# A Context-free Grammar for Peaks and Double Descents of Permutations 

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#### Abstract

This paper is concerned with the joint distribution of the number of exterior peaks and the number of proper double descents over permutations on $[n]=\{1,2, \ldots, n\}$. The notion of exterior peaks of a permutation was introduced by Aguiar, Bergeron and Nyman in their study of the peak algebra. Gessel obtained the generating function of the number of permutations on $[n]$ with a given number of exterior peaks. On the other hand, by establishing differential equations, Elizalde and Noy derived the generating function for the number of permutations on $[n]$ with a given number of proper double descents. Barry and Basset independently deduced the generating function of the number of permutations on $[n]$ with no proper double descents. We find a context-free grammar which can be used to compute the number of permutations on $[n]$ with a given number of exterior peaks and a given number of proper double descents. Based on the grammar, the recurrence relation of the number of permutations on $[n]$ with a give number of exterior peaks can be easily obtained. Moreover, we use the grammatical calculus to derive the generating function without solving differential equations. Our formula reduces to the formulas of Gessel, Elizalde-Noy, Barry, and Basset. Finally, from the grammar we establish a relationship between our generating function and the generating function of the joint distribution of the number of peaks and the number of double descent derived by Carlitz and Scoville.


Keywords: context-free grammars, grammatical labeling, exterior peaks, proper double descents.

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## 1 Introduction

The objective of this paper is to present a grammatical approach to the joint distribution of exterior peaks and proper double descents of permutations on $[n]$. The notion of exterior peaks was introduced by Aguiar, Bergeron and Nyman [1]. Given a permutation $\pi=\pi_{1} \pi_{2} \cdots \pi_{n}$ on [ $n$ ], an index $i$ is called an exterior peak if $\pi_{1}>\pi_{2}$ for $i=1$ or $\pi_{i-1}<\pi_{i}>\pi_{i+1}$ for $1<i<n$.

Let $T(n, k)$ be the number of permutations on $[n]$ with $k$ exterior peaks and let

$$
\begin{equation*}
T_{n}(x)=\sum_{k \geq 0} T(n, k) x^{k} . \tag{1.1}
\end{equation*}
$$

Gessel [19] obtained the generating function of $T_{n}(x)$.
Theorem 1.1 (Gessel [19]). We have

$$
\begin{equation*}
\sum_{n=0}^{\infty} \frac{T_{n}(x) t^{n}}{n!}=\frac{\sqrt{1-x}}{\sqrt{1-x} \cosh (t \sqrt{1-x})-\sinh (t \sqrt{1-x})} . \tag{1.2}
\end{equation*}
$$

The number of proper double descents is a classical statistic on permutations, which has been extensively studied. An index $i$ of a permutation $\pi=\pi_{1} \pi_{2} \ldots \pi_{n}$ on $[n]$ is called a proper double descent if $3 \leq i \leq n$ and $\pi_{i-2}>\pi_{i-1}>\pi_{i}$. Denote $U(n, k)$ the number of permutations on $[n]$ with $k$ proper double descents and let

$$
\begin{equation*}
U_{n}(y)=\sum_{k \geq 0} U(n, k) y^{k} . \tag{1.3}
\end{equation*}
$$

By establishing the following ordinary differential equations,

$$
f^{\prime \prime}+(1-y)\left(f^{\prime}+f\right)=0
$$

with $f(0)=1$ and $f^{\prime}(0)=-1$, Elizalde and Noy [10] derived the generating function of $U_{n}(y)$.

Theorem 1.2 (Elizalde and Noy [10]). We have

$$
\begin{equation*}
\sum_{n=0}^{\infty} \frac{U_{n}(y) t^{n}}{n!}=\frac{2 \sqrt{(y-1)(y+3)} e^{t / 2 \cdot(1-y+\sqrt{(y-1)(y+3)})}}{1+y+\sqrt{(y-1)(y+3)}-(1+y-\sqrt{(y-1)(y+3)}) e^{t \sqrt{(y-1)(y+3)}}} \tag{1.4}
\end{equation*}
$$

Barry [2] and Basset [3] also independently studied the generating function for the number of permutations on $[n]$ with no proper double descents.

