HEREDITARY SEMIORDERS AND ENUMERATION OF SEMIORDERS BY DIMENSION

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ABSTRACT. In 2010, Bousquet-Mélou et al. defined sequences of nonnegative integers called ascent sequences and showed that the ascent sequences of length n are in one-to-one correspondence with the interval orders, i.e., the posets not containing the poset $\mathbf{2+2}$. Through the use of generating functions, this provided an answer to the longstanding open question of enumerating the (unlabeled) interval orders. A semiorder is an interval order having a representation in which all intervals have the same length. In terms of forbidden subposets, the semiorders exclude $\mathbf{2+2}$ and $\mathbf{1+3}$. The number of unlabeled semiorders on n points has long been known to be the nth Catalan number. However, describing the ascent sequences that correspond to the semiorders under the bijection of Bousquet-Mélou et al. has proved difficult. In this paper, we discuss a major part of the difficulty in this area: the ascent sequence corresponding to a semiorder may have an initial subsequence that corresponds to an interval order that is not a semiorder.

We define the hereditary semiorders to be those corresponding to an ascent sequence for which every initial subsequence also corresponds to a semiorder. We provide a structural result that characterizes the hereditary semiorders and use this characterization to determine the ordinary generating function for hereditary semiorders. We also use our characterization of hereditary semiorders and the characterization of semiorders of dimension 3 given by Rabinovitch to provide a structural description of the semiorders of dimension at most 2. From this description, we are able to determine the ordinary generating for the semiorders of dimension at most 2.

1. BACKGROUND AND MOTIVATION

In this article, we investigate the bijective relationship between interval orders and ascent sequences introduced by Bousquet-Mélou et al. in [2]. In that paper, the authors answered a classic open question by providing an enumeration of interval orders through a bijection with nonnegative sequences of integers known as ascent sequences. The number of semiorders has long been known to be given by the Catalan numbers, but no one has yet given a description of the subclass of ascent sequences associated to the semiorders by the bijection of Bousquet-Mélou et al. in terms of ascents. Most problematic is the fact that it is possible for an ascent sequence to correspond to a semiorder while some initial subsequence of that ascent sequence corresponds to an interval order that is *not* a semiorder. To address this, we define the class of hereditary semiorders as those for which every initial subsequence of the corresponding ascent sequence corresponds to a semiorder. The hereditary semiorders can also be nicely described in terms of their interval representation, and this structure further allows us to give a characterization of the semiorders of dimension 2 in terms of this structure. (Rabinovitch proved in [19] that all semiorders have dimension

at most 3 and characterized those of dimension 3, and our analysis uses that characterization.) With these structural results in hand, we are able to enumerate both the hereditary semiorders and the semiorders of dimension 2.

Interval orders, semiorders, and dimension. Before proceeding to our discussion of interval orders and semiorders, we require a couple of definitions that apply to all posets. For a poset $P = (X, \leq_P)$ and $x \in X$, the (open) *down set* of x, denoted by $D(x)$, is $\{y \in X : y <_P x\}$. Dually, the *up set* of x, denoted by $U(x)$ is $\{y \in X : y >_P x\}$. For n a positive integer, $\mathbf{n}$ denotes the totally ordered poset with n elements. If n and m are positive integers, then $\mathbf{n} + \mathbf{m}$ denotes the disjoint union of the posets $\mathbf{n}$ and $\mathbf{m}$. The posets $\mathbf{2} + \mathbf{2}$ and $\mathbf{1} + \mathbf{3}$ are depicted in Figure 1.

2 + 2

1 + 3

FIGURE 1. The posets $\mathbf{2} + \mathbf{2}$ and $\mathbf{1} + \mathbf{3}$.

We call a poset $P = (X, \leq_P)$ an *interval order* provided that for each $x \in X$ there exists a closed, bounded interval $I(x) = [\mathbb{l}(x), \mathbb{r}(x)]$ of $\mathbb{R}$ such that $x <_P y$ if and only if $\mathbb{r}(x) < \mathbb{l}(y)$, i.e., the interval of x lies completely to the left of the interval of y. The collection of intervals associated to P is called an *interval representation* of P (or just a *representation*). An interval order P is called a *semiorder* provided that P has an interval representation in which all intervals have the same (typically unit) length. The first appearance of what we today recognize as an interval order is in a paper by Wiener [24]. It wasn't until 1970, however, that the following theorem was established by Fishburn.

Theorem 1.1 (Fishburn [8])**.** *Let $P = (X, \leq_P)$ be a poset. The following are equivalent:*

  (1) *P is an interval order.*
  (2) *P does not contain $\mathbf{2} + \mathbf{2}$ as a subposet.*
  (3) *If $x <_P y$ and $z <_P w$, then $x <_P w$ or $z <_P y$.*
  (4) *The collection of down sets of elements of X is totally ordered by inclusion.*
  (5) *The collection of up sets of elements of X is totally ordered by inclusion.*

The characterization of semiorders was actually arrived at earlier in the form of a result in mathematical logic by Scott and Suppes.

Theorem 1.2 (Scott and Suppes [21])**.** *A poset P is a semiorder if and only if P contains neither $\mathbf{2} + \mathbf{2}$ nor $\mathbf{1} + \mathbf{3}$ as a subposet.*

In [11], Greenough not only showed that when $P = (X, \leq_P)$ is an interval order, the number of distinct down sets of elements of X is equal to the number of distinct up sets of elements of X but also gave an algorithm for generating a unique interval representation using the smallest number of endpoints possible. Although they did not discuss it in this manner, the bijection of Bousquet-Mélou et al. between ascent sequences and interval orders (described in the next subsection) gives rise to such a representation, and such a representation will be central to our arguments. Thus, we briefly describe the algorithm and

its critical properties here. To produce the representation, list the down sets of elements of X as $D_0 \subsetneq D_1 \subsetneq D_2 \subsetneq \cdots \subsetneq D_{t-1}$, where t is the number of distinct down sets (and hence up sets). Also list the up sets of elements of X as $U_0 \supsetneq U_1 \supsetneq U_2 \supsetneq \cdots \supsetneq U_{t-1}$. For $x \in X$, we define $I(x) = [i, j]$ where $D(x) = D_i$ and $U(x) = U_j$. Note that this may map distinct elements $x, y \in X$ to the same interval, which is allowed by our definition of interval representation. This happens if and only if $D(x) = D(y)$ and $U(x) = U(y)$. In this case, we say that x and y have *duplicated holdings*. A poset in which no two elements have duplicated holdings is said to have *no duplicated holdings*, sometimes abbreviated NODH.

In this article, we shall refer to the representation produced by the algorithm described above as the *minimal endpoint representation* of an interval order $P = (X, \leq_P)$. Because of the manner in which the minimal endpoint representation is created, we know that for each $i \in \{0, \ldots, t-1\}$, there exist $x, y \in X$ such that $l(x) = i$ and $r(y) = i$. That is, in a minimal endpoint representation, every integer from 0 to $t-1$ occurs as both a left endpoint and a right endpoint.

Example 1.3. To illustrate the algorithm for finding the minimal endpoint representation of an interval order, consider the poset shown in Figure 2. The down sets and up sets as ordered by the algorithm are listed below.

$$
\begin{aligned}
D_0 &= \{\} & U_0 &= \{b, c, d, y\} \\
D_1 &= \{a\} & U_1 &= \{b, c, d\} \\
D_2 &= \{a, x\} & U_2 &= \{c, d\} \\
D_3 &= \{a, x, y\} & U_3 &= \{d\} \\
D_4 &= \{a, c, x, y\} & U_4 &= \{\}
\end{aligned}
$$

Since $D(x) = \{\}$ and $U(x) = \{b, c, d\}$, the algorithm tells us that in the minimal endpoint representation, $I(x) = [0, 1]$ by locating the subscripts corresponding to these sets. Similarly, $D(y) = \{a\}$ and $U(y) = \{c, d\}$, so $I(y) = [1, 2]$. The remaining four intervals of the minimal endpoint representation are found similarly, and the representation is depicted at the right in Figure 2.
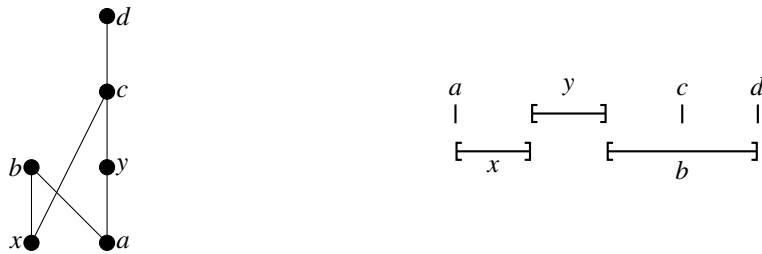

FIGURE 2. An interval order and its minimal endpoint representation

Notice that when P is a semiorder, its minimal endpoint representation is not necessarily one in which all intervals have the same length. The most straightforward example of this is $\mathbf{1} + \mathbf{2}$, which is shown in Figure 3 along with its minimal endpoint representation.

While the minimal endpoint representation of a semiorder does not have all intervals of the same length, there is a straightforward interval containment test to determine if a minimal endpoint representation of an interval order is one of a semiorder. We will frequently make use of the following lemma in this paper.

FIGURE 3. The poset $\mathbf{1}+\mathbf{2}$ and its minimal endpoint representation

Lemma 1.4. *An interval order P is a semiorder if and only if its minimal endpoint representation does not include intervals $[a_1, b_1]$ and $[a_2, b_2]$ such that $a_1 < a_2$ and $b_2 < b_1$.*

Proof. Let $P = (X, \leq_P)$ be an interval order. For the "only if" direction, we consider the contrapositive. The existence of intervals satisfying the conditions in the lemma means that the interval $[a_2, b_2]$ lies in the interior of $[a_1, b_1]$. Suppose that $I(x) = [a_1, b_1]$ and $I(z) = [a_2, b_2]$. Since the representation is minimal, we know that there are $y, w \in X$ such that $\mathbb{r}(y) = a_1$ and $\mathbb{l}(w) = b_1$. Then $\{x, y, z, w\}$ is a $\mathbf{1}+\mathbf{3}$ in P, so P is not a semiorder.

For the converse, suppose that P is an interval order that is not a semiorder and let $\{x, y, z, w\}$ be a $\mathbf{1}+\mathbf{3}$ in P with x incomparable to y, z, w and $y < z < w$. Notice that

$$\mathbb{l}(x) \leq \mathbb{r}(y) < \mathbb{l}(z) \leq \mathbb{r}(z) < \mathbb{l}(w) \leq \mathbb{r}(x),$$

which shows that $I(x)$ and $I(z)$ are the intervals we seek.   □

If $P = (X, \leq_P)$ is a poset, we say that a total order L on X is a *linear extension* of P provided that for all $x, y \in X$, if $x \leq_P y$, then $x \leq y$ in L. The *dimension* of P, denoted $\dim(P)$, is the least d such that there exist linear extensions $L_1, L_2, \ldots, L_d$ of P such that (as sets of ordered pairs)

$$\leq_P = L_1 \cap L_2 \cap \cdots \cap L_d.$$

In [1], Bogart et al. showed that for every positive integer d, there exists an interval order having dimension at least d. On the other hand, the situation for semiorders is much more restricted. Rabinovitch showed in [19] that if P is a semiorder, then $\dim(P) \leq 3$. Furthermore, $\dim(P) = 3$ if and only if P contains one of the posets shown in Figure 4.
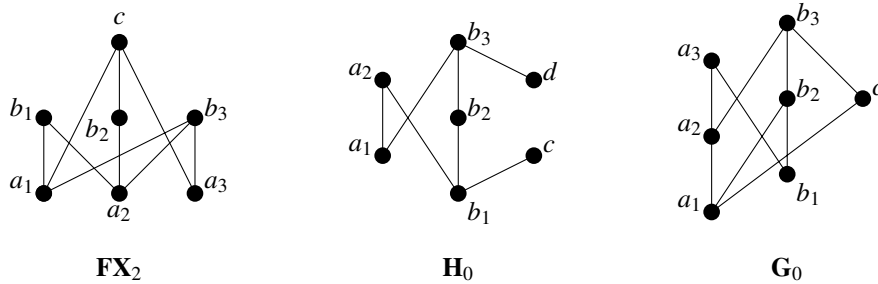

FIGURE 4. The three subposets that can force a semiorder to have dimension 3

For more information on interval orders and semiorders, Fishburn's monograph [9] is a classic while Trotter's survey article [23] provides a more recent look. The canonical work on dimension theory for posets is Trotter's monograph [22]. The labels on the posets in Figure 4 follow Trotter's notation, but we note that there is an error in his list of the forbidden subposets for a semiorder to have dimension at most 2. On page 197, he lists

FX$_1$ in addition to the three given here, but **FX**$_1$ is not a semiorder since $\{b_1, a_3, b_2, c\}$ is a $1 + 3$.

