# Congruences of Power Sums 

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#### Abstract

The following congruence for power sums, $S_{n}(p)$, is well known and has many applications: $$
1^{n}+2^{n}+\cdots+p^{n} \equiv \begin{cases}-1(\bmod p), & \text { if } p-1 \mid n ; \\ 0 \quad(\bmod p), & \text { if } p-1 \nmid n,\end{cases}
$$ where $n \in \mathbb{N}$ and $p$ is prime. We extend this congruence, in particular, to the case when $p$ is any power of a prime. We also show that the sequence $\left(S_{n}(m) \bmod k\right)_{m \geqslant 1}$ is periodic and determine its period.


## 1 Introduction

Sums of powers of integers defined below have captivated mathematicians for many centuries [1].

Definition 1. For $n, m \in \mathbb{N}$, let

$$
S_{n}(m)=\sum_{i=1}^{m} i^{n} .
$$

With their pebble experiments, the Pythagoreans were the first to discover a formula for the sum of the first powers. Formulas for the sums of second and third powers were proved geometrically by Aryabhatta and Archimedes, and Harriot later provided a generalizable form for these formulas. Faulhaber gave formulas for power sums up to the $17^{\text {th }}$ power, and Fermat, Pascal, and Bernoulli provided succinct formulas for them. Since then, different representations and number-theoretic properties of these power sums have been an object of study [5, 6]. Bernoulli numbers have been used to represent the coefficients of polynomial formulas for these power sums such as Faulhaber's formula [3, p. 107]. In a recent paper, Newsome et al. [10] have demonstrated symmetry properties of the power sum polynomials and their roots via a novel Bernoulli number identity.

One of the most well-known results concerning the number-theoretic properties of power sums is the following congruence relation:

Theorem 2. If $n \in \mathbb{N}$ and $p$ is prime, then

$$
S_{n}(p) \equiv \begin{cases}-1 \quad(\bmod p), & \text { if } p-1 \mid n ; \\ 0 \quad(\bmod p), & \text { if } p-1 \nmid n .\end{cases}
$$

The case $p-1 \mid n$ is an easy consequence of Fermat's little theorem. There are several different proofs for the case $p-1 \npreceq n$ in the literature. Some of the notable ones are by Rado $[4,8,11]$ using the theory of primitive roots, Zagier [7] using Lagrange's theorem, and MacMillan and Sondow [6] using Pascal's identity. Also, a proof of both cases by Carlitz [2] uses Bernoulli numbers.

This congruence is used to prove the von Staudt-Clausen theorem [4, 11] and its generalization [2], prove the Carlitz-von Staudt theorem [7], and study the Erdős-Moser equation $S_{n}(m-1)=m^{n}[7,8,9]$.

Our main goal in this paper is to generalize the well-known congruence result above and to present periodicity properties of the sequence $\left(S_{n}(m) \bmod \right.$ $k)_{m \geqslant 1}$. In Section 2, we extend Theorem 2 to the case when $p$ is a power of a prime. In Section 3, we prove that $\left(S_{n}(m) \bmod k\right)_{m \geqslant 1}$ is periodic and determine its period for different values of $k$ and $n$.

## 2 Generalization of Theorem 2

Theorem 3. (1) For $n \in \mathbb{N}$ and $p=2^{a}$ with $a \geqslant 2$,

$$
S_{n}(p) \equiv \begin{cases}\varphi(p)(\bmod p), & \text { if } n=1 \text { or } 2 \mid n ; \\ 0 \quad(\bmod p), & \text { if } n>1 \text { and } 2 \nmid n,\end{cases}
$$

where $\varphi$ is Euler's totient function.
(2) If $n \in \mathbb{N}$ and $p=q^{a}$ where $q$ is an odd prime and $a \geqslant 1$, then

$$
S_{n}(p) \equiv \begin{cases}\varphi(p)(\bmod p), & \text { if } q-1 \mid n \\ 0 \quad(\bmod p), & \text { if } q-1 \nmid n\end{cases}
$$

Proof. (1) The proof is by induction on $a$. For $a=2$,

$$
\begin{aligned}
S_{n}(4) & \equiv 1^{n}+2^{n}+3^{n}+4^{n} \equiv 1^{n}+2^{n}+(-1)^{n} \\
& \equiv\left\{\begin{array}{lll}
2 & (\bmod 4), & \text { if } n=1 \text { or } 2 \mid n ; \\
0 & (\bmod 4), & \text { if } n>1 \text { and } 2 \nmid n .
\end{array}\right.
\end{aligned}
$$

