant simplicial commutative G-algebra on G. Consider the maps $r_\bullet : B^{\mathrm{cy}}_\bullet G \to G_\bullet$, $(g_0, \ldots, g_p) \mapsto g_0 + \cdots + g_p$, and $p_\bullet : B^{\mathrm{cy}}_\bullet G \to B_\bullet G$, $(g_0, \ldots, g_p) \mapsto (g_1, \ldots, g_p)$. They assemble to a map

$$B^{\mathrm{cy}}_\bullet G \xrightarrow{(r_\bullet, p_\bullet)} G_\bullet \times B_\bullet G, \quad (g_0, \ldots, g_p) \mapsto (g_0 + \cdots + g_p, g_1, \ldots, g_p).$$

We also have maps $i_\bullet : G_\bullet \to B^{\mathrm{cy}}_\bullet G$, $g \mapsto (g, 0, \ldots, 0)$ and $s_\bullet : B_\bullet G \to B^{\mathrm{cy}}_\bullet G$, $(g_1, \ldots, g_p) \mapsto (-g_1 - \cdots - g_p, g_1, \ldots, g_p)$. We sum them up to a map

$$G_\bullet \times B_\bullet G \xrightarrow{i_\bullet + s_\bullet} B^{\mathrm{cy}}_\bullet G, \quad (g, g_1, \ldots, g_p) \mapsto (g - g_1 - \cdots - g_p, g_1, \ldots, g_p).$$

The maps $(r_\bullet, p_\bullet)$ and $i_\bullet + s_\bullet$ are morphisms of simplicial commutative G-algebras which are inverse to one another. (Note that the obvious isomorphisms $G \times G^p \cong G^{p+1}$ are not good, because they do not commute with the last face map.) Applying geometric realization we obtain the result. $\square$

A classical result (which we will not use) states that $B^{\mathrm{cy}}G$ is homotopy equivalent to the free loop space of BG (see e.g. [BHM93, Section 2]).

4.5. **Inverting an element in THH.** Let R be a cofibrant commutative $\mathbb{S}$-algebra and $x \in \pi_* R$. Denote by $\eta : R \to THH(R)$ the unit. Since $THH(R) = B^{\mathrm{cy}}(R)$ is a cofibrant commutative $\mathbb{S}$-algebra [SVW00, Lemma 3.6], Proposition 3.4 immediately gives a weak equivalence of commutative $R[x^{-1}]$-algebras

$$THH(R, R[x^{-1}]) \cong R[x^{-1}] \wedge_R THH(R) \xrightarrow{\sim} THH(R)[\pi_*\eta(x)^{-1}]. \tag{4.7}$$

For simplicity, denote the codomain of this arrow by $THH(R)[x^{-1}]$.

We now aim to prove that $THH(R, R[x^{-1}])$ and $THH(R[x^{-1}])$ are weakly equivalent commutative $R[x^{-1}]$-algebras. We will obtain this as a consequence of the following more general theorem, by taking the sequence (4.9) to be $\mathbb{S} \to R \to R[x^{-1}]$.

Theorem 4.8. *Let*

$$\mathbb{S} \to A \xrightarrow{f} B \tag{4.9}$$

be a sequence of cofibrations of commutative $\mathbb{S}$-algebras. Suppose that the multiplication map $\mu : B \wedge_A B \to B$ is a weak equivalence. Then the map of commutative B-algebras

$$B \wedge_A THH(A) \cong THH(A, B) \xrightarrow{THH(f,\mathrm{id})} THH(B, B) = THH(B) \tag{4.10}$$

is a weak equivalence.

This theorem is valid *mutatis mutandis* when $\mathbb{S}$ is replaced by some cofibrant commutative $\mathbb{S}$-algebra.

We draw inspiration from [Hön17, Lemma 2.4.10]. For $R \in \mathbb{S}$-**CAlg**, denote $R^e := R \wedge R$.

Proof. Consider A (resp. B) as a commutative A^e-algebra (resp. B^e-algebra) via the multiplication map $A^e \to A$ (resp. $B^e \to B$). Recall that $THH(A, B) \cong B \wedge_{A^e} B(A, A, A)$ and similarly for $THH(B)$.

Let $\tilde{B} \xrightarrow{\sim} B$ be a cofibrant replacement of B in the category of commutative B^e-algebras. There is a commutative diagram of $\mathbb{S}$-modules

$$\begin{tikzcd} THH(A,B) \arrow[d, "THH(f,\mathrm{id})"'] & \tilde{B} \wedge_{A^e} B(A,A,A) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "(\mathrm{id},f)"'] & \tilde{B} \wedge_{A^e} A \arrow[d, "\overline{f}"'] \\ THH(B) & \tilde{B} \wedge_{B^e} B(B,B,B) \arrow[l, "\sim"'] \arrow[r, "\sim"] & \tilde{B} \wedge_{B^e} B. \end{tikzcd} \tag{4.11}$$

Recall that the two-sided bar construction $B(A, A, A)$ induces a weak equivalence of commutative A^e-algebras $B(A, A, A) \to A$ [EKMM97, IV.7.5] and a cofibration in $\mathbb{S}$-**CAlg** $A^e \to B(A, A, A)$ given by inclusion of the first and last smash factors. See [Hön17, Proof of Lemma 2.4.8] for a proof of this last fact.

The arrow (id, f) in the middle is defined via the universal property for the coproduct in commutative A^e-algebras, using the canonical map $\tilde{B} \to \tilde{B} \wedge_{B^e} B(B, B, B)$ to the first factor, and the map $B(A,A,A) \to B(B,B,B)$ defined by smash powers of f at the simplicial level followed by the canonical map to the second factor.

The arrow $\overline{f}$ is described as follows. First note that there are isomorphisms

$$\tilde{B} \wedge_{A^e} A \cong \tilde{B} \wedge_{B^e} (B^e \wedge_{A^e} A) \cong \tilde{B} \wedge_{B^e} (B \wedge_A B).$$

The last step comes from the isomorphism of commutative B^e-algebras $B^e \wedge_{A^e} A \cong B \wedge_A B$ which appears e.g. in [Lin00, Lemma 2.1]. Then $\overline{f}$ is defined to be the composition

$$\tilde{B} \wedge_{A^e} A \cong \tilde{B} \wedge_{B^e} (B \wedge_A B) \xrightarrow{\mathrm{id} \wedge \mu} \tilde{B} \wedge_{B^e} B.$$

The previous diagram appears as the geometric realization of a diagram in simplicial $\mathbb{S}$-modules. The arrows in this latter diagram are very explicitely defined, and it is immediate that they make the diagram commute.

Therefore, to see that $THH(f, \mathrm{id})$ is a weak equivalence, it suffices to see that $\mathrm{id} \wedge \mu$ is a weak equivalence. This is the case: indeed, the functor $\tilde{B} \wedge_{B^e} -$ preserves weak equivalences between cofibrant commutative $\mathbb{S}$-algebras because $\tilde{B}$ is cofibrant as a commutative B^e-algebra. Now note that $B \wedge_A B$ is a cofibrant commutative $\mathbb{S}$-algebra because it is a cofibrant commutative A-algebra (it is a coproduct of two cofibrant commutative A-algebras). □

Lemma 3.2 allows us to apply Theorem 4.8 to $\mathbb{S} \to R \to R[x^{-1}]$. Putting this together with the weak equivalence (4.7), we obtain:

Corollary 4.12. *Let R be a cofibrant commutative $\mathbb{S}$-algebra, and let $x \in \pi_* R$. There are weak equivalences of commutative $R[x^{-1}]$-algebras*

$$THH(R)[x^{-1}] \xleftarrow{\sim} THH(R, R[x^{-1}]) \xrightarrow{\sim} THH(R[x^{-1}]).$$

Remark 4.13. We have recently become aware that, in [SVW00, Page 353], the authors state that “one can prove that THH commutes with localizations”, pointing to an article in preparation which never appeared.

Remark 4.14. We know three proofs of the fact that Hochschild homology commutes with localizations. Weibel [Wei94, 9.1.8(3)] proves it using the fact that Tor behaves well under flat base change. Brylinski [Bry89] (see also [Lod98, 1.1.17]) prove it by comparing the homological functors defined on A-bimodules $S^{-1}HH_n(A, -)$ and $HH_n(S^{-1}A, S^{-1}-)$, where S is a multiplicative subset of the commutative algebra A. In [WG91], Geller and Weibel prove the more general result that Hochschild homology behaves well with respect to étale maps of commutative algebras $A \to B$, of which a localization map is an example. Our proof of Theorem 4.8 is closer to the first of these approaches.

Remark 4.15. For a map $f : A \to B$ of commutative $\mathbb{S}$-algebras as in Theorem 4.8, the question of under what conditions is (4.10) a weak equivalence has been considered before. For example, in [MM03, Lemma 5.7] the authors prove that it holds when A and B are connective and the unit $B \to THH^A(B)$ is a weak equivalence. Mathew [Mat17, Theorem 1.3], working in the context of the E_∞-rings of Lurie, proved that a map $A \to B$ of E_∞-rings satisfies that (4.10) is an equivalence provided f is étale, with no hypotheses on connectivity. There is a notion of localization of E_∞-rings, and Lurie proved that localization maps are étale [Lur, 7.5.1.13]. This gives a one-line proof of Theorem 4.8 applied to $\mathbb{S} \to R \to R[x^{-1}]$ in the context of E_∞-rings.

5. Topological Hochschild homology of KU

5.1. **Topological Hochschild homology of $\mathbb{S}[G][x^{-1}]$.** Let G be a topological abelian group which is a CW-complex with a cellular addition map. As remarked in Section 4.4, this assumption guarantees that BG is again a CW-complex with a cellular multiplication map.

Let $x \in \pi_*\mathbb{S}[G]$. We prove the following theorem below.

Theorem 5.1. *The commutative $\mathbb{S}[G][x^{-1}]$-algebras $THH(\mathbb{S}[G][x^{-1}])$ and $\mathbb{S}[G][x^{-1}][BG]$ are weakly equivalent as $\mathbb{S}[G][x^{-1}]$-algebras.*

For any commutative $\mathbb{S}$-algebra A, the notation $A[BG]$ stands for the commutative A-algebra $A \wedge \mathbb{S}[BG]$: thus, its underlying A-module is $A \wedge (BG)_+$. No confusion should arise from the usage of square brackets for two different notions.

We first isolate the part of the proof that does not involve inverting an element.

Lemma 5.2. *There is an isomorphism of commutative $\mathbb{S}[G]$-algebras*

$$THH(\mathbb{S}[G]) \cong \mathbb{S}[G] \wedge \mathbb{S}[BG] = \mathbb{S}[G][BG].$$

Proof. It is an application of Propositions 4.4, 4.6 and 4.5:

$$THH(\mathbb{S}[G]) \xrightarrow{\cong} \mathbb{S}[B^{\mathrm{cy}}G] \xrightarrow{\cong} \mathbb{S}[G \times BG] \xleftarrow{\cong} \mathbb{S}[G] \wedge \mathbb{S}[BG]. \qquad \square$$

Proof of Theorem 5.1. By Corollary 4.12, we obtain a zig-zag of two weak equivalences of commutative $\mathbb{S}[G][x^{-1}]$-algebras

$$THH(\mathbb{S}[G][x^{-1}]) \simeq THH(\mathbb{S}[G])[x^{-1}].$$

Lemma 5.2 gives an isomorphism $THH(\mathbb{S}[G]) \cong \mathbb{S}[G] \wedge \mathbb{S}[BG]$ such that

$$THH(\mathbb{S}[G])[x^{-1}] \cong (\mathbb{S}[G] \wedge \mathbb{S}[BG])[(x \wedge 1)^{-1}].$$

Now Proposition 3.6 gives a weak equivalence of commutative $\mathbb{S}[G][x^{-1}]$-algebras

$$(\mathbb{S}[G] \wedge \mathbb{S}[BG])[(x \wedge 1)^{-1}] \simeq \mathbb{S}[G][x^{-1}] \wedge \mathbb{S}[BG] = \mathbb{S}[G][x^{-1}][BG]$$

finishing the proof. $\square$

5.2. **Snaith's theorem.** There is a cofibrant commutative $\mathbb{S}$-algebra KU of complex topological K-theory [EKMM97, VIII.4.3]. It is obtained by applying the localization theorem we reviewed in Theorem 3.1 to the cofibrant commutative $\mathbb{S}$-algebra ku of connective K-theory and its Bott element. Here ku is constructed by multiplicative infinite loop space theory.