Ascent sequences and enumeration. Given a sequence $(x_1, x_2, \ldots, x_i)$ of integers, the number of *ascents* in the sequence is defined to be

$$\mathrm{asc}(x_1, \ldots, x_i) = \left| \left\{ 1 \leq j < i \colon x_j < x_{j+1} \right\} \right|.$$

In [2], Bousquet-Mélou et al. defined an *ascent sequence* to be a sequence $(x_1, \ldots, x_n)$ of nonnegative integers such that $x_1 = 0$ and $x_i \in [0, 1 + \mathrm{asc}(x_1, \ldots, x_{i-1})]$ for all $2 \leq i \leq n$. They then defined a map Ψ from (unlabeled) interval orders on n points to ascent sequences of length n and showed that their function is a bijection. Here, we recast the inverse of that bijection as a way to construct the minimal endpoint representation of an interval order on n points from an ascent sequence of length n.

The process of constructing the interval order corresponding to an ascent sequence proceeds iteratively through the ascent sequence. The simplest ascent sequence, (0), corresponds to the minimal endpoint representation $[0, 0]$. To describe the algorithm, we assume that we have an ascent sequence $(x_1, \ldots, x_n)$ with $n \geq 2$ and have constructed the interval order Q corresponding to the ascent sequence $(x_1, \ldots, x_{n-1})$. We retain some of the notation from [2] by letting $\ell(Q)$ denote the *largest right* endpoint of an interval in the minimal endpoint representation of Q. We also let $\ell^*(Q)$ denote the *smallest left* endpoint of an interval *with right endpoint* $\ell(Q)$ (again, in the minimal endpoint representation of Q). Suppose now that $x_n = i$. We obtain the minimal endpoint representation of the interval order P corresponding to $(x_1, \ldots, x_n)$ by applying one of the following three moves:

`Move 1` If $i \leq \ell^*(Q)$ add the interval $[i, \ell(Q)]$.
`Move 2` If $i = \ell(Q) + 1$ add $[\ell(Q) + 1, \ell(Q) + 1]$.
`Move 3` If $\ell^*(Q) < i \leq \ell(Q)$,
- If $\lambda < i \leq \rho < \ell(Q)$, replace $[\lambda, \rho]$ with $[\lambda, \rho + 1]$.
- If $i \leq \lambda \leq \rho \leq \ell(Q)$, replace $[\lambda, \rho]$ with $[\lambda + 1, \rho + 1]$.
- If $\lambda < i$ and $\rho = \ell(Q)$, replace $[\lambda, \rho]$ with $[\lambda, i]$.
- Add the interval $[i, \ell(Q) + 1]$.

We know that the minimal endpoint representation of Q contains $[\ell^*(Q), \ell(Q)]$, and so `Move 1` adds another maximal element to the poset whose interval extends at least as far left as $\ell^*(Q)$ so that $\ell^*(P) = i$. If $i = \ell^*(Q)$, then `Move 1` merely adds another point to the poset that has $[\ell^*(Q), \ell(Q)]$ as its interval in the minimal endpoint representation. This gives rise to a pair of points with duplicated holdings. (This use of `Move 1` only occurs when $i = x_n = x_{n-1}$, and this is the only way to create duplicated holdings. We will frequently use this fact in our enumerative work later in the paper.) `Move 2` adds a new trivial interval that becomes the unique maximal element in P. `Move 3` is the problematic move when it comes to working with semiorders. Its effect is to increase the largest right endpoint by one so that $\ell(P) = \ell(Q) + 1$ while inserting a new endpoint at i. Any interval with its left endpoint less than i and its right endpoint at least i has its right endpoint moved one unit to the right. Any interval that had its left endpoint being i or larger is shifted to the right by one unit. Any interval that corresponds to a maximal element in Q (which is equivalent to having right endpoint $\ell(Q)$) is truncated by retaining its left endpoint but making its right endpoint i. A new interval $[i, \ell(Q) + 1]$ is then inserted, which ensures that the representation is minimal by having i used as both a left and right endpoint. We illustrate `Move 3` in Figure 5 with intervals in the representation of Q illustrated above and the corresponding intervals (in the same relative positions) and new interval (shown uppermost) of P below.
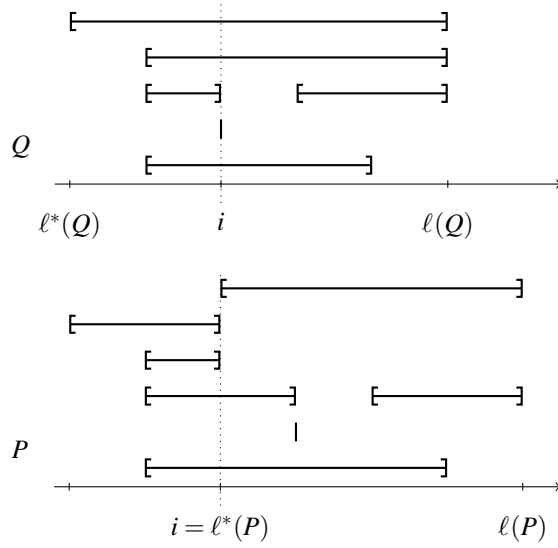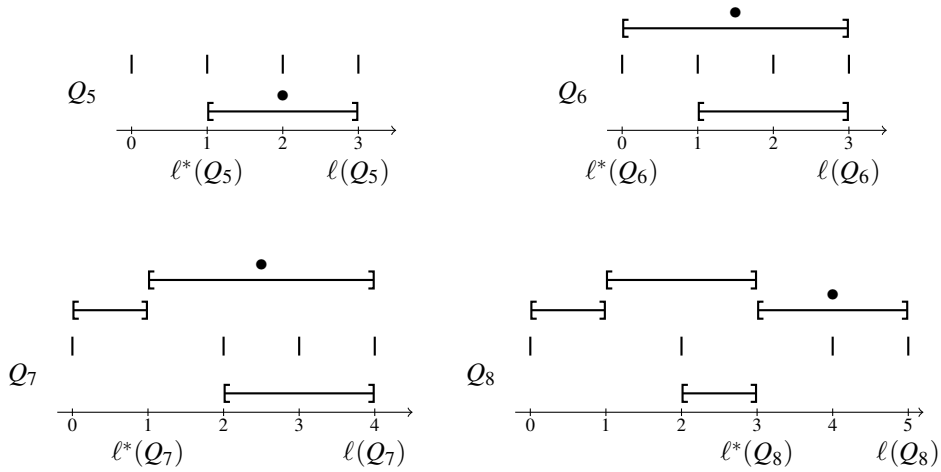
FIGURE 5. The effect of a `Move 3`

Example 1.5. We illustrate the process of constructing the minimal endpoint representation of the interval order corresponding to the ascent sequence $(0,1,2,3,1,0,1,3)$. We will denote by Q_i the interval order corresponding to the first i terms of the given ascent sequence. We know that Q_1 is represented by $\{[0,0]\}$ and $\ell(Q_1) = \ell^*(Q_1) = 0$. Thus, Q_2 is represented by $\{[0,0],[1,1]\}$ through a `Move 2`. The next two moves are also `Move 2`, which leads us to Q_4 being represented by $\{[0,0],[1,1],[2,2],[3,3]\}$. We have $\ell(Q_4) = \ell^*(Q_4) = 3$. Thus, to form Q_5, we apply `Move 1`, which adds the interval $[1,3]$, giving the representation illustrated in Figure 6. (At each stage, we place a $\bullet$ above the new interval added at that stage.) We also form Q_6 by using `Move 1` and have $\ell(Q_6) = 3$

FIGURE 6. Constructing the minimal endpoint representation of the interval order corresponding to ascent sequence $(0,1,2,3,1,0,1,3)$

and $\ell^*(Q_6) = 0$. This means that Q_7 must be formed by using `Move 3`, as depicted in the figure. The interval of length two and the three trivial intervals are shifted right, while the interval $[0,3]$ is truncated to become $[0,1]$ and the new interval is $[1,4]$. This leaves us with $\ell(Q_7) = 4$ and $\ell^*(Q_7) = 1$, so finishing requires another `Move 3`. This `Move 3` shifts two intervals, stretches no intervals, and truncates two intervals.

Since the work of Bousquet-Mélou et al., a variety of results building on their work have been published. Many of them relate to pattern-avoiding permutations and specialized classes of ascent sequences. However, it is worth highlighting some of those with connections to posets. For instance, in [16], Kitaev and Remmel enumerated interval orders by number of minimal elements (and other statistics). They also identified a subset of the ascent sequences that they termed the *restricted ascent sequences* and showed that the number of ascent sequences of length n is enumerated by the nth Catalan number. However, the bijection between ascent sequences and interval orders does not send the restricted ascent sequences to the semiorders, and the authors were unable to characterize the interval orders corresponding to the restricted ascent sequences. They did conjecture a refined version of their generating function for enumeration by number of minimal elements, which was proved independently by Levande in [17] and Yan in [27]. Dukes et al. looked at enumeration by the number of indistinguishable elements in [6], while Khamis enumerated the number of interval orders with no duplicated holdings by height in [15]. The focus of Jelínek's work in [14] was to enumerate the number of self-dual interval orders. Claesson and Linusson looked at connections between various classes of matchings and interval orders in [3]. Disanto et al. looked at some problems involving generating and enumerating series parallel interval orders and semiorders in [5].

As mentioned earlier, the number of semiorders on n points is the nth Catalan number, as shown by Wine and Freund in [26] and Dean and G. Keller in [4]. Greenough showed in [11] that the number of semiorders on n points with no duplicated holdings is given by

$$s(n) = \sum_{a=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{a, n-1-2a, a} - \sum_{a=2}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-1}{a, n+1-2a, a-2},$$

where the terms of the sums are multinomial coefficients. More recently, Lewis and Zhang were able to enumerate the number of graded posets (not interval orders) that do not contain $\mathbf{1+3}$ in [18], which was followed by the work of Guay-Paquet et al. in [12]. An enumeration of semiorders by length (one less than the number of elements in a maximum chain) was given by Hu in [13]. The only work we are aware of that enumerates any class of posets by dimension is the work of El-Zahar and Sauer in [7], where they provide an asymptotic enumeration of two-dimensional posets. Our work in the remainder of the paper will be restricted to unlabeled semiorders. We will proceed to define what we call the hereditary semiorders and characterize their structure. This structure then gives a way to access the semiorders of dimension 2. Our enumeration will proceed by looking at the ascent sequences corresponding to these classes of semiorders. The sequences of integers produced as a result were not in OEIS prior to this work.

