An approximate Jerusalem square whose side equals a Pell number

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Abstract

We take advantage of the properties of the Pell numbers to construct an integer version of the Jerusalem square fractal.

Keywords: Pell number, Jerusalem square, Jerusalem cube, fractal.

1 Introduction

Eric Baird [2] first introduced the Jerusalem square in 2011. This fractal object can be constructed as follows.

1. Start with a square.
2. Cut a cross through the square so that the corners then consist of four smaller scaled copies, of rank +1, of the original square, each pair of which being separated by a smaller square, of rank +2, centered along the edges of the original square. The scaling factor between the side length of the squares of consecutive rank is constrained to be constant.
3. Repeat the process on the squares of rank +1 and +2, see Figure 1.

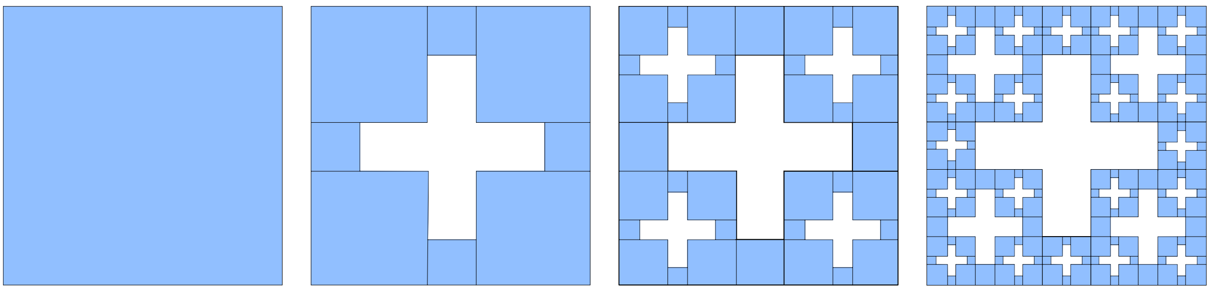


Figure 1: The iterative construction of the Jerusalem square.

Let n be a nonnegative integer, and let ℓ_n denote the side length of the square at the n-th iteration. Then,

$$\ell_n = 2\ell_{n+1} + \ell_{n+2}. \tag{1}$$

The scaling factor constraint implies

$$k = \frac{\ell_{n+1}}{\ell_n} = \frac{\ell_{n+2}}{\ell_{n+1}}. \tag{2}$$

Combining formulas (1) and (2), we obtain an irrational ratio $k = \sqrt{2} - 1$. This ratio suggests that the Jerusalem square cannot be built from a simple integer grid [1, 2].

However, a naive method is to consider a 5×5 square, and then remove a cross which consists of five unit squares as shown in Figure 2.

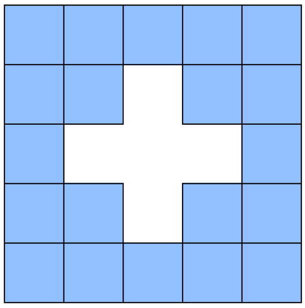


Figure 2: An integer grid approximation to the first iteration of the Jerusalem square.

We shall see in the next section that it is in fact a nice approximation of the actual fractal for the corresponding iteration. Indeed, the present paper is motivated by formulas (1), (2) and the observation of Figure 2. On the one hand, notice the similarities between the recurrence relation (1) and the definition of the Pell numbers A000129, and on the other hand, notice that the side lengths of the squares (of rank +1 and +2, as well as the original) in Figure 2 are exactly 1, 2 and 5, some of the first few terms of the Pell numbers.

2 Pell numbers in action

Firstly, recall that the Pell numbers are defined as the sequence of integers

$$p_0 = 0,\ p_1 = 1,\ \text{and } p_n = 2p_{n-1} + p_{n-2} \text{ for all } n \geq 2. \tag{3}$$

The first few Pell numbers are

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, \ldots \text{ (sequence A000129 in the OEIS [5]).}$$

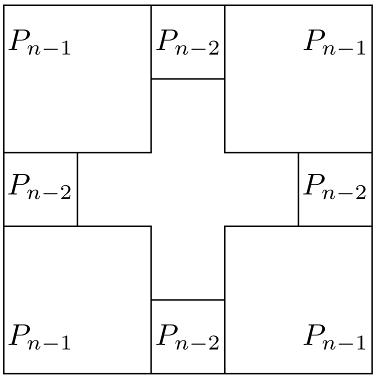
It is well-known that the integer ratio $\dfrac{p_n}{p_{n-1}}$ rapidly approach $1 + \sqrt{2}$ [4, p. 138].

Now, let us introduce the following informal notation.

