Congruences of Power Sums

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Abstract

The following congruence for power sums, $S_n(p)$, is well known and has many applications:

$$1^{n} + 2^{n} + \dots + p^{n} \equiv \begin{cases} -1 \pmod{p}, & \text{if } p - 1 \mid n; \\ 0 \pmod{p}, & \text{if } p - 1 \nmid n, \end{cases}$$

where $n \in \mathbb{N}$ and p is prime. We extend this congruence, in particular, to the case when p is any power of a prime. We also show that the sequence $(S_n(m) \mod k)_{m \ge 1}$ is periodic and determine its period.

1 Introduction

Sums of powers of integers defined below have captivated mathematicians for many centuries [1].

Definition 1. For $n, m \in \mathbb{N}$, let

$$S_n(m) = \sum_{i=1}^m i^n.$$

With their pebble experiments, the Pythagoreans were the first to discover a formula for the sum of the first powers. Formulas for the sums of second and third powers were proved geometrically by Aryabhatta and Archimedes, and Harriot later provided a generalizable form for these formulas. Faulhaber gave formulas for power sums up to the 17th power, and Fermat, Pascal, and Bernoulli provided succinct formulas for them. Since then, different representations and number-theoretic properties of these power sums have been an object of study [5, 6]. Bernoulli numbers have been used to represent the coefficients of polynomial formulas for these power sums such as Faulhaber's formula [3, p. 107]. In a recent paper, Newsome et al. [10] have demonstrated symmetry properties of the power sum polynomials and their roots via a novel Bernoulli number identity.

One of the most well-known results concerning the number-theoretic properties of power sums is the following congruence relation:

Theorem 2. If $n \in \mathbb{N}$ and p is prime, then

$$S_n(p) \equiv \begin{cases} -1 \pmod{p}, & \text{if } p-1 \mid n; \\ 0 \pmod{p}, & \text{if } p-1 \nmid n. \end{cases}$$

The case $p-1 \mid n$ is an easy consequence of Fermat's little theorem. There are several different proofs for the case $p-1 \nmid n$ in the literature. Some of the notable ones are by Rado [4, 8, 11] using the theory of primitive roots, Zagier [7] using Lagrange's theorem, and MacMillan and Sondow [6] using Pascal's identity. Also, a proof of both cases by Carlitz [2] uses Bernoulli numbers.

This congruence is used to prove the von Staudt-Clausen theorem [4, 11] and its generalization [2], prove the Carlitz-von Staudt theorem [7], and study the Erdős-Moser equation $S_n(m-1) = m^n$ [7, 8, 9].

Our main goal in this paper is to generalize the well-known congruence result above and to present periodicity properties of the sequence $(S_n(m) \mod k)_{m\geq 1}$. In Section 2, we extend Theorem 2 to the case when p is a power of a prime. In Section 3, we prove that $(S_n(m) \mod k)_{m\geq 1}$ is periodic and determine its period for different values of k and n.

2 Generalization of Theorem 2

Theorem 3. (1) For $n \in \mathbb{N}$ and $p = 2^a$ with $a \ge 2$,

$$S_n(p) \equiv \begin{cases} \varphi(p) \pmod{p}, & \text{if } n = 1 \text{ or } 2 \mid n; \\ 0 \pmod{p}, & \text{if } n > 1 \text{ and } 2 \nmid n, \end{cases}$$

where φ is Euler's totient function.

(2) If $n \in \mathbb{N}$ and $p = q^a$ where q is an odd prime and $a \ge 1$, then

$$S_n(p) \equiv \begin{cases} \varphi(p) \pmod{p}, & \text{if } q-1 \mid n; \\ 0 \pmod{p}, & \text{if } q-1 \nmid n. \end{cases}$$

Proof. (1) The proof is by induction on a. For a = 2,

$$S_n(4) \equiv 1^n + 2^n + 3^n + 4^n \equiv 1^n + 2^n + (-1)^n$$
$$\equiv \begin{cases} 2 \pmod{4}, & \text{if } n = 1 \text{ or } 2 \mid n; \\ 0 \pmod{4}, & \text{if } n > 1 \text{ and } 2 \nmid n. \end{cases}$$