2. BLOCK STRUCTURE OF HEREDITARY SEMIORDERS

It is straightforward to verify that the ascent sequence $(0,1,0,1,2,0)$ has minimal endpoint representation as an interval order as shown in Figure 7 under the bijection Ψ^{-1} of Bousquet-Mélou et al. By Lemma 1.4, we can tell that this is not a semiorder. Progressing to the ascent sequence $(0,1,0,1,2,0,2)$ requires a `Move 3`, however, which destroys the
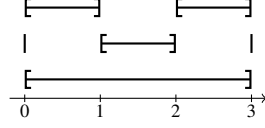
FIGURE 7. The interval order corresponding to $(0,1,0,1,2,0)$

$1+3$, giving us the minimal endpoint representation depicted in Figure 8. Since no interval is contained in the interior of any other interval, we know that this *is* a semiorder. This dilemma leads us to make the following definition.
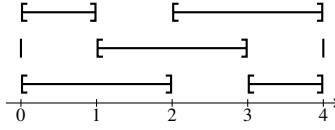
FIGURE 8. The semiorder corresponding to $(0,1,0,1,2,0,2)$

Definition 2.1. Let P be a semiorder on n points, and let $(x_1,\ldots,x_n)$ be the ascent sequence corresponding to P under Ψ. We say that P is *hereditary* provided that for every i with $1 \le i \le n$, $\Psi^{-1}((x_1,\ldots,x_i))$ is a semiorder.

As we will show, the hereditary semiorders have a particularly nice structure in terms of their minimal endpoint representations. We describe this structure as being built from certain fundamental blocks with three types of boundary options for how different blocks can be combined to form a larger hereditary semiorder.

Definition 2.2. The fundamental blocks we will use to characterize the hereditary semiorders are as given below. Throughout, b is a nonnegative integer and k is an integer.

$$T_0 = \{[0,0]\}$$
$$T_1^b = \{[b,b],[b+1,b+1]\}$$
$$W_k^b = \{[b,b],[b+k,b+k]\} \cup \bigcup_{i=0}^{k-1} \{[b+i,b+i+1]\} \quad \text{for } k \ge 1$$
$$C_2^b = \{[b,b],[b,b+1],[b,b+2],[b+1,b+2],[b+2,b+2]\}$$
$$U_k^b = \bigcup_{i=0}^{k-1} \{[b,b+i],[b+k-i,b+k]\} \quad \text{for } k \ge 3$$
$$C_k^b = \{[b,b+k]\} \cup U_k^b \quad \text{for } k \ge 3$$

We refer to T_0 as the *trivial block*. A *nontrivial block* is any block that is not T_0. We will occasionally omit the superscript and refer to T_1 if the position of the block is clear from context. If we wish to refer to a generic block of the form W_k^b, we will use $\mathcal{W}$. For U_k^b, we will write $\mathcal{U}$, and for C_k^b we will write $\mathcal{C}$. By $\mathcal{B}$, we will mean a block that could either be a $\mathcal{C}$ or a $\mathcal{U}$.

We give sample illustrations of some of the blocks defined in Definition 2.2 in Figure 9.
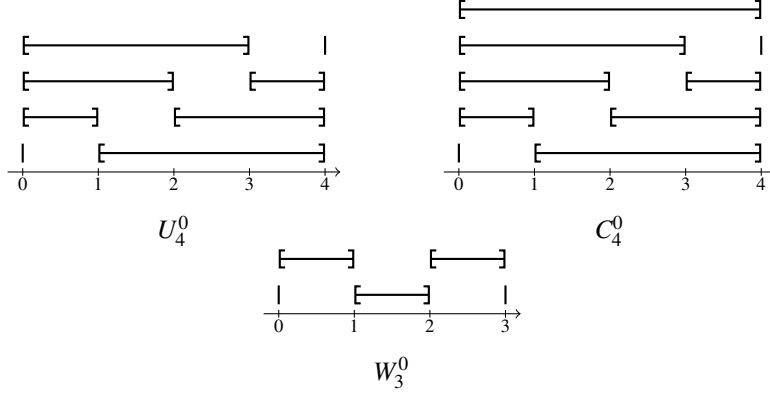
FIGURE 9. The blocks U_4^0, C_4^0, and W_3^0

Our definitions of the blocks is in terms of sets, but we have already discussed the fact that a semiorder with duplicated holdings will have multiple elements associated to the same interval. Because we know how duplicated holdings arise in terms of the ascent sequence, we will be able to disregard such issues in terms of the block structure, stating our results in terms of the intervals appearing in the interval representation and implicitly allowing multiple points of the semiorder to have the same interval associated. We will, however, be able to readily address duplicated holdings when we get to our enumerative results later in the paper.

Next we define ways in which blocks can be combined. Note that the sum of the subscript and superscript on any block gives the largest endpoint of an interval in the block (and the superscript is the smallest endpoint).

Definition 2.3. Let $A_{k_1}^b$ and $A_{k_2}^{k_1+b}$ be nontrivial blocks. We combine $A_{k_1}^b$ and $A_{k_2}^{k_1+b}$ with a *strong boundary*, denoted by $A_{k_1}^b | A_{k_2}^{k_1+b}$, by taking the set-theoretic union of the two blocks. The intersection between these two blocks is only the interval $[b+k_1, b+k_1]$. We can join two blocks (neither T_1^b and not both $\mathcal{W}$) with a *weak boundary*, denoted by $A_{k_1}^b \vdots A_{k_2}^{k_1+b}$, by removing $[b+k_1, b+k_1]$ from $A_{k_1}^b \cup A_{k_2}^{k_1+b}$. When $A_{k_1}^b$ is a $\mathcal{C}$ or $\mathcal{U}$ and $A_{k_2}^{k_1+b}$ is a $\mathcal{W}$ or $C_2^{k_1+b}$, we must also allow a *weak boundary with optional element* (or *optional interval*), which we denote by $A_{k_1}^b \vdots^\circ A_{k_2}^{k_1+b}$. The intervals in $A_{k_1}^b \vdots^\circ A_{k_2}^{k_1+b}$ are the same as with $A_{k-1}^b \vdots A_{k_2}^{k_1+b}$ with the addition of the interval $[k_1+b-1, k_1+b+1]$.

Note that weak boundaries are not permitted when one of the blocks is T_1^b, since we would not be left with a minimal endpoint representation. We also do not permit a weak boundary between two $\mathcal{W}$, since such a construction would simply produce a $\mathcal{W}$ with larger subscript. From the definitions alone, it is not clear that it is sufficient to define a weak boundary with optional element only in the restricted cases given in Definition 2.3. However, our argument will show that such a boundary cannot occur elsewhere. The block T_0 exists only to account for the antichain poset in which no two distinct points are comparable to one another, and T_0 cannot be combined with other blocks.

Example 2.4. In Figure 10, we now illustrate the intervals of a semiorder with block structure
$$C_3^0 \vdots^\circ W_2^3 | U_3^5 \vdots W_1^8 | T_1^9.$$

Notice that the interval $[3,3]$, which would be present in C_3^0, is absent because of the weak boundary. The interval $[8,8]$ is also omitted because of a weak boundary. The interval $[2,4]$ that bridges the boundary between the first two blocks is the optional interval.
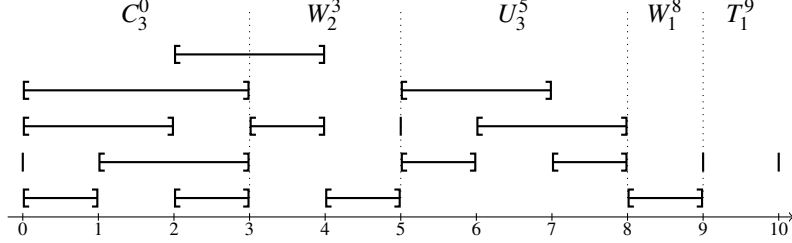

FIGURE 10. A sample block decomposition

We are now ready to state and prove our result about the structure of hereditary semiorders.

Theorem 2.5. *If P is a hereditary semiorder, then intervals in the minimal endpoint representation of P can be uniquely described using the blocks of Definition 2.2 combined according to the boundaries of Definition 2.3.*

Proof. Our proof is by induction on n, the number of points in P. Since points with duplicated holdings do not impact the intervals in the minimal endpoint representation, we will assume without loss of generality that P has no duplicated holdings. For $n = 1$, the only option is a single point, which has minimal endpoint representation of the interval $[0,0]$. This is T_0. Now suppose for some positive integer n that if Q is a hereditary semiorder on n points, then Q can be described in terms of blocks and boundaries. Let P be a hereditary semiorder on $n+1$ points, and let $(x_1, \ldots, x_n, x_{n+1})$ be the corresponding ascent sequence under the bijection Ψ of Bousquet-Mélou. Since P is hereditary, we know that $\Psi^{-1}((x_1, \ldots, x_n))$ is a semiorder Q on n points. Therefore, by the induction hypothesis, the intervals in the minimal endpoint representation of Q can be described in terms of blocks and boundaries. The proof is by cases based first on the last boundary and block in the block structure of Q and second on the value of $\alpha = x_{n+1}$.

When the last block is T_1^b (and hence $\ell(Q) = \ell^*(Q) = b+1$), the last boundary must be strong by definition. If $\alpha \leq b-1$, then $[b,b]$ lies in the interior of the new interval added to form P, and so P is not a semiorder by Lemma 1.4. When $\alpha = b$, we add the interval $[b, b+1]$, and thus the last boundary and block changes from $|T_1^b$ to $|W_1^b$. The case $\alpha = b+1$ results in duplicated holdings. Finally, when $\alpha = b+2$, a `Move 2` is used and the block structure of P ends in $|T_1^b|T_1^{b+1}$.

Before getting into the details of the other blocks and boundaries, note that when the final block has subscript a and superscript b, taking $\alpha = a+b+1$ always results in a `Move 2` that adds the trivial interval $[b+a+1, b+a+1]$. This adds $|T_1^{b+a}$ to the end of the block structure of Q to form the block structure of P. Therefore, we will not consider this situation below.