The presentation for KU which we will use is given by the following version of a theorem of Snaith [Sna79], [Sna81]:

Theorem 5.3. *KU is weakly equivalent as a commutative $\mathbb{S}$-algebra to the cofibrant commutative $\mathbb{S}$-algebra $\mathbb{S}[\mathbb{C}P^\infty][x^{-1}]$, where $x \in \pi_2(\mathbb{S}[\mathbb{C}P^\infty])$ is represented by the map induced from the inclusion $\mathbb{C}P^1 \to \mathbb{C}P^\infty$, i.e.*

$$\Sigma^\infty S^2 \cong \Sigma^\infty \mathbb{C}P^1 \to \mathbb{S} \vee \Sigma^\infty \mathbb{C}P^\infty \simeq \Sigma^\infty_+ \mathbb{C}P^\infty.$$

Remark 5.4. We thank Christian Schlichtkrull for pointing out the article [Art83] to us. In Theorems 5.1 and 5.2 therein, it is proven that if $t \in \pi_n(\mathbb{S}[K(\mathbb{Z}, n)])$ is a generator, then $\mathbb{S}[K(\mathbb{Z}, n)][t^{-1}]$ is contractible for n odd and is equivalent to $H\mathbb{Q}[t^{\pm 1}]$ for $n \geq 4$ even. So the case $n = 2$ which we treat here is the only interesting localization of $\mathbb{S}[K(\mathbb{Z}, n)]$.

5.3. **Rationalization.** In this short section we review some facts about rationalization that we will be using.

If X is an $\mathbb{S}$-module, we denote by $X_{\mathbb{Q}}$ its rationalization. Our model for $X_{\mathbb{Q}}$ is given by $H\mathbb{Q} \wedge X$, where $H\mathbb{Q}$ is any model for the Eilenberg-Mac Lane commutative $\mathbb{S}$-algebra of $\mathbb{Q}$ which is cofibrant as an $\mathbb{S}$-module. Therefore, the rationalization functor is $H\mathbb{Q} \wedge -$, and as such it is lax symmetric monoidal. The structure map $X_{\mathbb{Q}} \wedge Y_{\mathbb{Q}} \to (X \wedge Y)_{\mathbb{Q}}$ is a weak equivalence when X and Y are cofibrant $\mathbb{S}$-modules, since the multiplication of $H\mathbb{Q}$ is a weak equivalence. Note that we do not need to derive the functor $H\mathbb{Q} \wedge -$, since $H\mathbb{Q}$ is cofibrant.

Let n be any integer. The degree n map $n : \mathbb{S} \to \mathbb{S}$ induces a map $n : X \to X$ on any $\mathbb{S}$-module X. If $p : X \to X$ is a weak equivalence for every prime p then the homotopy groups of X are rational, since p induces the multiplication by p map on homotopy groups. Therefore X is rational, i.e. the rationalization map $X \to X_{\mathbb{Q}}$ is an equivalence.

We will also need the following fact concerning the rationalization of Eilenberg-Mac Lane spaces [FHT01, Page 202]: for $n \geq 2$,

$$K(\mathbb{Z}, n)_{\mathbb{Q}} \simeq \begin{cases} S^n_{\mathbb{Q}} & \text{if } n \text{ is odd,} \\ \Omega S^{n+1}_{\mathbb{Q}} & \text{if } n \text{ is even.} \end{cases}$$

Actually, the authors prove that for n even, $\Omega S^{n+1} \to K(\mathbb{Z}, n)$ is a rational equivalence, so that $K(\mathbb{Z}, n)_{\mathbb{Q}} \simeq (\Omega S^{n+1})_{\mathbb{Q}}$, which is not exactly what we wrote. But indeed, for any simply connected space X we have that $(\Omega X)_{\mathbb{Q}} \simeq \Omega(X_{\mathbb{Q}})$. This follows from comparing the rational cohomology Eilenberg-Moore spectral sequences [McC01, Corollary 7.16] for ΩX and for $\Omega X_{\mathbb{Q}}$ via the rationalization map $X \to X_{\mathbb{Q}}$: we obtain that the map $\Omega X \to \Omega X_{\mathbb{Q}}$ is a rationalization map, for $\Omega X_{\mathbb{Q}}$ is rational. Symbolically,

$$\begin{tikzcd} \mathrm{Tor}_{H^*(X;\mathbb{Q})}(\mathbb{Q}, \mathbb{Q}) \arrow[r, Rightarrow] \arrow[d, "\cong"'] & H^*(\Omega X; \mathbb{Q}) \arrow[d, "\cong"] \\ \mathrm{Tor}_{H^*(X_{\mathbb{Q}};\mathbb{Q})}(\mathbb{Q}, \mathbb{Q}) \arrow[r, Rightarrow] & H^*(\Omega X_{\mathbb{Q}}, \mathbb{Q}). \end{tikzcd}$$

5.4. **The main results.** As a particular case of Theorem 5.1, we have:

Theorem 5.5. *The commutative KU-algebras $THH(KU)$ and $KU[K(\mathbb{Z}, 3)]$ are weakly equivalent as commutative KU-algebras.*

Remark 5.6. Compare with what happens to $THH(MU)$: in [BCS10], the authors establish an equivalence of $\mathbb{S}$-modules $THH(MU) \simeq MU \wedge SU_+$. They actually prove the following more general result. Let BF denote the classifying space for stable spherical fibrations. If $f : X \to BF$ is a 3-fold loop map and $T(f)$ is its Thom spectrum, then there is a weak equivalence of $\mathbb{S}$-modules

$$THH(T(f)) \simeq T(f) \wedge BX_+. \tag{5.7}$$

Note that this result was improved to an equivalence of E_∞ $\mathbb{S}$-algebras by Schlichtkrull [Sch11, Corollary 1.2] in the case where X is a grouplike E_∞-space and f is an E_∞-map.

Our Theorem 5.5 gives in particular a weak equivalence of $\mathbb{S}$-modules $THH(KU) \simeq KU \wedge K(\mathbb{Z}, 3)_+$: by comparing this formula to (5.7), one is naturally led to conjecture that KU is the Thom spectrum of an ∞-loop map $K(\mathbb{Z}, 2) \simeq BU(1) \to BU$. However, this is not possible, since Thom spectra are connective. On the other hand, Sagave and Schlichtkrull [SS14] have introduced *graded* Thom spectra, and these can be non-connective. It also seems unlikely that KU will be the graded Thom spectrum of a map $BU(1) \to BU \times \mathbb{Z}$, since the image will be contained in one of the components of $BU \times \mathbb{Z}$. We would like to understand why does KU behave like a Thom spectrum, at least to the eyes of topological Hochschild homology. See also Remarks 7.21 and 8.4.

We will now describe the commutative KU-algebra $THH(KU)$ as the free commutative KU-algebra on the KU-module $\Sigma KU_{\mathbb{Q}}$, and we will prove this algebra is weakly equivalent to the split square-zero extension of KU by $\Sigma KU_{\mathbb{Q}}$. Let us first define this concept.

Let R be a commutative $\mathbb{S}$-algebra, let A be a commutative R-algebra and let M be a non-unital commutative A-algebra. Then $A \vee M$ (coproduct of A-modules) has a commutative A-algebra structure. Indeed, after distributing, a multiplication map

$$(A \vee M) \wedge_A (A \vee M) \to A \vee M$$

looks like

$$(A \wedge_A A) \vee (A \wedge_A M) \vee (M \wedge_A A) \vee (M \wedge_A M) \to A \vee M. \tag{5.8}$$

We may define a map like (5.8) by defining maps from each of the wedge summands to $A \vee M$. Define the maps to $A \vee M$ from $A \wedge_A A$, $A \wedge_A M$ and $M \wedge_A A$ to be the canonical isomorphisms followed by the canonical maps into the respective factor. Finally, consider the map $M \wedge_A M \to A \vee M$ given by the multiplication map of M followed by the canonical map to $A \vee M$. We have thus defined a multiplication map (5.8) such that $A \vee M$ is a commutative A-algebra with unit given by the canonical map $A \to A \vee M$. We say that $A \vee M$ is a *split extension of A by M*. If the multiplication of M is trivial, then $A \vee M$ is a *split square-zero extension of A by M*; in this case, M is no more than an A-module.

Conversely, if A is a commutative R-algebra with augmentation $\varepsilon : A \to R$, then there is a splitting in commutative augmented R-algebras $A \simeq R \vee \overline{A}$ where $\overline{A}$ is a non-unital commutative R-algebra fitting into a fiber sequence

$$\overline{A} \to A \to R.$$

More precisely, the underlying R-module of $\overline{A}$ fits into the following pullback square in R-**Mod**,

$$\begin{tikzcd} \overline{A} \arrow[r, "i"] \arrow[d] & A \arrow[d, "\varepsilon"] \\ 0 \arrow[r] & R \end{tikzcd}$$

and it gets a non-unital multiplication from the universal property of pullbacks, by considering the following commutative diagram in R-**Mod**. See [Bas99, Section 2] for further elaboration.

$$\begin{tikzcd} \overline{A} \wedge_R \overline{A} \arrow[r, "i \wedge i"] \arrow[d] & A \wedge_R A \arrow[r, "\mu"] \arrow[d, "\varepsilon \wedge \varepsilon"] & A \arrow[d, "\varepsilon"] \\ 0 \arrow[r] & R \wedge_R R \arrow[r, "\cong"'] & R \end{tikzcd}$$

In particular, there is a splitting of commutative augmented A-algebras

$$THH(A) \simeq A \vee \overline{THH}(A). \tag{5.9}$$

The rest of this section is devoted to the proof of the following

Theorem 5.10. *There is a morphism of commutative augmented KU-algebras*

$$\tilde{f} : F(\Sigma KU_{\mathbb{Q}}) \to THH(KU)$$

which is a weak equivalence. Here $F : KU\text{-}\mathbf{Mod} \to KU\text{-}\mathbf{CAlg}$ is the free commutative algebra functor.

Moreover, $F(\Sigma KU_{\mathbb{Q}})$ is weakly equivalent as an augmented commutative KU-algebra to the split square-zero extension $KU \vee \Sigma KU_{\mathbb{Q}}$.

The morphism $\tilde{f}$ is obtained by the universal property of F from a map of KU-modules $f : \Sigma KU_{\mathbb{Q}} \to THH(KU)$ to be described below (5.19).