Suppose the statement holds for some $a \geqslant 2$. Then, for $n=1$ we have

$$
S_{1}\left(2^{a+1}\right) \equiv \frac{2^{a+1}\left(2^{a+1}+1\right)}{2} \equiv 2^{a} \equiv \varphi\left(2^{a+1}\right) \quad\left(\bmod 2^{a+1}\right)
$$

and for $n \geqslant 2$,

$$
\begin{aligned}
S_{n}\left(2^{a+1}\right) & \equiv 1^{n}+\cdots+\left(2^{a}\right)^{n}+\left(2^{a}+1\right)^{n}+\cdots+\left(2^{a+1}\right)^{n} \\
& \equiv S_{n}\left(2^{a}\right)+\sum_{t=1}^{2^{a}}\left(2^{a}+t\right)^{n} \\
& \equiv S_{n}\left(2^{a}\right)+\sum_{t=1}^{2^{a}}\left(t^{n}+n 2^{a} t^{n-1}\right)
\end{aligned}
$$

(all other terms are divisible by $\left(2^{a}\right)^{2}$, thus divisible by $2^{a+1}$ )

$$
\equiv 2 S_{n}\left(2^{a}\right)+n 2^{a} \underbrace{S_{n-1}\left(2^{a}\right)}_{\text {even }}
$$

(since $a \geqslant 2, S_{n-1}\left(2^{a}\right)$ has an even number of odd terms) $\equiv 2 S_{n}\left(2^{a}\right) \quad\left(\bmod 2^{a+1}\right)$.

If $2 \mid n$, then

$$
S_{n}\left(2^{a}\right) \equiv \varphi\left(2^{a}\right) \quad\left(\bmod 2^{a}\right)
$$

so

$$
S_{n}\left(2^{a+1}\right) \equiv 2 S_{n}\left(2^{a}\right) \equiv 2 \varphi\left(2^{a}\right) \equiv \varphi\left(2^{a+1}\right) \quad\left(\bmod 2^{a+1}\right)
$$

If $2 \nmid n$, then

$$
S_{n}\left(2^{a}\right) \equiv 0 \quad\left(\bmod 2^{a}\right)
$$

so

$$
S_{n}\left(2^{a+1}\right) \equiv 2 S_{n}\left(2^{a}\right) \equiv 0 \quad\left(\bmod 2^{a+1}\right)
$$

(2) The proof is by induction on $a$. The case $a=1$ is Theorem 2 .

Suppose the statement holds for some $a \geqslant 1$. Then, for $n=1$ we have

$$
S_{1}\left(q^{a+1}\right)=\frac{q^{a+1}\left(q^{a+1}+1\right)}{2} \equiv 0 \quad\left(\bmod q^{a+1}\right),
$$

and for $n \geqslant 2$,
$S_{n}\left(q^{a+1}\right) \equiv\left(1^{n}+\cdots+\left(q^{a}\right)^{n}\right)+\cdots+\left(\left((q-1) q^{a}+1\right)^{n}+\cdots+\left(q^{a+1}\right)^{n}\right)$

$$
\equiv \sum_{i=0}^{q-1} \sum_{t=1}^{q^{a}}\left(i q^{a}+t\right)^{n}
$$

$$
\equiv \sum_{i=0}^{q-1} \sum_{t=1}^{q^{a}}\left(t^{n}+n i q^{a} t^{n-1}\right)
$$

(all other terms are divisible by $\left(q^{a}\right)^{2}$, thus divisible by $q^{a+1}$ )

$$
\begin{aligned}
& \equiv \sum_{i=0}^{q-1}\left(S_{n}\left(q^{a}\right)+n i q^{a} S_{n-1}\left(q^{a}\right)\right) \\
& \equiv q S_{n}\left(q^{a}\right)+n \frac{(q-1) q}{2} q^{a} S_{n-1}\left(q^{a}\right) \\
& \equiv q S_{n}\left(q^{a}\right) \quad\left(\bmod q^{a+1}\right)
\end{aligned}
$$

