Convergence of Pascal-Like Triangles in Parry–Bertrand Numeration Systems

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Abstract

We pursue the investigation of generalizations of the Pascal triangle based on binomial coefficients of finite words. These coefficients count the number of times a finite word appears as a subsequence of another finite word. The finite words occurring in this paper belong to the language of a Parry numeration system satisfying the Bertrand property, i.e., we can add or remove trailing zeroes to valid representations. It is a folklore fact that the Sierpiński gasket is the limit set, for the Hausdorff distance, of a convergent sequence of normalized compact blocks extracted from the classical Pascal triangle modulo 2. In a similar way, we describe and study the subset of $[0,1]\times[0,1]$ associated with the latter generalization of the Pascal triangle modulo a prime number.

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1 Introduction

Several generalizations and variations of the Pascal triangle exist and lead to interesting combinatorial, geometrical or dynamical properties [1, 2, 8, 9, 10]. This paper is inspired by a series of papers based on generalizations of Pascal triangles to finite words [10, 11, 12, 13].

1.1 Binomial coefficients of words and Pascal-like triangles

In this short subsection, we briefly introduce the concepts we use in this paper. For more definitions, see section 2. A *finite word* is a finite sequence of letters belonging to a finite set called the *alphabet*. The *binomial coefficient* $\binom{u}{v}$ of two finite words u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Let A be a totally ordered alphabet, and let $L \subset A^*$ be an infinite language over A. We order the words of L by increasing genealogical order and we write $L = \{w_0 < w_1 < w_2 < \cdots\}$. Associated with the language L, we define a Pascal-like triangle $\mathrm{P}_L : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ represented as an infinite table. The entry $\mathrm{P}_L(m,n)$ on the mth row and nth column of P_L is the integer $\binom{w_m}{w_n}$.

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1.2 Previous work

Let b be an integer greater than 1. We let $\mathrm{rep}_b(n)$ denote the (greedy) base-b expansion of $n \in \mathbb{N} \setminus \{0\}$ starting with a non-zero digit. We set $\mathrm{rep}_b(0)$ to be the empty word denoted by ε. We let

$$L_b = \{1, \dots, b-1\}\{0, \dots, b-1\}^* \cup \{\varepsilon\}$$

be the set of base-b expansions of the non-negative integers. In [10], we study the particular case of $L = L_b$. The increasing genealogical order thus coincides with the classical order in $\mathbb{N}$. For example, see Table 1 for the first few values[2] of P_2. Clearly, P_b contains several subtables corresponding to the usual Pascal triangle.

	ε	1	10	11	100	101	110	111
ε	**1**	0	0	0	0	0	0	0
1	**1**	**1**	0	0	0	0	0	0
10	1	1	1	0	0	0	0	0
11	**1**	**2**	0	**1**	0	0	0	0
100	1	1	2	0	1	0	0	0
101	1	2	1	1	0	1	0	0
110	1	2	2	1	0	0	1	0
111	**1**	**3**	0	**3**	0	0	0	**1**

Table 1: The first few values in the generalized Pascal triangle P_2 (A282714).

For instance, it contains $(b-1)$ copies of the usual Pascal triangle obtained when only considering words of the form a^m with $a \in \{1, \dots, b-1\}$ and $m \geq 0$ since $\binom{a^m}{a^n} = \binom{m}{n}$. In Table 1, a copy of the classical Pascal triangle is written in bold.

Considering the intersection of the lattice $\mathbb{N}^2$ with $[0, 2^n] \times [0, 2^n]$, the first 2^n rows and columns of the usual Pascal triangle modulo 2 $(\binom{i}{j} \bmod 2)_{0 \leq i,j < 2^n}$ provide a coloring of this lattice. If we normalize this compact set by a homothety of ratio $1/2^n$, we get a sequence of subsets of $[0,1] \times [0,1]$ which converges, for the Hausdorff distance, to the Sierpiński gasket when n tends to infinity. In the extended context described above, the case when $b = 2$ gives similar results and the limit set, generalizing the Sierpiński gasket, is described using a simple combinatorial property called $(\star)$ [10].

Inspired by [10], we study the sequence $(S_b(n))_{n \geq 0}$ which counts, on each row m of P_b, the number of words of L_b occurring as subwords of the mth word in L_b, i.e., $S_b(m) = \#\{n \in \mathbb{N} \mid \mathrm{P}_b(m, n) > 0\}$. This sequence is shown to be b-regular [11, 13]. We also consider the summatory function $(A_b(n))_{n \geq 0}$ of the sequence $(S_b(n))_{n \geq 0}$ and study its behavior [12, 13].

So far, the setting is the one of integer bases. As a first extension, we handle the case of the Fibonacci numeration system, i.e., with the language $L_F = \{\varepsilon\} \cup 1\{0, 01\}^*$ [11, 12]. It turns out that the sequence $(S_F(n))_{n \geq 0}$ counting the number of words in L_F occurring as subwords of the nth word in L_F has properties similar to those of $(S_b(n))_{n \geq 0}$. Finally, the summatory function $(A_F(n))_{n \geq 0}$ of the sequence $(S_F(n))_{n \geq 0}$ has a behavior similar to the one of $(A_b(n))_{n \geq 0}$.

1.3 Our contribution

The Fibonacci numeration system belongs to an extensively studied family of numeration systems called *Parry–Bertrand numeration systems*, which are based on particular sequences $(U(n))_{n \geq 0}$ (the precise definitions are given later). In this paper, we fill the gap between integer bases and the Fibonacci numeration systems by extending the results of [10] to every Parry–Bertrand numeration system. First, we generalize

[2] Some of the objects discussed here are stored in Sloane's *On-Line Encyclopedia of Integer Sequences* [20]. See sequences A007306, A282714, A282715, A282720, A282728, A284441, and A284442.

the construction of Pascal-like triangles to every Parry–Bertrand numeration system. For a given Parry–Bertrand numeration system based on a particular sequence $(U(n))_{n\geq 0}$, we consider the intersection of the lattice $\mathbb{N}^2$ with $[0,U(n)]\times[0,U(n)]$. Then the first $U(n)$ rows and columns of the corresponding generalized Pascal triangle modulo 2 provide a coloring of this lattice regarding the parity of the corresponding binomial coefficients. If we normalize this compact set by a homothety of ratio $1/U(n)$, we get a sequence in $[0,1]\times[0,1]$ which converges, for the Hausdorff distance, to a limit set when n tends to infinity. Again, the limit set is described using a simple combinatorial property extending the one from [10].

Compared to the integer bases, new technicalities have to be taken into account to generalize Pascal triangles to a large class of numeration systems. The numeration systems occurring in this paper essentially have two properties. The first one is that the language of the numeration system comes from a particular automaton. The second one is the Bertrand condition which allows to delete or add ending zeroes to valid representations.

This paper is organized as follows. In Section 2, we collect necessary background. Section 3 is devoted to a special combinatorial property that extends the $(\star)$ condition from [10]. This new condition allows us to define a sequence of compact sets, which is shown to be a Cauchy sequence in Section 4. In Section 5, using the property of the latter sequence, we define a limit set which is the analogue of the Sierpiński gasket in the classical framework. We show that the sequence of subblocks of the generalized Pascal triangle modulo 2 in a Parry–Bertrand numeration converges to this new limit set. As a final remark, we consider the latter sequence of compact sets modulo any prime number.

2 Background and particular framework

We begin this section with well-known definitions from combinatorics on words; see, for instance, [18]. Let A be an alphabet, i.e., a finite set. The elements of A are called *letters*. A finite sequence over A is called a *finite word*. The length of a finite word w, denoted by $|w|$, is the number of letters belonging to w. The only word of length 0 is the empty word ε. The set of finite words over the alphabet A including the empty word (resp., excluding the empty word) is denoted by A^* (resp., A^+). The set of words of length n over A is denoted by A^n. If u and v are two finite words belonging to A^*, the *binomial coefficient* $\binom{u}{v}$ of u and v is the number of occurrences of v as a subsequence of u, meaning as a scattered subword. The sequences over A indexed by $\mathbb{N}$ are the *infinite words* over A. If w is a finite non-empty word over A, we let $w^\omega := www\cdots$ denote the infinite word obtained by concatenating infinitely many copies of w. If $L\subset A^*$ is a set of finite words and $u\in A^*$ is a finite word, we let $u^{-1}.L$ denote the set of words $\{v\in A^*\mid uv\in L\}$. Let A be totally ordered. If $u,v\in A^*$ are two words, we say that *u is less than v in the genealogical order* and we write $u<v$ if either $|u|<|v|$, or if $|u|=|v|$ and there exist words $p,q,r\in A^*$ and letters $a,b\in A$ with $u=paq$, $v=pbr$ and $a<b$. By $u\leq v$, we mean that either $u<v$, or $u=v$.

In the first part of this section, we gather two results on binomial coefficients of finite words and integers. For a proof of the first lemma, we refer the reader to [14, Chap. 6].

Lemma 1. *Let A be a finite alphabet. Let $u,v\in A^*$ and let $a,b\in A$. Then we have*

$$\binom{ua}{vb}=\binom{u}{vb}+\delta_{a,b}\binom{u}{v}$$

where $\delta_{a,b}$ is equal to 1 if $a=b$, 0 otherwise.

Let us also recall Lucas' theorem relating classical binomial coefficients modulo a prime number p with base-p expansions. See [16, p. 230] or [7]. Note that in the following statement, if the base-p expansions of m and n are not of the same length, then we pad the shortest with leading zeroes.

Theorem 2. *Let m and n be two non-negative integers and let p be a prime number. If*

$$m=m_kp^k+m_{k-1}p^{k-1}+\cdots+m_1p+m_0$$

and

$$n = n_k p^k + n_{k-1} p^{k-1} + \cdots + n_1 p + n_0$$

with $m_i, n_i \in \{0, \ldots, p-1\}$ *for all* i*, then the following congruence relation holds*

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \bmod p,$$

using the following convention: $\binom{m}{n} = 0$ *if* $m < n$*.*

In the last part of this section, we introduce the setting of particular numeration systems that are used in this paper: the Parry–Bertrand numeration systems. First of all, we recall several definitions and results about representations of real numbers. For more details, see, for instance, [3, Chap. 2], [15, Chap. 7] or [19].

Definition 3. Let $\beta \in \mathbb{R}_{>1}$ and let $A_\beta = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$. Every real number $x \in [0, 1)$ can be written as a series

$$x = \sum_{j=1}^{+\infty} c_j \beta^{-j}$$

where $c_j \in A_\beta$ for all $j \geq 1$, and where $\lceil \cdot \rceil$ denotes the *ceiling function* defined by $\lceil x \rceil = \inf\{z \in \mathbb{Z} \mid z \geq x\}$. The infinite word $c_1 c_2 \cdots$ is called a *β-representation* of x. Among all the β-representations of x, we define the *β-expansion* $d_\beta(x)$ of x obtained in a greedy way, i.e., for all $j \geq 1$, we have $c_j \beta^{-j} + c_{j+1} \beta^{-j-1} + \cdots < \beta^{-j+1}$. We also make use of the following convention: if $w = w_n \cdots w_0$ is a finite word (resp., $w = w_1 w_2 \cdots$ is an infinite word) over A_β, the notation $0.w$ has to be understood as the real number $\sum_{j=0}^{n} w_j \beta^{j-n-1}$ (resp., $\sum_{j=1}^{+\infty} w_j \beta^{-j}$); it actually corresponds to the value of the word w in base β.

In an analogous way, the *β-expansion* $d_\beta(1)$ of 1 the following infinite word over A_β

$$d_\beta(1) := \begin{cases} (\beta - 1)^\omega, & \text{if } \beta \in \mathbb{N}; \\ (\lceil \beta \rceil - 1) d_\beta(1 - (\lceil \beta \rceil - 1)/\beta), & \text{otherwise.} \end{cases}$$

In other words, if β is not an integer, the first digit of the β-expansion of 1 is $\lceil \beta \rceil - 1$ and the other digits are derived from the β-expansion of $1 - (\lceil \beta \rceil - 1)/\beta$.

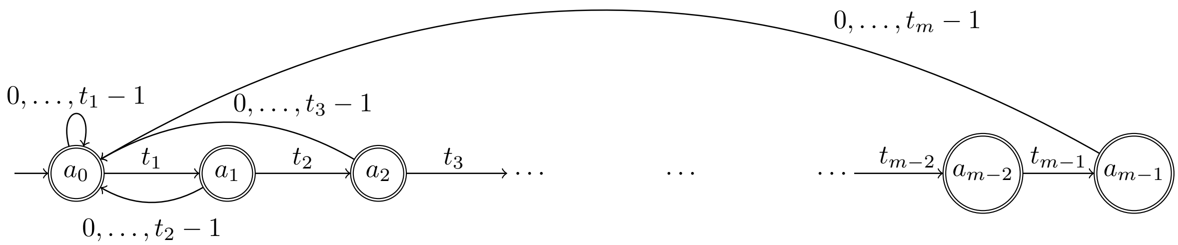
Let $d_\beta(1) = (t_n)_{n \geq 1}$ be the β-expansion of 1. Observe that $t_1 = \lceil \beta \rceil - 1$. We define the *quasi-greedy β-expansion* $d^*_\beta(1)$ of 1 as follows. If $d_\beta(1) = t_1 \cdots t_m$ is finite, i.e., $t_m \neq 0$ and $t_j = 0$ for all $j > m$, then $d^*_\beta(1) = (t_1 \cdots t_{m-1}(t_m - 1))^\omega$, otherwise $d^*_\beta(1) = d_\beta(1)$.

A real number $\beta > 1$ is a *Parry number* if $d_\beta(1)$ is ultimately periodic. If $d_\beta(1)$ is finite, then β is called a *simple Parry number*. In this case, Proposition 5 gives an easy way to decide if an infinite word is the β-expansion of a real number [17]. For more details, see, for instance, [15, Chap. 7]. First, let us recall the definition of a deterministic finite automaton.

