Jacob's Ladder: Prime numbers in 2d

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January 8, 2018

Abstract

Prime numbers are among the most intriguing figures in mathematics and, despite centuries of research, many questions remain still unsolved. Nowadays, computer simulations are playing a fundamental role in the study of such an immense variety of problems. In this work, we present a simple representation of prime numbers in two dimensions that brings about many interesting questions and allows us to formulate a number of conjectures that may lead to important avenues in the research area of prime numbers.

1 Introduction

Prime numbers have fascinated mathematicians since the beginnings of Mathematics [1–4]. Their distribution is intriguing and almost a mystery despite being non-chaotic. After centuries of research there are indeed many open problems still to be solved, and the exact details of the prime number distribution are yet to be understood [5, 6]. Today, the interest in prime numbers has received a new impulse after their unexpected appearance in different contexts ranging from cryptology [7] or quantum chaos [8, 9] to biology [10, 11]. Furthermore, with the advent of more and more powerful computers, different studies are being undertaken where methods traditionally used by physicists are being applied to the study of primes. For example, in [12] the multifractality of primes was investigated, whereas some appropriately defined Lyapunov exponents for the distribution of primes were calculated numerically in [13]. Not to mention one of the most studied problems in number theory, the Goldbach conjecture, first given about 1740, that is being continuously studied with the help of computers [14–16]. More examples con be found in [17].

2 Jacob's Ladder

One can argue that prime numbers present perplexing features, somewhere hybrid of local unpredictability and global regular behaviour. Is that interplay between randomness and regularity what motivated searches for local and global patterns that could perhaps be the signatures of fundamental mathematical properties.

Our work, as will be seen, is concerned with the long standing question of the prime number distribution, or more precisely with the gaps between primes [18, 19], a topic that has attracted much attention recently [16, 20, 21] after some massive advances [22]. Some problems related to prime gaps are well known, for instance Legendre's conjecture, stating that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n. The conjecture is one of Landau's problems on prime numbers [23] and up to date (2017) the conjecture waits to be proved or disproved. Or the famous twin primes conjecture [24]; Proving this conjecture seemed to be far out of reach until just recently. In 2013,Yitang Zhang demonstrated the existence of a finite bound $B = 70000000$, such that there are in finitely many pairs of distinct primes which di ffer by no more than B [22]. Thanks to this important breakthrough, proving the twin prime conjecture looked much more plausible. Immediately after, a cooperative team led by Terence Tao, by improving Zhang's techniques, was able to lower that bound to 4680 [25, 26]. Even more, also the same year, Maynard further slashed the value of B to 600 [27].

Displaying numbers in two dimensions has been a traditional approach towards primes visualization [28, 29]. We propose in this paper an original way of number arrangement that yields to a particularly appealing visual structure: an oscillating plot that increases and decreases according to the prime number distribution. We plot the integers from 1 to n in 2D (x, y) in the following way: starting with 1 and y = 0 (hence, the first point will be (1, 0)), next step is 2 and we move up in the y axis, so next point in terms of coordinates is (2, 1), and the next step goes up or down depending on whether the number n is a prime number or not. Number 2 is prime so we change and next step will be down and hence the third point is (3, 0), now, 3 is prime so next step is up again, and we move to (4, 1), and 4 is not prime so we continue up and so on and so forth. In Fig. 1 the sequence produced by the algorithm is shown up to $n = 50$. The blue dashed line stands for the $y = 0$ line (or x axis), that will be central in our study. We will refer the points (x, 0) as zeroes from now on. According to the resemblance of this numerical structure with a ladder one may call this set of points in the 2D plane as 'Jacob's Ladder', $J(n)$ for short, hereafter.

3 Methodology

The algorithm can be summarized (in pseudocode) as follows:

```
N = number of "turns" == number of primes !By definition
for i=1 to n
write
1 0
```