If $q-1 \mid n$, then

$$
S_{n}\left(q^{a}\right) \equiv \varphi\left(q^{a}\right) \quad\left(\bmod q^{a}\right),
$$

so

$$
S_{n}\left(q^{a+1}\right) \equiv q S_{n}\left(q^{a}\right) \equiv q \varphi\left(q^{a}\right) \equiv \varphi\left(q^{a+1}\right) \quad\left(\bmod q^{a+1}\right)
$$

If $q-1 \nmid n$, then

$$
S_{n}\left(q^{a}\right) \equiv 0 \quad\left(\bmod q^{a}\right)
$$

SO

$$
S_{n}\left(q^{a+1}\right) \equiv q S_{n}\left(q^{a}\right) \equiv 0 \quad\left(\bmod q^{a+1}\right) .
$$

Corollary 4. For any $a, n \in \mathbb{N}$ and prime $q$,

$$
S_{n}\left(q^{a}\right) \equiv 0 \quad\left(\bmod q^{a-1}\right) .
$$

The next few results will be used to extend Theorem 3 (2).
Lemma 5. If $n, i, j, k \in \mathbb{N}$ and $q$ is an odd prime such that $q^{i} \mid n$, then $q^{i+j} \left\lvert\,\binom{ n}{k}\left(q^{j}\right)^{k}\right.$. Moreover, for $k \geqslant 2, q^{i+j+1} \left\lvert\,\binom{ n}{k}\left(q^{j}\right)^{k}\right.$.

Proof. If $k=1$, then $\binom{n}{k}\left(q^{j}\right)^{k}=n q^{j}$ is divisible by $q^{i+j}$.
If $k \geqslant 2$, note that since $\binom{n}{k}=\frac{n(n-1) \cdots(n-k+1)}{k!}$ and the highest power of $q$ that divides $k$ ! is $q^{\alpha}$ where $\alpha=\left\lfloor\frac{k}{q}\right\rfloor+\left\lfloor\frac{k}{q^{2}}\right\rfloor+\cdots$, it is sufficient to show that

$$
i+j<i-\left(\left\lfloor\frac{k}{q}\right\rfloor+\left\lfloor\frac{k}{q^{2}}\right\rfloor+\cdots\right)+j k .
$$

Indeed,

$$
\begin{aligned}
i+j & \leqslant i+j+\frac{k}{2}-1 \\
& =i+j-\frac{k}{2}+k-1 \\
& \leqslant i+j-\frac{k}{q-1}+j(k-1) \\
& =i-\sum_{t=1}^{\infty} \frac{k}{q^{t}}+j k \\
& <i-\left(\left\lfloor\frac{k}{q}\right\rfloor+\left\lfloor\frac{k}{q^{2}}\right\rfloor+\cdots\right)+j k
\end{aligned}
$$

Corollary 6. If $i, j, n, t \in \mathbb{N}, q$ is an odd prime, and $q^{i} \mid n$, then

$$
\left(t+q^{j}\right)^{n} \equiv t^{n} \quad\left(\bmod q^{i+j}\right)
$$

Proof. We have

$$
\begin{aligned}
\left(t+q^{j}\right)^{n} & \equiv t^{n}+\sum_{k=1}^{n}\binom{n}{k}\left(q^{j}\right)^{k} t^{n-k} \\
& \equiv t^{n} \quad\left(\bmod q^{i+j}\right) . \quad(\text { by Lemma } 5)
\end{aligned}
$$

If $q$ is an odd prime and $g$ is invertible modulo $q$, then multiplication by $g$ permutes elements of $\mathbb{Z}_{q}^{*}$, that is,

$$
\begin{equation*}
\{g \cdot 1 \bmod q, \ldots, g(q-1) \bmod q\}=\{1, \ldots, q-1\} \tag{1}
\end{equation*}
$$

as sets.
The following theorem extends Theorem 3 (2).
Theorem 7. If $i \in \mathbb{Z}, i \geqslant 0, j, n \in \mathbb{N}$, $q$ is an odd prime, $q-1 \nmid n$, and $q^{i} \mid n$, then

$$
S_{n}\left(q^{j}\right) \equiv 0 \quad\left(\bmod q^{i+j}\right)
$$

Proof. The case $i=0$ is Theorem 3 (2).
For any fixed $i \geqslant 1$ we use induction on $j$. First consider $j=1$. Let $g$ be a generator of the multiplicative group $\mathbb{Z}_{q}^{*}$. Then

$$
\begin{aligned}
g^{n} S_{n}(q) & \equiv g^{n} \sum_{k=1}^{q} k^{n} \\
& \equiv \sum_{k=1}^{q}(g k)^{n} \\
& \equiv \sum_{k=1}^{q}(g k \bmod q)^{n} \quad(\text { by Corollary } 6) \\
& \equiv \sum_{k=1}^{q} k^{n} \quad(\text { by }(1)) \\
& \equiv S_{n}(q) \quad\left(\bmod q^{i+1}\right) .
\end{aligned}
$$