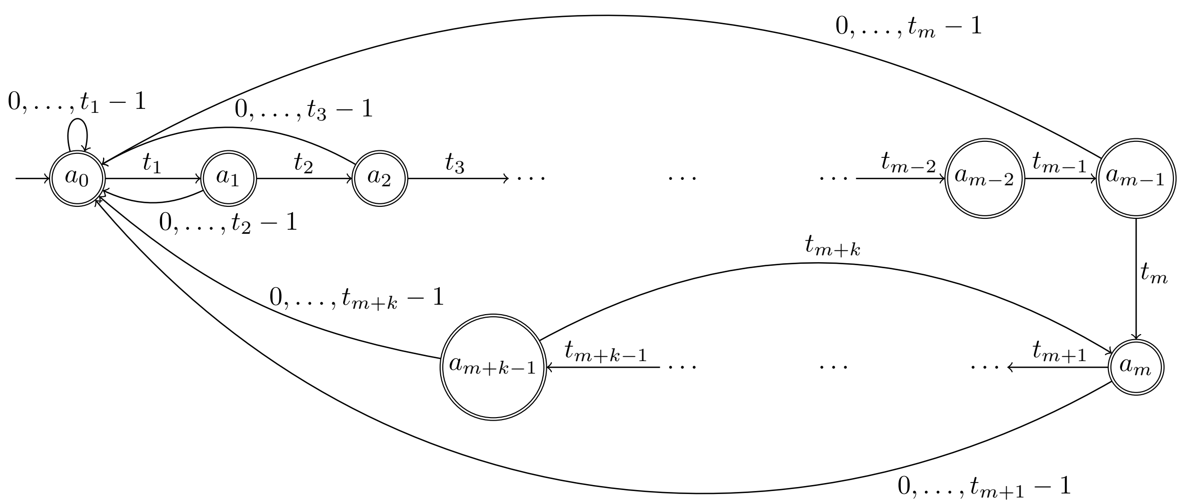
Definition 4. A deterministic finite automaton (DFA), over an alphabet A is given by a 5-tuple $\mathcal{A} = (Q, q_0, A, \delta, F)$ where Q is a finite set of states, $q_0 \in Q$ is the initial state, $\delta : Q \times A \mapsto Q$ is the transition function and $F \subset Q$ is the set of final states (graphically represented by two concentric circles). The map δ can be extended to $Q \times A^*$ by setting $\delta(q, \varepsilon) = q$ and $\delta(q, wa) = \delta(\delta(q, w), a)$ for all $q \in Q$, $a \in A$ and $w \in A^*$. We also say that a word w is *accepted* by the automaton if $\delta(q_0, w) \in F$.

Proposition 5. *Let* $\beta \in \mathbb{R}_{>1}$ *be a Parry number.*

(a) Suppose that $d_\beta(1) = t_1 \cdots t_m$ *is finite, i.e.,* $t_m \neq 0$ *and* $t_j = 0$ *for all* $j > m$*. Then an infinite word is the* β*-expansion of a real number in* $[0, 1)$ *if and only if it is the label of a path in the automaton* $\mathcal{A}_\beta = (\{a_0, \ldots, a_{m-1}\}, a_0, A_\beta, \delta, \{a_0, \ldots, a_{m-1}\})$ *depicted in Figure 1a.*



(a) The case when $d_\beta(1)$ is finite.



(b) The case when $d_\beta(1)$ is ultimately periodic but not finite.

Figure 1: The automaton $\mathcal{A}_\beta$ in function of the ultimately periodic word $d_\beta(1)$.

(a) (b)

Figure 2: The automaton $\mathcal{A}_\varphi$ (on the left) and the automaton $\mathcal{A}_{\varphi^2}$ (on the right).

(b) Suppose that $d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+k})^\omega$ *where* m, k *are taken to be minimal. Then an infinite word is the* β*-expansion of a real number* $[0, 1)$ *if and only if it is the label of a path in the automaton* $\mathcal{A}_\beta = (\{a_0, \ldots, a_{m+k-1}\}, a_0, A_\beta, \delta, \{a_0, \ldots, a_{m+k-1}\})$ *depicted in Figure 1b.*

Let us illustrate the previous proposition. For other examples, see, for instance, [5].

Example 6. If $\beta \in \mathbb{R}_{>1}$ is an integer, then $d_\beta(1) = d^*_\beta(1) = (\beta - 1)^\omega$. The automaton $\mathcal{A}_\beta$ consists of a single initial and final state a_0 with a loop of labels $0, 1, \ldots, \beta - 1$.

Example 7. Consider the golden ratio φ. Since $1 = 1/\varphi + 1/\varphi^2$, we have $d_\varphi(1) = 11$ and $d^*_\varphi(1) = (10)^\omega$. It is thus a Parry number. The automaton $\mathcal{A}_\varphi$ is depicted in Figure 2a.

The square φ^2 of the golden ratio is again a Parry number with $d_{\varphi^2}(1) = d^*_{\varphi^2}(1) = 21^\omega$. The automaton $\mathcal{A}_{\varphi^2}$ is depicted in Figure 2b.

With every Parry number is canonically associated a linear numeration system. Let us recall the definition of such numeration systems.

Definition 8. Let $U = (U(n))_{n \ge 0}$ be a sequence of integers such that U is increasing, $U(0) = 1$ and $\sup_{n \ge 0} \frac{U(n+1)}{U(n)}$ is bounded by a constant. We say that U is a *linear numeration system* if U satisfies a linear recurrence relation, i.e., there exist $k \ge 1$ and $a_0, \ldots, a_{k-1} \in \mathbb{Z}$ such that

$$\forall n \ge 0, \quad U(n+k) = a_{k-1}\, U(n+k-1) + \cdots + a_0\, U(n). \tag{1}$$

Let n be a positive integer. By successive Euclidean divisions, there exists $\ell \ge 1$ such that

$$n = \sum_{j=0}^{\ell-1} c_j\, U(j)$$

where the c_j's are non-negative integers and $c_{\ell-1}$ is non-zero. The word $c_{\ell-1} \cdots c_0$ is called the *normal* U*-representation* of n and is denoted by $\mathrm{rep}_U(n)$. In other words, the word $c_{\ell-1} \cdots c_0$ is the greedy expansion of n in the considered numeration system. We set $\mathrm{rep}_U(0) := \varepsilon$. Finally, we refer to $L_U := \mathrm{rep}_U(\mathbb{N})$ as the *language of the numeration* and we let A_U denote the minimal alphabet such that $L_U \subset A_U^*$. If $d_r \cdots d_0$ is a word over an alphabet of digits, then its U*-numerical value* is

$$\mathrm{val}_U(d_r \cdots d_0) := \sum_{j=0}^{r} d_j\, U(j).$$

Observe that, if $\mathrm{val}_U(d_r \cdots d_0) = n$, then the word $d_r \cdots d_0$ is a U-representation of n (but not necessarily its normal U-representation).

Definition 9. Let $\beta \in \mathbb{R}_{>1}$ be a Parry number. We define a particular linear numeration system $U_\beta := (U_\beta(n))_{n \ge 0}$ associated with β as follows.

If $d_\beta(1) = t_1 \cdots t_m$ is finite ($t_m \neq 0$), then we set $U_\beta(0) := 1$, $U_\beta(i) := t_1 U_\beta(i-1) + \cdots + t_i U_\beta(0) + 1$ for all $i \in \{1, \ldots, m-1\}$ and, for all $n \ge m$,

$$U_\beta(n) := t_1 U_\beta(n-1) + \cdots + t_m U_\beta(n-m).$$

If $d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+k})^\omega$ (m, k are minimal), then we set $U_\beta(0) := 1$, $U_\beta(i) := t_1 U_\beta(i-1) + \cdots + t_i U_\beta(0) + 1$ for all $i \in \{1, \ldots, m+k-1\}$ and, for all $n \ge m+k$,

$$\begin{aligned} U_\beta(n) :=\, & t_1 U_\beta(n-1) + \cdots + t_{m+k} U_\beta(n-m-k) + U_\beta(n-k) \\ & - t_1 U_\beta(n-k-1) - \cdots - t_m U_\beta(n-m-k). \end{aligned}$$

The linear numeration system U_β from Definition 9 has an interesting property: it is a Bertrand numeration system.

Definition 10. A linear numeration system $U = (U(n))_{n\geq 0}$ is a *Bertrand numeration system* if, for all $w \in A_U^+$, $w \in L_U \Leftrightarrow w0 \in L_U$.

Bertrand proved that the linear numeration system U_β associated with the Parry number β from Definition 9 is the unique linear numeration system associated with β that is also a Bertrand numeration system [4]. In that case [4], any word w in the set $0^*L_{U_\beta}$ of all normal U_β-representations with leading zeroes is the label of a path in the automaton $\mathcal{A}_\beta$ from Proposition 5.

Finally, every Parry number is a Perron number [15, Chap. 7]. A real number $\beta > 1$ is a *Perron number* if it is an algebraic integer whose conjugates have modulus less than β. Numeration systems based on Perron numbers are defined as follows and have the property (2), which is often used in this paper.

Definition 11. Let $U = (U(n))_{n\geq 0}$ be a linear numeration system. Consider the characteristic polynomial of the recurrence (1) given by $P(X) = X^k - a_{k-1}X^{k-1} - \cdots - a_1X - a_0$. If P is the minimal polynomial of a Perron number $\beta \in \mathbb{R}_{>1}$, we say that U is a *Perron numeration system.* In this case, the polynomial P can be factored as

$$P(X) = (X - \beta)(X - \alpha_2)\cdots(X - \alpha_k)$$

where the complex numbers $\alpha_2, \ldots, \alpha_k$ are the conjugates of β, and, for all $j > 1$, we have $|\alpha_j| < \beta$. Using a well-known fact regarding recurrence relations, we have

$$U(n) = c_1\beta^n + c_2\alpha_2^n + \cdots + c_k\alpha_k^n \quad \forall n \geq 0$$

where $c_1, \ldots, c_k$ are complex numbers depending on the initial values of U. Since $|\alpha_j| < \beta$ for all $j > 1$, we have

$$\lim_{n\to+\infty} \frac{U(n)}{\beta^n} = c_1. \tag{2}$$

Remark 12. Note that if two Perron numeration systems are associated with the same Perron number, then these two systems only differ by the choice of the initial values $U(0), \ldots, U(k-1)$. The choice of those initial values is of great importance. See, for instance, Example 14.

Example 13. The usual integer base system is a special case of a Perron–Bertrand numeration system.

Example 14. The golden ratio φ is a Perron number whose minimal polynomial is $P(X) = X^2 - X - 1$. A Perron–Bertrand numeration system associated with φ is the Fibonacci numeration system based on the Fibonacci numbers $(F(n))_{n\geq 0}$ defined by $F(0) = 1$, $F(1) = 2$ and $F(n+2) = F(n+1) + F(n)$. If we change the initial conditions and set $F'(0) = 1$, $F'(1) = 3$ and $F'(n+2) = F'(n+1) + F'(n)$, we again get a Perron numeration associated with φ which is not a Bertrand numeration system. Indeed, 2 is a greedy representation, but not 20 because $\mathrm{rep}_{F'}(\mathrm{val}_{F'}(20)) = 102$.

The particular setting of this paper is the following one: we let $\beta \in \mathbb{R}_{>1}$ be a Parry number and we constantly use the special Parry–Bertrand numeration U_β from Definition 9. From Definition 3 and Definition 8, the alphabet A_{U_β} is the set $\{0, 1, \ldots, \lceil\beta\rceil - 1\}$ and the language of the system of numeration U_β is $L_{U_\beta} \subset A_{U_\beta}^*$ (which is defined using the automaton $\mathcal{A}_\beta$ from Proposition 5). To end this section, we prove a useful lemma about binomial coefficients of words ending with blocks of zeroes.

Lemma 15. *For all non-empty words $u, v \in L_{U_\beta}$ and all $k \in \mathbb{N}$, we have*

$$\binom{u0^k}{v0^k} = \sum_{j=0}^{k} \binom{k}{j}\binom{u}{v0^j}.$$

Proof. We proceed by induction on $k \in \mathbb{N}$. If $k = 0$, the result is obvious. Suppose that the result holds true for all non-empty words $u, v \in L_{U_\beta}$ and for $0, \ldots, k$. We show that it still holds true for all non-empty words $u, v \in L_{U_\beta}$ and $k+1$. Using Lemma 1, we first have

$$\binom{u0^{k+1}}{v0^{k+1}} = \binom{u0^k}{v'0^k} + \binom{u0^k}{v0^k}$$

where $v' = v0 \in L_{U_\beta}$ since U_β is a Parry–Bertrand numeration system. By induction hypothesis, we get

$$\begin{aligned}
\binom{u0^{k+1}}{v0^{k+1}} &= \sum_{j=1}^{k+1} \binom{k}{j-1}\binom{u}{v0^j} + \sum_{j=0}^{k} \binom{k}{j}\binom{u}{v0^j} \\
&= \binom{k+1}{k+1}\binom{u}{v0^{k+1}} + \sum_{j=1}^{k} \left(\binom{k}{j-1} + \binom{k}{j}\right)\binom{u}{v0^j} + \binom{k+1}{0}\binom{u}{v} \\
&= \sum_{j=0}^{k+1} \binom{k+1}{j}\binom{u}{v0^j}.
\end{aligned}$$

□

3 The $(\star)$ condition

We let $w_n = \text{rep}_{U_\beta}(n)$ denote the nth word of the language L_{U_β} in the genealogical order. The generalized Pascal triangle $\text{P}_{U_\beta} : \mathbb{N} \times \mathbb{N} \to \mathbb{N} : (i,j) \mapsto \binom{w_i}{w_j}$ is represented as an infinite table[3] whose entry on the ith row and the jth column is the binomial coefficient $\binom{w_i}{w_j}$. For instance, when $\beta = \varphi$, the first few values in the generalized Pascal triangle P_{U_φ} are given in Table 2 below. Considering the intersection of the lattice $\mathbb{N}^2$ with $[0, U_\beta(n)] \times [0, U_\beta(n)]$, the first $U_\beta(n)$ rows and columns of the generalized Pascal triangle P_{U_β} modulo 2

$$\left(\binom{w_i}{w_j} \bmod 2\right)_{0 \le i,j < U_\beta(n)}$$

provide a coloring of this lattice, leading to a sequence of compact subsets of $\mathbb{R}^2$. If we normalize these sets respectively by a homothety of ratio $1/U_\beta(n)$, we define a sequence $(\mathcal{U}_n^\beta)_{n\ge0}$ of subsets of $[0,1] \times [0,1]$.