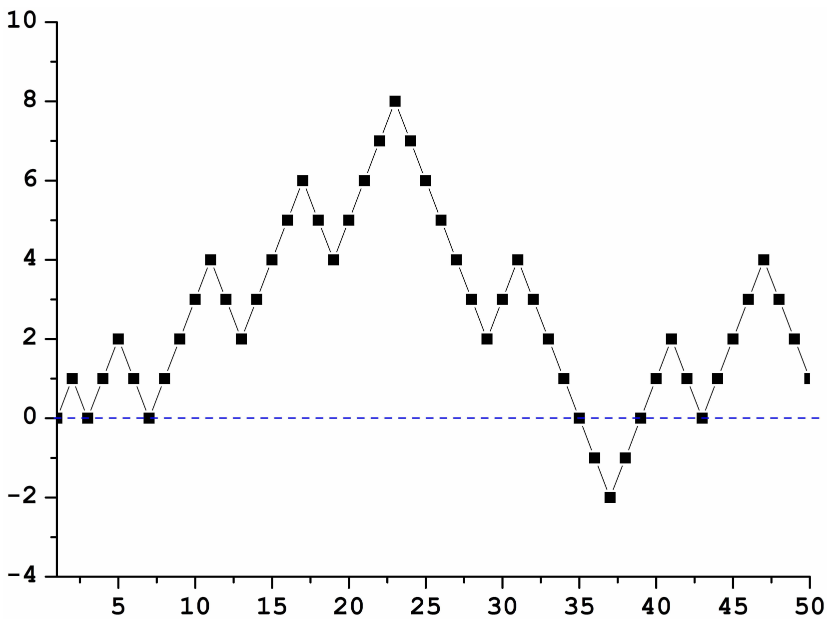

Figure 1: Illustrative plot of the first 50 points of the Jacob's Ladder sequence. The blue dashed line stands for $y = 0$.

```
2 1
for 2 y[2] = +1 !This is to say, we take the value +1 as y[2]
for i=2 count +1 !Start counting primes, 1 is not prime
for i>2
If i is prime: count ++(N) !Counting the number of primes
If i − 1 is prime: y[i] = y[i-1] - (-1) exp N
If i − 1 is not prime: y[i] = y[i-1] + (-1) exp N
write i y[i]
end
```

This algorithm can be implemented in any machine. We used a Fortran90 code that proved to be faster than the same version written in python. Our code is available under request.

4 Results

Next figures present our experimental results to illustrate our ideas. Fig. 2 (Top) shows Jacob's Ladder from 1 to 100. The blue dashed line signals the x axis for clarity shake. As it can be seen up to 100, the Ladder is almost positive, being most of its points above the x axis. However, this is misleading, as we will see in the next figure: Fig. 2 (bottom) shows the Jacob's Ladder from 1 to 10,000. As can be seen now most the Ladder is negative except for two regions. However, this fact changes again if we move to higher values. Fig. 3 (top) shows Jacob's Ladder from 1 to 100,000. Now, as said before, after $n \sim 50,000$

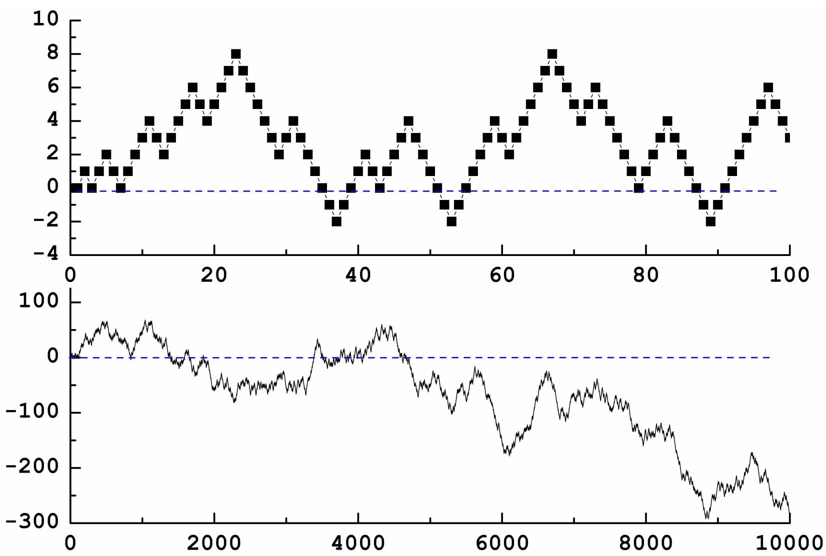

Figure 2: (Top) Jacob's Ladder from 1 to 100. (Bottom) Jacob's Ladder from 1 to 10,000. The blue dashed line stands for $y = 0$.

most the Ladder is again positive. Fig. 3 (bottom) shows the Jacob's Ladder from 1 to 1,000,000. Note that the Ladder presents a big region of negative values after $n \sim 150,000$. No more zeroes are present up to 16,500,000 (See Fig. 4 (Top)). Afterwards, the Ladder is mostly negative again.
Around 45 million, hundreds of zeroes are found (Fig. 4, (bottom)). Going up to 100 million more zeroes appear, totalling 2415. Below 45 million most of the Ladder lies in the negative part. However, from $70 \cdot 10^6$ to $100 \cdot 10^6$, most of it is positive (pointing that Conjectures I, II, III-A and III-B hold true for large numbers - see Next Section).

5 Conjectures

After examining the previous figures it is tempting to present a list of conjectures that despite their simplicity they may be difficult to prove.