Theorem 1.3 (Elizalde and Noy [10], Barry [2], Basset [3]). We have

$$
\begin{equation*}
\sum_{n=0}^{\infty} U(n, 0) \frac{t^{n}}{n!}=\frac{\sqrt{3}}{2} \frac{e^{t / 2}}{\cos (\sqrt{3} t / 2+\pi / 6)} . \tag{1.5}
\end{equation*}
$$

Note that Theorem 1.3 gives the explicit form of the following generating function

$$
\begin{equation*}
\sum_{n=0}^{\infty} U(n, 0) \frac{t^{n}}{n!}=\left(\sum_{j=0}^{\infty} \frac{t^{3 j}}{(3 j)!}-\sum_{j=0}^{\infty} \frac{t^{3 j+1}}{(3 j+1)!}\right)^{-1} \tag{1.6}
\end{equation*}
$$

which appears in [7, pp.156-157], [13, Example 3, pp. 51], [14, Exercise 5.2.17] and [18, pp. 126 and 260]. The right hand sides of (1.5) and (1.6) are equal, which can be seen from the following formula

$$
\begin{aligned}
\sum_{j=0}^{\infty} \frac{t^{3 j}}{(3 j)!}-\sum_{j=0}^{\infty} \frac{t^{3 j+1}}{(3 j+1)!} & =\frac{1}{3}\left(e^{x}+e^{\omega x}+e^{\omega^{2} x}\right)-\frac{1}{3}\left(e^{x}+\omega^{2} e^{\omega x}+\omega e^{\omega^{2} x}\right) \\
& =\frac{1}{3}\left(1-\omega^{2}\right) e^{\omega x}+\frac{1}{3}(1-\omega) e^{\omega^{2} x}
\end{aligned}
$$

where $\omega=e^{2 \pi \mathrm{i} / 3}$.
To consider the joint distribution of the number of exterior peaks and the number of proper double descents over permutations, we define $P_{n}(i, j)$ to be the number of permutations on $[n]$ with $i$ exterior peaks and $j$ proper double descents. Let

$$
\begin{equation*}
P_{n}(x, y)=\sum_{i, j} P_{n}(i, j) x^{i} y^{j}, \tag{1.7}
\end{equation*}
$$

where $0 \leq j \leq n-1$ and $2 i+j \leq n$. we find a context free grammar and a grammatical labeling of permutations to generate the polynomials $P_{n}(x, y)$. In fact, for the reason of the grammar, we need to define the polynomials $P_{n}(x, y, z, w)$ in four variables:

$$
\begin{equation*}
P_{n}(x, y, z, w)=\sum_{i, j} P_{n}(i, j) x^{i} y^{j} z^{i+1} w^{n-2 i-j}, \tag{1.8}
\end{equation*}
$$

where $i$ and $j$ are in the same range as in (1.7).
We also notice that this grammar can be used to investigate the joint distribution of the number of peaks and the number of double descents of permutations on $[n]$. To this end, we shall define polynomials $Q_{n}(x, y, z, w)$ and we shall show that the above grammar $G$ leads to a recurrence relation on the polynomials $P_{n}(x, y)$, which also involves the polynomials $Q_{n}(x, y, z, w)$.

Using the grammatical calculus, we deduce the following generating function of $P_{n}(x, y)$ without solving differential equations:

$$
\begin{equation*}
\sum_{n=0}^{\infty} P_{n}(x, y) \frac{t^{n}}{n!}=\frac{2 \sqrt{(1+y)^{2}-4 x} e^{t / 2 \cdot\left(1-y+\sqrt{(1+y)^{2}-4 x}\right)}}{1+y+\sqrt{(1+y)^{2}-4 x}-\left(1+y-\sqrt{(1+y)^{2}-4 x}\right) e^{t \sqrt{(1+y)^{2}-4 x}}} \tag{1.9}
\end{equation*}
$$

We also establish a relationship between $\operatorname{Gen}(z, t)$ and $\operatorname{Gen}(y, t)$, which leads to the generating function of the polynomials $Q_{n}(x, y, z, w)$. We show that

$$
\begin{equation*}
\operatorname{Gen}(y, t)=y+x z F(x, y, z, w ; t), \tag{1.10}
\end{equation*}
$$

where $F(x, y, z, w ; t)$ is the generating function of the joint distribution of the number of peaks, the number of valleys, the number of double descents and the number of double rises,
obtained by Carlitz and Scoville [4]. Thus we have given a grammatical treatment of the formula of Carlitz and Scoville.