Let P_n denote a $p_n \times p_n$ square whose edges result from the alignment of the squares P_{n-1}, P_{n-2} and P_{n-1} as illustrated in the following formula:

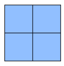
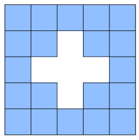
$P_0 := \varnothing$ (the square of side length 0), $P_1 :=$  (the unit square) and

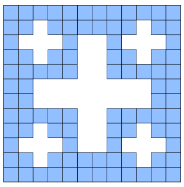
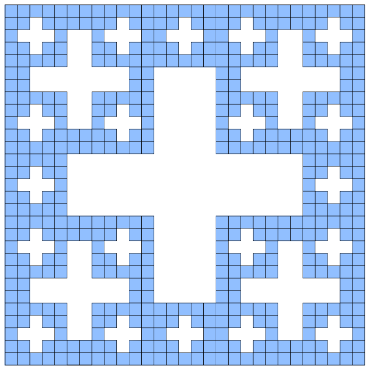
$$P_n := \begin{bmatrix} P_{n-1} & P_{n-2} & P_{n-1} \\ P_{n-2} & & P_{n-2} \\ P_{n-1} & P_{n-2} & P_{n-1} \end{bmatrix} =$$

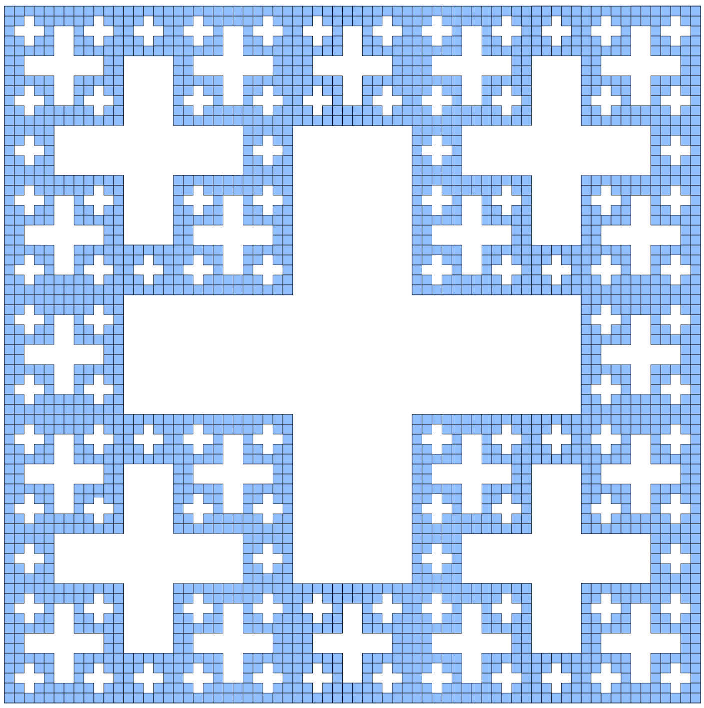


for all $n \geq 2$. (4)

The blank entry in the matrix representation in (4) is there to indicate the cross removal. For example, for $n = 2, 3, 4, 5, 6$ we have