Conjecture I:
The number of cuts (zeroes) in the x axis is infinite. In mathematical language, being $Z(n)$ the number of zeroes in the Ladder

$$\lim_{n \to \infty} Z(n) = \infty \tag{1}$$

Discussion:
Proving this conjecture is beyond the scope of this article, but in the following we describe the empirical motivation behind it. That is the main idea in our study and the base of the conjectures that follow. Fig. 5 presents the number of zeroes vs n, in Jacob's Ladder, from 1 to 100 million (logarithmic scale in both axes). Notice that the number of zeroes increases

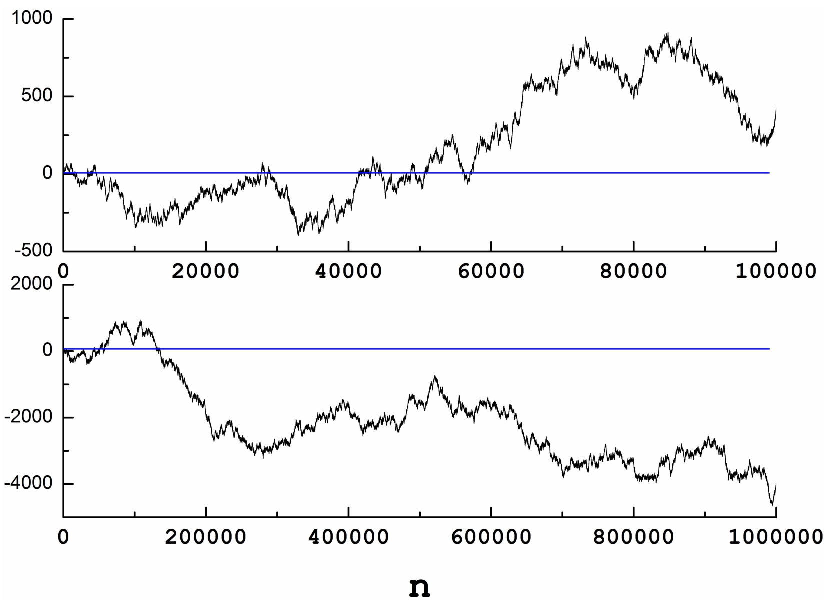

Figure 3: Jacob's Ladder from 1 to 100,000 (top) and from 1 to 1,000,000 (bottom). The blue line stands for $y = 0$.

(by construction it cannot decrease) with n in an apparently chaotic or unpredictable way. In some intervals of n the number of zeroes is constant, meaning that the ladder is above or below $y = 0$, and after those plateaus it increases again and again, and if our conjecture is true, it will increase forever as we move towards bigger values of n. However, we conjecture as well that the slope of the increase ratio will be lower and lower as n goes to infinite. The fact that the prime numbers become increasingly separated seems to indicate so, since the prime numbers are more and more separated and then, the ladder will present less zigzagging.

The sequence of cuts, or zeroes, can be denoted as $F_0(n)$ and the same way an infinite number of successions can be defined, i.e., $F_1(n)$, number of cuts in the $y = 1$ line up to n, $F_{-4}(n)$ cuts in $y = -4$ up to n, and so on. In other words, we conjecture that $F_0(n)$ contains an infinite number of elements, and, if Conjecture I is true, then it's likely that any other $F_y(n)$ will contain infinite terms when n tends to infinity. So, the Ladder allows to nontrivially define an infinite number of successions with infinite terms (and without repetitions) whose cardinal will be $\aleph_0$ and the sum of all of them is again $\aleph_0$.

An interesting question can be formulated here: how does the $Z(n)$ grow? What would be a good approximation for the number of zeroes given a value of n? The answer is not easy, the $Z(n)$ function is neither multiplicative nor additive. In Fig. 5 we present two simple functions that could be some upper and lower limit, $\sqrt{n}$ and $\sqrt[3]{n}$, represented by red lines. In blue, we plotted the counting function, $\pi(n)$, for the shake of comparison.

Here the idea is not to extract accurate upper and lower bounds with these functions, but to give a qualitative approximation of them. The reason why such a qualitative conclusion is relevant could be as follows: It is significant to mention the geometrical insight that can

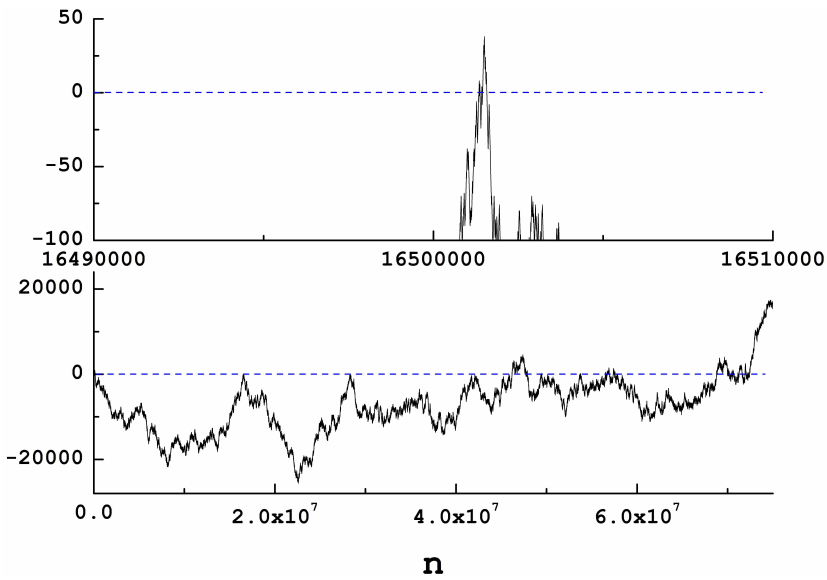

Figure 4: (Top) Jacob's Ladder goes up again at $\sim 16,000,000$. (Bottom). Jacob's Ladder shows a big positive peak and hundreds of zeroes around 44 million (and then again around 70 million). The blue dashed line stands for $y = 0$.