Remark 5.11. The functor F, or more generally, the free commutative algebra functor $F_R : R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{CAlg}$ where R is a commutative $\mathbb{S}$-algebra, is the left adjoint of the forgetful functor $U_R : R\text{-}\mathbf{CAlg} \to R\text{-}\mathbf{Mod}$, or alternatively, the free algebra functor for the monad $\mathbb{P}_R$ on $R\text{-}\mathbf{Mod}$ defined as

$$\mathbb{P}_R(M) = \bigvee_{n\geq 0} M^{\wedge_R n}/\Sigma_n = R \vee M \vee \bigvee_{n\geq 2} M^{\wedge_R n}/\Sigma_n, \tag{5.12}$$

where Σ_n is the symmetric group on n elements (see e.g. [EKMM97, II.7.1] or [Bas99, Section 1]). Note that $F_R M$ is augmented over R: the augmentation is the projection on the 0-th term.

As explained in Section 2, the functor $U_R : R\text{-}\mathbf{CAlg} \to R\text{-}\mathbf{Mod}$ is a right Quillen functor, so $F_R : R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{CAlg}$ is a left Quillen functor. In particular, it preserves weak equivalences between cofibrant R-modules.

Note as well that, if $M \in R\text{-}\mathbf{Mod}$ is cofibrant, then the arrow $\bigvee_{n\geq 0}(M^{\wedge_R n})_{h\Sigma_n} \to F_R(M)$ induced from the canonical arrows from the homotopy orbits to the orbits is a weak equivalence [EKMM97, III.5.1]. This is a step in the proof of the determination of the model structure on $R\text{-}\mathbf{CAlg}$.

Remark 5.13. A spectrum-level result related to Theorem 5.10 was obtained by McClure and Staffeldt in [MS93, Theorem 8.1]: they showed that $THH(L) \simeq L \vee \Sigma L_{\mathbb{Q}}$ as spectra, where L is the p-adic completion of the Adams summand of KU for a given odd prime p; the result was extended to $p = 2$ by Angeltveit, Hill and Lawson in [AHL10, 2.3]. Ausoni [Aus05, Proposition 7.13] formulated without proof the analogous theorem (for an odd p) for KU completed at p in place of L. In Corollary 7.9 of [AHL10], the authors show that $THH(KO) \simeq KO \vee \Sigma KO_{\mathbb{Q}}$ as KO-modules. The methods used in the proofs of the results just cited are different from ours.

We first prove a couple of results needed for the proof. Note that in the following statement we are considering $K(\mathbb{Z}, 3)$ as a pointed space: we are not adding a disjoint basepoint.

Proposition 5.14. *There is a weak equivalence $KU \wedge K(\mathbb{Z}, 3) \simeq \Sigma KU_{\mathbb{Q}}$ of KU-modules.*

Proof. Let p be a prime and consider the cofiber sequence of KU-modules

$$KU \xrightarrow{p} KU \longrightarrow KU/p. \tag{5.15}$$

If $p > 2$, then KU/p is equivalent to $\bigvee_{i=0}^{p-2} \Sigma^{2i} K(1)$ (see [Ada69, Lecture 4]), where $K(1) \simeq L/p$ is the first Morava K-theory at p. If $p = 2$, then $K(1) \simeq KU/2$.

The homology $K(1)_* K(\mathbb{Z}, 3)$ is trivial: see [RW80, Theorem 12.1] for the $p > 2$ case, and [JW85, Appendix] for the $p = 2$ case. Therefore, after smashing (5.15) with $K(\mathbb{Z}, 3)$, we get a weak equivalence of KU-modules

$$KU \wedge K(\mathbb{Z}, 3) \xrightarrow[p \wedge \mathrm{id}]{\sim} KU \wedge K(\mathbb{Z}, 3)$$

for all primes p. This means that $KU \wedge K(\mathbb{Z}, 3)$ is rational, and so

$$KU \wedge K(\mathbb{Z}, 3) \simeq (KU \wedge K(\mathbb{Z}, 3))_{\mathbb{Q}} \simeq KU_{\mathbb{Q}} \wedge K(\mathbb{Z}, 3)_{\mathbb{Q}} \simeq KU_{\mathbb{Q}} \wedge S^3_{\mathbb{Q}} \simeq \Sigma KU_{\mathbb{Q}}$$

by the results quoted in Section 5.3, plus Bott periodicity for the last step. □

Proposition 5.16. *Let R be a commutative $\mathbb{S}$-algebra and $F_R : R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{CAlg}$ be the free commutative algebra functor. The augmented commutative R-algebra $F_R(\Sigma R_{\mathbb{Q}})$ is weakly equivalent to the split square-zero extension $R \vee \Sigma R_{\mathbb{Q}}$.*

Remark 5.17. Note that we are applying F_R to a cofibrant R-module. Indeed, since $H\mathbb{Q}$ is a cofibrant $\mathbb{S}$-module and S^1 is a cofibrant based space, then $S^1 \wedge H\mathbb{Q}$ is a cofibrant $\mathbb{S}$-module. Now, the extension of scalars functor $R \wedge - : \mathbb{S}\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ is left Quillen: indeed, its right adjoint, the forgetful functor, is right Quillen since the model structure in R-**Mod** is created through it. Therefore, $R \wedge (S^1 \wedge H\mathbb{Q}) \cong \Sigma R_{\mathbb{Q}}$ is a cofibrant R-module.

Proof. Recall Remark 5.11 describing the functor F_R. Note that for an $\mathbb{S}$-module X, we have a natural isomorphism $F_R(R \wedge X) \cong R \wedge F_{\mathbb{S}}(X)$. Indeed,

$$F_R(R \wedge X) = \bigvee_{n \geq 0} {(R \wedge X)^{\wedge_R n}} / {\Sigma_n} \cong R \wedge \bigvee_{n \geq 0} {X^{\wedge n}} / {\Sigma_n} = R \wedge F_{\mathbb{S}}(X)$$

since the left adjoint functor $R \wedge - : \mathbb{S}\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ preserves colimits. Therefore, $F_R(\Sigma R_{\mathbb{Q}}) = F_R(R \wedge S^1_{\mathbb{Q}}) \cong R \wedge F_{\mathbb{S}}(S^1_{\mathbb{Q}})$. We have

$$F_{\mathbb{S}}(S^1_{\mathbb{Q}}) = \mathbb{S} \vee S^1_{\mathbb{Q}} \vee \bigvee_{n \geq 2} {(S^1_{\mathbb{Q}})^{\wedge n}} / {\Sigma_n} \simeq \mathbb{S} \vee S^1_{\mathbb{Q}} \vee \bigvee_{n \geq 2} \left({(S^1)^{\wedge n}} / {\Sigma_n} \right)_{\mathbb{Q}}$$

where we have used that the rationalization functor $H\mathbb{Q} \wedge -$ commutes with colimits and that the rationalization of a smash product of finitely many factors is weakly equivalent to the smash product of the rationalizations.

Now we claim that ${(S^1)^{\wedge n}} / {\Sigma_n}$ is contractible for $n \geq 2$, and this finishes the proof. To see this, let X be a based space and consider $\widetilde{SP}^n(X) := X^{\wedge n}/\Sigma_n$ and $SP^n(X) := X^{\times n}/\Sigma_n$. There is a map $f_n : SP^{n-1}(X) \to SP^n(X)$ given by inserting a basepoint, and its cofiber is $\widetilde{SP}^n(X)$. When $X = S^1$, the map f_n is a homotopy equivalence for any $n \geq 2$ [AGP02, 5.2.23], so that $\widetilde{SP}^n(S^1)$ is contractible for all $n \geq 2$. □

Proof of Theorem 5.10. First, we work additively, and then we will determine the multiplicative structure.

Recall that for any based space X, there is a weak equivalence $\Sigma^\infty_+ X \simeq \mathbb{S} \vee \Sigma^\infty X$ coming from the homotopy equivalence of spaces $\Sigma(X_+) \simeq S^1 \vee \Sigma X$. Combining this with Theorem 5.5 and Proposition 5.14, we obtain weak equivalences of KU-modules

$$\begin{aligned} THH(KU) &\simeq KU \wedge \Sigma^{\infty}_{+} K(\mathbb{Z},3) \simeq KU \wedge (\mathbb{S} \vee \Sigma^{\infty} K(\mathbb{Z},3)) \cong \\ &\simeq KU \vee (KU \wedge \Sigma^{\infty} K(\mathbb{Z},3)) \simeq KU \vee \Sigma KU_{\mathbb{Q}}. \end{aligned}$$

This splitting is compatible with the splitting (5.9) since both are just splitting off the unit, so we have a weak equivalence of KU-modules $\Sigma KU_{\mathbb{Q}} \xrightarrow{\sim} \overline{THH}(KU)$.

In particular, we have an isomorphism

$$\pi_* THH(KU) \cong \mathbb{Z}[t^{\pm 1}] \oplus \Sigma \mathbb{Q}[t^{\pm 1}] \tag{5.18}$$

of $\mathbb{Z}[t^{\pm 1}]$-graded modules.

We will now determine the multiplicative structure. Consider the map of KU-modules

$$f : \ \Sigma KU_{\mathbb{Q}} \xrightarrow{\ \sim\ } \overline{THH}(KU) \longrightarrow THH(KU). \tag{5.19}$$

In homotopy groups, this is an isomorphism $\Sigma\mathbb{Q}[t^{\pm 1}] \to \pi_* \overline{THH}(KU)$ followed by the inclusion into $\pi_* THH(KU)$.

By the universal property satisfied by F, we get that f induces a map of commutative KU-algebras

$$\tilde{f} : F(\Sigma KU_{\mathbb{Q}}) \to THH(KU).$$

Note that, by definition of f, we have that $\varepsilon \circ f$ is the trivial map, where $\varepsilon : THH(KU) \to KU$ is the augmentation. This implies that $\tilde{f}$ preserves the augmentation.

Now, $\tilde{f}$ is a weak equivalence. Indeed, after identifying $F(\Sigma KU_{\mathbb{Q}})$ with the split square-zero extension $KU \vee \Sigma KU_{\mathbb{Q}}$ (Proposition 5.16), $\tilde{f}$ amounts to the map of commutative KU-algebras $(\eta, f) : KU \vee \Sigma KU_{\mathbb{Q}} \to THH(KU)$. But (η, f) is a weak equivalence by construction of f. □

Corollary 5.20. *The map*

$$(\eta, f) : KU \vee \Sigma KU_{\mathbb{Q}} \to THH(KU)$$

is a weak equivalence of augmented commutative KU-algebras, where $\eta : KU \to THH(KU)$ is the unit, the map f was defined in (5.19), and $KU \vee \Sigma KU_{\mathbb{Q}}$ is a split square-zero extension.

5.5. **The morphism σ.** If R is a commutative $\mathbb{S}$-algebra, there is a natural transformation of $\mathbb{S}$-modules [MS93, Section 3], [EKMM97, IX.3.8], [AR05, 3.12]

$$\sigma : \Sigma R \to THH(R).$$

Consider the map

$$(\eta, \sigma) : KU \vee \Sigma KU \to THH(KU).$$

It is tempting to conjecture that its rationalization

$$(\eta_{\mathbb{Q}}, \sigma_{\mathbb{Q}}) : KU_{\mathbb{Q}} \vee \Sigma KU_{\mathbb{Q}} \to THH(KU)_{\mathbb{Q}}$$

is a weak equivalence, since by the results of the previous section, the $\mathbb{S}$-modules $KU_{\mathbb{Q}} \vee \Sigma KU_{\mathbb{Q}}$ and $THH(KU)_{\mathbb{Q}}$ are weakly equivalent.