This paper is organized as follows. In Section 2, we give an overview of the formal derivative with respect to a context-free grammar and the notion of a grammatical labeling. Then a present a grammar $G$ and give a proof of Theorem 2.1 on the joint distribution of the number of exterior peaks and the number of proper double descents. We also show that the same grammar $G$ also can be used to generate the polynomials $Q_{n}(x, y, z, w)$, which are the generating functions of the number of peaks and the numbers of double descents. Using the grammar, a recurrence relation of $P_{n}(x, y)$ is derived. In Section 3, we give a proof of Theorem 3.1 without solving differential equations.

## 2 A Grammatical Labeling of Permutations

In this section, we give an overview of the formal derivative with respect to a context-free grammar. Then we present a context-free grammar and a grammatical labeling of permutations, which can be employed to generate the number of permutations on $[n]$ with a given number of exterior peaks and a given number of proper double descents. Using a different grammatical labeling of permutations, we show that the same grammar can be used to generate the number of permutations on $[n]$ with a given number of peaks and a given number of double descents. These two statistics have been studied by Carlitz and Scoville [4].

A context-free grammar $G$ over a variable set $V$ is defined as a set of substitution rules replacing a variable in $V$ by a Laurent polynomial of variables in $V$. The polynomials considered in this paper are assumed to be over real numbers. The formal derivative $D$ with respect to $G$ is a linear operator acting on Laurent polynomials in variables in $V$ such that

$$
D(u v)=u D(v)+v D(u) .
$$

If $c$ is a constant, we define $D(c)=0$. Thus we have $D\left(u^{-1}\right)=-u^{-2} D(u)$ since $D\left(u u^{-1}\right)=0$. Clearly, we have the Leibniz formula:

$$
\begin{equation*}
D^{n}(u v)=\sum_{k=0}^{n}\binom{n}{k} D^{k}(u) D^{n-k}(v) . \tag{2.1}
\end{equation*}
$$

For a Laurent polynomial $w$ of variables in $V$, we define the generating function of $w$ by

$$
\operatorname{Gen}(w, t)=\sum_{n=0}^{\infty} D^{n}(w) \frac{t^{n}}{n!} .
$$

Then the following relations hold:

$$
\begin{align*}
\operatorname{Gen}(u+v, t) & =\operatorname{Gen}(u, t)+\operatorname{Gen}(v, t)  \tag{2.2}\\
\operatorname{Gen}(u v, t) & =\operatorname{Gen}(u, t) \operatorname{Gen}(v, t)  \tag{2.3}\\
\operatorname{Gen}^{\prime}(u, t) & =\operatorname{Gen}(D(u), t) \tag{2.4}
\end{align*}
$$

where $u, v$ are Laurent polynomials of variables in $V$.
The idea of using the formal derivative with respect to a context-free grammar to study combinatorial structures was initiated by Chen [5]. Dumont [8] found grammars for several classical combinatorial structures. For example, the grammar

$$
\begin{equation*}
x \rightarrow x y, \quad y \rightarrow x y \tag{2.5}
\end{equation*}
$$

can be used to generate the Eulerian polynomials. Moreover, Dumont [8] gave the grammar

$$
x \rightarrow x y, \quad y \rightarrow x
$$

to generate the André polynomials $E_{n}(x, y)$. Note that the generating function for the André polynomials $E_{n}(x, y)$ was first obtained by Foata and Schützenberger [12] by solving a differential equation. However, Foata and Han [11] later found a way to compute the generating function of $E_{n}(x, 1)$ without solving a differential equation. Dumont [9] also discovered the following grammar for the Ramanujan polynomials:

$$
x \rightarrow x^{3} y, \quad y \rightarrow x y^{2}
$$

Recently, the concept of a grammatical labeling was introduced in [6]. More precisely, a grammatical labeling is an assignment of the underlying elements of a combinatorial structure with constants or variables, which is consistent with the substitution rules of a grammar $G$. For example, given a context-free grammar

$$
\begin{equation*}
x \rightarrow x y, \quad y \rightarrow x^{2} \tag{2.6}
\end{equation*}
$$

we may label the elements of a permutation $\pi$ on $[n]$ by $x$ and $y$ based on the exterior peaks. Then it can be shown that $D^{n}(x)$ yields the generating function of the number of permutations on $[n]$ with a given number of exterior peaks. Reminiscent of (2.6), Ma [16] found a connection between the number of peaks of permutations and the relations $D_{z}(x)=x y$ and $D_{z}(y)=x^{2}$, where $x=\sec (z), y=\tan (z)$ and $D_{z}$ is the ordinary derivative with respect to $z$. Ma, Ma, Yeh and Zhu [17] also found grammars to generate several polynomials associated with Eulerian polynomials, including $q$-Eulerian polynomials, alternating run polynomials and derangement polynomials.