be taken from these simple approximations. If the numbers from 1 to n^2 are plotted in a square, (for instance forming a spiral, like in [28]) then, having approximately n zeroes (or any other special numbers) means that the number of those numbers we have is about the side of the square. In an analogous way, if we display the numbers from 1 to n^3 in a cube of side n, then, the numbers would be about n, the side of the cube. So, having a number of zeroes that is between, $\sqrt{n}$ and $\sqrt[3]{n}$ seems to point to some kind of fractality in the number of zeroes present in the sequence $F_0(n)$.

Remark: Pseudorandom Number Generation

Taking advantage of the definition of the $F_y(n)$ we may point out, as a practical application, a straightforward way of generating pseudorandom numbers. In particular, we are going to focus on a linear congruential generator (LCG) defined by the recurrence relation with parameters a, c and m [30]:

$$X_{i+1} = (aX_i + c) \bmod m \tag{2}$$

Since every horizontal line corresponding to ordinate y defines a subset of points of the Ladder, we can take c and m as the lowest and highest prime numbers in the subset, respectively, whereas parameter a is set to be equal to the ordinate y. If the goal were to find a particular selection of an LCG set of parameters (i.e. a, c and m) yielding a full period generation for all seed values,then the Hull-Dobell Theorem [31] must be satisfied. However, under the definition of the $F_y(n)$, the first condition of the theorem would be already satisfied (i.e. m and c must be relatively primes) for any y. An interesting matter

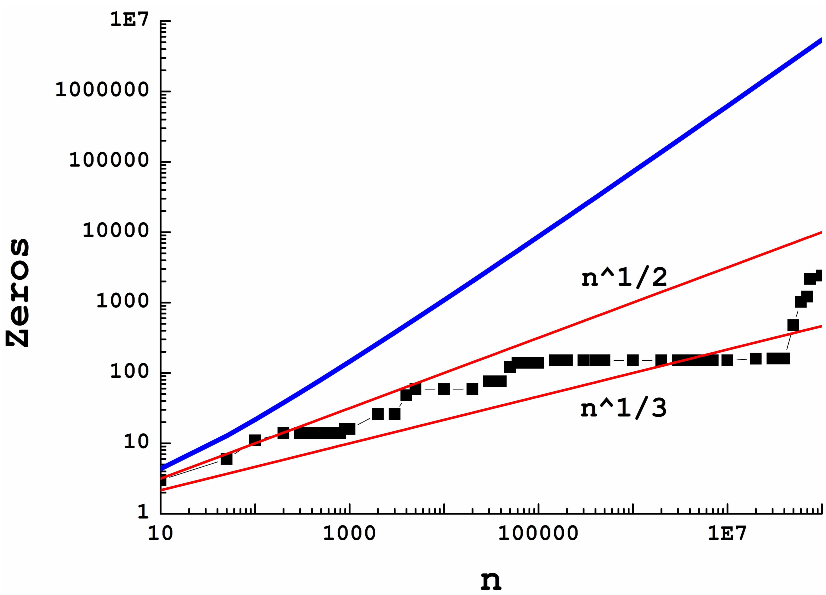

Figure 5: (Color online) $Z(n)$, the number of zeroes in Jacob's Ladder from 1 to 100,000,000. Note the logarithmic scale in both axis. Number of zeroes = 2415. Red lines stand for the $\sqrt{n}$ and $\sqrt[3]{n}$ functions. In blue, we plotted the counting function, $\pi(n)$, for the shake of comparison.

for future development is the question of which of the LCG set of parameters defined by each $F_y(n)$ subset are the fittest from the viewpoint of a pseudorandomness test.

Conjecture II:
The slope, $\epsilon(n)$, of the Ladder is zero in the limit when n goes to infinite.

Discussion:
The Ladder could have a finite (although we conjecture this is not the case) number of zeroes, then, after a certain number, X, all points would be above or below the x axis, and hence the slope would be positive or negative in consequence. Obviously, the slope will be always $<$ 1 or $>$ -1, but in the limit, it could be zero. Even if Conjecture I is false, the slope could decrease continuously when n increases. What is obvious is that if Conjecture I is true, then Conjecture II is more likely, but not necessarily true. In any case, we argue that Conjecture II is true even if Conjecture I is not. A different, more complex question is how fast $\epsilon(n) \to 0$.
The value of $\epsilon(n)$, obtained fitting the Ladder to $y = bx$, will depend, on a first approximation, on the number of primes found in the interval $[1, n]$, hence a simple model can be proposed. A first naive idea could be to assume that the slope could be of the same order of magnitude as $\frac{n/\pi(n)}{n}$, that is to say $1/\pi(n)$, where $\pi(n)$ is the prime-counting function as usual (See Fig. 6). It is to be mentioned that Fig. 6 plots the absolute value of b, but this value can be positive or negative. It goes without saying that that the slope will depend not only in the number of primes but also on their order, which is unknown a priori.