Suppose the statement holds for some $a \ge 2$. Then, for n = 1 we have

$$S_1(2^{a+1}) \equiv \frac{2^{a+1}(2^{a+1}+1)}{2} \equiv 2^a \equiv \varphi(2^{a+1}) \pmod{2^{a+1}},$$

and for $n \ge 2$,

$$S_n(2^{a+1}) \equiv 1^n + \dots + (2^a)^n + (2^a + 1)^n + \dots + (2^{a+1})^n$$
$$\equiv S_n(2^a) + \sum_{t=1}^{2^a} (2^a + t)^n$$
$$\equiv S_n(2^a) + \sum_{t=1}^{2^a} (t^n + n2^a t^{n-1})$$

(all other terms are divisible by $(2^a)^2$, thus divisible by 2^{a+1}) $\equiv 2S_n(2^a) + n2^a \underbrace{S_{n-1}(2^a)}_{\text{even}}$

(since $a \ge 2$, $S_{n-1}(2^a)$ has an even number of odd terms) $\equiv 2S_n(2^a) \pmod{2^{a+1}}.$ If $2 \mid n$, then

$$S_n(2^a) \equiv \varphi(2^a) \pmod{2^a},$$

 \mathbf{SO}

$$S_n(2^{a+1}) \equiv 2S_n(2^a) \equiv 2\varphi(2^a) \equiv \varphi(2^{a+1}) \pmod{2^{a+1}}.$$

If $2 \nmid n$, then

$$S_n(2^a) \equiv 0 \pmod{2^a},$$

SO

$$S_n(2^{a+1}) \equiv 2S_n(2^a) \equiv 0 \pmod{2^{a+1}}.$$

(2) The proof is by induction on a. The case a = 1 is Theorem 2. Suppose the statement holds for some $a \ge 1$. Then, for n = 1 we have

$$S_1(q^{a+1}) = \frac{q^{a+1}(q^{a+1}+1)}{2} \equiv 0 \pmod{q^{a+1}},$$

and for $n \ge 2$,

$$S_n(q^{a+1}) \equiv (1^n + \dots + (q^a)^n) + \dots + \left(((q-1)q^a + 1)^n + \dots + (q^{a+1})^n\right)$$
$$\equiv \sum_{i=0}^{q-1} \sum_{t=1}^{q^a} (iq^a + t)^n$$
$$\equiv \sum_{i=0}^{q-1} \sum_{t=1}^{q^a} \left(t^n + niq^a t^{n-1}\right)$$

(all other terms are divisible by $(q^a)^2$, thus divisible by q^{a+1})

$$\equiv \sum_{i=0}^{q-1} (S_n(q^a) + niq^a S_{n-1}(q^a))$$

$$\equiv qS_n(q^a) + n\frac{(q-1)q}{2}q^a S_{n-1}(q^a)$$

$$\equiv qS_n(q^a) \pmod{q^{a+1}}.$$

If $q-1 \mid n$, then

$$S_n(q^a) \equiv \varphi(q^a) \pmod{q^a},$$

 \mathbf{SO}

$$S_n(q^{a+1}) \equiv qS_n(q^a) \equiv q\varphi(q^a) \equiv \varphi(q^{a+1}) \pmod{q^{a+1}}.$$

If $q - 1 \nmid n$, then

$$S_n(q^a) \equiv 0 \pmod{q^a},$$

SO

$$S_n(q^{a+1}) \equiv qS_n(q^a) \equiv 0 \pmod{q^{a+1}}.$$

Corollary 4. For any $a, n \in \mathbb{N}$ and prime q,

$$S_n(q^a) \equiv 0 \pmod{q^{a-1}}.$$

The next few results will be used to extend Theorem 3 (2).

Lemma 5. If $n, i, j, k \in \mathbb{N}$ and q is an odd prime such that $q^i \mid n$, then $q^{i+j} \mid \binom{n}{k} (q^j)^k$. Moreover, for $k \ge 2$, $q^{i+j+1} \mid \binom{n}{k} (q^j)^k$. *Proof.* If k = 1, then $\binom{n}{k} (q^j)^k = nq^j$ is divisible by q^{i+j} . If $k \ge 2$, note that since $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$ and the highest |k| = |k|.

power of q that divides k! is q^{α} where $\alpha = \left\lfloor \frac{k}{q} \right\rfloor + \left\lfloor \frac{k}{q^2} \right\rfloor + \cdots$, it is sufficient to show that $i + j < i - \left(\left\lfloor \frac{k}{q} \right\rfloor + \left\lfloor \frac{k}{q} \right\rfloor + \cdots \right) + jk.$

$$i + j < i - \left(\left\lfloor \frac{k}{q} \right\rfloor + \left\lfloor \frac{k}{q^2} \right\rfloor + \cdots \right) + jk.$$