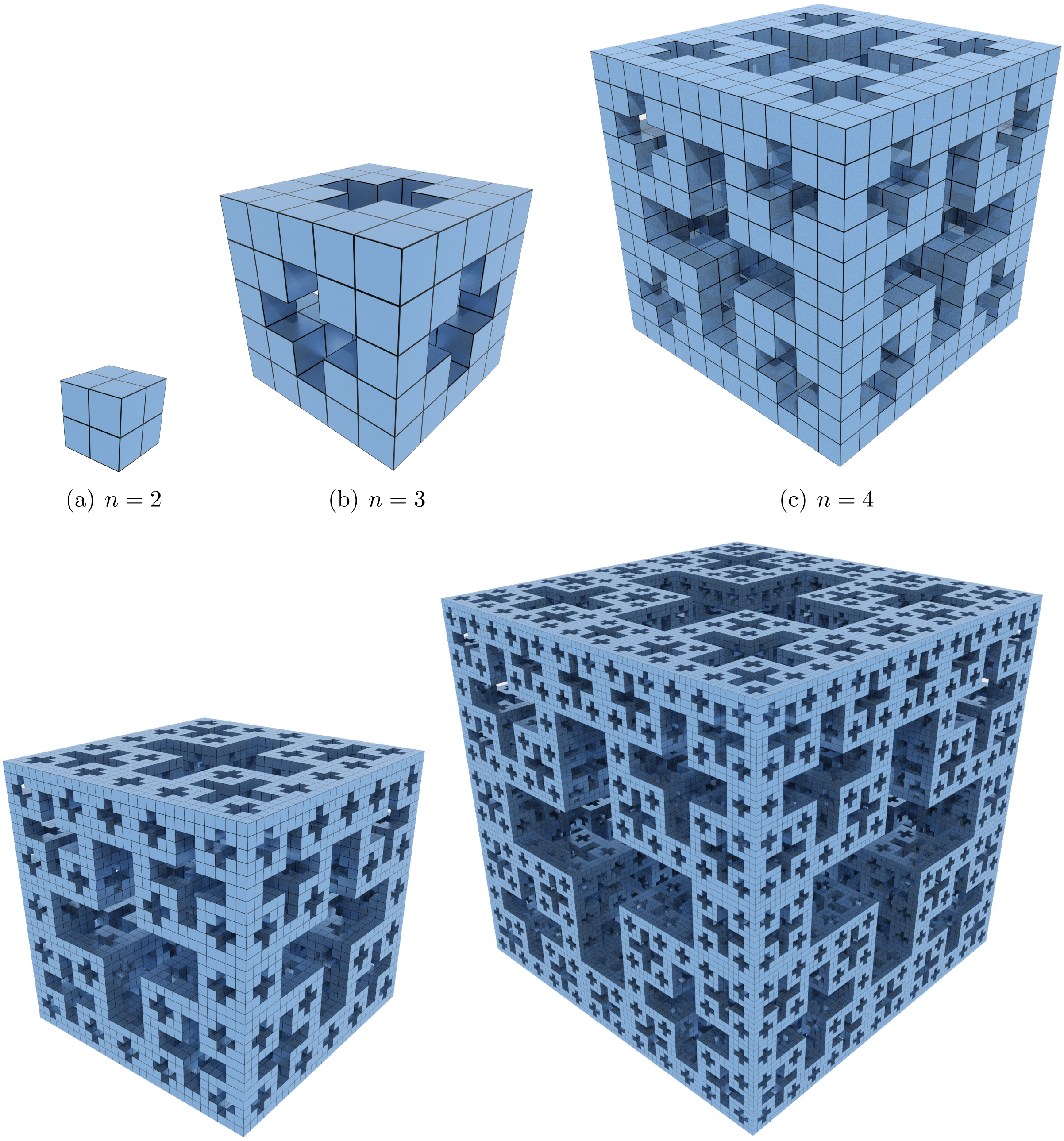
$$P_2 = \begin{bmatrix} P_1 & P_0 & P_1 \\ P_0 & & P_0 \\ P_1 & P_0 & P_1 \end{bmatrix} =$$ , $$P_3 = \begin{bmatrix} P_2 & P_1 & P_2 \\ P_1 & & P_1 \\ P_2 & P_1 & P_2 \end{bmatrix} =$$ ,

$$P_4 = \begin{bmatrix} P_3 & P_2 & P_3 \\ P_2 & & P_2 \\ P_3 & P_2 & P_3 \end{bmatrix} =$$ , $$P_5 = \begin{bmatrix} P_4 & P_3 & P_4 \\ P_3 & & P_3 \\ P_4 & P_3 & P_4 \end{bmatrix} =$$ ,

$$P_6 = \begin{bmatrix} P_5 & P_4 & P_5 \\ P_4 & & P_4 \\ P_5 & P_4 & P_5 \end{bmatrix} =$$ .

Since the square P_n is of length p_n, then by the property of the Pell numbers, the ratios $\frac{p_{n-1}}{p_n}$ and $\frac{p_{n-2}}{p_{n-1}}$ give a good approximation to the Jerusalem square ratio $\sqrt{2}-1$ when n is sufficiently large.

We can extend this method to the Jerusalem cube [1], and consider a construction of this three-dimensional case with the popular business card cube [3, p. 152]. For example, we see in Figure 3 the corresponding iterations for $n = 2, 3, 4, 5, 6$.



(a) $n = 2$ (b) $n = 3$ (c) $n = 4$

(d) $n = 5$ (e) $n = 6$

Figure 3: Some iterations of the approximation to the Jerusalem cube.

References

[1] Eric Baird, The Jerusalem Cube, http://alt-fractals.blogspot.fr/2011/08/jerusalem-cube.html, 2011.

[2] Eric Baird, The Jerusalem Square, http://alt-fractals.blogspot.com/2011/11/jerusalem-square.html, 2011.

[3] Thomas Hull, *Project Origami: Activities for Exploring Mathematics*, A.K Peters, Ltd., 2006.

[4] Thomas Koshy, *Pell and Pell–Lucas Numbers with Applications*, Springer, 2014.

[5] Neil J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, Published electronically at http://oeis.org, 2017.