However, this is not the case. I thank Geoffroy Horel and Thomas Nikolaus for pointing out this fact and the following proof to me. We will prove that $\sigma : \Sigma KU \to THH(KU)$

is zero in π_1, therefore it is still zero after rationalization. By naturality of σ, we have a commutative diagram

$$\begin{tikzcd} \Sigma\mathbb{S} \arrow[r, "\sigma"] \arrow[d, "\Sigma\iota"'] & THH(\mathbb{S}) \simeq \mathbb{S} \arrow[d, "THH(\iota)"] \\ \Sigma KU \arrow[r, "\sigma"'] & THH(KU) \end{tikzcd} \tag{5.21}$$

where ι is the unit of KU. After taking π_1, we obtain a commutative diagram of abelian groups

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"'] & \mathbb{Z}/2 \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathbb{Q}. \end{tikzcd} \tag{5.22}$$

Therefore, $\mathbb{Z} \to \mathbb{Q}$ must be the zero map, since only the abelian group map $\mathbb{Z}/2 \to \mathbb{Q}$ is the zero map.

Note that the same proof works for L (the p-adic completion of the Adams summand of KU, p a prime) instead of KU. Recall that $\pi_* L \cong \mathbb{Z}_{(p)}[(v_1)^{\pm 1}]$, with v_1 in degree $2p-2$. After replacing KU with L in (5.21) and taking π_1, we obtain a square which looks like (5.22) except with a $\mathbb{Z}_{(p)}$ on the lower left corner. The vertical map $\mathbb{Z} \to \mathbb{Z}_{(p)}$ is the unit of $\mathbb{Z}_{(p)}$: this still forces $\pi_1\sigma : \pi_1(\Sigma L) \to \pi_1(THH(L))$ to be zero.

This corrects an error in [MS93, 8.4] where it is claimed that there is a weak equivalence $L_{\mathbb{Q}} \vee \Sigma L_{\mathbb{Q}} \xrightarrow{\sim} THH(L)_{\mathbb{Q}}$ induced by (η, σ). As a positive result, we have Corollary 5.20 instead.

6. Iterated topological Hochschild homology of KU

Let A be a commutative $\mathbb{S}$-algebra. We denote by $THH^n(A)$ the *iterated topological Hochschild homology of* A, i.e. $THH(\dots(THH(A)))$ where THH is applied n times. Other expressions for $THH^n(A)$ include $T^n \otimes A$ or $\Lambda_{T^n}(A)$, where T^n is an n-torus and Λ is the Loday functor [CDD11].

We will now give two different descriptions of $THH^n(KU)$ for $n \geq 2$. The first one, given in Theorem 6.6, generalizes Theorem 5.5 which describes $THH(KU)$ via Eilenberg-Mac Lane spaces. The second one, given in Theorem 6.10, generalizes Theorem 5.10 which describes $THH(KU)$ as a free commutative KU-algebra on a KU-module.

We have also given a description of the commutative KU-algebra $THH(KU)$ as a split square-zero extension in Theorem 5.10. For $n \geq 2$, $THH^n(KU)$ is not a split square-zero extension of KU, as we shall see. However, it is a split extension: we will describe the non-unital commutative algebra structure of the homotopy groups of its augmentation ideal, which is rational as in the $n = 1$ case.

6.1. **Description via Eilenberg-Mac Lane spaces.** Let G be a topological abelian group which is a CW-complex with a cellular addition map. Applying Lemma 5.2 and Proposition 4.5, we obtain isomorphisms of commutative $\mathbb{S}[G]$-algebras:

$$\begin{aligned} THH^2(\mathbb{S}[G]) &\cong THH(\mathbb{S}[G] \wedge \mathbb{S}[BG]) \cong THH(\mathbb{S}[G \times BG]) \\ &\cong \mathbb{S}[G \times BG] \wedge \mathbb{S}[B(G \times BG)] \cong \mathbb{S}[G] \wedge \mathbb{S}[BG \times BG \times B^2G] \end{aligned}$$

which we have written as $\mathbb{S}[G][BG \times BG \times B^2G]$.

For general n, the same type of computation gives a description of $THH^n(\mathbb{S}[G])$: we obtain an isomorphism of commutative $\mathbb{S}[G]$-algebras

$$THH^n(\mathbb{S}[G]) \cong \mathbb{S}[G][B^{a_1}G \times \cdots \times B^{a_{2^n-1}}G]. \tag{6.1}$$

The numbers a_i can be described as follows. Let $v_0 = 0$. Define by induction

$$v_n = (v_{n-1}, v_{n-1} + (1, \ldots, 1)) = (a_0, \ldots, a_{2^n-1}) \in \mathbb{N}^{2^n} \tag{6.2}$$

for $n \geq 1$. For example, $v_1 = (0, 1)$, $v_2 = (0, 1, 1, 2)$ and $v_3 = (0, 1, 1, 2, 1, 2, 2, 3)$. This sequence of integers can be found in the On-Line Encyclopedia of Integer Sequences [Slo]. We can give an easier description. Let I_n be the multiset having as elements the numbers i with multiplicity $\binom{n}{i}$, for $i = 1, \ldots, n$. Denote the multiplicity of an element x of a multiset by $|x|$. Now note that the multiset underlying the sequence $(a_1, \ldots, a_{2^n-1})$ defined in (6.2) coincides with I_n, by Pascal's rule. Therefore, the isomorphism (6.1) can be reformulated as

$$THH^n(\mathbb{S}[G]) \simeq \mathbb{S}[G]\left[\prod_{i=1}^{n} (B^iG)^{\times\binom{n}{i}}\right]. \tag{6.3}$$

The following theorem generalizes Theorem 5.1 to higher iterations of THH.

Theorem 6.4. *Let $x \in \pi_*\mathbb{S}[G]$. There is a zig-zag of weak equivalences of commutative $\mathbb{S}[G][x^{-1}]$-algebras*

$$THH^n(\mathbb{S}[G][x^{-1}]) \simeq \mathbb{S}[G][x^{-1}][B^{a_1}G \times \cdots \times B^{a_{2^n-1}}G],$$

or alternatively,

$$THH^n(\mathbb{S}[G][x^{-1}]) \simeq \mathbb{S}[G][x^{-1}]\left[\prod_{i=1}^{n} (B^iG)^{\times\binom{n}{i}}\right].$$

Proof. The proof is by induction. The base case is Theorem 5.1. We do the induction step for $n = 2$ for simplicity: for higher n it is analogous, only with more indices to juggle around. By Theorem 5.1, there is a zig-zag of weak equivalences of commutative $\mathbb{S}[G][x^{-1}]$-algebras

$$THH^2(\mathbb{S}[G][x^{-1}]) = THH(THH(\mathbb{S}[G][x^{-1}])) \simeq THH(\mathbb{S}[G][x^{-1}][BG]).$$

Applying Propositions 3.6 and 4.5, we get

$$\begin{aligned} \mathbb{S}[G][x^{-1}][BG] = \mathbb{S}[G][x^{-1}] \wedge \mathbb{S}[BG] &\simeq (\mathbb{S}[G] \wedge \mathbb{S}[BG])[(x \wedge 1)^{-1}] \\ &\cong S[G \times BG][(x, e)^{-1}] \end{aligned} \tag{6.5}$$

where e is the unit of BG. Continuing the computation, we apply (6.5), Theorem 5.1 and (6.5) again, obtaining:

$$\begin{aligned} THH(\mathbb{S}[G][x^{-1}][BG]) &\simeq THH(\mathbb{S}[G \times BG][(x, e)^{-1}]) \\ &\simeq \mathbb{S}[G \times BG][(x, e)^{-1}][B(G \times BG)] \\ &\simeq (\mathbb{S}[G][x^{-1}] \wedge \mathbb{S}[BG])[BG \times B^2G] \\ &\cong \mathbb{S}[G][x^{-1}][BG \times BG \times B^2G]. \end{aligned}$$

□

As a corollary, we obtain:

Theorem 6.6. *There is a zig-zag of weak equivalences of commutative KU-algebras*

$$THH^n(KU) \simeq KU[K(\mathbb{Z}, a_1 + 2) \times \cdots \times K(\mathbb{Z}, a_{2^n-1} + 2)], \tag{6.7}$$

or alternatively,

$$THH^n(KU) \simeq KU\left[\prod_{i=1}^{n} K(\mathbb{Z}, i+2)^{\times \binom{n}{i}}\right]. \tag{6.8}$$

For example,

$$THH^2(KU) \simeq KU[K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 4)]. \tag{6.9}$$

The previous theorem generalizes the expression of Theorem 5.5 for $THH(KU)$ as $KU[K(\mathbb{Z}, 3)]$ to $THH^n(KU)$. We can also generalize the expression for $THH(KU)$ as $F(\Sigma KU_{\mathbb{Q}})$ of Theorem 5.10. Note that the proof uses results from Section 7 below.

Theorem 6.10. *Let $n \geq 2$. There is a zig-zag of weak equivalences of commutative KU-algebras*

$$F\left(\bigvee_{i=1}^{n} (S^i)^{\vee \binom{n}{i}} \wedge KU_{\mathbb{Q}}\right) \simeq T^n \otimes KU.$$

Proof. Since $- \wedge KU_{\mathbb{Q}}$ and $F : KU\text{-}\mathbf{Mod} \to KU\text{-}\mathbf{CAlg}$ are left adjoints, they preserve coproducts, so:

$$F\left(\bigvee_{i=1}^{n} (S^i)^{\vee \binom{n}{i}} \wedge KU_{\mathbb{Q}}\right) \cong F\left(\bigvee_{i=1}^{n} (\Sigma^i KU_{\mathbb{Q}})^{\vee \binom{n}{i}}\right) \cong \bigwedge_{\substack{KU\\ i=1}}^{n} F(\Sigma^i KU_{\mathbb{Q}})^{\wedge_{KU} \binom{n}{i}}.$$

Using (7.18), we continue:

$$\bigwedge_{\substack{KU\\ i=1}}^{n} F(\Sigma^i KU_{\mathbb{Q}})^{\wedge_{KU} \binom{n}{i}} \simeq \bigwedge_{\substack{KU\\ i=1}}^{n} (S^i \otimes KU)^{\wedge_{KU} \binom{n}{i}}.$$

Using Theorem 7.20,

$$\bigwedge_{\substack{KU\\ i=1}}^{n} (S^i \otimes KU)^{\wedge_{KU} \binom{n}{i}} \simeq \bigwedge_{\substack{KU\\ i=1}}^{n} KU[K(\mathbb{Z}, i+2)]^{\wedge_{KU} \binom{n}{i}} \cong KU\left[\prod_{i=1}^{n} K(\mathbb{Z}, i+2)^{\times \binom{n}{i}}\right]$$

which is equivalent to $T^n \otimes KU$ by Theorem 6.6. □

Remark 6.11. We might be tempted to prove the previous theorem more directly, arguing from the weak equivalence of spaces

$$\Sigma T^n \simeq \Sigma \bigvee_{i=1}^{n} (S^i)^{\vee \binom{n}{i}}. \tag{6.12}$$

However, we do not know a priori whether this guarantees that $T^n \otimes KU$ is equivalent to $\bigvee_{i=1}^{n} (S^i)^{\vee \binom{n}{i}} \otimes KU$ (which we can easily compute using the description from Theorem 7.20 of $S^i \otimes KU$ for all $i \geq 1$ and the fact that $- \otimes KU$ preserves coproducts). Indeed, there are counterexamples to the statement that if A is a commutative $\mathbb{S}$-algebra, then $X \otimes A \simeq Y \otimes A$ provided $\Sigma X \simeq \Sigma Y$ [DT16]. After having proved the theorem, though, we have that KU does satisfy this for the special case of (6.12). We are led to ask ourselves the question, as [DT16] did for $A = H\mathbb{F}_p$, of whether more generally KU is such that

$X \otimes KU \simeq Y \otimes KU$ provided $\Sigma X \simeq \Sigma Y$. More ambitiously, it would be interesting to find conditions on any commutative $\mathbb{S}$-algebra A that guarantee this property.