Indeed,

$$i+j \leq i+j+\frac{k}{2}-1$$

$$= i+j-\frac{k}{2}+k-1$$

$$\leq i+j-\frac{k}{q-1}+j(k-1)$$

$$= i-\sum_{t=1}^{\infty}\frac{k}{q^t}+jk$$

$$< i-\left(\left\lfloor\frac{k}{q}\right\rfloor+\left\lfloor\frac{k}{q^2}\right\rfloor+\cdots\right)+jk.$$

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Corollary 6. If $i, j, n, t \in \mathbb{N}$, q is an odd prime, and $q^i \mid n$, then

$$(t+q^j)^n \equiv t^n \pmod{q^{i+j}}.$$

Proof. We have

$$(t+q^{j})^{n} \equiv t^{n} + \sum_{k=1}^{n} \binom{n}{k} (q^{j})^{k} t^{n-k}$$
$$\equiv t^{n} \pmod{q^{i+j}}. \qquad \text{(by Lemma 5)}$$

If q is an odd prime and g is invertible modulo q, then multiplication by g permutes elements of \mathbb{Z}_q^* , that is,

$$\{g \cdot 1 \bmod q, \dots, g(q-1) \bmod q\} = \{1, \dots, q-1\}$$
(1)

as sets.

The following theorem extends Theorem 3 (2).

Theorem 7. If $i \in \mathbb{Z}$, $i \ge 0$, $j, n \in \mathbb{N}$, q is an odd prime, $q - 1 \nmid n$, and $q^i \mid n$, then

$$S_n(q^j) \equiv 0 \pmod{q^{i+j}}.$$

Proof. The case i = 0 is Theorem 3 (2).

For any fixed $i \ge 1$ we use induction on j. First consider j = 1. Let g be a generator of the multiplicative group \mathbb{Z}_q^* . Then

$$g^{n}S_{n}(q) \equiv g^{n}\sum_{k=1}^{q}k^{n}$$
$$\equiv \sum_{k=1}^{q}(gk)^{n}$$
$$\equiv \sum_{k=1}^{q}(gk \mod q)^{n} \quad \text{(by Corollary 6)}$$
$$\equiv \sum_{k=1}^{q}k^{n} \quad \text{(by (1))}$$
$$\equiv S_{n}(q) \pmod{q^{i+1}}.$$

Thus

$$(g^n - 1)S_n(q) \equiv 0 \pmod{q^{i+1}}.$$

But $g^n \not\equiv 1 \pmod{q}$ since g is a generator of \mathbb{Z}_q^* and $q - 1 \not\nmid n$. Therefore

$$S_n(q) \equiv 0 \pmod{q^{i+1}}.$$

Now assume that $S_n(q^j) \equiv 0 \pmod{q^{i+j}}$ for some $j \ge 1$. Then

$$S_{n}(q^{j+1}) \equiv \sum_{t=0}^{q-1} \sum_{r=1}^{q^{j}} (tq^{j} + r)^{n}$$

$$\equiv \sum_{t=0}^{q-1} \sum_{r=1}^{q^{j}} \left(r^{p} + ntq^{j}r^{n-1} + \sum_{k=2}^{n} \binom{n}{k} (tq^{j})^{k}r^{n-k} \right)$$

$$\equiv \sum_{t=0}^{q-1} \sum_{r=1}^{q^{j}} \left(r^{n} + ntq^{j}r^{n-1} + 0 \right) \quad \text{(by Lemma 5)}$$

$$\equiv \sum_{t=0}^{q-1} \left(S_{n}(q^{j}) + ntq^{j}S_{n-1}(q^{j}) \right)$$

$$\equiv qS_{n}(q^{j}) + n\frac{(q-1)q}{2}q^{j}S_{n-1}(q^{j})$$

$$\equiv 0 \pmod{q^{i+j+1}}. \quad \text{(since } q^{i} \mid n)$$

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3 Periodicity

In this section, we first establish the periodicity of the sequence of sequences $((S_n(m) \mod k)_{n \ge 1})_{m \ge 1}$ for any $k \in \mathbb{N}$. An immediate implication of this result is that the sequence $(S_n(m) \mod k)_{m \ge 1}$ is periodic for all values of k and n. We then provide formulas for the length of the period when k is a power of a prime.