We now consider when the block structure of Q ends $|C_a^b$, which implies $\ell^*(Q) = b$. When $\alpha = b$, the result is duplicated holdings. When $\alpha < b$, `Move 1` is used, adding the interval $[\alpha, a+b]$. This places the interval $[b,b]$, which exists because of the strong boundary, in the interior of the new interval, and so P is not a semiorder. When α satisfies $b+1 \leq \alpha \leq a+b$, a `Move 3` is applied to construct P. For $\alpha = b+1$, this converts the $|C_a^b$

at the end of the block structure of Q into $|U_{a+1}^b$ as depicted in Figure 11. When α satisfies $b+2 \leq \alpha < b+a$, the Move 3 results in an interval order that is not a semiorder. This is because Move 3 extends the interval $[b, \alpha]$ in Q to the interval $[b, \alpha+1]$ in P and truncates the interval $[b+1, a+b]$ in Q to become the interval $[b+1, \alpha]$ in P. This results in one interval contained in the interior of another, violating Lemma 1.4. When $\alpha = b+a$, no intervals are truncated but the interval $[b+a, b+a]$ becomes $[b+a+1, b+a+1]$ and the interval $[b+a, b+a+1]$ is added. This results in the block structure of P ending $C_a^b \vdots W_1^{b+a}$.
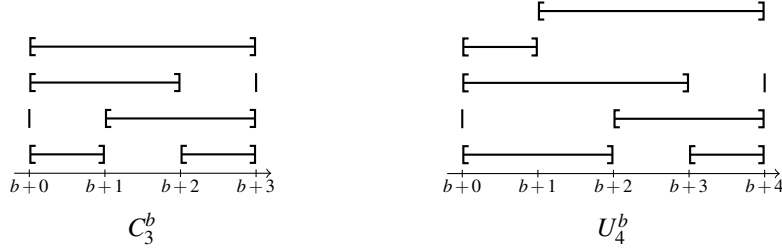

FIGURE 11. $\alpha = b+1$ when Q ends $|C_3^b$

The next case is that the block structure of Q ends $|U_a^b$, which means $\ell^*(Q) = b+1$. When $\alpha < b$, we use a Move 1, which adds the interval $[\alpha, b+a]$. This interval contains $[b, b+1]$ in its interior, and so P would not be a semiorder. For $\alpha = b$, this adds the interval required to convert the U_a^b to a C_a^b while retaining the strong boundary. For $\alpha = b+1$, we produce duplicated holdings. The remaining cases involve Move 3. The case $\alpha = b+a$ is as with $|C_a^b$, resulting in the block structure of P ending in $U_a^b \vdots W_1^{b+a}$. Because of the definition of U_b^a, we know that $a \geq 3$, and thus we must consider α satisfying $b+2 \leq \alpha \leq a+b-1$. Here we again use Lemma 1.4 by noting that the interval $[b+1, \alpha]$ is contained in the interior of $[b, \alpha+1]$.

The final case involving a strong boundary before the last block of Q is when Q's block structure ends with $|W_a^b$. In this case, $\ell^*(Q) = b+a-1$. When $\alpha = b+a$, the $\mathcal{W}$ at the end grows to become $|W_{a+1}^b$. If $\alpha = b+a-1$, we create duplicated holdings. If $a \geq 3$, having $\alpha \leq b+a-3$ is a Move 1 which results in the interval $[b+a-2, b+a-1]$ being contained in the interior of $[\alpha, b+a]$, which takes us out of the class of semiorders. For $a \in \{1, 2\}$, $\alpha < b$ places the interval $[b, b]$ in the interior of $[\alpha, b+a]$, so P would not be a semiorder. Thus, it remains only to consider $a \geq 2$ and $\alpha = b+a-2$. Here, we have a Move 1 that adds the interval $[b+a-2, b+a]$. When $a = 2$, this converts the $|W_2^b$ at the end of the block structure of Q into $|C_2^b$ at the end of the block structure of P. For $a > 2$, the block structure of P ends $|W_{a-2}^b \vdots C_2^{b+a-2}$. This is illustrated in Figure 12.
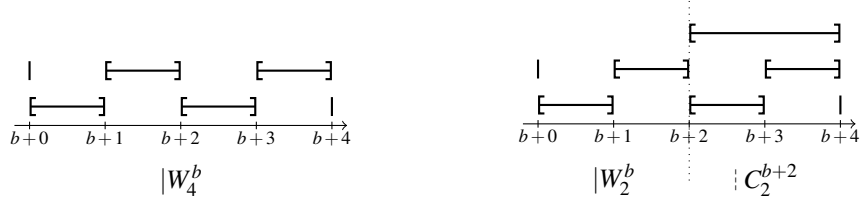

FIGURE 12. Converting $|W_4^b$ to $|W_2^b \vdots C_2^{b+2}$

To begin consideration of where the last boundary is weak, we assume that Q's block structure ends $\vdots\, C_a^b$. Here $\ell^*(Q) = b$. The `Move 1` cases are $\alpha \leq b$. When $\alpha = b$, we have duplicated holdings. If $\alpha < b$, then since $a \geq 2$ the interval $[b, b+a-1]$ exists and is contained in the interior of $[\alpha, b+a]$; therefore, P is not a semiorder. For $\alpha = b+1$, the situation is just as with a strong boundary, and the last block of P's block structure is $\vdots\, U_{a+1}^b$. When α satisfies $b+2 \leq \alpha \leq a+b$, the two possibilities are just as with $|C_a^b$.

When Q's block structure ends $\vdots\, U_a^b$, the argument is identical to the $|U_a^b$ case. Thus, we proceed to assume that the last boundary and block of Q is $\vdots\, W_a^b$, which means that $\ell^*(Q) = b+a-1$. For $a \geq 2$ or $\alpha \geq b$, the situation is just as when the final boundary is strong. Thus, we must only consider when $a = 1$ and $\alpha < b$. When $\alpha \leq b-2$, we note that the existence of a weak boundary means that the block preceding our W_1^b must be a $\mathcal{C}$ or a $\mathcal{U}$, and thus the minimal endpoint representation of Q contains the interval $[b-1, b]$. This interval is contained in the interior of the new interval $[\alpha, b+1]$, and P is not a semiorder. The final case is $\alpha = b-1$, which leads us to the necessity of the optional interval, since the new interval is $[b-1, b+1]$. Thus, the final block and boundary of P's block structure is $\vdots^{\mathrm{o}} W_1^b$.

The previous case has forced us now to consider the situation where the final boundary and block of Q is $\vdots^{\mathrm{o}} W_1^b$, in which case $\ell^*(Q) = b-1$. Here $a = 1$, so $\alpha = b+a+1 = b+2$ is the same as all other cases. For $\alpha = b+1$, we extend the last $\mathcal{W}$ to become $\vdots^{\mathrm{o}} W_2^b$. The case $\alpha \leq b-2$ creates a nonsemiorder by Lemma 1.4 as in the case of a weak boundary. For $\alpha = b-1$, we have duplicated holdings. It remains only to consider the case $\alpha = b$. Note that the weak boundary preceding the last block of Q must be preceded by a $\mathcal{C}$ or a $\mathcal{U}$. In either case, the interval $[b-2, b]$ must be present because of the minimum size of such blocks. In P, this interval becomes $[b-2, b+1]$, and the optional interval is truncated to $[b-1, b]$. We now have one interval in the interior of another, and so P is not a semiorder by Lemma 1.4.

We now must consider the case where Q's final boundary and block are $\vdots^{\mathrm{o}} W_2^b$, which gives $\ell^*(Q) = b+1$. Here we have that $\alpha = b+a+1$ when $\alpha = b+3$. For $\alpha = b+2$, we extend the final $\mathcal{W}$ to $\vdots^{\mathrm{o}} W_3^b$. When $\alpha = b+1$, we have duplicated holdings. If $\alpha = b$, the $\mathcal{W}$ at the end of Q's block structure becomes $\vdots^{\mathrm{o}} C_2^b$ in P. For $\alpha \leq b-1$, the new interval is $[\alpha, b+2]$, which contains in its interior the interval $[b, b+1]$. Therefore, P is not a semiorder.

When $a \geq 3$ and Q's block structure ends $\vdots^{\mathrm{o}} W_a^b$, the argument proceeds as it did with $\vdots\, W_a^b$. Thus, the only case we must still address is when the final block and boundary of Q's block structure is $\vdots^{\mathrm{o}} C_2^b$. Here $\ell^*(Q) = b$. The case $\alpha = b+3$ is taken care of because $\alpha = b+a+1$ here. When $\alpha = b+2$, this is the same as the $\alpha = b+a$ case for $\vdots\, W_a^b$, and we have that the block structure for P ends $\vdots^{\mathrm{o}} C_2^b \vdots W_1^{b+2}$. This is illustrated in Figure 13. For $\alpha = b+1$, note that the optional interval $[b-1, b+1]$ extends to become $[b-1, b+2]$, which contains in its interior the interval $[b, b+1]$ that results from the `Move 3` truncating $[b, b+2]$. Thus, P is not a semiorder by Lemma 1.4. When $\alpha = b$, we have duplicated holdings. When $\alpha \leq b-1$, the interval $[\alpha, b+2]$ contains the interval $[b, b+1]$ in its interior, violating Lemma 1.4. Since this case did not require us to permit a weak boundary with optional element before any other types of blocks, our proof of the existence of the block structure is complete.

It remains to show that the block structure of a hereditary semiorder P is unique. To do so, we will identify the location and type of each boundary between blocks. Once this is done, the blocks between the boundaries are uniquely defined. To identify the boundaries,
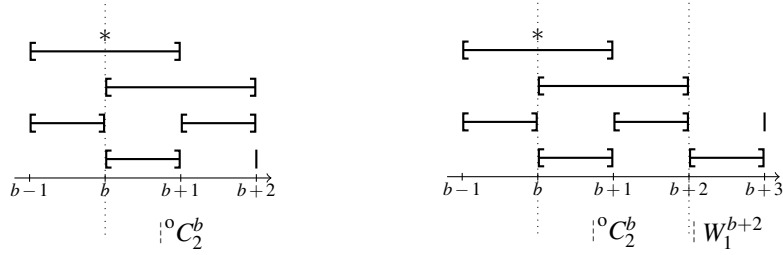
FIGURE 13. Moving on from $\vdots^{\text{o}}C_2^b$

we begin by labeling all of the integers between 0 and $\ell(P)$ as follows:

$$
t(i) = \begin{cases}
s & \text{if } [i,i] \text{ is in the representation} \\
w & \text{if } i \text{ is the endpoint of at least 3 intervals and not in the interior of an interval} \\
z & \text{if } i \text{ is the endpoint of at least 3 intervals and in the interior of an interval} \\
x & \text{otherwise.}
\end{cases}
$$

We have that $t(i) = s$ if and only if i is the location of a strong boundary, since the block and boundary definitions only allow intervals of length 0 at strong boundaries (including the implicit strong boundaries at the ends). Next, note that $t(i) = w$ if and only if i is the location of a weak boundary. (This holds because we do not allow weak boundaries between $\mathcal{W}$.) We observe that the definition of t tells us that if $t(i) = x$, then i is not a boundary. To finish our argument, we will redefine $t(i)$ for those integers i with $t(i) = z$. We wish to have $t(i) = o$ if and only if i is the location of a weak boundary with optional interval and will define the other integers j for which $t(j) = z$ to have $t(j) = x$. Let i be the smallest integer in a maximal sequence of consecutive integers with label z. Since a weak boundary with optional interval must be preceded by a $\mathcal{U}$ or a $\mathcal{C}$, we know that i cannot be the location of a weak boundary with optional interval. This is because the left endpoint of an optional interval must be the left endpoint of two intervals of the preceding $\mathcal{U}$ or $\mathcal{C}$ as well as being in the interior of at least one interval of the preceding block. Thus, we let $t(i) = x$, which then tells us that there is a weak boundary with optional element at $i+1$, so we let $t(i+1) = o$. If $t(i+2) = z$, then we must change $t(i+2)$ to x, since we cannot have two weak boundaries with optional intervals at consecutive integers. This process continues until no integers in $[0, \ell(P)]$ have label z, which means the boundaries have all been uniquely determined because all decisions are forced. $\qquad\square$

A careful reading of the preceding proof will show why the definition of a weak boundary with optional element is so restrictive. In particular, the optional interval is only introduced when absolutely necessary, and then the argument proceeds to consider what can develop following an optional interval. The fact that an optional interval can only be preceded by a $\mathcal{C}$ or a $\mathcal{U}$ comes from the fact that our first optional interval arises in the $\vdots W_1^b$ case, which requires a $\mathcal{C}$ or $\mathcal{U}$ before it because of the prohibition against weak boundaries between $\mathcal{W}$s. The only other weak boundaries with optional elements arise as a consequence of building up from the $\vdots^{\text{o}}W_1^b$ case, and thus cannot be preceded by a $\mathcal{W}$ either.