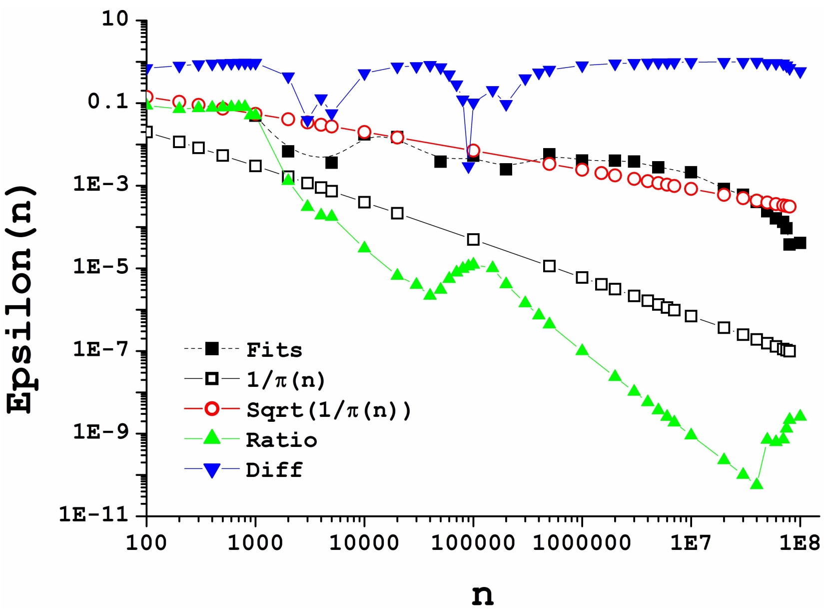

Figure 6: (Color online) Slope, $\epsilon(n)$, of Jacob's Ladder from 100 to 100,000,000 in logarithmic scale. Black symbols are numerical values (after fitting the Ladder to $y = bx$), red empty circles and black empty squares are simple models, green and blue triangles are the slope estimates using two different models.

The $1/\pi(n)$ model seems however to be an underestimate of $\epsilon(n)$, suggesting that $\sqrt{1/\pi(n)}$ could be a better approximation. In fact, this simple expression fits reasonably well the $\epsilon(n)$ values for $n > 10,000$ (See Fig. 6 (red open circles)). It seems clear that $1/\pi(n)$ decreases faster than the slope up to where we could compute. However, this could change in the limit $n \to \infty$. In blue (green) triangles a somewhat crude estimation obtained dividing the difference (ratio) between points up and down by n is presented for comparison. We conjecture that the ratio will tend to 1 so the slope calculated that way will tend to 0. Interestingly, $1/\pi(n)$ seems to be a reasonable fit to the slope calculated that way. The difference divided by n is a clear overestimate of the actual slope as can be seen.

Conjecture III-A

If Conjecture I is true the area below the upper part of the Ladder is equal to the area above the lower part of the Ladder when n tends to infinite, or more precisely, the ratio between both areas, tends to 1 when n tends to infinite.

$$\lim_{n\to\infty} \frac{A_{\text{pos}}(n)}{A_{\text{neg}}(n)} = 1 \tag{3}$$

Discussion:

It is natural to think that if no particular order is found, then the ratio of both areas in the limit will be 1. This is similar but not equivalent to say that the number of points above and below $y = 0$ (positive and negative respectively, or just C_{pos} and C_{neg}) will be the same in the limit n going to infinite, or better said, the ratio will tend to 1. So, a similar conjecture can be formulated:

Conjecture III-B
If Conjecture I is true the ratio between the number of points above and below $y = 0$ will tend to 1.

$$\lim_{n \to \infty} \frac{C_{\text{pos}}(n)}{C_{\text{neg}}(n)} = 1 \tag{4}$$

Discussion:
The area depends not only on the number of points up or down but also on the ordinates of those points, so it is straightforward to see that Conjecture III-A can be true and Conjecture III-B false, or the other way around. However, we presume that in the limit both the area and the ratio $C_{\text{pos}}/C_{\text{neg}}$ will tend to 1. Next graph, Fig. 7 (bottom), shows the number of positive (black) and negative (red) points in Jacob's Ladder up to 100 million.
Importantly, note that Conjecture III (A or B) is not equivalent to Conjecture I. If Conjecture III (A or B) is true then Conjecture I is true but it could be that Conjecture I is true and Conjecture III (A or B) is not[1].
Fig. 7 (top) shows the ratio between the number of positive and negative points in Jacob's Ladder (presented in Fig. 6) up to 100 million. As can be seen, some approximate periodicity over a decreasing behaviour seems apparent. Nonetheless, we think is a misleading result and that the ratio, whenever n is sufficiently big, will be one, and the approximate periodicity in Fig. 7 (top) will disappear.