Theorem 8. For each $k \in \mathbb{N}$, the sequence of sequences

 $((S_1(m) \mod k, S_2(m) \mod k, S_3(m) \mod k, \ldots))_{m \ge 1}$

is periodic. If $k = q_1^{a_1} q_2^{a_2} \cdots q_r^{a_r}$ where q_i 's are distinct primes, then the period is $q_1^{a_1+1} q_2^{a_2+1} \cdots q_r^{a_r+1}$.

Proof. We will first prove that

$$S_n(m+q^{a+1}) \equiv S_n(m) \pmod{q^a}$$

for all prime q and natural n, m, and a. We have

$$S_n(m + q^{a+1}) \equiv S_n(q^{a+1}) + (q^{a+1} + 1)^n + \dots + (q^{a+1} + m)^n$$

$$\equiv S_n(q^{a+1}) + S_n(m)$$

$$\equiv S_n(m) \pmod{q^a}$$

since $S_n(q^{a+1}) \equiv 0 \pmod{q^a}$ by Corollary 4. Thus, the sequence of sequences

$$((S_1(m) \mod q^a, S_2(m) \mod q^a, S_3(m) \mod q^a, \dots))_{m \ge 1}$$

repeats every q^{a+1} terms. Thus it is periodic with period being a factor of q^{a+1} . To show that the period is not less than q^{a+1} , it is sufficient to show that the sequence does not repeat every q^a terms. More precisely, we will show that $S_n(q^a) \not\equiv S_n(q^{a+1}) \pmod{q^a}$ for at least one value of n.

Consider n = q - 1 (or, in fact, any *n* divisible by q - 1 if *q* is odd). By Theorem 2 in the case a = 1, and by Theorem 3 otherwise, and using Corollary 4,

$$S_n(q^a) \equiv \varphi(q^a) \neq 0 \equiv S_n(q^{a+1}) \pmod{q^a}.$$

Thus the sequence does not repeat every q^a terms, which implies the period is exactly q^{a+1} .

Next, if $k = q_1^{a_1} q_2^{a_2} \cdots q_r^{a_r}$ where q_i 's are distinct primes, then from the case proved above and the Chinese Remainder Theorem, it follows that the period of the sequence is $q_1^{a_1+1} q_2^{a_2+1} \cdots q_r^{a_r+1}$.

It follows from Theorem 8 that given any values of k and n, the sequence

$$(S_n(m) \mod k)_{m \ge 1}$$

is periodic with period not exceeding the one given in Theorem 8. However, for some values of k and n the period is smaller.

Theorem 9. For $k, n \in \mathbb{N}$, let $\ell(k, n)$ denote the period of the sequence $(S_n(m) \mod k)_{m \ge 1}$. Then

(1)
$$\ell(2,n) = 4$$
 for all n .

(2) for $a \ge 2$,

$$\ell(2^{a}, n) = \begin{cases} 2^{a+1}, & \text{if } n = 1 \text{ or } 2 \mid n; \\ 2^{a}, & \text{otherwise.} \end{cases}$$

(3) for q an odd prime and $a \ge 1$,

$$\ell(q^{a}, n) = \begin{cases} q^{a+1}, & \text{if } q-1 \mid n; \\ q^{a-i}, & \text{if } q-1 \nmid n, \ \nu_{q}(n) = i, \ 0 \leq i \leq a-2; \\ q, & \text{if } q-1 \nmid n \text{ and } q^{a-1} \mid n, \end{cases}$$

where $\nu_q(n)$ is the exponent of the highest power of q that divides n.

Proof. (1) Theorem 8 implies that $\ell(2, n)$ is a factor of 4. Since

$$1^{n} \equiv 1 \pmod{2},$$

$$1^{n} + 2^{n} \equiv 1 \pmod{2},$$

$$1^{n} + 2^{n} + 3^{n} \equiv 0 \pmod{2},$$

$$1^{n} + 2^{n} + 3^{n} = 0 \pmod{2},$$

$$1^{n} + 2^{n} + 3^{n} + 4^{n} \equiv 0 \pmod{2},$$

 $\ell(2,n) = 4.$

(2) Let $a \ge 2$.

If n = 1 or $2 \mid n$, by Theorem 3 (1) we have $S_n(2^a) \equiv \varphi(2^a) \pmod{2^a}$. However, Theorem 8 implies that $\ell(2^a, n)$ is a factor of 2^{a+1} , and hence must be 2^{a+1} .