6.2. **The augmentation ideal.** We first need a generalization of Proposition 5.14:

Proposition 6.13. *Let $r \geq 3$. There are weak equivalences of KU-modules*

$$KU \wedge K(\mathbb{Z}, r) \simeq \begin{cases} \Sigma KU_{\mathbb{Q}} & \text{if } r \text{ is odd,} \\ \bigvee_{m\geq 1} KU_{\mathbb{Q}} & \text{if } r \text{ is even.} \end{cases}$$

Proof. When r is odd, the proof of Proposition 5.14 works just as well, and when r is even it gives us

$$KU \wedge K(\mathbb{Z}, r) \simeq KU_{\mathbb{Q}} \wedge K(\mathbb{Z}, r)_{\mathbb{Q}}.$$

So let r be even. As noted in Section 5.3, $K(\mathbb{Z}, r)_{\mathbb{Q}} \simeq \Omega S^{r+1}_{\mathbb{Q}}$. Now we use the James splitting which says that, for X a connected pointed CW-complex, $\Sigma\Omega\Sigma X \simeq \Sigma \bigvee_{m\geq 1} X^{\wedge m}$. Therefore, $\Sigma^\infty\Omega\Sigma X \simeq \Sigma^\infty \bigvee_{m\geq 1} X^{\wedge m}$. Rationalizing it and applying it to $X = S^r$, we obtain

$$\Sigma^\infty K(\mathbb{Z}, r)_{\mathbb{Q}} \simeq \Sigma^\infty \Omega S^{r+1}_{\mathbb{Q}} \simeq \Sigma^\infty \bigvee_{m\geq 1} S^{rm}_{\mathbb{Q}}.$$

Since r is even, Bott periodicity gives the result. $\square$

Corollary 6.14. *The augmentation ideal $\overline{THH}^n(KU)$ is rational.*

Proof. The expression (6.7) gives, after splitting off the units of the spherical group rings, a weak equivalence of KU-modules

(6.15) $THH^n(KU) \simeq KU \wedge (\mathbb{S} \vee \Sigma^\infty[K(\mathbb{Z}, a_1 + 2)]) \wedge \cdots \wedge (\mathbb{S} \vee \Sigma^\infty[K(\mathbb{Z}, a_{2^n-1} + 2)])$.

Observe that if T is a rational $\mathbb{S}$-module and X is any $\mathbb{S}$-module, then $T \wedge X \simeq T \wedge X_{\mathbb{Q}}$. Distributing the terms in the previous expression and applying Proposition 6.13 gives the result. $\square$

6.2.1. *The homotopy algebra $\overline{THH}^n_*(KU)$.* From what we have just observed, we have that $\overline{THH}^n(KU)$ is a non-unital commutative $KU_{\mathbb{Q}}$-algebra. Thus, its homotopy groups are a non-unital commutative $\mathbb{Q}[t^{\pm 1}]$-algebra, which we now aim to describe.

Since THH commutes with rationalization, we get a weak equivalence

$$\overline{THH}^n(KU) \xrightarrow{\sim} \overline{THH}^n(KU_{\mathbb{Q}})$$

of non-unital commutative KU-algebras. We aim to describe

$$\overline{THH}^n_*(KU) \cong \overline{THH}^n_*(KU_{\mathbb{Q}}).$$

We will describe the latter. To do so, we look at $THH^n_*(KU_{\mathbb{Q}})$.

By rationalizing (6.7), we obtain a weak equivalence of commutative $KU_{\mathbb{Q}}$-algebras

$$THH^n(KU_{\mathbb{Q}}) \simeq KU_{\mathbb{Q}} \wedge K(\mathbb{Z}, a_1 + 2)_+ \wedge \cdots \wedge K(\mathbb{Z}, a_{2^n-1} + 2)_+.$$

Its homotopy is isomorphic to its rational homology, and rational homology satisfies a Künneth isomorphism. By using the identification of the rationalized Eilenberg-Mac Lane spaces of Section 5.3, we obtain

Proposition 6.16. *There is an isomorphism of commutative $\mathbb{Q}[t^{\pm 1}]$-algebras*

$$THH_*^n(KU_{\mathbb{Q}}) \cong \mathbb{Q}[t^{\pm 1}] \otimes \bigotimes_{a_i \text{ odd}} E(\sigma^i t) \otimes \bigotimes_{a_j \text{ even}} \mathbb{Q}[\sigma^j t] \tag{6.17}$$

where $|\sigma^r t| = a_r + 2$ and $i, j \in \{1, \dots, 2^n - 1\}$.

For example,

$$THH_*^2(KU_{\mathbb{Q}}) \cong \mathbb{Q}[t^{\pm 1}] \otimes E(\sigma t) \otimes E(\sigma t) \otimes \mathbb{Q}[\sigma^2 t]$$

with $|\sigma t| = 3$ and $|\sigma^2 t| = 4$.

We can recognize the expression (6.17) as an iterated Hochschild homology algebra:

$$THH_*^n(KU_{\mathbb{Q}}) \cong HH_*^{\mathbb{Q},n}(\mathbb{Q}[t^{\pm 1}]). \tag{6.18}$$

Indeed, $HH_*^{\mathbb{Q}}(\mathbb{Q}[t^{\pm 1}]) \cong \mathbb{Q}[t^{\pm 1}] \otimes E(\sigma t)$, and $HH_*^{\mathbb{Q}}(E(\sigma t)) \cong E(\sigma t) \otimes \mathbb{Q}[\sigma^2 t]$. These Hochschild homology calculations are classical and can be found e.g. in [MS93, Section 2] and [AR05, 2.4]. We use that localization commutes with Hochschild homology [Wei94, Theorem 9.1.8(3)]. Also note that in general, the Hochschild homology of an exterior algebra is isomorphic to the tensor product of this same exterior algebra with a divided power algebra, but over $\mathbb{Q}$ such algebras are polynomial.

We can also arrive at such an iterated Hochschild homology expression by a spectral sequence computation in rational homology. First, note that if A is a rational commutative $\mathbb{S}$-algebra, then there is a weak equivalence $THH(A) \xrightarrow{\sim} THH^{H\mathbb{Q}}(A)$. Indeed, this can be checked simplicially, the multiplication map $H\mathbb{Q} \wedge H\mathbb{Q} \to H\mathbb{Q}$ being a weak equivalence.

There are strongly convergent Bökstedt spectral sequences [EKMM97, IX.1.9], [AR05, 4.1]

$$E^2_{p,q}(n) = HH^{\mathbb{Q}}_{p,q}(H\mathbb{Q}_*(THH^{H\mathbb{Q},n-1}(KU_{\mathbb{Q}}))) \Rightarrow H\mathbb{Q}_{p+q}(THH^{H\mathbb{Q},n}(KU_{\mathbb{Q}}))$$

which we can express as

$$E^2_{p,q}(n) = HH^{\mathbb{Q}}_{p,q}(THH_*^{n-1}(KU_{\mathbb{Q}})) \Rightarrow THH^n_{p+q}(KU_{\mathbb{Q}}).$$

These are spectral sequences of commutative $\pi_*(KU_{\mathbb{Q}}) \cong \mathbb{Q}[t^{\pm 1}]$-algebras, and by induction on n they collapse, since the algebra generators are in filtration degree 0 and 1. Thus we obtain an isomorphism $E^2_{p,q}(n) \cong HH^{\mathbb{Q},n}_{p,q}(\mathbb{Q}[t^{\pm 1}])$.

Denote by $\overline{HH}_*^{\mathbb{Q},n}(B)$ the kernel of the augmentation $HH_*^{\mathbb{Q},n}(B) \to B$. In conclusion,

Theorem 6.19. *There is an isomorphism of non-unital commutative $\mathbb{Q}[t^{\pm 1}]$-algebras*

$$\overline{THH}_*^n(KU) \cong \overline{HH}_*^{\mathbb{Q},n}(\mathbb{Q}[t^{\pm 1}]).$$

Of course, this is also the kernel of the augmentation of the right-hand side of (6.17) over $\mathbb{Q}[t^{\pm 1}]$, but alas, we do not see a slick notational device for it.

7. $\Sigma Y \otimes KU$

In this section, we evaluate the commutative KU-algebra $\Sigma Y \otimes KU$ when Y is a based CW-complex, by comparing it with $Y \otimes_{KU} (S^1 \otimes KU)$. We are very grateful to Bjørn Dundas for suggesting this line of argument.

Recall that if R is a commutative $\mathbb{S}$-algebra, the category R-**CAlg** is tensored over **Top** [EKMM97, VII.2.9]. If $A \in R$-**CAlg**, then the tensor $S^1 \otimes_R A$ is naturally isomorphic to $THH^R(A)$ as a commutative augmented A-algebra [MSV97], [EKMM97, IX.3.3], [AR05, Section 3]. Therefore, in this section we will identify $S^1 \otimes_R A$ and $THH^R(A)$ without further notice.

7.1. **The morphism ν.** Let $\mathcal{C}$ be a category enriched and tensored over **Top**. Denote its tensor by $\otimes$. Fix a pointed space (Z, z_0). We denote by ν^Z the natural transformation

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "Z\otimes -"'] \arrow[r, phantom, "\Downarrow\nu^Z"] & \mathcal{C} \end{tikzcd} \tag{7.1}$$

whose component in $C \in \mathcal{C}$ is given by

$$\nu^Z_C := \eta^C_Z(z_0) : C \to Z \otimes C. \tag{7.2}$$

Here $\eta^C_Z : Z \to \mathcal{C}(C, Z \otimes C)$ is the unit at Z of the adjunction

$$\begin{tikzcd} \mathbf{Top} \arrow[r, bend left, "-\otimes C"] & \mathcal{C}. \arrow[l, bend left, "\mathcal{C}(C,-)"] \end{tikzcd} \tag{7.3}$$

Let us now highlight the naturality properties of ν^Z_C at C and at Z. Let $\varphi : C \to C'$ be a morphism in $\mathcal{C}$. The naturality of the isomorphism

$$\mathcal{C}(Z \otimes C, Z \otimes -) \cong \mathbf{Top}(Z, \mathcal{C}(C, Z \otimes -))$$

gives the commutativity of the following diagram

$$\begin{tikzcd} C \arrow[r, "\nu^Z_C"] \arrow[d, "\varphi"'] & Z \otimes C \arrow[d, "\mathrm{id}\otimes\varphi"] \\ C' \arrow[r, "\nu^Z_{C'}"'] & Z \otimes C' \end{tikzcd} \tag{7.4}$$

Let $u : Z \to Z'$ be a morphism of based spaces. The naturality of η^C gives the commutativity of the following diagram.

$$\begin{tikzcd} C \arrow[r, "\nu^Z_C"] \arrow[dr, "\nu^{Z'}_C"'] & Z \otimes C \arrow[d, "u\otimes\mathrm{id}"] \\ & Z' \otimes C \end{tikzcd} \tag{7.5}$$

Example 7.6. Let $\mathcal{C} = \mathbf{Top}_*$ be the category of pointed objects in **Top**. It is tensored over **Top**: if $X \in \mathbf{Top}_*$ and $Y \in \mathbf{Top}$, then $Y \otimes X$ is defined as $Y_+ \wedge X$. When (Y, y_0) is pointed, we denote by

$$n^Y_X : X \to Y_+ \wedge X \tag{7.7}$$

the map ν^Y_X of (7.2) applied to $\mathcal{C} = \mathbf{Top}_*$. More explicitely, the map n^Y_X takes X to the copy of X lying over y_0 in $Y_+ \wedge X$.