Definition 16. Let $Q := [0,1] \times [0,1]$. Consider the sequence $(\mathcal{U}_n^\beta)_{n\ge0}$ of sets in $[0,1] \times [0,1]$ defined for all $n \ge 0$ by

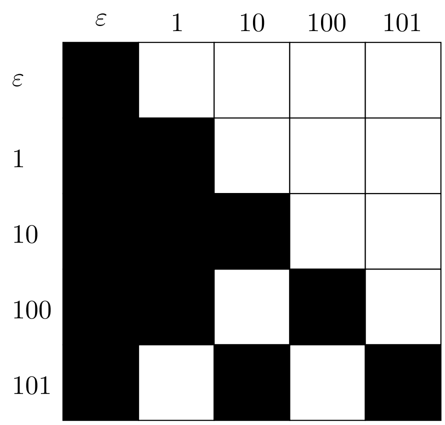
$$\mathcal{U}_n^\beta := \frac{1}{U_\beta(n)} \bigcup \left\{ (\text{val}_{U_\beta}(v), \text{val}_{U_\beta}(u)) + Q \mid u, v \in L_{U_\beta}, \binom{u}{v} \equiv 1 \bmod 2 \right\} \subset [0,1] \times [0,1].$$

Each $\mathcal{U}_n^\beta$ is a finite union of squares of size $1/U_\beta(n)$ and is thus compact.

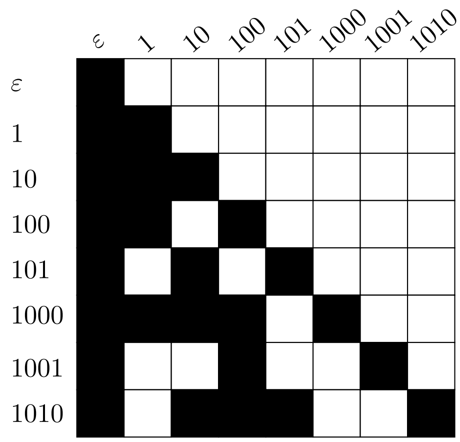
Example 17. When $\beta = \varphi$ is the golden ratio, the first values in the generalized Pascal triangle P_{U_φ} are given in Table 2. The sets $\mathcal{U}_3^\varphi$, $\mathcal{U}_4^\varphi$ and $\mathcal{U}_5^\varphi$ are depicted in Figure 3. The set $\mathcal{U}_9^\varphi$ is depicted in Figure 14 given in the appendix.

Remark 18. Each pair (u,v) of words of length at most n with an odd binomial coefficient gives rise to a square region in $\mathcal{U}_n^\beta$. More precisely, we have the following situation. Let $n \ge 0$ and $u, v \in L_{U_\beta}$ such that $0 \le |v| \le |u| \le n$ and $\binom{u}{v} \equiv 1 \bmod 2$. We have

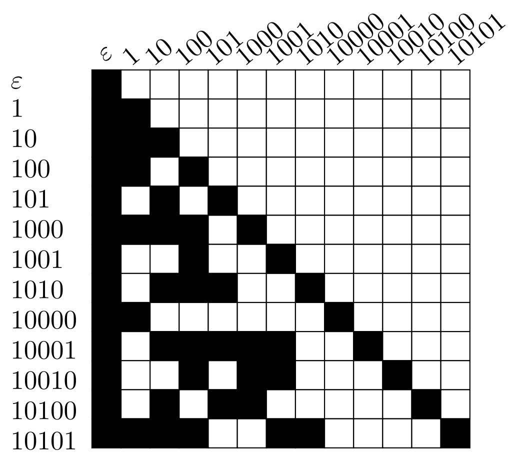
$$((\text{val}_{U_\beta}(v), \text{val}_{U_\beta}(u)) + Q)/U_\beta(n) \subset \mathcal{U}_n^\beta$$



(a) The set $\mathcal{U}_3^{\varphi}$.



(b) The set $\mathcal{U}_4^{\varphi}$.



(c) The set $\mathcal{U}_5^{\varphi}$.

Figure 3: The sets $\mathcal{U}_3^{\varphi}$, $\mathcal{U}_4^{\varphi}$ and $\mathcal{U}_5^{\varphi}$ when $\beta = \varphi$ is the golden ratio.

						j			
	$\binom{w_i}{w_j}$	ε	1	10	100	101	1000	1001	1010
	ε	1	0	0	0	0	0	0	0
	1	1	1	0	0	0	0	0	0
	10	1	1	1	0	0	0	0	0
i	100	1	1	2	1	0	0	0	0
	101	1	2	1	0	1	0	0	0
	1000	1	1	3	3	0	1	0	0
	1001	1	2	2	1	2	0	1	0
	1010	1	2	3	1	1	0	0	1

Table 2: The first few values in the generalized Pascal triangle P_{U_φ}.

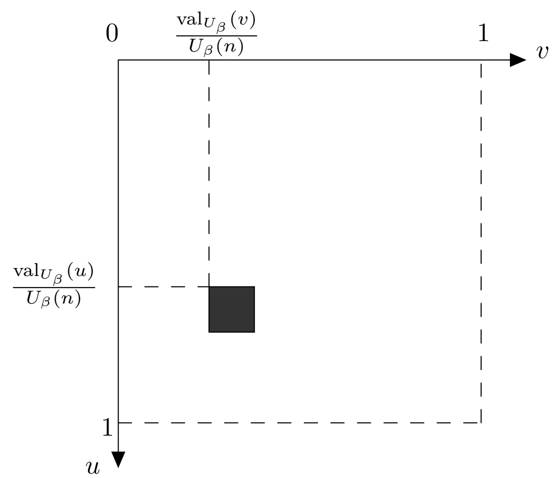


Figure 4: Visualization of a square region in $\mathcal{U}_n^\beta$.

as depicted in Figure 4.

We consider the space $(\mathcal{H}(\mathbb{R}^2), d_h)$ of the non-empty compact subsets of $\mathbb{R}^2$ equipped with the Hausdorff metric d_h induced by the Euclidean distance d on $\mathbb{R}^2$. It is well known that $(\mathcal{H}(\mathbb{R}^2), d_h)$ is complete [6]. We let $B(x, \epsilon)$ denote the open ball of radius $\epsilon \geq 0$ centered at $x \in \mathbb{R}^2$ and, if $S \subset \mathbb{R}^2$, we let

$$[S]_\epsilon := \bigcup_{x \in S} B(x, \epsilon)$$

denote *ϵ-fattening* of S.

Our aim is to show that the sequence $(\mathcal{U}_n^\beta)_{n \geq 0}$ of compact subsets of $[0,1] \times [0,1]$ is converging and to provide an elementary description of its limit set. The idea is the following one. Let $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ be a pair of words having an odd binomial coefficient. On the one hand, some of those pairs are such that $\binom{ua}{va} \equiv 0 \bmod 2$ for all letters a such that $ua, va \in L_{U_\beta}$. In other words, those pairs of words create a black square region in $\mathcal{U}_{|u|}^\beta$ while the corresponding square region in $\mathcal{U}_{|u|+1}^\beta$ is white. As an example, take $\beta = \varphi$, $u = 1010$ and $v = 101$. We have $\binom{u0}{v0} = 2$ (see Figure 3). On the other hand, some of those pairs create a more stable pattern, i.e., $\binom{uw}{vw} \equiv 1 \bmod 2$ for all words w such that $uw, vw \in L_{U_\beta}$. Roughly, those

[3] Using the notation $\binom{u}{v}$, the rows (resp., columns) of P_{U_β} are indexed by the words u (resp., v).

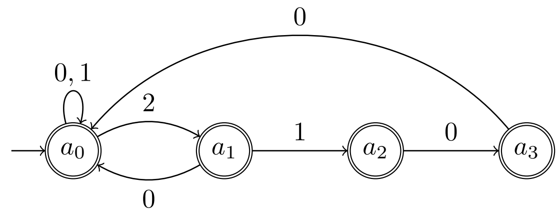


Figure 5: The automaton $\mathcal{A}_\beta$ for the dominant root β of the polynomial $P(X) = X^4 - 2X^3 - X^2 - 1$.

pairs create a diagonal of square regions in $(\mathcal{U}_n^\beta)_{n\geq 0}$. For instance, take $\beta = \varphi$, $u = 101$ and $v = 10$. In this case, $\binom{uw}{vw} \equiv 1 \bmod 2$ for all admissible words w. In particular, the pairs of words (u, v), $(u0, v0)$ and $(u00, v00)$, $(u01, v01)$ have odd binomial coefficients (see Figure 3) and create a diagonal of square regions. With the second type of pairs of words, we define a new sequence of compact subsets $(\mathcal{A}_n^\beta)_{n\geq 0}$ of $[0, 1] \times [0, 1]$ which converges to some well-defined limit set $\mathcal{L}^\beta$. Then, we show that the first sequence of compact sets $(\mathcal{U}_n^\beta)_{n\geq 0}$ also converges to this limit set. The remaining of this paper is dedicated to formalize and prove those statements.

To reach that goal, for all non-empty words $u, v \in L_{U_\beta}$, we first define the least integer p such that $u0^pw, v0^pw$ belong to L_{U_β} for all words $w \in 0^*L_{U_\beta}$. In other terms, any word w can be read after $u0^p$ and $v0^p$ in the automaton $\mathcal{A}_\beta$. Then, some pairs of words $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ have the property that not only $\binom{u}{v} \equiv 1 \bmod 2$ but also $\binom{u0^pw}{v0^pw} \equiv 1 \bmod 2$ for all words $w \in 0^*L_{U_\beta}$; see Corollary 28. Such a property creates a particular pattern occurring in $\mathcal{U}_n^\beta$ for all sufficiently large n, as shown in Remark 30.

Proposition 19. *For all non-empty words $u, v \in L_{U_\beta}$, there exists a smallest nonnegative integer $p(u, v)$ such that*
$$(u0^{p(u,v)})^{-1}.L_{U_\beta} = (v0^{p(u,v)})^{-1}.L_{U_\beta} = 0^*L_{U_\beta}.$$

Proof. Using Proposition 5, take $p(u, v)$ to be the least nonnegative integer p such that $\delta(a_0, u0^p) = a_0 = \delta(a_0, v0^p)$. Then, for any word $w \in 0^*L_{U_\beta}$, the words $u0^{p(u,v)}w, v0^{p(u,v)}w$ are labels of paths in $\mathcal{A}_\beta$. Consequently, they are words in L_{U_β}. Conversely, if the words $u0^{p(u,v)}w, v0^{p(u,v)}w$ are labels of paths in $\mathcal{A}_\beta$, then $w \in 0^*L_{U_\beta}$. □

In the following, we will be using $p(\varepsilon, \varepsilon)$. Observe that, using Proposition 5, $\delta(a_0, \varepsilon) = a_0$. We naturally set $p(\varepsilon, \varepsilon) := 0$ and we thus have $(\varepsilon 0^{p(\varepsilon,\varepsilon)})^{-1}.L_{U_\beta} = L_{U_\beta}$.

Example 20. If $\beta > 1$ is an integer, then $p(u, v) = 0$ for all $u, v \in L_{U_\beta}$. See Example 6.

Example 21. If $\beta = \varphi$ is the golden ratio, then $p(u, v) = 0$ if and only if u and v end with 0 or $u = v = \varepsilon$, otherwise $p(u, v) = 1$.

The integer of Proposition 19 can be greater than 1 as illustrated in the following example.

Example 22. Let β be the dominant root of the polynomial $P(X) = X^4 - 2X^3 - X^2 - 1$. Then $\beta \approx 2.47098$ is a Parry number with $d_\beta(1) = 2101$ and $d_\beta^*(1) = (2100)^\omega$. The automaton $\mathcal{A}_\beta$ is depicted in Figure 5. For instance, $p(101, 21) = 2$.

Definition 23. Let $(u, v) \in L_{U_\beta} \times L_{U_\beta}$. We say that (u, v) satisfies the $(\star)$ condition if either $u = v = \varepsilon$, or $|u| \geq |v| > 0$ and
$$\binom{u0^{p(u,v)}}{v0^{p(u,v)}} \equiv 1 \bmod 2 \quad \text{and} \quad \binom{u0^{p(u,v)}}{v0^{p(u,v)}a} = 0 \quad \forall\, a \in A_{U_\beta}$$
where $p(u, v)$ is defined by Proposition 19. Observe that, if $(u, v) \neq (\varepsilon, \varepsilon)$, then $v0^{p(u,v)}a \in L_{U_\beta}$ for all $a \in A_{U_\beta}$.

Remark 24. Observe that if only one of the two words u or v is empty, then the pair (u,v) never satisfies $(\star)$.

The next lemma shows that all diagonal elements of $\mathcal{U}_n^\beta$ satisfy $(\star)$.

Lemma 25. *For any word $u \in L_{U_\beta}$, the pair (u,u) satisfies $(\star)$.*

Proof. If $u = \varepsilon$, the result is clear using Definition 23. Suppose u is non-empty and let $p := p(u,u)$ denote the integer from Proposition 19. Then, for all $a \in A_{U_\beta}$, we have

$$\binom{u0^p}{u0^p} = 1 \equiv 1 \bmod 2 \quad \text{and} \quad \binom{u0^p}{u0^p a} = 0$$

since $|u0^p a| > |u0^p|$. □

If a pair of words satisfies $(\star)$, it has the following two properties. First, its binomial coefficient is odd, as stated in the following proposition. Secondly, it creates a special pattern in $\mathcal{U}_n^\beta$ for all large enough n; see Proposition 27, Corollary 28 and Remark 30.