6 Discussion

While writing the paper, we found similar studies like those presented in [32–34]. For instance, in [34] a one-dimensional random walk (RW), where steps up and down are performed according to the occurrence of special primes (twins and cousins), was defined. If there is an infinite amount of twins and cousins (as suggested by the Hardy-Littlewood conjecture), then the RW defined there will continue to perform steps forever, in contrast to the RW considered in [32] or [33], where random walks were finite. In our case, we prefer not to talk about random walks since the distribution of primes despite being mysterious is not random. On the other side, the Jacob's ladder idea is simple and beautiful, since, because of the infinity of prime numbers, its intrinsic complexity and the zigzagging will continue indefinitely. Now, by definition, its properties depend both on the number of primes in a given interval and on the separation between them. It is known that the gaps between consecutive prime numbers cluster on multiples of 6 [35,36]. Because of this fact 6

[1]Conjecture III (A and B) could be stated as Conjecture I, and hence we could talk of Conjecture I (infinite zeroes) as Corollary, since if Conjecture III (A, B or both) is true, then, that demonstrates the number of zeroes is infinite, being the result a corollary of the Conjecture I. However we consider more natural, at a first glance of Jacob's ladder, to think of the number of possible cuts (zeroes) when n tends to infinite, so we keep this order.

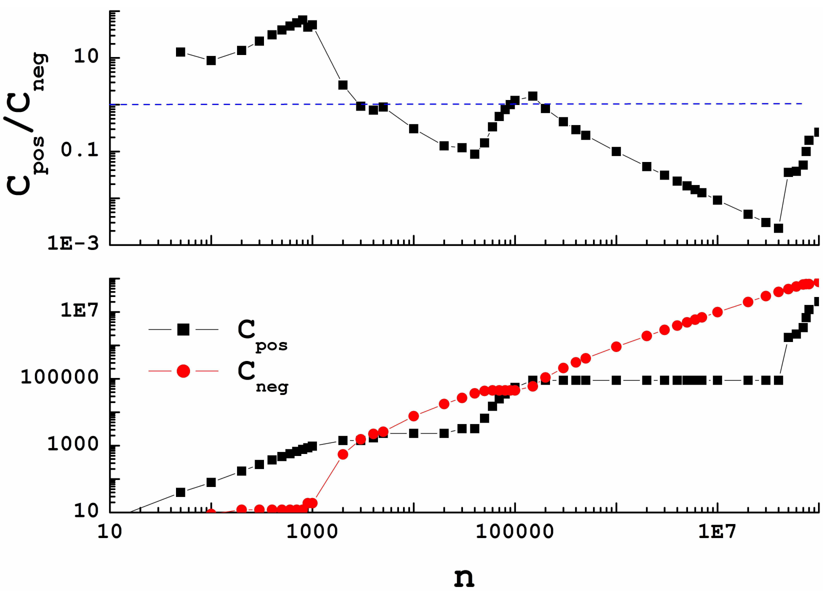

Figure 7: (Color online) *Bottom)* Number of positive (black squares) and negative (red circles) points in Jacob's Ladder up to 100,000,000. *Top)* Ratio between number of positive and negative points up to 100,000,000. The blue dashed line marks $y = 1$, the conjectured ratio. Note the logarithmic scale in both axes.

is sometimes called the jumping champion, and it is conjectured that it stays the champion all the way up to about 1035 [35, 36].
Beyond 1035, and until 10425, the jumping champion becomes 30 (=$2 \cdot 3 \cdot 5$), and beyond that the most frequent gap is 210 (=$2 \cdot 3 \cdot 5 \cdot 7$) [36]. It is a natural conjecture that after some big number then the jumping champion will be 2310 (=$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$) and so on and so forth. Further interesting results on some statistical properties between gaps have been recently found [37, 38]. However, all the aforementioned numerical observations, despite revealing intriguing properties of the prime sequence, are not easily put into our problem in order to know whether or not the Ladder will have or not infinite zeroes or not. On the other side, according to Ares et al. [39], the apparent regularities observed in some works [34, 40, 41] do not reveal any structure in the sequence of primes, and that it is precisely a consequence of its randomness [31]. However, there seems to be some controversy. Recent computational work points that "after appropriate rescaling, the statistics of spacings between adjacent prime numbers follows the Poisson distribution" [42]. See [42, 43] and references therein for more on the statistics of the gaps between consecutive prime numbers.
The problem seems to be how to know if after a given number all turns in the Ladder will have an "order". By order we mean that $J(n)$ will go up (or down) after the next prime, and then down (or up) and so on but always keeping a property: that in "average" the curve will continue forever in the same direction without going back to the x axis. To be precise, this rises the question: is it possible that, after an unknown number X, the sum of all intervals between primes "up" minus the sum of all intervals "down" (or the other way

around) will be positive (or negative) for all possible Y values when n tends to infinite?

$$\sum_{X}^{Y} (I[\uparrow] - I[\downarrow]) > 0\,(< 0)\;\forall Y \tag{5}$$

Where $I[\uparrow]$ ($I[\downarrow]$) means the intervals between two primes $[p_n - p_{n-1}]$ that make to Ladder go up (down)[2]. That looks unlikely and would be some interesting order property in case that it exists (and would prove Conjecture I to be false). If so, it would be interesting to find that number X distinctly.