7.2. **In commutative algebras.** Let R be a commutative $\mathbb{S}$-algebra. Let A be a commutative R-algebra and (X, x_0) be a based space. The map (7.2) in this scenario is a map of commutative R-algebras

$$\nu^X_A : A \to X \otimes_R A$$

which gives $X \otimes_R A$ the structure of a commutative A-algebra. In particular, when $X = S^1$, this is the usual structure of an A-algebra of $THH^R(A)$.

Now, take $R = \mathbb{S}$ and $A = KU$. Let (Y, y_0) be a based space. We use the symbol $\otimes$ to denote the tensor of $\mathbb{S}$-$\mathbf{CAlg}$ over $\mathbf{Top}$. The following diagram in KU-$\mathbf{CAlg}$ commutes. Here the map $e : S^1 \to *$ collapses the circle into its basepoint, and we have identified $F(* \wedge KU_{\mathbb{Q}})$ and $* \otimes KU$ with KU.

$$\begin{tikzcd}
Y \otimes_{KU} F(S^1 \wedge KU_{\mathbb{Q}}) \arrow[d, "\mathrm{id}\otimes\tilde{f}", "\sim"'] & F(S^1 \wedge KU_{\mathbb{Q}}) \arrow[l, "\nu^Y_{F(S^1\wedge KU_{\mathbb{Q}})}"'] \arrow[r, "F(e\wedge \mathrm{id})"] \arrow[d, "\tilde{f}", "\sim"'] & KU \arrow[d, "\mathrm{id}"] \\
Y \otimes_{KU} (S^1 \otimes KU) & S^1 \otimes KU \arrow[l, "\nu^Y_{S^1\otimes KU}"] \arrow[r, "e\otimes\mathrm{id}"] & KU
\end{tikzcd} \tag{7.8}$$

Indeed, the right square commutes because the $\tilde{f}$ of Theorem 5.10 is a morphism of augmented KU-algebras, and the commutativity of the left square is an application of the commutativity of (7.4). Note that $\mathrm{id} \otimes \tilde{f}$ is a weak equivalence because $Y \otimes_{KU} -$ is a left Quillen functor, assuming Y is a CW-complex.

We will now identify the members of the left column.

Proposition 7.9. *Let (X, x_0) and (Y, y_0) be based spaces, and let A be a commutative R-algebra.*

(1) There is an isomorphism of commutative A-algebras

$$Y \otimes_A (X \otimes_R A) \cong (Y_+ \wedge X) \otimes_R A$$

where $\otimes_R$ (resp. $\otimes_A$) denotes the tensoring of R-$\mathbf{CAlg}$ (resp. A-$\mathbf{CAlg}$) over $\mathbf{Top}$.

Moreover, the isomorphism makes the following diagram in A-$\mathbf{CAlg}$ commute. The morphism $n^Y_X : X \to Y_+ \wedge X$ was defined in (7.7).

$$\begin{tikzcd}
X \otimes_R A \arrow[r, "\nu^Y_{X\otimes_R A}"] \arrow[dr, "n^Y_X\otimes\mathrm{id}"'] & Y \otimes_A (X \otimes_R A) \arrow[d, "\cong"] \\
& (Y_+ \wedge X) \otimes_R A
\end{tikzcd} \tag{7.10}$$

(2) Let M be an A-module. Let $F : A$-$\mathbf{Mod} \to A$-$\mathbf{CAlg}$ be the free commutative algebra functor. There is an isomorphism

$$Y \otimes_A F(X \wedge M) \cong F(Y_+ \wedge X \wedge M)$$

making the following diagram commute.

$$\begin{tikzcd} F(X \wedge M) \arrow[r, "\nu^Y_{F(X\wedge M)}"] \arrow[dr, "F(n^Y_X \wedge \mathrm{id})"'] & Y \otimes_A F(X \wedge M) \arrow[d, "\cong"] \\ & F(Y_+ \wedge X \wedge M) \end{tikzcd}$$

*In the expression $Z \wedge M$ for a based space Z we are using the standard tensoring of A-**Mod** over $\mathbf{Top}_*$, i.e. $Z \wedge M = \Sigma^\infty Z \wedge M$.*

Proof. (1) Let B be a commutative A-algebra with unit $\varphi : A \to B$. Using the defining adjunction for $Y \otimes_A -$, we get a homeomorphism

(7.11) $$A\text{-}\mathbf{CAlg}(Y \otimes_A (X \otimes_R A), B) \cong \mathbf{Top}(Y, A\text{-}\mathbf{CAlg}(X \otimes_R A, B)).$$

The morphisms of commutative A-algebras $X \otimes_R A \to B$ are the morphisms of commutative R-algebras $g : X \otimes_R A \to B$ making the following diagram commute:

$$\begin{tikzcd} & A \arrow[dl, "\nu^X_A"'] \arrow[dr, "\varphi"] & \\ X \otimes_R A \arrow[rr, "g"'] & & B. \end{tikzcd}$$

Recalling the definition of ν, this means that

(7.12) $$g \circ \eta^A_X(x_0) = \varphi.$$

The adjoint map of g by the defining adjunction of $- \otimes_R A$ is the map in **Top**

(7.13) $$X \xrightarrow{\eta^A_X} R\text{-}\mathbf{CAlg}(A, X \otimes_R A) \xrightarrow{g_*} R\text{-}\mathbf{CAlg}(A, B).$$

Let the space $R\text{-}\mathbf{CAlg}(A, B)$ be pointed by $\varphi : A \to B$. The condition (7.12) on the map g is then translated to the adjoint (7.13) by stating that it is a pointed map, i.e. it takes x_0 to φ. Thus, continuing (7.11),

(7.14) $$\mathbf{Top}(Y, A\text{-}\mathbf{CAlg}(X \otimes_R A, B)) \cong \mathbf{Top}(Y, U\mathbf{Top}_*(X, R\text{-}\mathbf{CAlg}(A, B))),$$

where $U : \mathbf{Top}_* \to \mathbf{Top}$ is the functor forgetting the basepoint. It is the right adjoint to the functor $(-)_+ : \mathbf{Top} \to \mathbf{Top}_*$ which adds a disjoint basepoint, so we continue:

$$\mathbf{Top}(Y, U\mathbf{Top}_*(X, R\text{-}\mathbf{CAlg}(A, B))) \cong U\mathbf{Top}_*(Y_+, \mathbf{Top}_*(X, R\text{-}\mathbf{CAlg}(A, B))).$$

Since $\mathbf{Top}_*(X, -) : \mathbf{Top}_* \to \mathbf{Top}_*$ is the right adjoint to $- \wedge X$, we get:

$$U\mathbf{Top}_*(Y_+, \mathbf{Top}_*(X, R\text{-}\mathbf{CAlg}(A, B))) \cong U\mathbf{Top}_*(Y_+ \wedge X, R\text{-}\mathbf{CAlg}(A, B)).$$

By the same argument proving (7.14), we get

$$U\mathbf{Top}_*(Y_+ \wedge X, R\text{-}\mathbf{CAlg}(A, B)) \cong A\text{-}\mathbf{CAlg}((Y_+ \wedge X) \otimes_R A, B).$$

In conclusion, we have a homeomorphism

$$A\text{-}\mathbf{CAlg}(Y \otimes_A (X \otimes_R A), B) \cong A\text{-}\mathbf{CAlg}((Y_+ \wedge X) \otimes_R A, B),$$

and the Yoneda lemma finishes the proof.

The isomorphism was established using a chain of adjunctions. Following this chain, one observes that both n^Y_X and $\nu^Y_{X \otimes_R A}$, which are defined via units of adjunctions by analogous procedures, make the diagram (7.10) commute.

(2) The functor F is defined via a *continuous* monad in A-**Mod** (i.e. it is enriched over **Top**), see [EKMM97, proof of VII.2.9]. Therefore, the functor F preserves tensors over **Top**, so we get the desired isomorphism. □

Applying the previous proposition to $R = \mathbb{S}$, $A = KU$, $X = S^1$ and $M = KU_{\mathbb{Q}}$, the diagram (7.8) can be replaced with the following one.

(7.15)
$$\begin{tikzcd} F(Y_+ \wedge S^1 \wedge KU_{\mathbb{Q}}) \arrow[d, "\sim"] & F(S^1 \wedge KU_{\mathbb{Q}}) \arrow[l, "F(n^Y_{S^1} \wedge \mathrm{id})"'] \arrow[r, "F(e \wedge \mathrm{id})"] \arrow[d, "\sim"', "\tilde{f}"] & KU \arrow[d, "\mathrm{id}"] \\ (Y_+ \wedge S^1) \otimes KU & S^1 \otimes KU \arrow[l, "n^Y_{S^1} \otimes \mathrm{id}"'] \arrow[r, "e \otimes \mathrm{id}"] & KU \end{tikzcd}$$

When Y is a based CW-complex, the left map is a weak equivalence.

Now, note that the following is a pushout square of based or unbased spaces.

$$\begin{tikzcd} S^1 \arrow[r, "e"] \arrow[d, "n^Y_{S^1}"'] & * \arrow[d] \\ Y_+ \wedge S^1 \arrow[r] & Y \wedge S^1 \end{tikzcd}$$

Since the functors $- \otimes KU : \mathbf{Top} \to KU\text{-}\mathbf{CAlg}$ and $F(- \wedge KU_{\mathbb{Q}}) : \mathbf{Top}_* \to KU\text{-}\mathbf{CAlg}$ are left adjoints, they preserve pushouts, hence we get an induced map

$$\tau_Y : F(Y \wedge S^1 \wedge KU_{\mathbb{Q}}) \to (Y \wedge S^1) \otimes KU.$$

This is the component in Y of a natural transformation

(7.16)
$$\begin{tikzcd} \mathbf{Top}_* \arrow[r, bend left, "F(- \wedge S^1 \wedge KU_{\mathbb{Q}})"{name=U}] \arrow[r, bend right, "(- \wedge S^1) \otimes KU"'{name=D}] & KU\text{-}\mathbf{CAlg} \arrow[from=U, to=D, phantom, "\Downarrow\tau"] \end{tikzcd}$$

as follows from the naturality of $n^Y_{S^1}$ in Y (7.5).

Suppose Y is a CW-complex. The three vertical maps of (7.15) are weak equivalences. The horizontal maps pointing left are cofibrations: indeed, $n^Y_{S^1}$ is a cofibration, $KU_{\mathbb{Q}}$ is a cofibrant KU-module (similarly as in Remark 5.17) so $KU_{\mathbb{Q}} \wedge -$ is left Quillen, F is left Quillen and $- \otimes KU$ is left Quillen. Moreover, all the objects are cofibrant in KU-**CAlg**. Therefore, as in any model category, the induced map of pushouts τ_Y is a weak equivalence. This proves the following theorem.

Theorem 7.17. *There is a weak equivalence of commutative KU-algebras*

$$\tau_Y : F(Y \wedge S^1 \wedge KU_{\mathbb{Q}}) \to (Y \wedge S^1) \otimes KU$$

natural in the based CW-complex Y.

This determines $\Sigma Y \otimes KU$ as the free commutative KU-algebra on the KU-module $\Sigma Y \wedge KU_{\mathbb{Q}}$, up to weak equivalence. In particular, we have a weak equivalence of commutative KU-algebras

(7.18)
$$F(\Sigma^n KU_{\mathbb{Q}}) \to S^n \otimes KU$$

for every $n \geq 1$.