Thus

$$
\left(g^{n}-1\right) S_{n}(q) \equiv 0 \quad\left(\bmod q^{i+1}\right) .
$$

But $g^{n} \not \equiv 1(\bmod q)$ since $g$ is a generator of $\mathbb{Z}_{q}^{*}$ and $q-1 \nmid n$. Therefore

$$
S_{n}(q) \equiv 0 \quad\left(\bmod q^{i+1}\right)
$$

Now assume that $S_{n}\left(q^{j}\right) \equiv 0\left(\bmod q^{i+j}\right)$ for some $j \geqslant 1$. Then

$$
\begin{aligned}
S_{n}\left(q^{j+1}\right) & \equiv \sum_{t=0}^{q-1} \sum_{r=1}^{q^{j}}\left(t q^{j}+r\right)^{n} \\
& \equiv \sum_{t=0}^{q-1} \sum_{r=1}^{q^{j}}\left(r^{p}+n t q^{j} r^{n-1}+\sum_{k=2}^{n}\binom{n}{k}\left(t q^{j}\right)^{k} r^{n-k}\right) \\
& \equiv \sum_{t=0}^{q-1} \sum_{r=1}^{q^{j}}\left(r^{n}+n t q^{j} r^{n-1}+0\right) \quad(\text { by Lemma }) \\
& \equiv \sum_{t=0}^{q-1}\left(S_{n}\left(q^{j}\right)+n t q^{j} S_{n-1}\left(q^{j}\right)\right) \\
& \equiv q S_{n}\left(q^{j}\right)+n \frac{(q-1) q}{2} q^{j} S_{n-1}\left(q^{j}\right) \\
& \equiv 0 \quad\left(\bmod q^{i+j+1}\right) . \quad\left(\text { since } q^{i} \mid n\right)
\end{aligned}
$$

## 3 Periodicity

In this section, we first establish the periodicity of the sequence of sequences $\left(\left(S_{n}(m) \bmod k\right)_{n \geqslant 1}\right)_{m \geqslant 1}$ for any $k \in \mathbb{N}$. An immediate implication of this result is that the sequence $\left(S_{n}(m) \bmod k\right)_{m \geqslant 1}$ is periodic for all values of $k$ and $n$. We then provide formulas for the length of the period when $k$ is a power of a prime.

Theorem 8. For each $k \in \mathbb{N}$, the sequence of sequences

$$
\left(\left(S_{1}(m) \bmod k, S_{2}(m) \bmod k, S_{3}(m) \bmod k, \ldots\right)\right)_{m \geqslant 1}
$$

is periodic. If $k=q_{1}^{a_{1}} q_{2}^{a_{2}} \cdots q_{r}^{a_{r}}$ where $q_{i}$ 's are distinct primes, then the period is $q_{1}^{a_{1}+1} q_{2}^{a_{2}+1} \cdots q_{r}^{a_{r}+1}$.

Proof. We will first prove that

$$
S_{n}\left(m+q^{a+1}\right) \equiv S_{n}(m) \quad\left(\bmod q^{a}\right)
$$

for all prime $q$ and natural $n, m$, and $a$. We have

$$
\begin{aligned}
S_{n}\left(m+q^{a+1}\right) & \equiv S_{n}\left(q^{a+1}\right)+\left(q^{a+1}+1\right)^{n}+\cdots+\left(q^{a+1}+m\right)^{n} \\
& \equiv S_{n}\left(q^{a+1}\right)+S_{n}(m) \\
& \equiv S_{n}(m) \quad\left(\bmod q^{a}\right)
\end{aligned}
$$

since $S_{n}\left(q^{a+1}\right) \equiv 0\left(\bmod q^{a}\right)$ by Corollary 4.
Thus, the sequence of sequences

$$
\left(\left(S_{1}(m) \bmod q^{a}, S_{2}(m) \bmod q^{a}, S_{3}(m) \bmod q^{a}, \ldots\right)\right)_{m \geqslant 1}
$$

repeats every $q^{a+1}$ terms. Thus it is periodic with period being a factor of $q^{a+1}$. To show that the period is not less than $q^{a+1}$, it is sufficient to show that the sequence does not repeat every $q^{a}$ terms. More precisely, we will show that $S_{n}\left(q^{a}\right) \not \equiv S_{n}\left(q^{a+1}\right)\left(\bmod q^{a}\right)$ for at least one value of $n$.