3. BLOCK CHARACTERIZATION OF DIMENSION 2 SEMIORDERS

We are now prepared to use the blocks and boundaries introduced above to provide a characterization of the semiorders of dimension 2, which we will eventually use to enumerate them. We begin with a straightforward lemma that links the moves used to construct an interval order from an ascent sequence to subposet structure. This will be useful in connecting to Rabinovitch's forbidden subposet characterization of the dimension 2 semiorders.

Lemma 3.1. *Let P be a poset. If Q is a subposet of P and P' is a poset obtained from P by* Move 1 *or* Move 2*, then Q is a subposet of P'.*

Proof. Since neither Move 1 nor Move 2 changes any of the existing comparabilities in P to form P', P is a subposet of P'. Thus, Q is a subposet of P' as well.   □

A full description of the block structure of semiorders of dimension 2 will be accomplished through a few steps. We begin by showing that all semiorders of dimension 2 are hereditary.

**Theorem 3.2.** *Let P be a semiorder. If P is not hereditary, then $\dim(P) = 3$.*

Proof. Let P be a semiorder on n points and let $(x_1, \ldots, x_n) = \Psi(P)$ be the ascent sequence corresponding to P. Without loss of generality, we may assume that P has no duplicated holdings. Since P is not hereditary, there is some largest positive integer $k < n$ such that $Q = \Psi^{-1}((x_1, \ldots, x_k))$ is not a semiorder. Since we know that $Q' = \Psi^{-1}((x_1, \ldots, x_{k+1}))$ is a semiorder, Q' does not contain $\mathbf{1} + \mathbf{3}$. However, Q must contain $\mathbf{1} + \mathbf{3}$, since Q is an interval order that is not a semiorder. Therefore, by Lemma 3.1, Q' is not obtained from Q by Move 1 or Move 2. We consider the minimal endpoint representation of Q. By Lemma 1.4, this representation has two intervals $[a, b]$ and $[c, d]$ with $[c, d]$ contained in the interior of $[a, b]$. Since the Move 3 that obtains Q' from Q destroys the $\mathbf{1} + \mathbf{3}$, we must have that $b = \ell(Q)$ and $a < x_{k+1}$. If $x_{k+1} \leq c$, then the minimal endpoint representation of Q' contains the interval $[c+1, d+1]$ and the interval $[x_{k+1}, b+1]$, which implies that Q' is not a semiorder by Lemma 1.4. If $x_{k+1} > d$, then the minimal endpoint representation of Q' contains the interval $[a, x_{k+1}]$, which contains $[c, d]$ in its interior. This would force Q' to not be a semiorder. Thus, we must have that x_{k+1} is an integer with $c < x_{k+1} \leq d$, forcing $c \neq d$.

Since $d < b$ and we are considering the minimal endpoint representation of Q, there exists an interval $[d, f]$ in the representation. Moreover, if $f < b$, then the minimal endpoint representation of Q' contains the interval $[d+1, f+1]$, and this interval is contained in the interior of $[x_{k+1}, b+1]$. This would prevent Q' from being a semiorder, so $f = b$. Also note that there must be an interval $[g, c]$. If $g > a$, then the minimal endpoint representation of Q' contains the interval $[g, c]$ and the interval $[a, x_{k+1}]$ with $x_{k+1} > c$. This again violates Lemma 1.4. The structural information we have gleaned so far is depicted in Figure 14. Using what we know about x_{k+1}, we can draw Figure 15 to reflect intervals that must exist in Q'. It is straightforward to verify that these intervals give us the three-dimensional semiorder $\mathbf{FX_2}$ from Figure 4.

By assumption, once we have obtained Q' from $(x_1, \ldots, x_{k+1})$, each of the posets obtained from $(x_1, \ldots, x_m)$ with $m \geq k+1$ is a semiorder. We will show that it is impossible to eliminate all copies of $\mathbf{FX_2}$ subject to this constraint, and thus we must have $\dim(P) = 3$. To do so, assume that m is such that $R = \Psi^{-1}((x_1, \ldots, x_m))$ contains $\mathbf{FX_2}$ and that for all $m > m'$, $\Psi^{-1}((x_1, \ldots, x_{m'}))$ does not contain $\mathbf{FX_2}$. We cannot be as precise about the endpoints as we were above at the first occurrence of $\mathbf{FX_2}$, but we do have the configuration
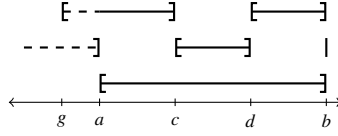
FIGURE 14. Intervals that must exist before eliminating $\mathbf{1+3}$
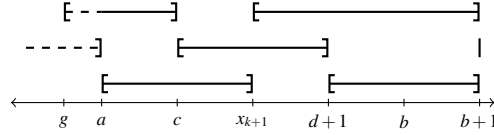
FIGURE 15. Intervals that must exist after eliminating $\mathbf{1+3}$

shown in Figure 16. Note that we do not necessarily have that the endpoints shown as equal (such as $\mathrm{r}(a_1)$ and $\mathbb{l}(b_2)$) are equal. Instead, we merely require that the intervals overlap.
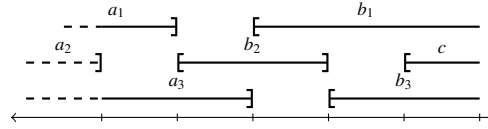
FIGURE 16. Intervals forming an $\mathbf{FX}_2$ in R

By Lemma 3.1, we only need to consider the effect of Move 3. If $x_{m+1} \leq \mathbb{l}(b_1)$, then the only impact of the Move 3 on these seven intervals is stretching or shifting that does not impact their relationship to one another, and thus the $\mathbf{FX}_2$ is not removed. Note that if none of b_1, c, b_3 is maximal in R, then this copy of $\mathbf{FX}_2$ cannot be removed. If b_1 is maximal in R, then a Move 3 with $\mathbb{l}(b_1) < x_{m+1} \leq \mathbb{l}(b_3)$ truncates b_1 and moves the right endpoint of b_2's interval one unit right. This places b_1 in the interior of b_2, violating the requirement that we must obtain a semiorder. (If b_1 is not maximal in R, a Move 3 with x_{m+1} in this range does not remove the $\mathbf{FX}_2$.) A Move 3 with $x_{m+1} > \mathrm{r}(b_2)$ leaves a $\mathbf{FX}_2$, either consisting of the same points (possibly with truncated intervals) or with the new interval playing the role of c (and possibly with the intervals for b_1 and b_3 being truncated). A Move 3 with x_{m+1} satisfying $\mathbb{l}(b_3) < x_{m+1} \leq \mathrm{r}(b_2)$ must truncate at least one of b_1 and b_3 if the $\mathbf{FX}_2$ is to be eliminated. However, then the truncated interval lies in the interior of the stretched interval for b_2, and the resulting poset would not be a semiorder.

Having shown that we cannot eliminate the last occurence of an $\mathbf{FX}_2$ after the last occurrence of a $\mathbf{1+3}$, we can therefore conclude that if P is not hereditary, then $\dim(P) = 3$ as claimed. $\qquad\square$

We now know that our search for semiorders of dimension at most 2 can be restricted to the hereditary semiorders. Thus, we will proceed to consider the three forbidden subposets of Figure 4 and what restrictions we must place upon the block structure of a hereditary semiorder in order to exclude them.

Lemma 3.3. *Let P be a hereditary semiorder. If P contains $\mathbf{FX}_2$, then the block structure of P requires an optional interval.*

Proof. First note that b_2 is incomparable to a_1, a_3, b_1, and b_3, but $\{a_1, a_3, b_1, b_2, b_3\}$ is not a 5-element antichain. Therefore, the interval corresponding to b_2 in the minimal endpoint representation of P must have positive length. Since a_3, b_1, and b_2 are pairwise incomparable, their intervals must overlap. Let x be an integer in the intersection of the intervals for a_3, b_1, and b_2. Since $a_3 < b_3$, we have that $x < \mathbb{l}(b_3)$. Similarly, $\mathbb{r}(a_1) < x$. Thus $\mathbb{r}(a_1)$, x, and $\mathbb{l}(b_3)$ are all distinct points in the interval for b_2. Hence, this interval has length at least 2. If b_2 is an optional interval in the block structure, then we are done. If b_2 is not an optional interval, then since its length is at least 2, it must lie in a $\mathcal{C}$ or a $\mathcal{U}$. Furthermore, at least one endpoint of b_2 must be the endpoint of the block containing b_2. Since b_1 is incomparable to c and $b_2 < c$, we must have that the interval of b_1 extends to the right of $\mathbb{r}(b_2)$. Since a_3 is incomparable to a_2 and $a_2 < b_2$, we must also have that the interval of a_3 extends to the left of $\mathbb{l}(b_2)$. Since x lies in both the interval of a_3 and that of b_1, this forces one of b_1 and a_3 to have its endpoints in two different blocks, and therefore, there must be an optional interval. $\square$

Lemma 3.4. *Let P be a hereditary semiorder. If P contains $\mathbf{H}_0$, then the block structure of P requires an optional interval.*

Proof. As before, we will assume that we are working with the minimal endpoint representation of P. Since b_2 is incomparable to a_1, a_2, c, and d, but $a_1 < a_2$, we must have that the length of b_2's interval is at least 1. Since $b_1 < b_2$ but b_1 is incomparable to d, we must have $\mathbb{l}(d) < \mathbb{l}(b_2)$. Similarly, since $b_2 < b_3$ and b_3 is incomparable to c, we must have $\mathbb{r}(b_2) < \mathbb{r}(c)$. Since $a_1 < a_2$ but both a_1 and a_2 are incomprable to d, we must have that the interval of d extends left of the interval of a_2. This gives $\mathbb{l}(a_2) \leq \mathbb{r}(d)$. Further, the interval of a_2 must leave room for the interval of a_1 to intersect that of b_2, which requires $\mathbb{l}(b_2) < \mathbb{l}(a_2)$. Combining these inequalities gives $\mathbb{l}(b_2) < \mathbb{l}(a_2) \leq \mathbb{r}(d)$. We may now conclude that $\mathbb{l}(b_2)$ lies in the interior of d. By the dual argument, we have that $\mathbb{r}(b_2)$ lies in the interior of c. By the minimality of the representation, this forces the intervals of c and d to each have length at least 2. If either of these is an optional interval, then we are done. If not, then they cannot belong to a $\mathcal{W}$ because of their intervals' lengths. Thus, the endpoints of b_2 lie in the interiors of two different blocks, which is only possible if b_2 is an optional interval. $\square$

Lemma 3.5. *If P is a hereditary semiorder containing a $\mathcal{C}$ somewhere other than the first or last block, then at least one of the blocks adjacent to the $\mathcal{C}$ is T_1 or P contains $\mathbf{G}_0$.*

Proof. Suppose that P is a hereditary semiorder containing C_n^b with $b \neq 0$ and at least one following block. We also assume that neither neighboring block is T_1. Then the C_n^b contains the intervals $[b, b+n]$, $[b, b+1]$, and $[b+1, b+n]$. The preceding block contains an interval containing the interval $[b-1, b]$ and an interval with right endpoint $b-1$. The succeeding block contains an interval containing $[b+n, b+n+1]$ and an interval with left endpoint $b+n+1$. These intervals, which are depicted in Figure 17, form a copy of $\mathbf{G}_0$ in P. $\square$

Lemma 3.6. *If P is a hereditary semiorder and the block structure of P requires an optional interval, then $\dim(P) = 3$.*

*Proof.* The proof is by straightforward case analysis based on what the blocks on either side of a weak boundary with optional interval can be. Recall that a weak boundary with optional interval must be preceded by a $\mathcal{C}$ or a $\mathcal{U}$ and must be followed by a $\mathcal{W}$ or a C_2^b, which limits the cases required. The cases and which forbidden subposet is produced are listed below.
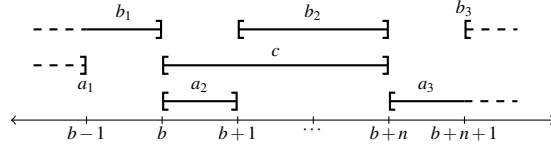
FIGURE 17. A $\mathcal{C}$ with neighbors other than T_1 forcing $\mathbf{G}_0$

(1) $C_n^b{}_!^{\mathrm{o}}C_2^{b+n}$ for $n \geq 2$ and $U_n^b{}_!^{\mathrm{o}}C_2^{b+n}$ for $n \geq 3$ both produce $\mathbf{FX}_2$.
(2) $C_n^b{}_!^{\mathrm{o}}W_1^{b+n}$ for $n \geq 2$ and $U_n^b{}_!^{\mathrm{o}}W_1^{b+n}$ for $n \geq 3$ both produce $\mathbf{H}_0$.
(3) $C_2^b{}_!^{\mathrm{o}}W_m^{b+2}$ with $m \geq 2$ produces $\mathbf{H}_0$.
(4) $C_n^b{}_!^{\mathrm{o}}W_m^{b+n}$ and $U_n^b{}_!^{\mathrm{o}}W_m^{b+n}$ with $n \geq 3$ and $m \geq 2$ both produce $\mathbf{FX}_2$.