Proposition 26. *Let $(u,v) \in L_{U_\beta} \times L_{U_\beta}$ satisfying $(\star)$. Then*

$$\binom{u}{v} \equiv 1 \bmod 2.$$

Proof. If $u = v = \varepsilon$, the result is clear by definition. Suppose that u and v are non-empty. Let us proceed by contradiction and suppose that $\binom{u}{v}$ is even. Let us set $p := p(u,v)$ from Proposition 19. On the one hand, by Definition 23, we know that

$$\binom{u0^p}{v0^p} \equiv 1 \bmod 2$$

and, on the other hand, Lemma 15 states that

$$\binom{u0^p}{v0^p} = \sum_{j=1}^{p} \binom{p}{j}\binom{u}{v0^j} + \binom{u}{v}.$$

Consequently, we have

$$\sum_{j=1}^{p} \binom{p}{j}\binom{u}{v0^j} \equiv 1 \bmod 2 > 0$$

and there must exist $i \in \{1,\dots,p\}$ such that $\binom{u}{v0^i} > 0$. Using again Lemma 15, we also have

$$\binom{u0^p}{v0^p0} = \sum_{j=1}^{p+1} \binom{p}{j-1}\binom{u}{v0^j} \geq \binom{p}{i-1}\binom{u}{v0^i} > 0,$$

which contradicts Definition 23. □

Proposition 27. *Let $u,v \in L_{U_\beta}$ be two non-empty words such that (u,v) satisfies $(\star)$. For any letter $a \in A_{U_\beta}$, the pair of words $(u0^{p(u,v)}a, v0^{p(u,v)}a) \in L_{U_\beta} \times L_{U_\beta}$ satisfies $(\star)$.*

Proof. For the sake of clarity, set $p := p(u,v)$. Let a be a letter in A_{U_β} and also set $p' := p(u0^p a, v0^p a)$. By Lemma 1 and Lemma 15,

$$\binom{u0^p a0^{p'}}{v0^p a0^{p'}} = \sum_{j=1}^{p'} \binom{p'}{j}\binom{u0^p a}{v0^p a0^j} + \binom{u0^p}{v0^p a} + \binom{u0^p}{v0^p}.$$

Since (u,v) satisfies $(\star)$, all the coefficients $\binom{u0^pa}{v0^pa0^j}$, for $j=1,\ldots,p'$, and $\binom{u0^p}{v0^pa}$ are equal to 0. Otherwise, it means that the word $v0^pa$ appears as a subword of the word $u0^p$, which contradicts $(\star)$. Consequently, using Definition 23, we get

$$\binom{u0^pa0^{p'}}{v0^pa0^{p'}} = \binom{u0^p}{v0^p} \equiv 1 \bmod 2.$$

Using the same argument, for any letter $b\in A_{U_\beta}$, we have

$$\binom{u0^pa0^{p'}}{v0^pa0^{p'}b} = 0.$$

□

The next corollary extends Lemma 25 when $(u,v)\neq(\varepsilon,\varepsilon)$. Indeed, recall that $p(\varepsilon,\varepsilon)=0$.

Corollary 28. *Let $u,v\in L_{U_\beta}$ be two non-empty words such that (u,v) satisfies $(\star)$. Then*

$$\binom{u0^{p(u,v)}w}{v0^{p(u,v)}w} \equiv 1 \bmod 2 \quad \forall\, w\in 0^*L_{U_\beta}.$$

Proof. Set $p:=p(u,v)$. From Proposition 19, $u0^pw, v0^pw$ belong to L_{U_β} for any word $w\in 0^*L_{U_\beta}$. Now proceed by induction on the length of $w\in 0^*L_{U_\beta}$. If $|w|=0$, then $w=\varepsilon$ is the empty word and the statement is true using Definition 23. If $|w|=1$, then $w=a$ is a letter belonging to A_{U_β}. Then, by Proposition 27, we know that $(u0^pa, v0^pa)$ satisfies $(\star)$. Using Proposition 26, we have

$$\binom{u0^pa}{v0^pa} \equiv 1 \bmod 2.$$

Now suppose that $|w|\geq 2$ and write $w=aw'b\in 0^*L_{U_\beta}$ where a,b are letters. From Lemma 1, we deduce that

$$\binom{u0^pw}{v0^pw} = \binom{u0^paw'}{v0^paw'b} + \binom{u0^paw'}{v0^paw'}.$$

By induction hypothesis, $\binom{u0^paw'}{v0^paw'}\equiv 1 \bmod 2$ since $aw'\in 0^*L_{U_\beta}$ and $|aw'|<|w|$. Furthermore, $\binom{u0^paw'}{v0^paw'b}$ must be 0, otherwise it means that the word $v0^pa$ occurs as a subword of the word $u0^p$, which contradicts the fact that (u,v) satisfies $(\star)$. This ends the proof. □

The next lemma is useful to characterize the pattern created in $\mathcal{U}_n^\beta$, for all sufficiently large n, by pairs of words satisfying $(\star)$, see Remark 30. In this result, we make use of the convention given in Definition 3.

Lemma 29. *Let $(u,v)\in L_{U_\beta}\times L_{U_\beta}$ satisfying $(\star)$.*

(a) The sequence

$$\left(\left(\frac{\mathrm{val}_{U_\beta}(v0^{p(u,v)+n})}{U_\beta(|u|+p(u,v)+n)}, \frac{\mathrm{val}_{U_\beta}(u0^{p(u,v)+n})}{U_\beta(|u|+p(u,v)+n)}\right)\right)_{n\geq 0}$$

converges to the pair of real numbers $(0.0^{|u|-|v|}v, 0.u)$.

(b) For all $n\geq 0$, let $w=d_n$ denotes the prefix of length n of $d_\beta^*(1)$. Then the sequence

$$\left(\left(\frac{\mathrm{val}_{U_\beta}(v0^{p(u,v)}d_n)}{U_\beta(|u|+p(u,v)+n)}, \frac{\mathrm{val}_{U_\beta}(u0^{p(u,v)}d_n)}{U_\beta(|u|+p(u,v)+n)}\right)\right)_{n\geq 0}$$

converges to the pair of real numbers $(0.0^{|u|-|v|}v0^{p(u,v)}d_\beta^*(1), 0.u0^{p(u,v)}d_\beta^*(1))$.

Proof. Let $(u,v) \in L_{U_\beta} \times L_{U_\beta}$ satisfying $(\star)$ and set $p := p(u,v)$. We prove the first item as the proof of the second one is similar. The result is trivial if $u = v = \varepsilon$. Suppose that u and v are non-empty words. Let us write $u = u_{|u|-1}u_{|u|-2}\cdots u_0$ where $u_i \in A_{U_\beta}$ for all i. By definition, we have

$$\frac{\mathrm{val}_{U_\beta}(u0^{p+n})}{U_\beta(|u|+p+n)} = \sum_{i=0}^{|u|-1} u_i \, \frac{U_\beta(i+p+n)}{U_\beta(|u|+p+n)}.$$

Using (2), $U_\beta(i+p+n)/U_\beta(|u|+p+n)$ tends to $\beta^i/\beta^{|u|}$ when n tend to infinity. Consequently,

$$\lim_{n\to+\infty} \frac{\mathrm{val}_{U_\beta}(u0^{p+n})}{U_\beta(|u|+p+n)} = \sum_{i=0}^{|u|-1} u_i\beta^{i-|u|} = 0.u.$$

Using the same reasoning on the word v, we conclude that the sequence

$$\left(\left(\frac{\mathrm{val}_{U_\beta}(v0^{p(u,v)+n})}{U_\beta(|u|+p(u,v)+n)}, \frac{\mathrm{val}_{U_\beta}(u0^{p(u,v)+n})}{U_\beta(|u|+p(u,v)+n)}\right)\right)_{n\geq 0}$$

converges to the pair of real numbers $(0.0^{|u|-|v|}v, 0.u)$. □

Remark 30. Let $(u,v) \in L_{U_\beta} \times L_{U_\beta}$ satisfying $(\star)$ and set $p := p(u,v)$. Suppose that u and v are non-empty (the case when $u = v = \varepsilon$ is similar: in the following, replace $0^*L_{U_\beta}$ by L_{U_β} where needed). Using Corollary 28, the pair of words $(u0^pw, v0^pw)$ has an odd binomial coefficient for any word $w \in 0^*L_{U_\beta}$. In particular, the pair of words $(u0^pw, v0^pw)$ corresponds to a square region in $\mathcal{U}^\beta_{|u|+p+n}$ for all $w \in 0^*L_{U_\beta}$ such that $|w| = n \geq 0$. Using Remark 18, this region is

$$\left(\frac{\mathrm{val}_{U_\beta}(v0^pw)}{U_\beta(|u|+p+n)}, \frac{\mathrm{val}_{U_\beta}(u0^pw)}{U_\beta(|u|+p+n)}\right) + \frac{Q}{U_\beta(|u|+p+n)} \subset \mathcal{U}^\beta_{|u|+p+n}.$$

Using Lemma 29, when $w = 0^n$ (the smallest word of length n in $0^*L_{U_\beta}$), the sequence

$$\left(\left(\frac{\mathrm{val}_{U_\beta}(v0^{p+n})}{U_\beta(|u|+p+n)}, \frac{\mathrm{val}_{U_\beta}(u0^{p+n})}{U_\beta(|u|+p+n)}\right)\right)_{n\geq 0}$$

converges to the pair of real numbers $(0.0^{|u|-|v|}v, 0.u)$. This point will be the first endpoint of a segment associated with u and v. See Definition 32. Analogously, using Lemma 29, when $w = d_n$ is the prefix of length n of $d^*_\beta(1)$ (the greatest word of length n in $0^*L_{U_\beta}$), then the sequence

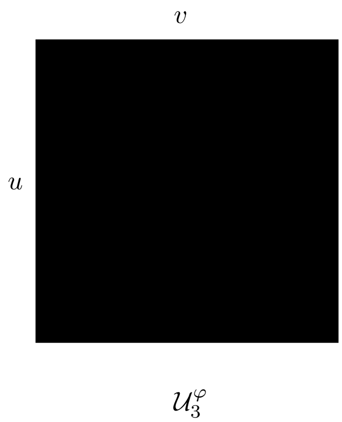
$$\left(\left(\frac{\mathrm{val}_{U_\beta}(v0^pd_n)}{U_\beta(|u|+p+n)}, \frac{\mathrm{val}_{U_\beta}(u0^pd_n)}{U_\beta(|u|+p+n)}\right)\right)_{n\geq 0}$$

converges to the pair of real numbers $(0.0^{|u|-|v|}v0^pd^*_\beta(1), 0.u0^pd^*_\beta(1))$. This point will be the second endpoint of the same segment associated with u and v. See Definition 32. As a consequence, the sequence of sets whose nth term is defined by

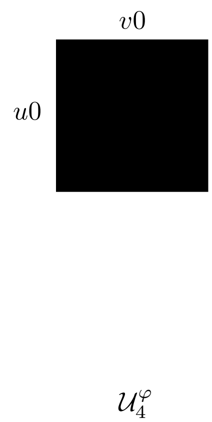
$$\bigcup_{\substack{|w|=n\\ w\in 0^*L_{U_\beta}}} \left(\left(\frac{\mathrm{val}_{U_\beta}(v0^pw)}{U_\beta(|u|+p+n)}, \frac{\mathrm{val}_{U_\beta}(u0^pw)}{U_\beta(|u|+p+n)}\right) + \frac{Q}{U_\beta(|u|+p+n)}\right) \tag{3}$$

converges, for the Hausdorff distance, to the diagonal of the square $(0.0^{|u|-|v|}v, 0.u) + Q/\beta^{|u|+p}$.

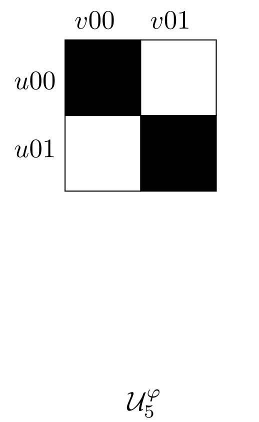
Example 31. As a first example, when $\beta = 2$, we find back the construction in [10]. As a second example, let us take $\beta = \varphi$ to be the golden ratio. Let $u = 101$ and $v = 10$ (resp., $u' = 100 = v'$). Then $p(u,v) = 1$ (resp., $p(u',v') = 0$); see Example 21. Those pairs of words satisfy $(\star)$. The first few terms of the sequence of sets (3) are respectively depicted in Figure 6 and Figure 7. Observe that when n tends to infinity, the union of black squares in $\mathcal{U}^\varphi_{n+4}$ (resp., $\mathcal{U}^\varphi_{n+3}$) converges to the diagonal of $(0.0v, 0.u)+Q/\varphi^4$ (resp., $(0.v', 0.u')+Q/\varphi^3$).



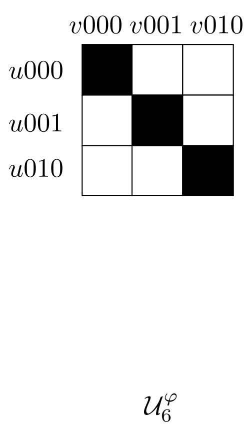
(a) A subset of $\mathcal{U}_3^\varphi$.



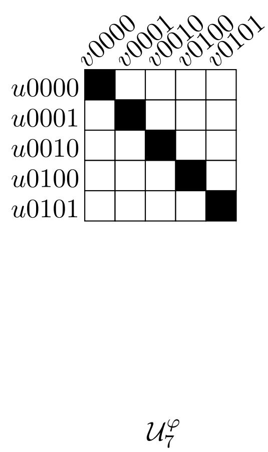
(b) The element $n = 0$ of (3).



(c) The element $n = 1$ of (3).