Consider $n=q-1$ (or, in fact, any $n$ divisible by $q-1$ if $q$ is odd). By Theorem 2 in the case $a=1$, and by Theorem 3 otherwise, and using Corollary 4,

$$
S_{n}\left(q^{a}\right) \equiv \varphi\left(q^{a}\right) \not \equiv 0 \equiv S_{n}\left(q^{a+1}\right) \quad\left(\bmod q^{a}\right)
$$

Thus the sequence does not repeat every $q^{a}$ terms, which implies the period is exactly $q^{a+1}$.
Next, if $k=q_{1}^{a_{1}} q_{2}^{a_{2}} \cdots q_{r}^{a_{r}}$ where $q_{i}$ 's are distinct primes, then from the case proved above and the Chinese Remainder Theorem, it follows that the period of the sequence is $q_{1}^{a_{1}+1} q_{2}^{a_{2}+1} \cdots q_{r}^{a_{r}+1}$.

It follows from Theorem 8 that given any values of $k$ and $n$, the sequence

$$
\left(S_{n}(m) \bmod k\right)_{m \geqslant 1}
$$

is periodic with period not exceeding the one given in Theorem 8. However, for some values of $k$ and $n$ the period is smaller.

Theorem 9. For $k, n \in \mathbb{N}$, let $\ell(k, n)$ denote the period of the sequence $\left(S_{n}(m) \bmod k\right)_{m \geqslant 1}$. Then
(1) $\ell(2, n)=4$ for all $n$.
(2) for $a \geqslant 2$,

$$
\ell\left(2^{a}, n\right)= \begin{cases}2^{a+1}, & \text { if } n=1 \text { or } 2 \mid n ; \\ 2^{a}, & \text { otherwise }\end{cases}
$$

(3) for $q$ an odd prime and $a \geqslant 1$,

$$
\ell\left(q^{a}, n\right)= \begin{cases}q^{a+1}, & \text { if } q-1 \mid n ; \\ q^{a-i}, & \text { if } q-1 \nmid n, \nu_{q}(n)=i, 0 \leqslant i \leqslant a-2 ; \\ q, & \text { if } q-1 \nmid n \text { and } q^{a-1} \mid n,\end{cases}
$$

where $\nu_{q}(n)$ is the exponent of the highest power of $q$ that divides $n$.
Proof. (1) Theorem 8 implies that $\ell(2, n)$ is a factor of 4 . Since

$$
\begin{aligned}
1^{n} & \equiv 1 \quad(\bmod 2), \\
1^{n}+2^{n} & \equiv 1 \quad(\bmod 2), \\
1^{n}+2^{n}+3^{n} & \equiv 0 \quad(\bmod 2), \\
1^{n}+2^{n}+3^{n}+4^{n} & \equiv 0 \quad(\bmod 2),
\end{aligned}
$$

$\ell(2, n)=4$.
(2) Let $a \geqslant 2$.

If $n=1$ or $2 \mid n$, by Theorem $3(1)$ we have $S_{n}\left(2^{a}\right) \equiv \varphi\left(2^{a}\right)\left(\bmod 2^{a}\right)$. However, Theorem 8 implies that $\ell\left(2^{a}, n\right)$ is a factor of $2^{a+1}$, and hence must be $2^{a+1}$.

If $n>1$ and $2 \nmid n$, Theorem $3(1)$ implies that $S_{n}\left(2^{a}\right) \equiv 0\left(\bmod 2^{a}\right)$.
We have

$$
\begin{aligned}
S_{n}\left(m+2^{a}\right) & \equiv S_{n}\left(2^{a}\right)+\left(2^{a}+1\right)^{n}+\cdots+\left(2^{a}+m\right)^{n} \\
& \equiv S_{n}\left(2^{a}\right)+S_{n}(m) \\
& \equiv S_{n}(m) \quad\left(\bmod 2^{a}\right) .
\end{aligned}
$$