The first case is illustrated in Figure 18. Note that if the C_2^b is followed by a weak boundary, there is some interval from the next block with its left endpoint at $b+n$ that can be used as c. A similar situation applies if the block before the weak boundary is C_2^{b-2} preceded by a weak boundary. Figure 18 is drawn to be general enough to encompass a $\mathcal{C}$ or $\mathcal{U}$ as the preceding block, and note that some intervals not involved in the $\mathbf{FX}_2$ are omitted.
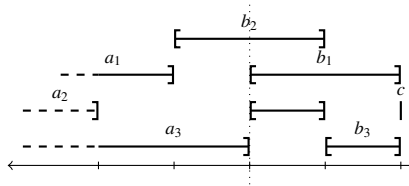
FIGURE 18. Intervals forming an $\mathbf{FX}_2$ with ${}_!^{\mathrm{o}}C_2^{b+n}$

The third case is illustrated in Figure 19. Again, a weak boundary before the C_2^b is not a problem, since there must be an interval from the previous block with its right endpoint at b. Figure 19 can be readily extended to the left in the style of Figure 18 to cover the second case as well, provided that one turns b_3 into an interval of length 0 (or uses an interval from the next block if the following boundary is weak).
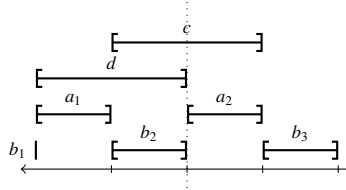
FIGURE 19. Intervals forming a $\mathbf{H}_0$ with $C_2^b{}_!^{\mathrm{o}}W_m^{b+2}$ and $m \geq 2$

The final case is not illustrated, but it is straightforward to verify after noting that the optional interval and the first two intervals of length 1 from the $\mathcal{W}$ are b_1, b_3, and c (in order by increasing left endpoint). $\qquad\square$

We are now ready to assemble the preceding results to prove the block characterization of semiorders of dimension at most 2.

Theorem 3.7. *Let P be a semiorder. The dimension of P is at most 2 if and only if all of the following hold:*

(1) *P is hereditary,*

(2) *the block structure of P does not require optional elements, and*

(3) *if the block structure of P contains a $\mathcal{C}$ somewhere other than the first or last block, then at least one of the blocks adjacent to the $\mathcal{C}$ is T_1.*

Proof. We first assume that P is a semiorder and $\dim(P) \leq 2$ and will prove that the three statements hold. The first statement is the contrapositive of Theorem 3.2. The second statement is the contrapositive of Lemma 3.6. Since $\dim(P) \leq 2$, P does not contain $\mathbf{G}_0$, and therefore the third statement follows from Lemma 3.5.

Now suppose that P is a semiorder for which all three statements hold. We will show that $\dim(P) \leq 2$. The first two statements and the contrapositives of Lemmas 3.3 and 3.4 show that P does not contain $\mathbf{FX}_2$ or $\mathbf{H}_0$. For a contradiction, we now assume that P has dimension 3. By what we've already shown, this means that P must contain $\mathbf{G}_0$. We will now show that the third statement must be violated by finding a $\mathcal{C}$ with two neighboring blocks that are not T_1. Since c is incomparable to a_2, a_3, b_1, and b_2 but $a_2 < a_3$, we know that the length of c's interval in the minimal endpoint representation must be at least 1. Since there are no duplicated holdings amongst the seven points of $\mathbf{G}_0$, all intervals must be distinct. Furthermore, since an interval contained in the interior of a $\mathcal{W}$ is incomparable to only two other intervals but c is incomparable to 4 points from $\mathbf{G}_0$, we know that one endpoint of c's interval is on the boundary between two blocks. By duality, we may assume without loss of generality that this endpoint is $\mathbb{l}(c)$.

Since $a_1 < c$ and b_1 is incomparable to both c and a_1, we must have that a_1 and b_1 both lie in a block before the one containing c and that $\mathbb{r}(b_1) = \mathbb{l}(c)$ because there are no optional intervals. Since b_1 has a larger up set than a_2 but is incomparable to a_2, we know that $\mathbb{r}(b_1) < \mathbb{r}(a_2)$. Thus, a_2 belongs to the same block as c and $\mathbb{l}(c) = \mathbb{l}(a_2)$ because of the lack of optional intervals. Since a_2 has a larger up set than c, we know that $\mathbb{r}(a_2) < \mathbb{r}(c)$. Since b_2 is incomparable to a_2, we can thus conclude that $\mathbb{l}(b_2) < \mathbb{r}(c)$. If $\mathbb{r}(b_2) > \mathbb{r}(c)$, then we know by the lack of optional elements that $\mathbb{r}(b_2)$ must be the right end of the block containing c, a_2, and b_2. Since $b_2 < b_3$, this means that b_3 must lie in a subsequent block. We now notice that a_3 is incomparable to b_3 and c, and thus $\mathbb{r}(b_2)$ lies in the interior of a_3's interval, which requires that a_3 cross the boundary of a block, contradicting the lack of optional intervals. Thus, we must have $\mathbb{r}(b_2) = \mathbb{r}(c)$ is the right boundary of the block containing c. Since c has both its endpoints on the boundary of its block, we must have that c, a_2, and b_2 lie in a $\mathcal{C}$. Recognizing that a_3 is incomparable to both c and b_3 and b_1 is incomparable to both c and a_1 shows that neither of the adjacent blocks to the $\mathcal{C}$ can be T_1, and our proof is complete. $\qquad\square$

4. ENUMERATION OF HEREDITARY SEMIORDERS

We are now prepared to use the block structure in order to enumerate the hereditary semiorders. The difficulting in building a generating function to complete this task is that there are restrictions on how the blocks can be combined using the various boundaries. In particular, we recall that a T_1^b may not be combined using a weak boundary, two consecutive $\mathcal{W}$ may not be combined using a weak boundary, a $\mathcal{W}$ may not be followed by a weak boundary with optional element, and a weak boundary with optional element may only preceed a $\mathcal{W}$ or a C_2^b. We will use notation inspired by regular expressions to give a compact way of describing the ways in which blocks are arranged. The components of our

notation are $+$, $*$, and ?, used as superscripts. A superscript $+$ will denote one or more consecutive occurrences of the entity to which the $+$ is attached. A $*$ means that zero or more consecutive occurrences of the entity are allowed. A ? means that at most one occurrence of the entity is allowed. When a $+$ is used between two strings (rather than a superscript), each pattern is allowed. All of our boundaries will be assumed to be weak unless explicitly shown in the notation. Recall from Definition 2.2 that we will use $\mathcal{B}$ to refer to a block that could be either a $\mathcal{C}$ or a $\mathcal{U}$.

We will break up the block structure of a hereditary semiorder based upon the occurrences of strong boundaries and the occurences of weak boundaries with optional intervals. Because a strong boundary is determined based on the presence of an interval of length 0 and such an interval must be present at the left of the first block of a hereditary semiorder and at the right of the last block, we will treat the the two ends of a block structure as if there are strong boundaries there. If we first consider the situation where no weak boundaries with optional intervals are allowed, then it suffices to break the full block structure up into the pieces between strong boundaries. We represent this as $\mathcal{X}_{ss} = \mathcal{W}^?(\mathcal{B}^+\mathcal{W})^*\mathcal{B}^*$. Essentially, between two strong boundaries, we can view the blocks as divided further by the occurrences of the $\mathcal{W}$, which may not be adjacent (since all boundaries inside this string are weak). Between $\mathcal{W}$s, we must have at least one $\mathcal{B}$. Notice that this structure allows for there to be *no* blocks between two strong boundaries, which is what creates a T_1^b. We may further repeat the pattern $\mathcal{X}_{ss}$ as many times as required, which then introduces strong boundaries into the overall block structure.

When an optional interval is present, we may trace backward from that weak boundary with optional interval until we reach either another weak boundary with optional interval or a strong boundary (including the beginning of the block structure). Thus, we will now describe two further subpatterns, one to cover what can occur between a strong boundary and the first ensuing weak boundary with optional interval (denoted $\mathcal{X}_{so}$) and the other to cover what occurs between two weak boundaries with optional intervals (denoted $\mathcal{X}_{oo}$). To construct $\mathcal{X}_{so}$, note that the block before a weak boundary with optional element must be a $\mathcal{B}$, and certainly many of them are permitted, so $\mathcal{X}_{so}$ must end with $\mathcal{B}^+$. Other than needing to end with a $\mathcal{B}$, this case looks much like $\mathcal{X}_{ss}$, in that we see isolated $\mathcal{W}$ with strings of $\mathcal{B}$ in between, and an initial $\mathcal{W}$ may or may not occur. Thus, $\mathcal{X}_{so} = W^?(\mathcal{B}^+\mathcal{W})^*\mathcal{B}^+$. When both ends of a string of blocks joined by weak boundaries are weak boundaries with optional intervals, the situation is more complicated. The weak boundary with optional interval may be followed by a $\mathcal{W}$, in which case the structure proceeds just as with $\mathcal{X}_{so}$, since the final block of the pattern must be a $\mathcal{B}$ to allow for the trailing weak boundary with optional interval. This means $\mathcal{X}_{oo}$ must allow $\mathcal{W}(\mathcal{B}^+\mathcal{W})^*\mathcal{B}^+$. We may also follow the weak boundary with optional interval with C_2^b. Since this block is itself a $\mathcal{B}$, this could be the end of the pattern, proceeding immediately to another weak boundary with optional interval. If not, we then see the remainder divided up by $\mathcal{W}$, ensuring that the last block before the weak boundary with optional interval is a $\mathcal{B}$. This gives us $C_2^b\mathcal{B}^*(\mathcal{W}\mathcal{B}^+)^*$, which combines with the case where the first block after the weak boundary with optional interval is $\mathcal{W}$ to give us

$$\mathcal{X}_{oo} = \mathcal{W}(\mathcal{B}^+\mathcal{W})^*\mathcal{B}^+ + C_2^b\mathcal{B}^*(\mathcal{W}\mathcal{B}^+)^*.$$