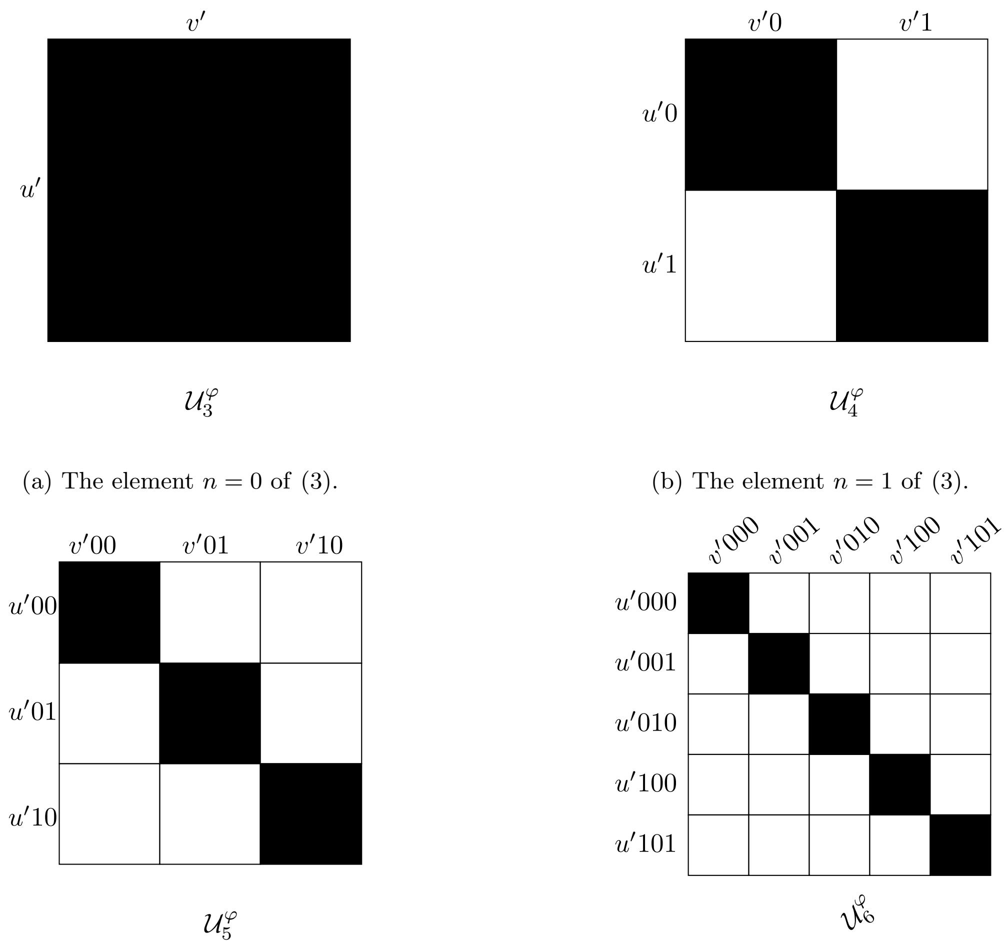


(d) The element $n = 2$ of (3).



(e) The element $n = 3$ of (3).

Figure 6: The first few terms of sequence of sets (3) converging to the diagonal of the square $(0.0v, 0.u) + Q/\varphi^4$ for $u = 101$ and $v = 10$.



(a) The element $n = 0$ of (3).

(b) The element $n = 1$ of (3).

(c) The element $n = 2$ of (3).

(d) The element $n = 3$ of (3).

Figure 7: The first few terms of sequence of sets (3) converging to the diagonal of the square $(0.v', 0.u') + Q/\varphi^3$ for $u' = 100$ and $v' = 100$.

4 The sequence of compact sets $(\mathcal{A}_n^\beta)_{n\geq 0}$

The observation made in Remark 30 leads to the definition of an initial set $\mathcal{A}_0^\beta$. The same technique is applied in [10]. At first, let us define a segment associated with a pair of words.

Definition 32. Let (u, v) in $L_{U_\beta} \times L_{U_\beta}$ such that $|u| \geq |v| \geq 0$. We define a closed segment $S_{u,v}$ of slope 1 and of length $\sqrt{2} \cdot \beta^{-|u|-p(u,v)}$ in $[0, 1] \times [0, 1]$. The endpoints of $S_{u,v}$ are given by $A_{u,v} := (0.0^{|u|-|v|}v, 0.u)$ and

$$B_{u,v} := A_{u,v} + (\beta^{-|u|-p(u,v)}, \beta^{-|u|-p(u,v)}) = (0.0^{|u|-|v|}v0^{p(u,v)}d_\beta^*(1), 0.u0^{p(u,v)}d_\beta^*(1)).$$

Observe that, if $u = v = \varepsilon$, the associated segment of slope 1 has endpoints $(0, 0)$ and $(1, 1)$. Otherwise, the segment $S_{u,v}$ lies in $[0, 1] \times [1/\beta, 1]$.

Definition 33. Let us define the following compact set which is the closure of a countable union of segments

$$\mathcal{A}_0^\beta := \overline{\bigcup_{\substack{(u,v)\\ \text{satisfying}(\star)}} S_{u,v}}.$$

Notice that Definition 32 implies that $\mathcal{A}_0^\beta \subset [0, 1] \times [0, 1]$. More precisely, $\mathcal{A}_0^\beta \setminus S_{\varepsilon,\varepsilon} \subset [0, 1] \times [1/\beta, 1]$. Furthermore, observe that we take the closure of a union to ensure the compactness of the set.

Example 34. Let $\beta = \varphi$ be the golden ratio. In Figure 8, the segment $S_{u,v}$ is represented for all (u, v) satisfying $(\star)$ and such that $0 \leq |v| \leq |u| \leq 10$.

In the following definition, we introduce another sequence of compact sets obtained by transforming the initial set $\mathcal{A}_0^\beta$ under iterations of two maps. This new sequence, which is shown to be a Cauchy sequence in Proposition 36, allows us to define properly the limit set $\mathcal{L}^\beta$.

Definition 35. We let c denote the homothety of center $(0, 0)$ and ratio $1/\beta$ and we consider the map $h : (x, y) \mapsto (x, \beta y)$. We define a sequence of compact sets by setting, for all $n \geq 0$,

$$\mathcal{A}_n^\beta := \bigcup_{\substack{0\leq i\leq n\\ 0\leq j\leq i}} h^j(c^i(\mathcal{A}_0^\beta)).$$

In Figure 9, we apply c and h at most twice from $\mathcal{A}_0^\beta \setminus S_{\varepsilon,\varepsilon}$. Let m, n with $m \leq n$. Using Figure 9, observe that

$$\mathcal{A}_m^\beta \cap ([1/\beta^{m+1}, 1] \times [0, 1]) = \mathcal{A}_n^\beta \cap ([1/\beta^{m+1}, 1] \times [0, 1]). \tag{4}$$

Proposition 36. *The sequence $(\mathcal{A}_n^\beta)_{n\geq 0}$ is a Cauchy sequence.*

Proof. Let $\epsilon > 0$ and take $n > m$. We must show that $\mathcal{A}_m^\beta \subset [\mathcal{A}_n^\beta]_\epsilon$ and $\mathcal{A}_n^\beta \subset [\mathcal{A}_m^\beta]_\epsilon$. The first inclusion is easy. Indeed, since $\mathcal{A}_m^\beta \subset \mathcal{A}_n^\beta$, we directly have that $[\mathcal{A}_n^\beta]_\epsilon$ contains $\mathcal{A}_m^\beta$. Let us show the second inclusion. From (4), $\mathcal{A}_m^\beta$ and consequently $[\mathcal{A}_m^\beta]_\epsilon$ both contain $\mathcal{A}_n^\beta \cap ([1/\beta^{m+1}, 1] \times [0, 1])$. Now we show that $[\mathcal{A}_m^\beta]_\epsilon$ contains $[0, 1/\beta^{m+1}) \times [0, 1]$ if m is sufficiently large, which ends the proof. By Definition 33, $\mathcal{A}_0^\beta$ contains the segment $S_{\varepsilon,\varepsilon}$ of slope 1 with endpoints $(0, 0)$ and $(1, 1)$. Thus, by Definition 35, $\mathcal{A}_m^\beta$ contains the segment $h^m(c^m(S_{\varepsilon,\varepsilon}))$ of slope β^m with endpoints $(0, 0)$ and $(1/\beta^m, 1)$. Let $(x, y) \in [0, 1/\beta^{m+1}) \times [0, 1]$. Then $(y/\beta^m, y)$ belongs to $h^m(c^m(S_{\varepsilon,\varepsilon})) \subset \mathcal{A}_m^\beta$. Consequently,

$$d((x, y), \mathcal{A}_m^\beta) \leq d((x, y), (y/\beta^m, y)) \leq x + y/\beta^m < \epsilon$$

if m is sufficiently large. $\square$

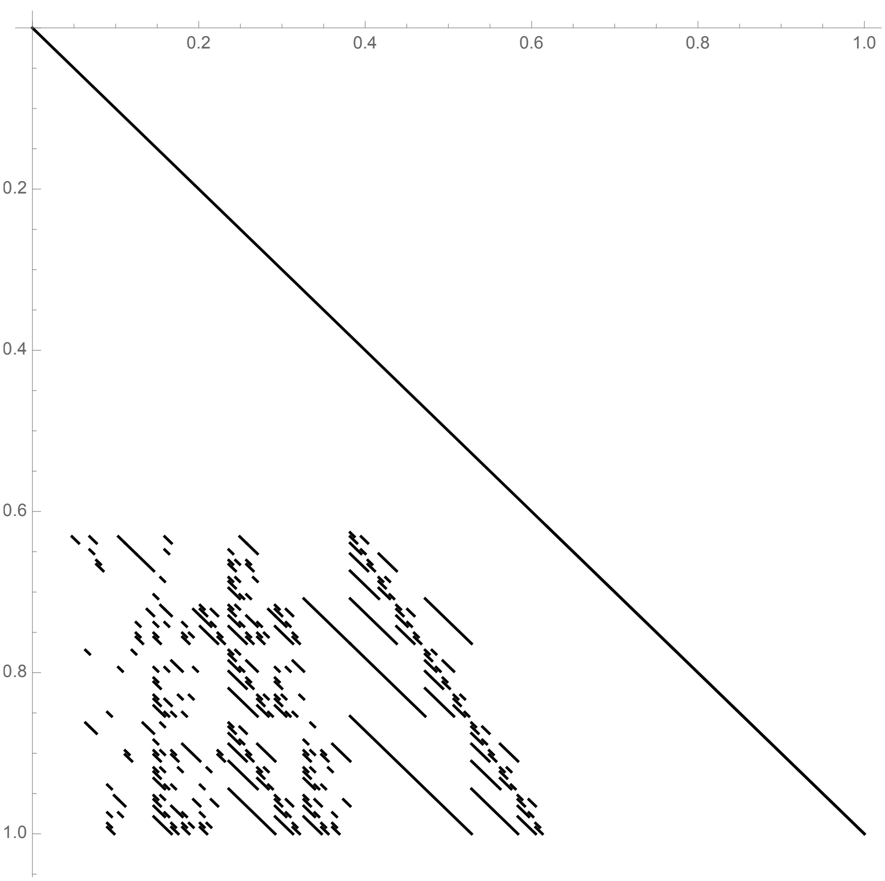


Figure 8: An approximation of $\mathcal{A}_0^{\varphi}$ computed with words of length ≤ 10.

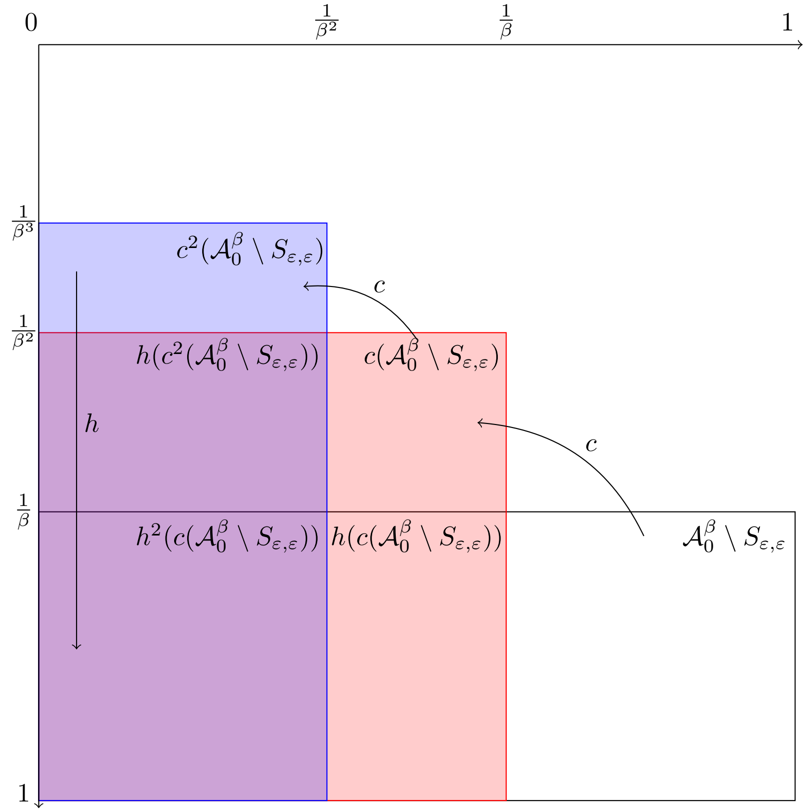


Figure 9: Two applications of c and h from $\mathcal{A}_0^\beta \setminus S_{\varepsilon,\varepsilon}$.

Definition 37. Since the sequence $(\mathcal{A}_n^\beta)_{n\geq 0}$ is a Cauchy sequence in the complete metric space $(\mathcal{H}(\mathbb{R}^2), d_h)$, its limit is a well-defined compact set denoted by $\mathcal{L}^\beta$.

Example 38. Let φ be the golden ratio. We have represented in Figure 10 all the segments of $\mathcal{A}_0^\varphi$ for words of length at most 10 and we have applied the maps $h^j(c^i(\cdot))$ to this set of segments for $0 \leq j \leq i \leq 4$. Thus we have an approximation of $\mathcal{A}_4^\varphi$.

5 The limit of the sequence of compact sets $(\mathcal{U}_n^\beta)_{n\geq 0}$

In this section, we show that the sequence $(\mathcal{U}_n^\beta)_{n\geq 0}$ of compact subsets of $[0,1]\times[0,1]$ also converges to $\mathcal{L}^\beta$. The proofs of Lemma 39, Lemma 44 are essentially the same as the ones from [10] ([10, Lemma 27, Lemma 28, Theorem 29]). However we recall them so that the paper is self-contained. The first part is to show that, when ϵ is a positive real number, then $\mathcal{U}_n^\beta \subset [\mathcal{L}^\beta]_\epsilon$ for all sufficiently large n.