Further questions: Are the zeroes randomly distributed, or following some kind of distribution? For instance, do the terms in $F_0(n)$ follow Benford's law [44]? It doesn't seem to be the case although a proof is beyond the scope of this paper (See Fig. 8). Note that despite examining the Ladder up to 30 million, the number of zeroes was only 160. After checking up to 70 million we counted around 1200 zeroes, which gave us a different behaviour, clearly not following Bendford's law. It was tempting to believe that the zeroes will present a Gaussian (or similar) distribution centred in $d = 5$, as shown by the data presented in Fig. 8. However, we suspected that it was just an artifact due to the number of zeroes found around 46 million. Examining the Ladder up to 100 million, more zeroes were found, which gave a clear non-Gaussian distribution (See green triangles in Fig. 8) confirming our initial judgement. In any case it is more than clear that statistics on such a small sample are not conclusive, and even with a bigger sample there would not be a proof.

Up to here we focused on the sequence of zeroes, $F_0(n)$. We conjectured that the number of terms in $F_y(n)$ tends to infinite for all y. However, from looking at Fig. 4 (bottom) we infer, that if we take the results in the interval $[1,\, 75 \cdot 10^6]$, some sequences will have more terms than others, for instance $F_y(n)$ with $y < 0$ will have in average much more terms than those with $y > 0$. Nevertheless, the asymmetry is believed to be just an artifact (let us say, a size effect due to the small interval examined) and, as discussed before, we conjectured that in the limit the number of points above and below 0 will be the same (the ratio will tend to 1). It is natural to assume that the distribution presented by $F_0(n)$ will be the same for all values of y. Leaded by this assumption we carried out the same analysis for $F_y(n)$ with $y =$ -3,000, -8,000, and –10,000, sequences that contain around 5 times more terms than $F_0(n)$. Comparing those results to Fig. 8 a clear departure from Benford's law was observed for the three $F_y(n)$ selected. So, while being aware that the numbers are still very small from a statistically point of view, this analysis of the terms in more crowded sequences can provide further evidence of the random distribution of $F_y(n)$ for all y.

And to conclude a few notes about the zeroes obtained insofar: First thing to be noticed is that they are all odd numbers. This is not a surprise and can be easily demonstrated that, it must be this way. Since, after 3, (the first zero excluding 1) all zeroes must be the sum of an even number of even numbers (since every prime number is separated by an even number), then the gaps between zeroes will be always an even number, and hence, all of them will be odd.

[2] *Sensu stricto* this could be true for a number $n = X$ and the sum represented by Eq. (1) could be $> z$ (or $< z$) without crossing the x axis because we start from some point up (or down) the axis.

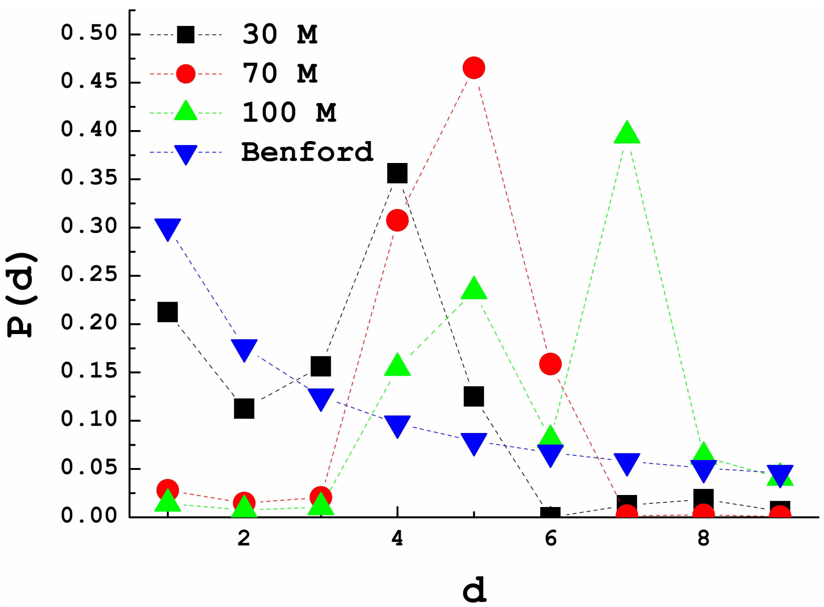

Figure 8: (Color online) Leading digit histogram of the zeroes sequence $F_0(n)$. Proportion of the different leading digits found in $30 \cdot 10^6$ (black squares), $70 \cdot 10^6$ (red circles), $100 \cdot 10^6$ (green triangles) and the expected values according to the Benford's law (blue inverted triangles).