As in Remark 5.17, the KU-modules $\Sigma^n KU_{\mathbb{Q}}$ are cofibrant for $n \geq 0$. Since F is a left Quillen functor, Bott periodicity implies that we have weak equivalences

$$S^n \otimes KU \xleftarrow{\sim} \begin{cases} F(\Sigma KU_{\mathbb{Q}}) & \text{if } n \text{ is odd,} \\ F(KU_{\mathbb{Q}}) & \text{if } n \text{ is even} \end{cases} \tag{7.19}$$

for every $n \geq 1$.

The weak equivalence (7.18) generalizes the expression of Theorem 5.10 for $THH(KU)$ via free commutative KU-algebras. The following generalizes the expression of Theorem 5.5 for $THH(KU)$ via Eilenberg-Mac Lane spaces.

Theorem 7.20. *Let $n \geq 1$. Then $S^n \otimes KU \simeq KU[K(\mathbb{Z}, n+2)]$ as commutative KU-algebras.*

Proof. In [HHL+16, Proof of Theorem 3.1] the authors attribute the following result to [Vee14]: if $\mathbb{S} \to A \to B$ are cofibrations of commutative $\mathbb{S}$-algebras, then

$$S^{n+1} \otimes_A B \simeq B^A(B, S^n \otimes_A B, B)$$

where the term on the right side is a two-sided bar construction. Here $\otimes_A$ denotes the tensor of commutative A-algebras over **Top**. Let us give a proof. Since the functor $- \otimes_A B$ is left Quillen, it preserves pushouts and cofibrations, so we have a pushout of commutative A-algebras where the arrows $S^n \otimes_A B \to D^{n+1} \otimes_A B$ are cofibrations:

$$\begin{tikzcd} S^n \otimes_A B \arrow[r, tail] \arrow[d, tail] & D^{n+1} \otimes_A B \arrow[d] \\ D^{n+1} \otimes_A B \arrow[r] & S^{n+1} \otimes_A B. \end{tikzcd}$$

Therefore,

$$\begin{aligned} S^{n+1} \otimes_A B &\cong (D^{n+1} \otimes_A B) \wedge_{S^n \otimes_A B} (D^{n+1} \otimes_A B) \\ &\simeq B^A(D^{n+1} \otimes_A B, S^n \otimes_A B, D^{n+1} \otimes_A B) \\ &\simeq B^A(B, S^n \otimes_A B, B) \end{aligned}$$

where the weak equivalence in the middle is an application of [EKMM97, IX.2.3], and the last one is an application of [EKMM97, VII.7.2].

We use this to prove the result by induction. The result is true for $n = 1$ (Theorem 5.5); suppose it is true for some $n \geq 1$. Then

$$\begin{aligned} S^{n+1} \otimes KU &\simeq B^{\mathbb{S}}(KU, KU[K(\mathbb{Z}, n+2)], KU) \\ &\simeq B^{\mathbb{S}}(KU, KU, KU) \wedge B^{\mathbb{S}}(\mathbb{S}, \mathbb{S}[K(\mathbb{Z}, n+2)], \mathbb{S}) \\ &\simeq KU \wedge \mathbb{S}[K(\mathbb{Z}, n+3)] = KU[K(\mathbb{Z}, n+3)]. \end{aligned}$$

Here we have used that $B^{\mathbb{S}}(\mathbb{S}, \mathbb{S}[G], \mathbb{S}) \cong \mathbb{S}[BG]$ for G a topological commutative monoid. This result is proven in the same fashion as Proposition 4.4, which deals with the analogous result for the *cyclic* bar construction. ☐

Remark 7.21. In Remark 5.6 we remarked that KU behaves like a Thom spectrum to the eyes of topological Hochschild homology. Comparing Theorems 6.6 and 7.20 with [Sch11, Theorem 1.1], we see that, more generally, KU behaves like a Thom spectrum to the eyes of $X \otimes -$ when X is a torus or a sphere. See also Remark 8.4.

8. Topological André-Quillen homology of KU

If $A \to B$ is a morphism of commutative $\mathbb{S}$-algebras, one can define its *cotangent complex* $\Omega_{B|A} \in B$-$\mathbf{Mod}$, also known as its *topological André-Quillen* B-module, $TAQ(B|A)$, see [Bas99]. We adopt the latter notation. When $A = \mathbb{S}$, we delete it from the notation.

Theorem 8.1. *There is a weak equivalence of KU-modules*

$$TAQ(KU) \simeq KU \wedge \mathbf{K}(\mathbb{Z}, \mathbf{2}).$$

Here $\mathbf{K}(\mathbb{Z}, \mathbf{2})$ is the $\mathbb{S}$-module associated to the topological abelian group $K(\mathbb{Z}, 2)$: it is a model for $\Sigma^2 H\mathbb{Z}$. As explained in [BM05] before Theorem 5, for a topological abelian group G there is an $\mathbb{S}$-module associated to G whose zeroth space is G. We denote it by $\mathbf{G}$. More generally, we denote by $\mathbf{X}$ the $\mathbb{S}$-module associated to an E_∞-space X whose zeroth space is the group completion of X.

In the next proof we will use the localization of a module, which we have not used before. For the purposes of this section, if R is a commutative $\mathbb{S}$-algebra, $x \in \pi_* R$ and M is an R-module, then we define the $R[x^{-1}]$-module $M[x^{-1}]$ by $R[x^{-1}] \wedge_R M$ [EKMM97, VII.4].

Proof. Basterra [Bas99, Proposition 4.2] proved that, if $A \to B \to C$ are maps of cofibrant commutative $\mathbb{S}$-algebras, then

$$TAQ(B|A) \wedge_B C \to TAQ(C|A) \to TAQ(C|B)$$

is a homotopy cofiber sequence of C-modules. We apply it to

$$\mathbb{S} \to \mathbb{S}[K(\mathbb{Z}, 2)] \to \mathbb{S}[K(\mathbb{Z}, 2)][x^{-1}]$$

to obtain a homotopy cofiber sequence of KU-modules

$$TAQ(\mathbb{S}[K(\mathbb{Z}, 2)]) \wedge_{\mathbb{S}[K(\mathbb{Z},2)]} KU \to TAQ(KU) \to TAQ(KU|\mathbb{S}[K(\mathbb{Z}, 2)]). \tag{8.2}$$

Now, $TAQ(KU|\mathbb{S}[K(\mathbb{Z}, 2)])$ is trivial, since $\mathbb{S}[K(\mathbb{Z}, 2)] \to KU$ is a localization map [MM03, Remark 3.4]. Since $TAQ(\mathbb{S}[K(\mathbb{Z}, 2)]) \wedge_{\mathbb{S}[K(\mathbb{Z},2)]} KU \simeq TAQ(\mathbb{S}[K(\mathbb{Z}, 2)])[x^{-1}]$, the sequence (8.2) gives a weak equivalence of KU-modules

$$TAQ(\mathbb{S}[K(\mathbb{Z}, 2)])[x^{-1}] \xrightarrow{\sim} TAQ(KU). \tag{8.3}$$

But [BM05, Theorem 5] gives that if G is a topological abelian group, then there is a weak equivalence of $\mathbb{S}[G]$-modules

$$TAQ(\mathbb{S}[G]) \simeq \mathbb{S}[G] \wedge \mathbf{G}.$$

Taking $G = K(\mathbb{Z}, 2)$, localizing this equivalence at x and combining it with the map (8.3), we get a weak equivalence of KU-modules

$$KU \wedge \mathbf{K}(\mathbb{Z}, \mathbf{2}) \simeq TAQ(\mathbb{S}[K(\mathbb{Z}, 2)])[x^{-1}] \xrightarrow{\sim} TAQ(KU). \qquad \square$$

Remark 8.4. Compare this with the reformulation of [Kuh04, Page 230] and [Sch11, Page 164] of a result of [BM05]. It states that if $f : X \to BF$ is a map of ∞-loop spaces, then $TAQ(T(f)) \simeq T(f) \wedge \mathbf{X}$. Just as in Remark 7.21, the result for $TAQ(KU)$ coincides with the result we would obtain if we knew that KU was somehow the Thom spectrum of a map $K(\mathbb{Z}, 2) \to BU$.

8.1. **Some general TAQ-theory.** In order to say more, we first present some general theory about TAQ. Let R be a commutative $\mathbb{S}$-algebra, let $A \in R$-**CAlg** and $X \in \mathbf{Top}_*$. Then $X \otimes_R A \in A$-**CAlg**. Denote by $X \tilde{\otimes}_R A$ the cofiber of the unit map $A \to X \otimes_R A$: it is a non-unital commutative R-algebra. Denote by ρ the induced map on cofibers

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{id}"'] & X_+ \wedge A \arrow[r] \arrow[d, "\omega"'] & X \wedge A \arrow[d, dotted, "\rho"'] \\ A \arrow[r] & X \otimes_R A \arrow[r] & X \tilde{\otimes}_R A. \end{tikzcd}$$

Note that the upper cofiber sequence is induced from the cofiber sequence $S^0 \to X_+ \to X$ in $\mathbf{Top}_*$. Here the map ω is defined e.g. in [EKMM97, VII.2.11]: it is the image of the identity by the following map induced from the forgetful functor R-**CAlg** $\to$ R-**Mod**:

$$\begin{aligned} R\text{-}\mathbf{CAlg}(X \otimes_R A, X \otimes_R A) &\cong \mathbf{Top}(X, R\text{-}\mathbf{CAlg}(A, X \otimes_R A)) \\ &\to \mathbf{Top}(X, R\text{-}\mathbf{Mod}(A, X \otimes_R A)) \\ &\cong R\text{-}\mathbf{Mod}(X_+ \wedge A, X \otimes_R A). \end{aligned}$$

Remark 8.5. We can forget the unit of A and consider it as a non-unital commutative R-algebra. Then $R \vee A$ is an augmented commutative R-algebra. The category of augmented commutative R-algebras, R-$\mathbf{CAlg}^{\mathrm{aug}}$, is enriched and tensored over $\mathbf{Top}_*$ [Kuh04, Section 4]. Denote by $\odot_R$ its tensor. The object $X \tilde{\otimes}_R A$ can be identified with $X \odot_R (R \vee A)$, and the map which coincides with ρ under this identification is the image of the identity under the following map induced from the forgetful functor R-$\mathbf{CAlg}^{\mathrm{aug}} \to R$-**Mod**:

$$\begin{aligned} R\text{-}\mathbf{CAlg}^{\mathrm{aug}}(X \odot_R A, X \odot_R A) &\cong \mathbf{Top}_*(X, R\text{-}\mathbf{CAlg}^{\mathrm{aug}}(A, X \odot_R A)) \\ &\to \mathbf{Top}_*(X, R\text{-}\mathbf{Mod}(A, X \odot_R A)) \\ &\cong R\text{-}\mathbf{Mod}(X \wedge A, X \odot_R A). \end{aligned}$$

In particular, if $X, Y \in \mathbf{Top}_*$, then $\rho : X \wedge (Y \tilde{\otimes}_R A) \to X \tilde{\otimes}_R (Y \otimes_R A) \cong (X \wedge Y) \tilde{\otimes}_R A$. When $X = S^1$ and $Y = S^n$, this gives maps $\Sigma(S^n \tilde{\otimes}_R A) \to S^{n+1} \tilde{\otimes}_R A$, with adjoints

$$\tilde{\rho} : S^n \tilde{\otimes}_R A \to \Omega(S^{n+1} \tilde{\otimes}_R A).$$

It can be proven that
(8.6)

$$TAQ(A|R) \simeq \mathrm{hocolim}(\; A = S^0 \tilde{\otimes}_R A \xrightarrow{\tilde{\rho}} \Omega(S^1 \tilde{\otimes}_R A) \xrightarrow{\Omega\tilde{\rho}} \Omega^2(S^2 \tilde{\otimes}_R A) \xrightarrow{\Omega^2\tilde{\rho}} \cdots \;),$$

see [Kuh04, Page 230], [Sch11, Page 164], [BM02, Section 3], [BM05, Section 3].