Lemma 39. *Let $\epsilon > 0$. For all sufficiently large $n \in \mathbb{N}$, we have*

$$\mathcal{U}_n^\beta \subset [\mathcal{L}^\beta]_\epsilon.$$

Proof. Let $\epsilon > 0$. Take $n \in \mathbb{N}$ and let $(x,y) \in \mathcal{U}_n^\beta$. From Remark 18, there exists $(u,v) \in L_{U_\beta} \times L_{U_\beta}$ such that $\binom{u}{v} \equiv 1 \bmod 2$, $0 \leq |v| \leq |u| \leq n$ and the point (x,y) belongs to the square region

$$((\mathrm{val}_{U_\beta}(v), \mathrm{val}_{U_\beta}(u)) + Q)/U_\beta(n) \subset \mathcal{U}_n^\beta. \tag{5}$$

Let us set

$$A := \left(\frac{\mathrm{val}_{U_\beta}(v)}{U_\beta(n)}, \frac{\mathrm{val}_{U_\beta}(u)}{U_\beta(n)}\right)$$

to be the upper-left corner of the square region (5) in $\mathcal{U}_n^\beta$.

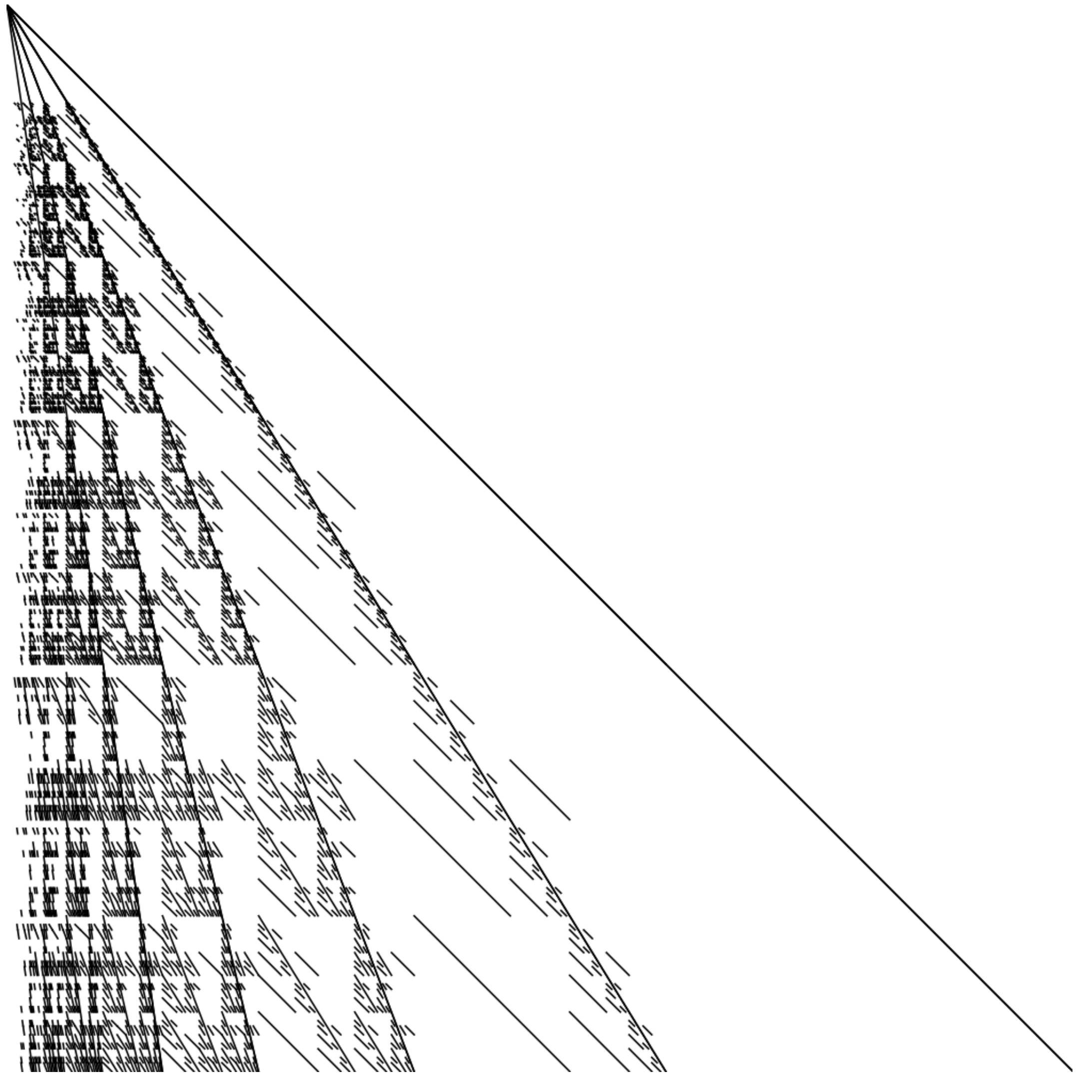


Figure 10: An approximation of the limit set $\mathcal{L}^{\varphi}$.

Assume first that (u,v) satisfies $(\star)$. The segment $S_{u,v}$ of length $\sqrt{2}\cdot\beta^{-|u|-p(u,v)}$ having $A_{u,v} = (0.0^{|u|-|v|}v, 0.u)$ as endpoint belongs to $\mathcal{A}_0^{\beta}$. Now apply $n-|u|$ times the homothety c to this segment. So the segment $c^{n-|u|}(S_{u,v})$ of length $\sqrt{2}\cdot\beta^{-n-p(u,v)}$ of endpoint $B_1 := (0.0^{n-|v|}v, 0.0^{n-|u|}u)$ belongs to $\mathcal{A}^{\beta}_{n-|u|}$ and thus to $\mathcal{L}^{\beta}$. Using (2) (the reasoning is similar to the one developed in the proof of Lemma 29), there exists $N_1 \in \mathbb{N}$ such that, for all $n \geq N_1$, $d(A, B_1) < \epsilon/2$. Hence, for all $n \geq N_1$ such that $\sqrt{2}/U_\beta(n) < \epsilon/2$, we have

$$d((x,y),\mathcal{L}^{\beta}) \leq d((x,y),B_1) \leq d((x,y),A) + d(A,B_1) \leq \sqrt{2}/U_\beta(n) + d(A,B_1) < \epsilon.$$

Now assume that (u,v) does not satisfy $(\star)$. Since $\binom{u}{v} \equiv 1 \bmod 2$, then either u and v are non-empty words, or u is non-empty and $v=\varepsilon$. Suppose that u and v are non-empty. By assumption, we have an odd number r of occurrences of v in u. For each occurrence of v in u, we count the total number of zeroes after it. We thus define a sequence of non-negative integer indices

$$|u| \geq i_1 \geq i_2 \geq \cdots \geq i_r \geq 0$$

corresponding to the number of zeroes following the first, the second, ..., the rth occurrence of v in u. Now let k be a non-negative integer such that $k > \lceil \log_2 |u| \rceil$ and $2^k > p(u,v)$. By definition of $p(u,v)$, the words $u0^{2^k}1$ and $v0^{2^k}1$ belong to L_{U_β}. We get

$$\binom{u0^{2^k}1}{v0^{2^k}1} = \sum_{\ell=1}^{r} \binom{2^k + i_\ell}{2^k}.$$

Indeed, for each $\ell \in \{1, \ldots, r\}$, consider the ℓth occurrence of v in u: we have the factorization $u = pw$ where the last letter of p is the last letter of the ℓth occurrence of v and $|w|_0 = i_\ell$. With this particular occurrence of v, we obtain occurrences of $v0^{2^k}1$ in $u0^{2^k}1$ by choosing 2^k zeroes among the $2^k + i_\ell$ zeroes available in $w0^{2^k}1$. Moreover, with the long block of 2^k zeroes, it is not possible to have any other occurrence of $v0^{2^k}1$ than those obtained from occurrences of v in u.

Then, for each $\ell \in \{1, \ldots, r\}$, we have

$$\binom{2^k + i_\ell}{2^k} \equiv 1 \bmod 2$$

from Theorem 2. Since r is odd, we get

$$\binom{u0^{2^k}1}{v0^{2^k}1} \equiv 1 \bmod 2.$$

Now, for all $k \in \mathbb{N}$ such that $k > \lceil \log_2 |u| \rceil$ and $2^k > p(u,v)$, it is easy to check that the pair of words $(u0^{2^k}1, v0^{2^k}1)$ satisfies $(\star)$. For the sake of simplicity, define $u_k := u0^{2^k}1$, $v_k := v0^{2^k}1$ and $p_k := p(u_k, v_k)$. As in the first part of the proof, the segment S_{u_k,v_k} of length $\sqrt{2} \cdot \beta^{-|u|-2^k-1-p_k}$ having $A_{u_k,v_k} = (0.0^{|u|-|v|}v0^{2^k}1, 0.u0^{2^k}1)$ as endpoint belongs to $\mathcal{A}_0^\beta$. Now apply $n-|u|$ times the homothety c to this segment. So the segment $c^{n-|u|}(S_{u_k,v_k})$ of length $\sqrt{2} \cdot \beta^{-n-2^k-1-p_k}$ of endpoint $B_2 := (0.0^{n-|v|}v0^{2^k}1, 0.0^{n-|u|}u0^{2^k}1)$ belongs to $\mathcal{A}_{n-|u|}^\beta$ and thus to $\mathcal{L}^\beta$. Using again (2) and a reasoning similar to the one from the proof of Lemma 29, there exists $N_2 \in \mathbb{N}$ such that, for all $n \geq N_2$, $d(A, B_2) < \epsilon/2$. Hence, for all $n \geq N_2$ such that $\sqrt{2}/U_\beta(n) < \epsilon/2$, we have

$$d((x,y), \mathcal{L}^\beta) \leq d((x,y), B_2) \leq d((x,y), A) + d(A, B_2) \leq \sqrt{2}/U_\beta(n) + d(A, B_2) < \epsilon.$$

Assume now that u is non-empty and $v = \varepsilon$. In this case, the point A is on the vertical line of equation $x = 0$. By Definition 33, $\mathcal{A}_0^\beta$ contains the segment $S_{\varepsilon,\varepsilon}$ of slope 1 with endpoints $(0,0)$ and $(1,1)$. Thus, by Definition 35, $\mathcal{A}_n^\beta$ contains the segment $h^n(c^n(S_{\varepsilon,\varepsilon}))$ of slope β^n with endpoints $(0,0)$ and $(1/\beta^n, 1)$. This segment also lies in $\mathcal{L}^\beta$. There exists $N_3 \in \mathbb{N}$ such that, for all $n \geq N_3$, $d(A, h^n(c^n(S_{\varepsilon,\varepsilon}))) \leq 1/\beta^n < \epsilon/2$. Consequently, for all $n \geq N_3$ such that $\sqrt{2}/U_\beta(n) < \epsilon/2$, we have

$$\begin{aligned} d((x,y), \mathcal{L}^\beta) &\leq d((x,y), h^n(c^n(S_{\varepsilon,\varepsilon}))) \leq d((x,y), A) + d(A, h^n(c^n(S_{\varepsilon,\varepsilon}))) \\ &\leq \sqrt{2}/U_\beta(n) + d(A, h^n(c^n(S_{\varepsilon,\varepsilon}))) < \epsilon. \end{aligned}$$

In each of the three cases, we conclude that $(x,y) \in [\mathcal{L}^\beta]_\epsilon$, which proves that $\mathcal{U}_n^\beta \subset [\mathcal{L}^\beta]_\epsilon$ for all sufficiently large n. □

If $\epsilon > 0$, it remains to show that $\mathcal{L}^\beta \subset [U_n]_\epsilon$ for all sufficiently large $n \in \mathbb{N}$. To that aim, we need to bound the number of consecutive words, in the genealogical order, that end with 0 in L_{U_β}.

Definition 40. We let $C_\beta \in \mathbb{N}$ denote the maximal number of consecutive 0 in $d_\beta^*(1)$, i.e.,

$$C_\beta := \max\{n \in \mathbb{N} \mid 0^n \text{ is a factor of } d_\beta^*(1)\}.$$

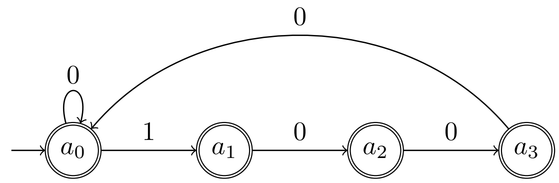


Figure 11: The automaton $\mathcal{A}_\beta$ for the dominant root β of the polynomial $P(X) = X^4 - X^3 - 1$.

In the next proposition, we show that the maximal number of consecutive words ending with 0 in L_{U_β} is $C_\beta + 1$.

Proposition 41. *If we order the words in L_{U_β} by the genealogical order, the maximal number of consecutive words ending with* 0 *in L_{U_β}, i.e., the maximal number of consecutive normal U_β-representations ending with* 0*, is $C_\beta + 1$.*

Proof. Let $n \in \mathbb{N}$ be such that $\operatorname{rep}_{U_\beta}(n)$ ends with 0. We can suppose that $\operatorname{rep}_{U_\beta}(n-1)$ does not end with 0, otherwise we translate n. If $|\operatorname{rep}_{U_\beta}(n+1)| = |\operatorname{rep}_{U_\beta}(n)|$, then $\operatorname{rep}_{U_\beta}(n+1)$ does not end with 0 because $U_\beta(m) \geq 2$ for all $m \geq 1$. Indeed, if a single digit (not the least significant one) is changed, then the value is increased by at least 2. Let $C \geq 1$ be such that, for all $k \in \{0, \ldots, C\}$, $|\operatorname{rep}_{U_\beta}(n+k)| = |\operatorname{rep}_{U_\beta}(n)| + k$ and $|\operatorname{rep}_{U_\beta}(n+C+1)| = |\operatorname{rep}_{U_\beta}(n+C)|$. The normal-$U$ representation preserves the order, i.e., for all integers m_1 and m_2, $m_1 \leq m_2$ if and only if $\operatorname{rep}_{U_\beta}(m_1) \leq \operatorname{rep}_{U_\beta}(m_2)$ (see, for instance, [3]). Thus, the words $\operatorname{rep}_{U_\beta}(n), \ldots, \operatorname{rep}_{U_\beta}(n+C-1)$ are prefixes of $d^*_\beta(1)$, respectively of length $|\operatorname{rep}_{U_\beta}(n)|, |\operatorname{rep}_{U_\beta}(n)| + 1, \ldots, |\operatorname{rep}_{U_\beta}(n)| + C - 1$ (the prefixes of $d^*_\beta(1)$ are the maximal words of different length in L_{U_β}). By Definition 40, we deduce that $C \leq C_\beta$. Consequently, there are at most $C_\beta + 1$ consecutive words ending with 0 in L_{U_β}. □

Let us illustrate the previous proposition.