As we proceed through the block structure, we must eventually reach an occurrence of a weak boundary with optional interval where the next meaningful boundary is a strong boundary (possibly the one at the end of the block structure). Thus, we need a pattern to describe what happens in such a case, which we denote by $\mathcal{X}_{os}$. Again, the weak boundary with optional interval may be followed by a $\mathcal{W}$ or a C_2^b. The former case gives rise to

$\mathcal{W}(\mathcal{B}^+\mathcal{W})^*\mathcal{B}^*$, much like in $\mathcal{X}_{oo}$, but here we end with $\mathcal{B}^*$ because the next boundary is strong, and so we may end with a $\mathcal{W}$. When beginning with a C_2^b, the situation is also analogous to $\mathcal{X}_{oo}$, but we must allow a $\mathcal{W}$ at the end, which gives $\hat{C}_2^b\mathcal{B}^*(\mathcal{W}\mathcal{B}^+)^*\mathcal{W}^?$. Combining these yields

$$\mathcal{X}_{os} = \mathcal{W}(\mathcal{B}^+\mathcal{W})^*\mathcal{B}^* + C_2^b\mathcal{B}^*(\mathcal{W}\mathcal{B}^+)^*\mathcal{W}^?.$$

We now have all the pieces necessary to create a pattern that describes the block structure of all hereditary semiorders. We first note that a weak boundary with optional interval may occur in the form $\mathcal{X}_{so}\vert^{\mathrm{o}}\mathcal{X}_{os}$, or we may place several copies of $\mathcal{X}_{oo}$ (with weak boundaries with optional intervals on each side) in between the $\mathcal{X}_{so}$ and the $\mathcal{X}_{os}$. This means we will need to see $\mathcal{X}_{so}\vert^{\mathrm{o}}(\mathcal{X}_{oo}\vert^{\mathrm{o}})^*\mathcal{X}_{os}$ in the overall pattern. Since there may be multiple strong boundaries before the first weak boundary with optional interval, the overall pattern must begin $\mathcal{X}_{ss}^*$. We need another occurrence of $\mathcal{X}_{ss}^*$ along with the pattern containing weak boundaries with optional intervals in order to allow weak boundaries with optional intervals to be separated by a combination of strong and weak boundaries. Thus, the pattern that accounts for all hereditary semiorders is

$$\mathcal{H} = (\mathcal{X}_{ss}\vert)^*(\mathcal{X}_{so}\vert^{\mathrm{o}}(\mathcal{X}_{oo}\vert^{\mathrm{o}})^*\mathcal{X}_{os}\vert(\mathcal{X}_{ss}\vert)^*)^*.$$

Note that $\mathcal{H}$ allows for the empty pattern, which is how we will account for T_0 when converting this pattern into a generating function.

Translation of the $+$, $*$, and ? used in our patterns into generating functions is relatively straightforward. For readers unfamiliar with the use of generating functions to enumerate strings or sequences in this manner, a good introduction is provided by Wilf in [25]. If $\mathcal{F}$ is a pattern with generating function $F(x)$, then $\mathcal{F}^*$ has generating function $1/(1-F(x))$, $\mathcal{F}^+$ has generating function $F(x)/(1-F(x))$, and $\mathcal{F}^?$ has generating function $(1+F(x))$. The other piece that will require attention is the boundaries, but first we will proceed to determine the generating functions for $\mathcal{W}$, $\mathcal{U}$, $\mathcal{C}$, $\mathcal{B}$, and C_2^b, since those are the atomic pieces of the patterns here. (The patterns developed so far do not involve $\mathcal{C}$ or $\mathcal{U}$ alone, but we will require these when considering the case of dimension at most 2 in the next section.)

Because our patterns above are built on the assumption of weak boundaries between blocks unless we specify a strong boundary or weak boundary with optional interval, we will build our generating functions for the blocks by assuming weak boundaries on each end. This then has the effect of making each of our blocks appear to have two fewer intervals in them than they would when occurring in isolation. For example, the smallest $\mathcal{C}$ is C_2^b, which has 5 intervals. However, the lowest order term in $C(x)$, the generating function for $\mathcal{C}$, will be x^3. Throughout the following, we will use $F(x)$ as the generating function for the block or pattern $\mathcal{F}$.

Recall that an interval order has duplicated holdings if and only if two points of the interval order have the same interval in its minimal endpoint representation. Also, the only way to create duplicated holdings in an ascent sequence is to have $x_i = x_{i+1}$. Thus, we may proceed to think about the blocks on the basis of no duplicated holdings and then form the generating function by allowing repetition of terms in the ascent sequence to allow for duplicated holdings. For conciseness as we do this, we will let $f(x) = x/(1-x)$ for the remainder of the paper.

When W_k^b is preceded and followed by a weak boundary, we do not have length 0 intervals to concern ourselves with. The one that would be present at the left with a strong boundary is simply never created, and the one that would be at the right with a strong boundary is created by the ascent sequence from an earlier block and subsequently moved

into a later block. Thus, we are concerned with a subsequence of length k when we work without duplicated holdings. The subsequence we must have is $b, b+1, b+2, b+3, \ldots, b+k-1$, since each move $b+i$ takes the interval of length 0 and shifts it to the right while adding the interval $[b+i, b+i+1]$. Since $k \geq 1$, we know that the subsequence is not empty. Thus, the generating function component we need here (before allowing for duplicated holdings) is $f/(1-f)$, since there is only one way to do things. Substituting $f(x)$ for f takes care of duplicated holdings for us, since we may repeat the integers from b to $b+k-1$ provided they remain in increasing order and each integer appears at least one time. Therefore, $W(x) = f(x)/(1 - f(x))$.

Note that a $\mathcal{U}$ must have at least six intervals (and thus at least four must be accounted for in our generating function). Also, the intervals appear in pairs. A U_k^b following a weak boundary is created by the subsequence $b, b+1, b, b+1, \ldots, b, b+1$, where there are $k-1$ appearances of $b, b+1$. This is because the b creates an interval with its left endpoint at b and its right endpoint as far right as possible, and then the subsequent $b+1$ produces a Move 3 that truncates the interval created at the previous step to end at $b+1$ and stretches/shifts the other intervals of the block. Since $k \geq 3$, before duplicated holdings here we have $f^4/(1-f^2)$, with the f^4 accounting for the initial $b, b+1, b, b+1$ and the $1/(1-f^2)$ providing the subsequent pairs $b, b+1$. Substituting $f(x)$ for f takes care of duplicated holdings and gives us $U(x) = (f(x))^4/(1 - (f(x))^2)$.

The situation for $\mathcal{C}$ is a slight modification of what we did for $\mathcal{U}$ above, since if the subsequence ended with a b instead of a $b+1$, we would have the interval that spans the length of the $\mathcal{C}$. Thus, the subsequence correspondint to C_k^b must start $b, b+1, b$ and then have $k-2$ pairs $b+1, b$ following it. Since $k \geq 2$, we have $f^3/(1-f^2)$ before addressing duplicated holdings. Therefore $C(x) = (f(x))^3/(1 - (f(x))^2)$. The generating function of C_2^b, which is required in $\mathcal{X}_{oo}$ and $\mathcal{X}_{os}$, is $(f(x))^3$. Since $\mathcal{B}$ merely stands for a $\mathcal{C}$ or a $\mathcal{U}$, $B(x) = U(x) + C(x)$, which simplifies to $(f(x))^3/(1 - f(x))$.

Assembling the generating function is now a matter of introducing additional factors of $f(x)$ for each strong boundary and each weak boundary with optional element, since we known exactly what number must appear in the ascent sequence to produce the required interval, but we may repeat it as many times as we like to account for duplicated holdings. Therefore, we have the following:

$$X_{ss}(x) = f(x)(1 + W(x)) \frac{1}{1 - \frac{B(x)}{1 - B(x)} W(x)} \frac{1}{1 - B(x)}$$

$$X_{so}(x) = f(x)(1 + W(x)) \frac{1}{1 - \frac{B(x)}{1 - B(x)} W(x)} \frac{B(x)}{1 - B(x)}$$

$$X_{oo}(x) = f(x) W(x) \frac{1}{1 - \frac{B(x)}{1 - B(x)} W(x)} \frac{B(x)}{1 - B(x)} + f(x) \cdot (f(x))^3 \cdot \frac{1}{1 - B(x)} \frac{1}{1 - W(x) \frac{B(x)}{1 - B(x)}}$$

$$X_{os}(x) = f(x) W(x) \frac{1}{1 - \frac{B(x)}{1 - B(x)} W(x)} \frac{1}{1 - B(x)}$$
$$+ f(x) \cdot (f(x))^3 \cdot \frac{1}{1 - B(x)} \frac{1}{1 - W(x) \frac{B(x)}{1 - B(x)}} (1 + W(x))$$

$$H(x) = f(x) \frac{1}{1 - X_{ss}(x)} \frac{1}{X_{so}(x) \frac{1}{1 - X_{oo}(x)} X_{os}(x) \frac{1}{1 - X_{ss}(x)}}$$

After fully substituting, we conclude that this section has proved the following theorem:

Theorem 4.1. *The generating function for the number of hereditary semiorders with n points is*

$$H(x) = \frac{x^5 - 9x^4 + 12x^3 - 6x^2 + x}{1 - x^5 + 14x^4 - 29x^3 + 23x^2 - 8x}.$$

A table of values and discussion of asymptotics will be deferred to section 6, after we have completed our enumeration of the semiorders of dimension at most 2.

5. ENUMERATION OF DIMENSION 2 SEMIORDERS

The previous section has completed much of the work required for the enumeration of the semiorders of dimension at most 2, since we have the necessary components to address each of the block types. However, the rules for combining the blocks in this case are different. On one level, things get simpler, because we no longer are allowed to have weak boundaries with optional elements. However, the third statement of Theorem 3.7 places significant restrictions on how a $\mathcal{C}$ may appear in the block structure of a semiorder of dimension at most 2. We can use this to our advantage, however, since an interior $\mathcal{C}$ must have a T_1^b as a neighbor on (at least) one side, which means that interior $\mathcal{C}$ must appear adjacent to a strong boundary.

We proceed by considering what can happen between occurrences of T_1. The first pattern we consider represents when there are no strong boundaries between two appearances of T_1 (other than the strong boundaries necessitated by the T_1s). We call this pattern $\mathcal{A}_0$. Because the blocks on either side of $\mathcal{A}_0$ are T_1, we are allowed the option of a $\mathcal{C}$ as the first block of $\mathcal{A}_0$ or as the last block of $\mathcal{A}_0$, but we may not have a $\mathcal{C}$ anywhere else inside $\mathcal{A}_0$. What appears between these two possible $\mathcal{C}$ must be a mix of $\mathcal{W}$ and $\mathcal{U}$, all combined by weak boundaries. Thus, the interior must take the form $\mathcal{U}^*(\mathcal{W}\mathcal{U}^+)^*\mathcal{W}^?$. It is tempting to sandwich this pattern between two $\mathcal{C}^?$ and be done, but that would give us two distinct ways of getting a $\mathcal{C}$ by itself, which we cannot allow. Thus, our definition is

$$\mathcal{A}_0 = \mathcal{C}^? \left\langle \mathcal{U}^*(\mathcal{W}\mathcal{U}^+)^*\mathcal{W}^? \right\rangle \mathcal{C}^? + \mathcal{C}\mathcal{C}^?,$$

where the $\langle \cdot \rangle$ indicates that we do not allow the enclosed portion of the pattern to be empty. (This is readily accomplished in the generating function by subtracting 1 from the factor that would otherwise be present.)