8.2. **Application to KU.** Using the expression $S^n \otimes KU \simeq KU[K(\mathbb{Z}, n+2)]$ of Theorem 7.20, we obtain

$$S^n \tilde{\otimes} KU \simeq KU \wedge K(\mathbb{Z}, n+2). \tag{8.7}$$

We can now easily compute the homotopy groups of $TAQ(KU)$:

Proposition 8.8. *For $i \in \mathbb{Z}$,*

$$TAQ_i(KU) \cong \begin{cases} 0 & \text{if } i \text{ is odd}, \\ \mathbb{Q} & \text{if } i \text{ is even}. \end{cases}$$

Proof. Combining (8.6) and (8.7), we get:

$$TAQ_i(KU) \cong \operatorname{colim}(\pi_i(KU) \to \pi_{i+1}(KU \wedge K(\mathbb{Z},3)) \to \pi_{i+2}(KU \wedge K(\mathbb{Z},4)) \to \cdots).$$

Recall Proposition 6.13: $KU \wedge K(\mathbb{Z}, r)$ is weakly equivalent to $\Sigma KU_{\mathbb{Q}}$ when $r \geq 3$ is odd, and to $\bigvee_{m\geq 1} KU_{\mathbb{Q}}$ when r is even.

When i is odd, all terms in the colimit computing $TAQ_i(KU)$ except the first one are zero, so $TAQ_i(KU)$ is zero.

When i is even, we get

$$TAQ_i(KU) \cong \operatorname{colim}(\ \mathbb{Z} \longrightarrow \mathbb{Q} \xrightarrow{i_1} \bigoplus_{m\geq 1} \mathbb{Q} \xrightarrow{\pi_1} \mathbb{Q} \xrightarrow{i_1} \bigoplus_{m\geq 1} \mathbb{Q} \xrightarrow{\pi_1} \cdots\ ).$$

Therefore, $TAQ_i(KU) \cong \mathbb{Q}$. □

Corollary 8.9. *There is a weak equivalence of KU-modules $TAQ(KU) \simeq KU_{\mathbb{Q}}$.*

Proof. Recall that an $\mathbb{S}$-module with rational homotopy groups is weakly equivalent to its corresponding generalized Eilenberg-Mac Lane $\mathbb{S}$-module, see e.g. [Sch, Theorem 9.6]. Since both $TAQ(KU)$ and $KU_{\mathbb{Q}}$ have the same homotopy groups and they are rational, this implies the result. □

Remark 8.10. Compare to the Hochschild-Kostant-Rosenberg theorem in [MM03, Theorem 1.1]: if A is a connective smooth commutative $\mathbb{S}$-algebra, there is a weak equivalence of commutative A-algebras $F(\Sigma TAQ(A)) \simeq THH(A)$, where $F : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{CAlg}$ is the free commutative algebra functor. This theorem does not apply to KU since KU is not connective (we have not checked the smoothness condition), but the conclusion is true (Theorem 5.10). Just as in Remarks 5.6, 7.21 and 8.4, here is an example of a theorem that does not apply to KU because KU is not connective, but whose conclusion is nonetheless true. We speculate that there should be a version of the HKR theorem for E_∞-ring spectra which dispenses with the connectiveness hypothesis. I would like to thank Tomasz Maszczyk for asking me about the HKR theorem in relation to Theorem 5.10, thus inciting me to make these reflections.

References

[Ada69] J. F. Adams. Lectures on generalised cohomology. In *Category Theory, Homology Theory and their Applications, III (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Three)*, pages 1–138. Springer, Berlin, 1969.

[AGP02] Marcelo Aguilar, Samuel Gitler, and Carlos Prieto. *Algebraic topology from a homotopical viewpoint*. Universitext. Springer-Verlag, New York, 2002. Translated from the Spanish by Stephen Bruce Sontz.

[AHL10] Vigleik Angeltveit, Michael A. Hill, and Tyler Lawson. Topological Hochschild homology of ℓ and ko. *Amer. J. Math.*, 132(2):297–330, 2010.

[AR05] Vigleik Angeltveit and John Rognes. Hopf algebra structure on topological Hochschild homology. *Algebr. Geom. Topol.*, 5:1223–1290 (electronic), 2005.

[AR12] Christian Ausoni and John Rognes. Rational algebraic K-theory of topological K-theory. *Geom. Topol.*, 16(4):2037–2065, 2012.

[Art83] R. D. Arthan. Localization of stable homotopy rings. *Math. Proc. Cambridge Philos. Soc.*, 93(2):295–302, 1983.

[Aus05] Christian Ausoni. Topological Hochschild homology of connective complex K-theory. *Amer. J. Math.*, 127(6):1261–1313, 2005.

[Bas99] M. Basterra. André-Quillen cohomology of commutative S-algebras. *J. Pure Appl. Algebra*, 144(2):111–143, 1999.

[BCS10] Andrew J. Blumberg, Ralph L. Cohen, and Christian Schlichtkrull. Topological Hochschild homology of Thom spectra and the free loop space. *Geom. Topol.*, 14(2):1165–1242, 2010.

[BHM93] M. Bökstedt, W. C. Hsiang, and I. Madsen. The cyclotomic trace and algebraic K-theory of spaces. *Invent. Math.*, 111(3):465–539, 1993.

[BM02] Maria Basterra and Randy McCarthy. Γ-homology, topological André-Quillen homology and stabilization. *Topology Appl.*, 121(3):551–566, 2002.

[BM05] Maria Basterra and Michael A. Mandell. Homology and cohomology of E_∞ ring spectra. *Math. Z.*, 249(4):903–944, 2005.

[Bök85] Marcel Bökstedt. Topological Hochschild homology, 1985. Universität Bielefeld. Unpublished.

[Bre78] Lawrence Breen. Extensions du groupe additif. *Inst. Hautes Études Sci. Publ. Math.*, (48):39–125, 1978.

[Bry89] Jean-Luc Brylinski. Central localization in Hochschild homology. *J. Pure Appl. Algebra*, 57(1):1–4, 1989.

[CDD11] Gunnar Carlsson, Christopher L. Douglas, and Bjørn Ian Dundas. Higher topological cyclic homology and the Segal conjecture for tori. *Adv. Math.*, 226(2):1823–1874, 2011.

[DGM13] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy. *The local structure of algebraic K-theory*, volume 18 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2013.

[DT16] Bjørn Ian Dundas and Andrea Tenti. Higher hochschild homology is not a stable invariant. *arXiv:1612.05175*, December 2016.

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[FHT01] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. *Rational homotopy theory*, volume 205 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[HHL+16] Gemma Halliwell, Eva Höning, Ayelet Lindenstrauss, Birgit Richter, and Inna Zakharevich. Relative loday constructions and applications to higher thh-calculations. *arXiv:1609.02397*, September 2016.

[HM97] Lars Hesselholt and Ib Madsen. On the K-theory of finite algebras over Witt vectors of perfect fields. *Topology*, 36(1):29–101, 1997.

[Hön17] Eva Höning. The topological hochschild homology of algebraic K-theory of finite fields. *PhD Thesis, Université Paris 13*, 2017.

[JW85] David Copeland Johnson and W. Stephen Wilson. The Brown-Peterson homology of elementary p-groups. *Amer. J. Math.*, 107(2):427–453, 1985.

[Kuh04] Nicholas J. Kuhn. The McCord model for the tensor product of a space and a commutative ring spectrum. In *Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001)*, volume 215 of *Progr. Math.*, pages 213–236. Birkhäuser, Basel, 2004.

[Lin00] Ayelet Lindenstrauss. A relative spectral sequence for topological Hochschild homology of spectra. *J. Pure Appl. Algebra*, 148(1):77–88, 2000.

[Lod98] Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1998. Appendix E by María O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili.

[Lur] Jacob Lurie. Higher algebra. September 2017 version. Available at http://www.math.harvard.edu/~lurie/papers/HA.pdf.

[Mat17] Akhil Mathew. THH and base-change for Galois extensions of ring spectra. *Algebr. Geom. Topol.*, 17(2):693–704, 2017.

[McC01] John McCleary. *A user's guide to spectral sequences*, volume 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[Mil67] R. James Milgram. The bar construction and abelian H-spaces. *Illinois J. Math.*, 11:242–250, 1967.

[MM03] Randy McCarthy and Vahagn Minasian. HKR theorem for smooth S-algebras. *J. Pure Appl. Algebra*, 185(1-3):239–258, 2003.

[MP12] J. P. May and K. Ponto. *More concise algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2012. Localization, completion, and model categories.

[MS93] J. E. McClure and R. E. Staffeldt. On the topological Hochschild homology of bu. I. *Amer. J. Math.*, 115(1):1–45, 1993.

[MSV97] J. McClure, R. Schwänzl, and R. Vogt. $THH(R) \cong R \otimes S^1$ for E_∞ ring spectra. *J. Pure Appl. Algebra*, 121(2):137–159, 1997.

[RW80] Douglas C. Ravenel and W. Stephen Wilson. The Morava K-theories of Eilenberg-Mac Lane spaces and the Conner-Floyd conjecture. *Amer. J. Math.*, 102(4):691–748, 1980.

[Sch] Stefan Schwede. Symmetric spectra. Unfinished book project, available electronically from `www.math.uni-bonn.de/people/schwede/SymSpec-v3.pdf`.

[Sch11] Christian Schlichtkrull. Higher topological Hochschild homology of Thom spectra. *J. Topol.*, 4(1):161–189, 2011.

[Shi00] Brooke Shipley. Symmetric spectra and topological Hochschild homology. *K-Theory*, 19(2):155–183, 2000.

[Slo] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. Sequence A000120. URL:`https://oeis.org/A000120`.

[Sna79] Victor P. Snaith. Algebraic cobordism and K-theory. *Mem. Amer. Math. Soc.*, 21(221):vii+152, 1979.

[Sna81] Victor Snaith. Localized stable homotopy of some classifying spaces. *Math. Proc. Cambridge Philos. Soc.*, 89(2):325–330, 1981.

[SS14] Steffen Sagave and Christian Schlichtkrull. Virtual vector bundles and graded Thom spectra, 2014. arXiv:1410.4492.

[Sto18] Bruno Stonek. Graded multiplications on iterated bar constructions, 2018. To appear in *Proceedings of the Alpine Algebraic and Applied Topology Conference*, Contemporary Mathematics, AMS. arXiv:1702.02984v2.

[SVW00] R. Schwänzl, R. M. Vogt, and F. Waldhausen. Topological Hochschild homology. *J. London Math. Soc. (2)*, 62(2):345–356, 2000.

[Vee14] Torleif Veen. Detecting periodic elements in higher topological Hochschild homology. *arXiv:1312.5699v2*, March 2014.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[WG91] Charles A. Weibel and Susan C. Geller. Etale descent for Hochschild and cyclic homology. *Comment. Math. Helv.*, 66(3):368–388, 1991.

Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS, UMR 7539, 93430 Villetaneuse, France.

E-mail address: `stonek@math.univ-paris13.fr`