Example 42. Let φ be the golden ratio. Then $C_\varphi = 1$ since $d^*_\varphi(1) = (10)^\omega$. The first few words of L_{U_φ} are $\varepsilon, 1, 10, 100, 101, 1000, 1001, 1010, 10000, 10001, \ldots$. The maximal number of consecutive words ending with 0 in L_{U_φ} is $C_\varphi + 1 = 2$.

Example 43. Let β be the dominant root of the polynomial $P(X) = X^4 - X^3 - 1$. Then $\beta \approx 1.38028$ is a Parry number with $d_\beta(1) = 1001$ and $d^*_\beta(1) = (1000)^\omega$. The automaton $\mathcal{A}_\beta$ is depicted in Figure 11. In this example, $C_\beta = 3$. The first few words of L_{U_β} are $\varepsilon, 1, 10, 100, 1000, 10000, 10001, \ldots$. The maximal number of consecutive words ending with 0 in L_{U_β} is $C_\beta + 1 = 4$.

Lemma 44. *Let $\epsilon > 0$. For all $(x, y) \in \mathcal{L}^\beta$, $d((x, y), \mathcal{U}^\beta_n) < \epsilon$ for all sufficiently large n.*

Proof. Let $\epsilon > 0$ and let $(x, y) \in \mathcal{L}^\beta$. Since $(\mathcal{A}^\beta_n)_{n\geq 0}$ converges to $\mathcal{L}^\beta$, there exists N_1 and $(x', y') \in \mathcal{A}^\beta_{N_1}$ such that,

$$d((x, y), (x', y')) < \epsilon/4.$$

By definition of $\mathcal{A}^\beta_{N_1}$, there exist i, j such that $0 \leq j \leq i \leq N_1$ and $(x'_0, y'_0) \in \mathcal{A}^\beta_0$ such that

$$h^j(c^i((x'_0, y'_0))) = (x', y').$$

By definition of $\mathcal{A}^\beta_0$, there exists a pair $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ satisfying $(\star)$ and $(x''_0, y''_0) \in S_{u,v}$ such that

$$d((x'_0, y'_0), (x''_0, y''_0)) < \epsilon/4.$$

Notice that, since $j \leq i$,

$$\begin{aligned} d((x', y'), h^j(c^i((x''_0, y''_0)))) &= d(h^j(c^i((x'_0, y'_0))), h^j(c^i((x''_0, y''_0)))) \\ &\leq d((x'_0, y'_0), (x''_0, y''_0)) < \epsilon/4. \end{aligned}$$

Consequently, we get that

$$d((x,y), h^j(c^i((x_0'',y_0'')))) < \epsilon/2.$$

In the second part of the proof, we will show that $d(h^j(c^i((x_0'',y_0''))),\mathcal{U}_n^\beta) < \epsilon/2$ for all sufficiently large n. We will make use of the constants i,j, the words u,v given above and the integer $p := p(u,v)$.

Set

$$L_{u,v} := \begin{cases} L_{U_\beta}, & \text{if } u = v = \varepsilon; \\ 0^* L_{U_\beta}, & \text{otherwise.} \end{cases}$$

Since $(u,v) \in L_{U_\beta} \times L_{U_\beta}$ satisfies $(\star)$, the pair of words $(u0^p w, v0^p w)$ has an odd binomial coefficient, for all words $w \in L_{u,v}$, using Lemma 25 and Corollary 28. In particular, this is the case when $w \in L_{u,v}$ is of length n. We can choose n sufficiently large such that $U_\beta(n) \geq C_\beta + 3$ using Proposition 41. In this case, there exist at least two words $w \in L_{u,v}$ with $|w| = n$ and not ending with 0. Furthermore, as soon as w does not end with 0, Lemma 1 shows that

$$\binom{u0^p w0^k}{v0^p w} = \binom{u0^p w}{v0^p w} \equiv 1 \bmod 2$$

for all $k \geq 0$. By definition of the sequence U_β, we also have

$$\#\{z \in 0^* L_{U_\beta} \mid u0^p wz \in L_{U_\beta} \text{ and } |z| = k\} \leq U_\beta(k).$$

Thus, for all $j \leq i$, we conclude that at least one of the $U_\beta(j)$ binomial coefficients of the form $\binom{u0^p wz}{v0^p w}$ with w not ending with 0 and $|z| = j$ is odd (indeed, choose $z = 0^j$ for instance). Otherwise stated, at least one of the square regions

$$\left(\frac{\mathrm{val}_{U_\beta}(v0^p w)}{U_\beta(n+i+|u|+p)}, \frac{\mathrm{val}_{U_\beta}(u0^p wz)}{U_\beta(n+i+|u|+p)}\right) + \frac{Q}{U_\beta(n+i+|u|+p)}, \text{ with } |z| = j, \tag{6}$$

is a subset of $\mathcal{U}_{n+i+|u|+p}^\beta$, since $|v0^p w|, |u0^p wz| \leq n+i+|u|+p$. This can be visualized in Figure 12.

Now observe that, for any word $w \in L_{u,v}$, each square region of the form (6) is intersected by $h^j(c^i(S_{u,v}))$. Indeed, the latter segment has $A := (0.0^{i+|u|-|v|}v, 0.0^{i-j}u)$ and $B := (0.0^{i+|u|-|v|}v0^p d_\beta^*(1), 0.0^{i-j}u0^p d_\beta^*(1))$ as endpoints and slope β^j. Using (2), if n is sufficiently large, the points

$$\left(\frac{\mathrm{val}_{U_\beta}(v0^p 0^n)}{U_\beta(n+i+|u|+p)}, \frac{\mathrm{val}_{U_\beta}(u0^p 0^{n+j})}{U_\beta(n+i+|u|+p)}\right) \left(\text{resp., } \left(\frac{\mathrm{val}_{U_\beta}(v0^p d_n)}{U_\beta(n+i+|u|+p)}, \frac{\mathrm{val}_{U_\beta}(u0^p d_{n+j})}{U_\beta(n+i+|u|+p)}\right)\right)$$

and A (resp., B) are close for all $j \leq i$, where d_n denotes the prefix of length n of $d_\beta^*(1)$ for all $n \geq 0$. When u and v are non-empty, this can be seen in Figure 13 where each rectangular gray region contains at least one square region from $\mathcal{U}_{n+i+|u|+p}^\beta$ (to draw this picture, we take the particular case of the golden ratio φ and $i = 2$). When $u = v = \varepsilon$, Figure 13 is modified in the following way: simply replace each word of the forms $u0^\ell$, $v0^\ell$ by ε.

Consequently, every point of $h^j(c^i(S_{u,v}))$ is at distance at most

$$\frac{2 \cdot (C_\beta + 2) \cdot U_\beta(j)}{U_\beta(n+i+|u|+p)}$$

from a point in $\mathcal{U}_{n+i+|u|+p}^\beta$ when n is sufficiently large. Indeed, either the point falls into a gray region from Figure 13, or not. In the first case, the point is at distance at most $U_\beta(j)/U_\beta(n+i+|u|+p)$ from a square region in $\mathcal{U}_{n+i+|u|+p}^\beta$; see Figure 12. Observe that this square region is of the form (6) where w does not end with 0. Otherwise, the point falls into a (white) square region of the form

$$\left(\frac{\mathrm{val}_{U_\beta}(v0^p w)}{U_\beta(n+i+|u|+p)}, \frac{\mathrm{val}_{U_\beta}(u0^p w'z)}{U_\beta(n+i+|u|+p)}\right) + \frac{Q}{U_\beta(n+i+|u|+p)}, \text{ with } |w| = |w'| = n, |z| = j.$$

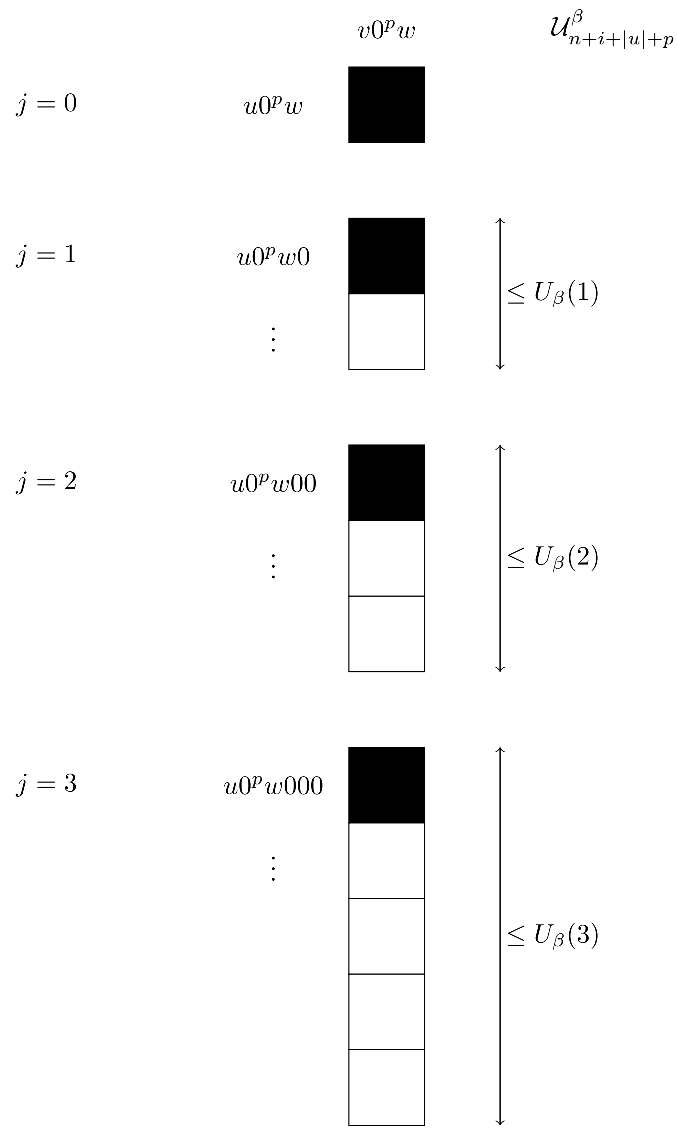


Figure 12: If w does not end with 0 and is such that $|w| = n$, then $\binom{u0^p w0^j}{v0^p w}$ being odd creates a square region in $\mathcal{U}^{\beta}_{n+i+|u|+p}$.

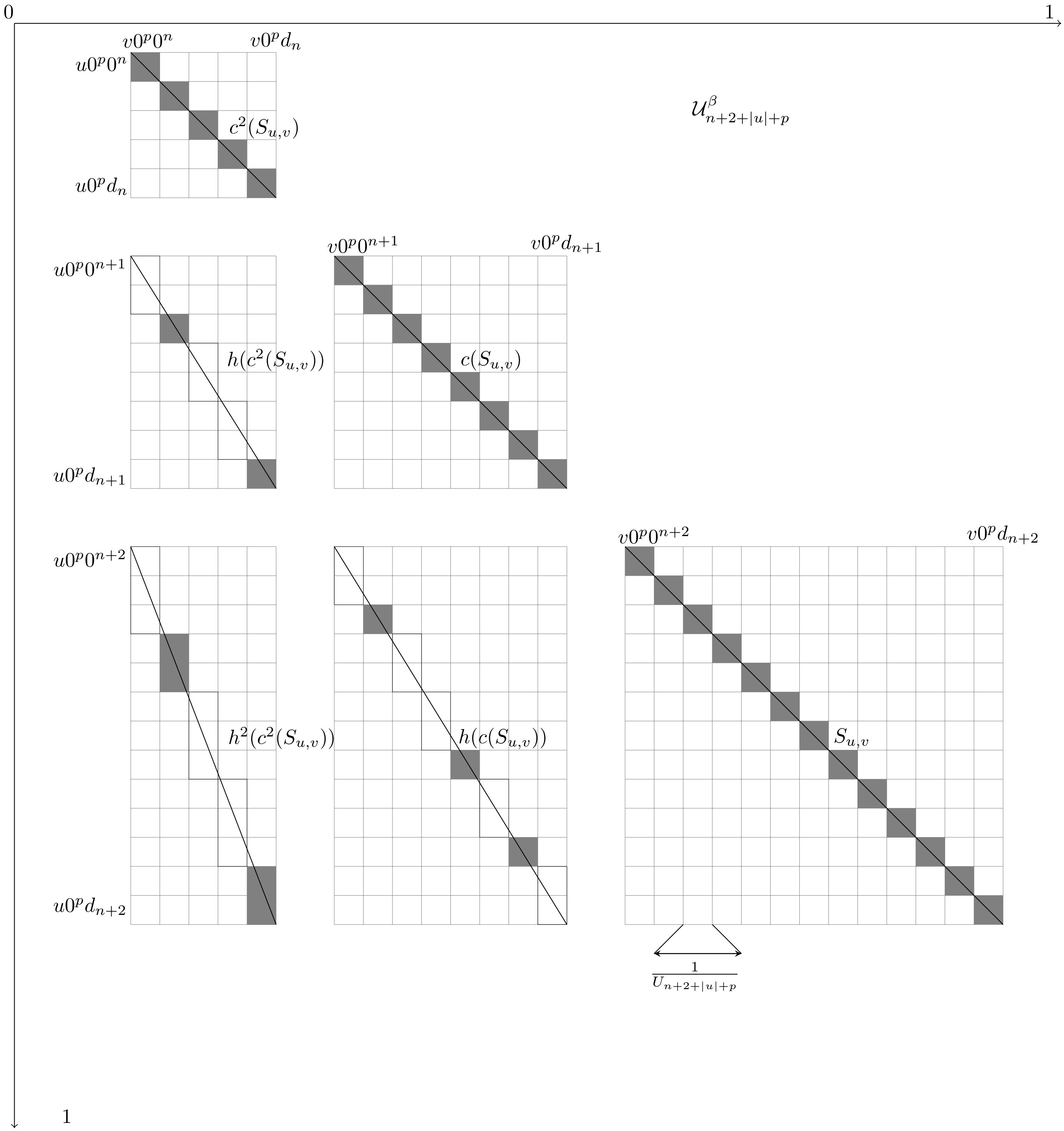


Figure 13: The situation occurring in the proof of Lemma 44, where we choose β to be the golden ratio.