Thus, $\ell\left(2^{a}, n\right)$ is a factor of $2^{a}$. We now show that $\ell\left(2^{a}, n\right)$ is not smaller than $2^{a}$. Assume to the contrary that $\ell\left(2^{a}, n\right)$ is a factor of $2^{a-1}$. Then

$$
S_{n}\left(2^{a-1}\right) \equiv S_{n}\left(2^{a}\right) \equiv 0 \quad\left(\bmod 2^{a}\right),
$$

but then

$$
\begin{aligned}
S_{n}\left(2^{a-1}+1\right) & \equiv S_{n}\left(2^{a-1}\right)+\left(2^{a-1}+1\right)^{n} \\
& \equiv 0+1^{n}+n 2^{a-1}+\sum_{k=2}^{n}\binom{n}{k}\left(2^{a-1}\right)^{k} 1^{n-k} \\
& \equiv 1^{n}+n 2^{a-1} \\
& \not \equiv 1^{n} \quad(\text { since } n \text { is odd }) \\
& \equiv S_{n}(1) \quad\left(\bmod 2^{a}\right)
\end{aligned}
$$

which is a contradiction.
(3) Let $q$ be an odd prime and $a \geqslant 1$.

The case $q-1 \mid n$ follows from the proof of Theorem 8 .
If $q-1 \nmid n$ and $q^{i} \mid n$ for $0 \leqslant i \leqslant a-1$, then by Theorem 7

$$
S_{n}\left(q^{a-i}\right) \equiv 0 \quad\left(\bmod q^{a}\right)
$$

Then

$$
\begin{aligned}
S_{n}\left(m+q^{a-i}\right) & \equiv S_{n}\left(q^{a-i}\right)+\sum_{r=1}^{m}\left(q^{a-i}+r\right)^{n} \\
& \equiv 0+\sum_{r=1}^{m}\left(r^{n}+\sum_{k=1}^{n}\binom{n}{k}\left(q^{a-i}\right)^{k} r^{n-k}\right) \\
& \equiv \sum_{r=1}^{m} r^{n} \quad(\text { by Lemma } 5) \\
& \equiv S_{n}(m) \quad\left(\bmod q^{a}\right)
\end{aligned}
$$

so $\ell\left(q^{a}, n\right)$ is a factor of $q^{a-i}$.
We will show that if $q^{i+1} \nmid n$ for $0 \leqslant i \leqslant a-2$, then $\ell\left(q^{a}, n\right)$ is not smaller than $q^{a-i}$. Assume to the contrary that $\ell\left(q^{a}, n\right)$ is a factor of $q^{a-i-1}$. Then

$$
S_{n}\left(q^{a-i-1}\right) \equiv S_{n}\left(q^{a-i}\right) \equiv 0 \quad\left(\bmod q^{a}\right)
$$

but then

$$
\begin{aligned}
S_{n}\left(q^{a-i-1}+1\right) & \equiv S_{n}\left(q^{a-i-1}\right)+\left(q^{a-i-1}+1\right)^{n} \\
& \equiv 0+1^{n}+n q^{a-i-1}+\sum_{k=2}^{n}\binom{n}{k}\left(q^{a-i-1}\right)^{k} 1^{n-k} \\
& \equiv 1^{n}+n q^{a-i-1}+0 \quad(\text { by Lemma } 5) \\
& \not \equiv 1^{n} \quad\left(\operatorname{since} q^{i+1} \nmid n\right) \\
& \equiv S_{n}(1) \quad\left(\bmod q^{a}\right),
\end{aligned}
$$

which is a contradiction.
Thus we have shown that for $0 \leqslant i \leqslant a-2$, if $q-1 \nmid n$, and $\nu_{q}(n)=i$, then $\ell\left(q^{a}, n\right)$ is a factor of $q^{a-i}$ but not a factor of $q^{a-i-1}$. Therefore, $\ell\left(q^{a}, n\right)=q^{a-i}$.
In the last case $\left(q-1 \nmid n\right.$ and $\left.q^{a-1} \mid n\right)$, we have shown above that $\ell\left(q^{a}, n\right)$ is a factor of $q$. However,

$$
S_{n}(1) \equiv 1 \not \equiv 0 \equiv S_{n}(q) \quad\left(\bmod q^{a}\right),
$$

so $\ell\left(q^{a}, n\right) \neq 1$. Therefore, $\ell\left(q^{a}, n\right)=q$.

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