If n > 1 and $2 \nmid n$, Theorem 3 (1) implies that $S_n(2^a) \equiv 0 \pmod{2^a}$. We have

$$S_n(m+2^a) \equiv S_n(2^a) + (2^a+1)^n + \dots + (2^a+m)^n$$

$$\equiv S_n(2^a) + S_n(m)$$

$$\equiv S_n(m) \pmod{2^a}.$$

Thus, $\ell(2^a, n)$ is a factor of 2^a . We now show that $\ell(2^a, n)$ is not smaller than 2^a . Assume to the contrary that $\ell(2^a, n)$ is a factor of 2^{a-1} . Then

$$S_n(2^{a-1}) \equiv S_n(2^a) \equiv 0 \pmod{2^a},$$

but then

$$S_n(2^{a-1}+1) \equiv S_n(2^{a-1}) + (2^{a-1}+1)^n$$

$$\equiv 0 + 1^n + n2^{a-1} + \sum_{k=2}^n \binom{n}{k} (2^{a-1})^k 1^{n-k}$$

$$\equiv 1^n + n2^{a-1}$$

$$\not\equiv 1^n \quad (\text{since } n \text{ is odd})$$

$$\equiv S_n(1) \pmod{2^a},$$

which is a contradiction.

(3) Let q be an odd prime and $a \ge 1$.

The case $q - 1 \mid n$ follows from the proof of Theorem 8. If $q - 1 \nmid n$ and $q^i \mid n$ for $0 \leq i \leq a - 1$, then by Theorem 7

$$S_n(q^{a-i}) \equiv 0 \pmod{q^a}.$$

Then

$$S_n(m+q^{a-i}) \equiv S_n(q^{a-i}) + \sum_{r=1}^m (q^{a-i}+r)^n$$
$$\equiv 0 + \sum_{r=1}^m \left(r^n + \sum_{k=1}^n \binom{n}{k} (q^{a-i})^k r^{n-k}\right)$$
$$\equiv \sum_{r=1}^m r^n \quad \text{(by Lemma 5)}$$
$$\equiv S_n(m) \pmod{q^a},$$

so $\ell(q^a, n)$ is a factor of q^{a-i} .

We will show that if $q^{i+1} \not\mid n$ for $0 \leq i \leq a-2$, then $\ell(q^a, n)$ is not smaller than q^{a-i} . Assume to the contrary that $\ell(q^a, n)$ is a factor of q^{a-i-1} . Then

$$S_n(q^{a-i-1}) \equiv S_n(q^{a-i}) \equiv 0 \pmod{q^a},$$

but then

$$S_n(q^{a-i-1}+1) \equiv S_n(q^{a-i-1}) + (q^{a-i-1}+1)^n$$

$$\equiv 0 + 1^n + nq^{a-i-1} + \sum_{k=2}^n \binom{n}{k} (q^{a-i-1})^k 1^{n-k}$$

$$\equiv 1^n + nq^{a-i-1} + 0 \quad \text{(by Lemma 5)}$$

$$\not\equiv 1^n \quad (\text{since } q^{i+1} \not\mid n)$$

$$\equiv S_n(1) \quad (\text{mod } q^a),$$

which is a contradiction.

Thus we have shown that for $0 \leq i \leq a-2$, if $q-1 \neq n$, and $\nu_q(n) = i$, then $\ell(q^a, n)$ is a factor of q^{a-i} but not a factor of q^{a-i-1} . Therefore, $\ell(q^a, n) = q^{a-i}$.

In the last case $(q - 1 \not\mid n \text{ and } q^{a-1} \mid n)$, we have shown above that $\ell(q^a, n)$ is a factor of q. However,

$$S_n(1) \equiv 1 \neq 0 \equiv S_n(q) \pmod{q^a},$$

so $\ell(q^a, n) \neq 1$. Therefore, $\ell(q^a, n) = q$.

4 Acknowledgi	ment
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The authors would like to thank the College of Science and Mathematics at California State University, Fresno for supporting this work.

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2010 Mathematics Subject Classification Primary 11A07; Secondary 11B50, 11A25, 11B83.

Keywords: number theory, power sum, congruence, periodicity, period, Euler phi-function.

(Concerned with sequences A000010, A000217, A026729, A027641, and A027642.)