For one more test about some possible unexpected nonrandom properties of the zeroes in the Ladder we can check the number of zeroes ending in a given odd digit (since all the even ones will not be in $F_0(n)$). Doing so we observe a possible trend in the terms ending in 1, 3, 5, 7 and 9, being the number of them 459, 479, 484, 487 and 506 respectively in the first 2415 zeroes, i.e. showing a clear linear increase. A linear fit ($y = a + bx$) to the data (in probabilities, given as %) gives the following parameters: a=18.9441 (0.21452) and b=0.21118 (0.03734), however whether this will be true or not in the limit $n \to \infty$ is an open question.

To finish, how many of these zeroes are primes? In the first 2415 zeroes, we find 313 primes. Is this number predictable? Since in the interval $[1, n]$ we find $\sim n/\log n$ primes, should we expect, in a set of X consecutive zeroes (but not consecutive numbers!), a number of primes approximately equal to $X/\log X$? In our case that would give us 310 in remarkable agreement with our numerical result. In Fig. 9, we present the number of primes as a function of the number of zeroes found in $F_0(n)$. However, in the sequence $F_0(n)$ all numbers are odd, hence, instead of $n/\log n$, should we expect the number of primes to be $2n/\log n$?

And two final remarks; we checked the sequence of zeroes in OEIS [45] and no match was found, as expected. Also, inspired by the idea of the Pisano periods, we checked if some period exists in the last digit of the sequence of the zeroes. If so, the period is larger than the number of zeroes found insofar.

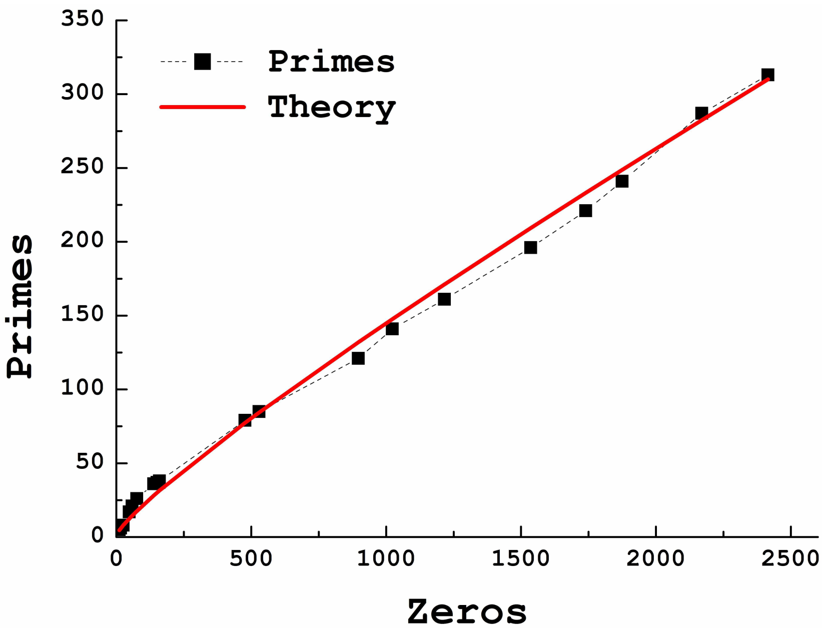

Figure 9: (Color online) Number of primes as a function of the number of terms (zeroes) in the sequence $F_0(n)$. The red line stands for the conjectured counting function $\pi(n) = n/\log n$.

7 Conclusions

In this paper, an original idea has been proposed with the aim of arranging integers in 2D according to the occurrence of primality. This arrangement of numbers resembles a 'ladder' and displays peaks and valleys at the positions of the primes. Numerical studies have been undertaken in order to extract qualitative information from such a representation. As a result of those studies it is possible to promote four observations to the category of conjectures. One may easily state that the ladder, from its own construction, will cross the horizontal axis an indefinite number of times; namely, the ladder will have infinite 'zeroes'. The number of primes in $F_0(n)$ is very well fitted by $n/\log n$ but, interestingly, this series of zeroes does not seem to follow Benford's Law, at least up to the range of $n = 10^8$. The number of zeroes grows in a chaotic way somewhere between $\sqrt{n}$ and $\sqrt[3]{n}$ and the sequence of zeroes, $F_0(n)$, apart of being, by construction odd numbers all of them, does not seem to follow any distribution in particular other than a possible linear trend in the last digit, being more likely to be 9 and less likely to be 1.

Our theoretical predictions are largely based on intuition, but we provide qualitative support from our preliminary results. Although the behaviors reported here have been validated only for n as large as 10^8, it is reasonable to expect that they would be observed for arbitrary n. Mathematical proofs for our conjectures are, likely, to be extremely difficult since they are inextricably intertwined with the mysterious prime numbers distribution.

As final and most important conclusion we can speculate that the main feature of the sequence of prime numbers, namely, its randomness, will fulfil the conjectures. Or, the other way around, that the conjectures will be true only if the sequence is completely random. Thus, new and interesting phenomena can be derived from our study and novel applications may be settled.

Author Contributions
The authors were responsible for all aspects of this study.

Conflicts of Interest
The authors declare no conflict of interest.

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