Let $P_{n}(i, j)$ denote the number of permutations on $[n]$ with $i$ exterior peaks and $j$ proper double descents. We now give a grammar to generate $P_{n}(i, j)$. Let $P_{n}(x, y, z, w)$ be defined
as in (1.8), that is,

$$
P_{n}(x, y, z, w)=\sum_{i, j} P_{n}(i, j) x^{i} y^{j} z^{i+1} w^{n-2 i-j},
$$

where $0 \leq j \leq n-1$ and $2 i+j \leq n$. The first few values of $P_{n}(x, y, z, w)$ are listed below:

$$
\begin{aligned}
P_{1}(x, y, z, w) & =z w, \\
P_{2}(x, y, z, w) & =z w^{2}+x z^{2}, \\
P_{3}(x, y, z, w) & =z w^{3}+4 x z^{2} w+x y z^{2}, \\
P_{4}(x, y, z, w) & =z w^{4}+11 x z^{2} w^{2}+6 x y z^{2} w+5 x^{2} z^{3}+x y^{2} z^{2}, \\
P_{5}(x, y, z, w)= & z w^{5}+26 x z^{2} w^{3}+23 x y z^{2} w^{2}+43 x^{2} z^{3} w+8 x y^{2} z^{2} w \\
& \quad+18 x^{2} y z^{3}+x y^{3} z^{2} .
\end{aligned}
$$

Denote by $G$ the context-free grammar

$$
\begin{equation*}
G: x \rightarrow x y, \quad y \rightarrow x z, \quad z \rightarrow z w, \quad w \rightarrow x z \tag{2.7}
\end{equation*}
$$

and let $D$ be the formal derivative with respect to $G$. This grammar can be viewed as a unification of grammars (2.5) and (2.6). Substituting $z, y$ by $x$ and substituting $x, w$ by $y, G$ becomes the grammar (2.5). Substituting $z, x$ by $x$, and substituting $w, y$ by $y, G$ becomes the grammar (2.6).

Theorem 2.1. For $n \geq 0$,

$$
\begin{equation*}
D^{n}(z)=P_{n}(x, y, z, w) \tag{2.8}
\end{equation*}
$$

For example, for $n=4$, we have

$$
D^{4}(z)=z w^{4}+11 x z^{2} w^{2}+6 x y z^{2} w+5 x^{2} z^{3}+x y^{2} z^{2}
$$

The coefficient of $x y z^{2} w$ in $D^{4}(z)$ is 6 , corresponding to the six permutations on $\{1,2,3,4\}$ with one exterior peak and one proper double descent: 1432, 2431, 3214, 3421, 4213 and 4312.

To prove Theorem 2.1, we need a grammatical labeling of a permutation $\pi=\pi_{1} \pi_{2} \cdots \pi_{n}$ on $[n]$ by four variables $x, y, z, w$. We first add an element 0 at the end of $\pi$ and label it by $z$. If $i$ is an exterior peak, we label $\pi_{i}$ by $x$ and label $\pi_{i+1}$ by $z$, if $i$ is a proper double descent, we label $\pi_{i}$ by $y$. The rest of the elements in $\pi$ are labeled by $w$. Define the weight of $\pi$ by

$$
w(\pi)=x^{\# \text { exterior peaks }} y^{\# \text { proper double descents }} z^{\# \text { exterior peaks }+1} w^{\# \text { otherwise }}
$$

For example, let $\pi=356412$. The labeling of $\pi$ is as follows:

$$
\begin{array}{ccccccc}
3 & 5 & 6 & 4 & 1 & 2 & 0 \\
w & w & x & z & y & w & z
\end{array},
$$

and $w(\pi)=x y z^{2} w^{3}$.
Proof of Theorem 2.1. We proceed by induction on $n$. For $n=1$, the grammatical labeling $\pi=1$ is given by

$$
\begin{array}{cc}
1 & 0 \\
w & z
\end{array} .
$$

This yields $P_{1}(x, y, z, w)=w z$. On the other hand, with respect to the grammar $G$, we have $D(z)=z w$. Hence the theorem is valid for $n=1$.