Next, we consider what happens when there are strong boundaries that occur between the T_1s. The argument is essentially the same as before, giving rise to the pattern

$$\mathcal{A}_s = \left\langle \mathcal{C}^?\mathcal{U}^*(\mathcal{W}\mathcal{U}^+)^*\mathcal{W}^? \right\rangle | \left\langle \mathcal{U}^*(\mathcal{W}\mathcal{U}^+)^*\mathcal{W}^? | \right\rangle^* \left\langle \mathcal{W}^?(\mathcal{U}^+\mathcal{W})^*\mathcal{U}^*\mathcal{C}^? \right\rangle.$$

If we define $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_s$, then $\mathcal{A}$ represents whatever can occur between two non-adjacent T_1 in a semiorder of dimension at most 2. This tells us that the pattern $\mathcal{D}$ that represents all semiorders of dimension at most 2 is

$$\mathcal{D} = (\mathcal{A}|)^? (T_1^+|\mathcal{A})^* T_1^*.$$

The conversion to a generating function proceeds as in the previous section, including the introduction of an initial factor of $f(x)$ to account for the interval $[0,0]$. Since the pattern $\mathcal{D}$ can be empty, this factor will account for T_0 (and duplicated holdings). We do need the generating function to introduce for T_1^+ and T_1^*. Because the subsequence required is prescribed and does not involve any repetitive structure, we conclude that the former is $f(x)/(1 - f(x))$, while the latter is $1 + (f(x)/(1 - f(x)))$. After fully substituting and simplifying, we can therefore conclude the following theorem.

Theorem 5.1. *The generating function for the number of semiorders of dimension at most* 2 *with n points is*

$$D(x) = \frac{-(5x^8 - 41x^7 + 101x^6 - 129x^5 + 96x^4 - 42x^3 + 10x^2 - x)}{7x^8 - 66x^7 + 197x^6 - 311x^5 + 294x^4 - 172x^3 + 61x^2 - 12x + 1}.$$

6. CONCLUSION

Exact and Asymptotic Values. Recalling that the number of semiorders on n points is the nth Catalan number, we can use SageMath [20] and the generating functions from Theorems 4.1 and 5.1 to calculate the number of semiorders on n points, the number of hereditary semiorders on n points, the number of semiorders of dimension at most 2 on n points, and the number of semiorders of dimension 3 on n points. These values are shown in Table 1, with the second line of each column header giving the sequence number in the Online Encyclopedia of Integer Sequences.

n	Semiorders A000108	Hereditary A293499	dim $\leq$ 2 A293498	dim $=$ 3 A293501
1	1	1	1	0
2	2	2	2	0
3	5	5	5	0
4	14	14	14	0
5	42	42	42	0
6	132	132	132	0
7	429	428	426	3
8	1,430	1,415	1,390	40
9	4,862	4,730	4,544	318
10	16,796	15,901	14,822	1,974
11	58,786	53,593	48,183	10,603
12	208,012	180,809	156,118	51,894
13	742,900	610,157	504,487	238,413
14	2,674,440	2,058,962	1,627,000	1,047,440
15	9,694,845	6,947,145	5,240,019	4,454,826
16	35,357,670	23,437,854	16,861,453	18,496,217
17	129,644,790	79,067,006	54,228,190	75,416,600
18	477,638,700	266,717,300	174,351,450	303,287,250
19	1,767,263,190	899,693,960	560,481,708	1,206,781,482
20	6,564,120,420	3,034,814,143	1,801,653,769	4,762,466,651
21	24,466,267,020	10,236,853,534	5,791,301,311	18,674,965,709
22	91,482,563,640	34,530,252,629	18,615,976,402	72,866,587,238
23	343,059,613,650	116,475,001,757	59,841,686,254	283,217,927,396
24	1,289,904,147,324	392,885,252,033	192,366,897,839	1,097,537,249,485
25	4,861,946,401,452	1,325,253,166,761	618,392,292,337	4,243,554,109,115

TABLE 1. Exact counts of the various classes of semiorders

An asymptotic analysis of the coefficients of the rational generating functions derived above is a straightforward application of the techniques of section IV.5 of [10] by Flajolet and Sedgewick. The poles of $H(x)$ are 1 and approximately 0.29646, 11.681, and $0.51131 \pm 0.16533i$. Thus, we have that the number of hereditary semiorders on n points

is asymptotically $0.08346 \cdot 3.373133^n$. The poles of $D(x)$ are approximately 0.311065, 5.60822, $0.456557 \pm 0.123792i$, $0.536649 \pm 0.24759i$, and $0.761438 \pm 0.68404i$. Thus, we have that the number of semiorders of dimension at most 2 on n points is asymptotically $0.12958 \cdot 3.2148^n$. For comparison, recall that the Catalan numbers are asymptotically $4^n/(n^{3/2}\sqrt{\pi})$.

No Duplicated Holdings. As discussed in the arguments that led to Theorems 4.1 and 5.1, we use $f(x) = x/(1-x)$ in the construction of the generating functions to allow for consecutive appearances of an integer in the ascent sequences and therefore duplicated holdings in the poset. If, instead, we write those generating functions in terms of the variable f (replacing any explicit occurrence of $f(x)$ by f), we then have the following corollary.

Corollary 6.1. *The ordinary generating functions for the number of hereditary semiorders with no duplicated holdings ($H_N(f)$) and the number of semiorders of dimension at most* 2 *with no duplicated holdings ($D_N(f)$) are*

$$H_N(f) = \frac{f^5 - f^4 + 2f^2 - f}{2f^4 - 2f^3 - f^2 + 3f - 1}$$

and

$$D_N(f) = \frac{-f^8 + f^7 - f^6 + f^4 - 3f^3 + 3f^2 - f}{f^8 - f^7 + f^6 - f^5 + f^4 + 2f^3 - 5f^2 + 4f - 1}.$$

Restricted Ascent Sequences. As mentioned in the introduction, Kitaev and Remmel showed in [16] that the Catalan numbers enumerate a nicely-defined subset of ascent sequences. They called an ascent sequence $(x_1,\ldots,x_n)$ a *restricted ascent sequences* if $x_1 = 0$ and for all i with $2 \le i \le n$, $m - 1 \le x_i \le \mathrm{asc}((x_1,\ldots,x_{i-1}))$, where m is the largest term in $(x_1,\ldots,x_{i-1})$. However, they also showed that the restricted ascent sequences do not correspond to the semiorders under the bijection Ψ. The ascent sequence $(0,1,0,1,2,0,2)$ of Figure 8 corresponds to a semiorder, but the ascent sequence is not restricted. The sequence $(0,1,0,1,0,1,2)$ is a restricted ascent sequence, but it is easy to verify that it does not correspond to a semiorder. While we are not able at this time to fully characterize the interval orders corresponding to restricted ascent sequences, we do have the following theorem as fairly direct consequence of our earlier work.

Theorem 6.2. *Let P be a semiorder and $(x_1,\ldots,x_n) = \Psi(P)$ the corresponding ascent sequence. The sequence $(x_1,\ldots,x_n)$ is a restricted ascent sequence if and only if P is hereditary.*

Proof. When P is hereditary, the fact that $(x_1,\ldots,x_n)$ is a restricted ascent sequence follows primarily from the proof of the block structure in Theorem 2.5 and the proof of the enumeration of hereditary semiorders in Theorem 4.1. We have given the values of $\ell^*(Q_i)$, where $Q_i = \Psi^{-1}((x_1,\ldots,x_i))$ in the proof of Theorem 2.5. Using the proof of Theorem 4.1, It is straightforward to verify that if the last block is T_1^b, C_a^b, or U_a^b, then the maximum value m in the ascent sequence is $b + 1$. If the last block is W_a^b and $a > 1$, then $m = b + a - 1$. If the last block is W_1^b, then $m = b + 1$ if the preceding boundary is strong and $m = b$ if the preceding boundary is weak. Since the only time we can add an interval that extends to the left of b in a hereditary semiorder in any of these cases is when the last block is W_1^b and we are adding the optional interval, we thus can see by induction that $(x_1,\ldots,x_n)$ is a restricted ascent sequence.

For the converse, we consider a minimal counterexample. That is, we assume that $(x_1, \ldots, x_n)$ is a restricted ascent sequence corresponding to a semiorder but that it is not hereditary. Hence there is an integer $k < n$ such that for all $i \le k$, $Q_i = \Psi^{-1}((x_1, \ldots, x_i))$ is a semiorder but $Q_{k+1} = \Psi^{-1}((x_1, \ldots, x_{k+1}))$ is not a semiorder. By what we have assumed, we know that Q_k is a hereditary semiorder. Therefore, Theorem 2.5 describes its block structure. If the last block is T_1^b, C_a^b, or U_a^b, then $\max_{i:\ 1 \le i \le k} x_i = b + 1$. Thus, $x_{k+1} \ge b$, since we are working with a restricted ascent sequence. If $x_{k+1} = b$, then `Move 1` is used, which cannot create a $\mathbf{1} + \mathbf{3}$ here. From the proof of Theorem 2.5, we also know that if $x_{k+1} \in \{b+1, b+a, b+a+1\}$, then Q_{k+1} is a semiorder. This leaves us to consider $a \ge 3$, $b + 2 \le x_{k+1} < b + a$, and the last block U_a^b or C_a^b. Here, the `Move 3` leaves us with $[b+1, b+2]$ in the interior of $[b, b+a]$, which results from stretching $[b, b+a-1]$. Since neither of these intervals reaches to the largest right endpoint of the minimal endpoint representation, this containment relationship cannot be changed, and $\Psi^{-1}((x_1, \ldots, x_n))$ cannot correspond to a semiorder. When the last block is W_a^b with $a > 1$, then the fact that $m = b + a - 1$ prevents us from adding an interval that contains another in its interior. When the block structure of Q_k ends $|W_1^b$, we have that $m = b + 1$, and thus we cannot add an interval creating a $\mathbf{1} + \mathbf{3}$. When the block structure of Q_k ends with W_1^b preceded by a weak boundary (with or without optional interval), $m = b$. The largest interval we can thus add, given we have a restricted ascent sequence, is $[b-1, b+1]$, which does not create a $\mathbf{1} + \mathbf{3}$ because the interval $[b, b]$ is not present. Therefore, a counterexample cannot exist, and our proof is complete. $\qquad\square$

Open Questions. We close with some possible interesting directions for future work. One would be to consider other classes of combinatorial objects equinumerous to interval orders and ascent sequences to see if there is another natural way to construct a bijection between interval orders and ascent sequences in such a way that every initial subsequence of an ascent sequence corresponding to a semiorder is one that also corresponds to a semiorder. Put another way, can we find a bijection Φ from interval orders to ascent sequences so that replacing Ψ by Φ in Definition 2.1 leads to all semiorders being hereditary?

Theorem 6.2 shows that the restricted ascent sequences defined by Kitaev and Remmel in [16] that correspond to semiorders give rise to precisely the hereditary semiorders. We have left open the question of characterizing all interval orders that correspond to restricted ascent sequences.

Another direction of interest would be to discover more enumerative results involving more global poset statistics. Most of the recent restricted enumeration results focus on statistics that do not appear frequently in the poset literature. (The exceptions being the work of Khamis in [15] and Hu in [13], where height was the driving statistic.) Given an interval representation, the width of an interval order is easy to calculate. An enumeration of interval orders (or semiorders) by width would be of interest. Dimension would be another natural parameter to attempt enumeration by, but since the dimension of interval orders is unbounded, the problem is likely very hard.

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