Since n is large enough, there exists a word w'' not ending with 0 with $|w''| = n$, which is within a distance of $2 \cdot (C_\beta + 2)$ of w and w'. Then, applying the argument from the previous case proves the statement.

In particular, the result holds for the point $h^j(c^i((x_0'', y_0'')))$ belonging to $h^j(c^i(S_{u,v}))$. Hence, for all sufficiently large n,

$$d(h^j(c^i((x_0'', y_0''))), \mathcal{U}_n^\beta) < \epsilon/2.$$

The conclusion follows. □

Corollary 45. *Let $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ satisfying $(\star)$ and let $0 \le j \le i$. For every point (f, g) of the segment $h^j(c^i(S_{u,v}))$, there exists a sequence $((f_n, g_n))_{n\ge0}$ converging to (f, g) and such that $(f_n, g_n) \in \mathcal{U}_n^\beta$ for all n.*

Proof. Let (f, g) be a point of the segment $h^j(c^i(S_{u,v}))$. From the proof of Lemma 44, we have

$$d((f, g), \mathcal{U}_m^\beta) \le \frac{2 \cdot (C_\beta + 2) \cdot U_\beta(j)}{U_\beta(m)}$$

for all sufficiently large m. Consequently, there exists a sequence $((f_n, g_n))_{n\ge0}$ converging to (f, g) and such that $(f_n, g_n) \in \mathcal{U}_n^\beta$ for all n. □

We are now ready to prove the main result of this paper.

Theorem 46. *The sequence $(\mathcal{U}_n^\beta)_{n\ge0}$ converges to $\mathcal{L}^\beta$.*

Proof. Let $\epsilon > 0$. From Lemma 39, it suffices to show that $\mathcal{L}^\beta \subset [\mathcal{U}_n^\beta]_\epsilon$ for all sufficiently large $n \in \mathbb{N}$. For all $(x, y) \in \mathcal{L}^\beta$, using Corollary 45, there exists a (Cauchy) sequence $((f_i(x, y), g_i(x, y))_{i\ge0}$ such that $(f_i(x, y), g_i(x, y)) \in \mathcal{U}_i^\beta$ for all i, and there exists $N_{(x,y)}$ such that, for all $i, j \ge N_{(x,y)}$,

$$d((f_i(x, y), g_i(x, y)), (f_j(x, y), g_j(x, y))) < \epsilon/2 \tag{7}$$

and

$$d((f_i(x, y), g_i(x, y)), (x, y)) < \epsilon/2.$$

We trivially have

$$\mathcal{L}^\beta \subset \bigcup_{(x,y)\in\mathcal{L}^\beta} B((f_{N_{(x,y)}}(x, y), g_{N_{(x,y)}}(x, y)), \epsilon/2).$$

Since $\mathcal{L}^\beta$ is compact, we can extract a finite covering: there exist $(x_1, y_1), \dots, (x_k, y_k)$ in $\mathcal{L}^\beta$ such that

$$\mathcal{L}^\beta \subset \bigcup_{j=1}^{k} B((f_{N_{(x_j,y_j)}}(x_j, y_j), g_{N_{(x_j,y_j)}}(x_j, y_j)), \epsilon/2).$$

Let $N = \max_{j=1,\dots,k} N_{(x_j,y_j)}$. From (7), we deduce that, for all $j \in \{1, \dots, k\}$ and all $n \ge N$,

$$B((f_{N_{(x_j,y_j)}}(x_j, y_j), g_{N_{(x_j,y_j)}}(x_j, y_j)), \epsilon/2) \subset B((f_n(x_j, y_j), g_n(x_j, y_j), \epsilon)$$

and therefore

$$\mathcal{L}^\beta \subset \bigcup_{j=1}^{k} B((f_n(x_j, y_j), g_n(x_j, y_j)), \epsilon) \subset [\mathcal{U}_n^\beta]_\epsilon.$$

□

Remark 47. As in [10], the results mentioned above can be extended to any prime number. Let q be a prime number and r be a positive residue in $\{1, \dots, q-1\}$. We can extend Definition 16 to

$$\mathcal{U}_{n,r}^{\beta} := \frac{1}{U_\beta(n)} \bigcup \left\{ (\mathrm{val}_{U_\beta}(v), \mathrm{val}_{U_\beta}(u)) + Q \mid u, v \in L_{U_\beta}, \binom{u}{v} \equiv r \bmod q \right\} \subset [0,1] \times [0,1].$$

Since we make use of Lucas' theorem, we limit ourselves to congruences modulo a prime number. We just sketch the main differences with the case $q = 2$.

See, for instance, Figure 15 for the case $\beta = \varphi$, $q = 3$ and $r = 2$.

The $(\star)$ condition from Definition 23 becomes $(\star)_r$. We say that $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ satisfies the $(\star)_r$ condition if either $u = v = \varepsilon$ and $\binom{u}{v} \equiv r \bmod q$, or $|u| \geq |v| > 0$ and

$$\binom{u0^{p(u,v)}}{v0^{p(u,v)}} \equiv r \bmod q \quad \text{and} \quad \binom{u0^{p(u,v)}}{v0^{p(u,v)}a} = 0 \quad \forall\, a \in A_{U_\beta}$$

where $p(u, v)$ is defined using Proposition 19. In this extended context, Proposition 26, Proposition 27, Corollary 28, Lemma 29 and Remark 30 are easy to adapt. Note that the pairs (u, v) satisfying this condition depend on the choice of q and r. The sets $\mathcal{A}_n^{\beta}$ are defined as before. The pair (u, u) satisfies $(\star)_r$ if and only if $r = 1$; see Lemma 25. Thus, the segment of slope 1 with endpoints $(0, 0)$ and $(1, 1)$ belongs to $\mathcal{A}_0^{\beta}$ if and only if $r = 1$. An alternative proof of Proposition 36 follows the same lines as in [10].

6 Appendix

Example 48. We have represented the set $\mathcal{U}_9^{\varphi}$ in Figure 14.

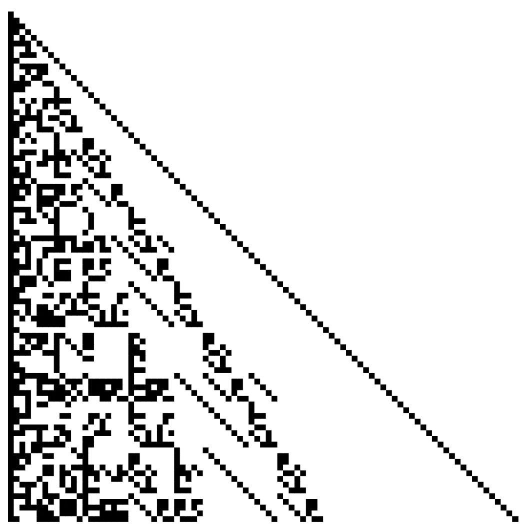


Figure 14: The set $\mathcal{U}_9^{\varphi}$.

Example 49. Let us consider the case when $\beta = \varphi$ is the golden ratio. We have represented in Figure 15 the set $\mathcal{U}_{9,2}^{\varphi}$ when considering binomial coefficients congruent to 2 modulo 3 and an approximation of the limit set $\mathcal{L}^{\varphi}$ proceeding as in Example 38.

In this last example, we give an approximation of the limit object $\mathcal{L}^{\beta}$ for several different values of β. A real number $\beta > 1$ is a *Pisot number* if it is an algebraic integer whose conjugates have modulus less than 1.

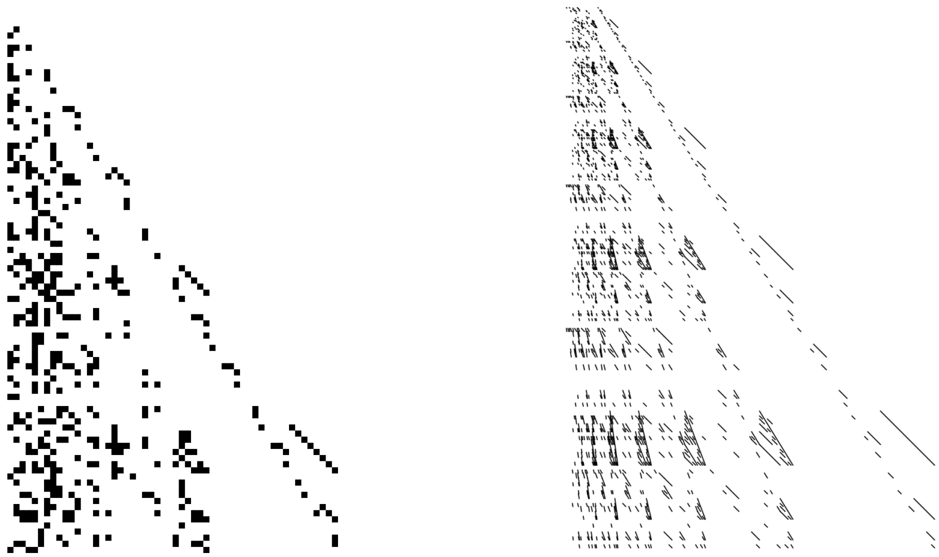


Figure 15: The set $\mathcal{U}_{9,2}^{\varphi}$ (on the left) and an approximation of the corresponding limit set $\mathcal{L}^{\varphi}$ (on the right).

Example 50. Let us define several Parry numbers. Let $\beta_1 \approx 2.47098$ be the dominant root of the polynomial $P(X) = X^4 - 2X^3 - X^2 - 1$, which is a Parry and Pisot number; see Example 22. Let $\beta_2 \approx 2.47098$ be the dominant root of the polynomial $P(X) = X^4 - X^3 - 1$, which is a Parry and Pisot number; see Example 43. Let $\beta_3 \approx 2.80399$ be the dominant root of the polynomial $P(X) = X^4 - 2X^3 - 2X^2 - 2$. We can show that β_3 is a Parry number, but not a Pisot number. Let $\beta_4 \approx 1.32472$ be the dominant root of the polynomial $P(X) = X^5 - X^4 - 1$. We can show that β_4 is a Parry number and also the smallest Pisot number. In Figure 16, we depict an approximation of $\mathcal{L}^{\beta}$ for β in $\{\varphi^2, \beta_1, \ldots, \beta_4\}$.

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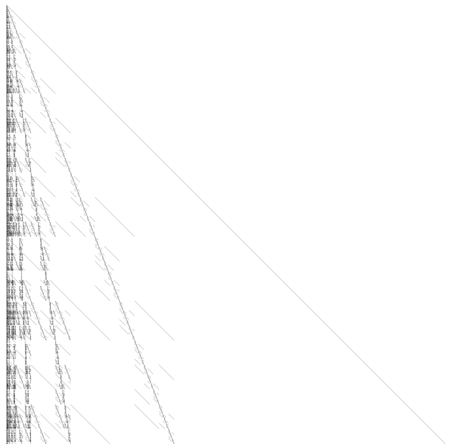
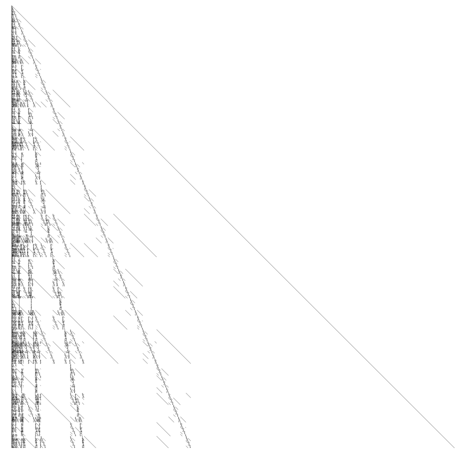
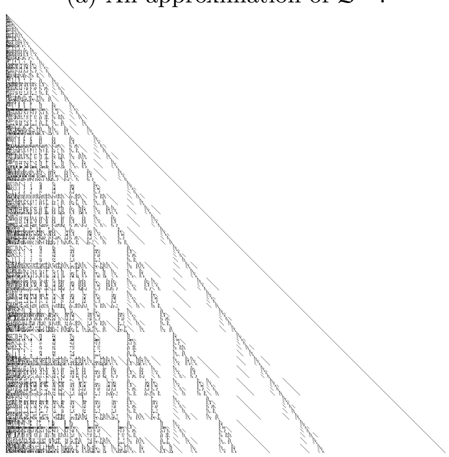


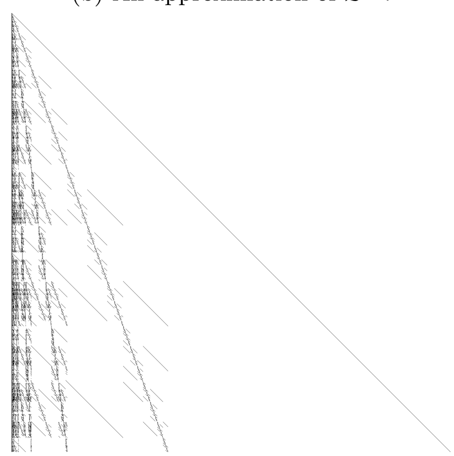
(a) An approximation of $\mathcal{L}^{\varphi^2}$.



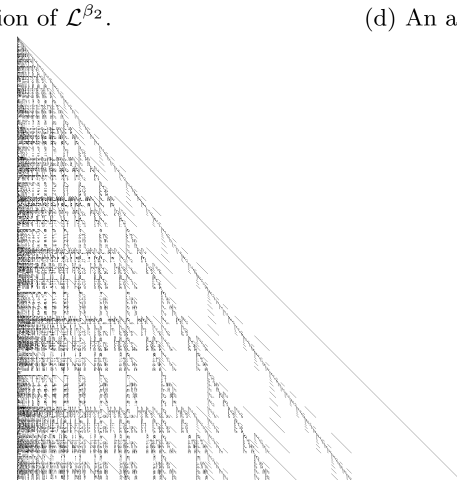
(b) An approximation of $\mathcal{L}^{\beta_1}$.



(c) An approximation of $\mathcal{L}^{\beta_2}$.



(d) An approximation of $\mathcal{L}^{\beta_3}$.



(e) An approximation of $\mathcal{L}^{\beta_4}$.

Figure 16: An approximation of the limit object $\mathcal{L}^{\beta}$ for different values of β.