Assume that the theorem holds for $n$, that is, $D^{n}(z)=P_{n}(x, y, z, w)$. Let $\pi=\pi_{1} \pi_{2} \cdots \pi_{n}$ be a permutation on $[n]$ with $i$ exterior peaks and $j$ proper double descents. Clearly, the weight $w(\pi)$ is $x^{i} y^{j} z^{i+1} w^{n-2 i-j}$. We add a zero at the end of $\pi$. Then we insert $n+1$ into $\pi$ to generate a new permutation on $[n+1]$ with a zero at the end. According to where $n+1$ is inserted, there are four cases to label $n+1$ and relabel some elements in $\pi$.

Case 1: $n+1$ is inserted immediately before an exterior peak $k$ such that $\pi_{k}$ labeled by $x$ and $2 \leq k \leq n-1$. Then we label $n+1$ by $x$ and relabel $\pi_{k}$ by $z$ and $\pi_{k+1}$ by $y$. The change of labelings is illustrated as follows:

$$
\begin{array}{cccccccccc}
\pi_{k-1} & <\pi_{k} & >\pi_{k+1} \\
x & z
\end{array} \Longrightarrow \begin{array}{ccc}
\pi_{k-1} & < & n+1 \\
x & > & \pi_{k} \\
z & > & \pi_{k+1} \\
y
\end{array}
$$

For example, if we insert 7 before 6 in the above example, we get

$$
\begin{array}{cccccccc}
3 & 5 & 7 & 6 & 4 & 1 & 2 & 0 \\
w & w & x & z & y & y & w & z
\end{array} .
$$

Similarly, for $i=1$ and $\pi_{1}>\pi_{2}$, the relabeling is shown below:

$$
\begin{array}{ccc}
\pi_{1} \\
x
\end{array}>\begin{gathered}
\pi_{2} \\
z
\end{gathered} \quad \Longrightarrow \quad \begin{gathered}
n+1 \\
x
\end{gathered}>\begin{array}{ccc}
\pi_{1} & > & \pi_{2} \\
z
\end{array} .
$$

Therefore, the insertion in this case always corresponds to the rule $x \rightarrow x y$ and produces $i$ permutations on $[n+1]$ with $i$ exterior peaks and $j+1$ proper double descents. The sum of weight of these $i$ permutations equals $i x^{i} y^{j+1} z^{i+1} w^{n-2 i-j}$.

Case 2: $n+1$ is inserted immediately before $\pi_{k}$ with labeling $y$, where $3 \leq k \leq n$. We label $n+1$ by $x$ and relabel $\pi_{k}$ by $z$, as illustrated below:

$$
\pi_{k-2}>\pi_{k-1}>\pi_{k} \Longrightarrow \begin{gathered}
\pi_{k-2}>\pi_{k-1}<n+1 \\
y
\end{gathered}>\pi_{k}
$$

For example, if we insert 7 before 2 in the above example, we obtain

$$
\begin{array}{cccccccc}
3 & 5 & 6 & 4 & 1 & 7 & 2 & 0 \\
w & w & x & z & y & x & z & z
\end{array} .
$$

Notice that this insertion corresponds to the rule $y \rightarrow x z$, and produces $j$ permutations on [ $n+1$ ] with $i+1$ exterior peaks and $j-1$ proper double descents. The sum of weights of these permutations equals $j x^{i+1} y^{j-1} z^{i+2} w^{n-2 i-j}$.

Case 3: $n+1$ is inserted immediately before an element with label $z$. Here are two subcases. If $n+1$ is inserted before zero, we just label $n+1$ by $w$ :

$$
\begin{array}{cl}
\pi_{n} & 0 \\
& z
\end{array} \quad \Longrightarrow \quad \pi_{n}<\begin{array}{cc}
n+1 & 0 \\
w & z
\end{array} .
$$

For example, if we insert 7 before 0 , we get

$$
\begin{array}{cccccccc}
3 & 5 & 6 & 4 & 1 & 2 & 7 & 0 \\
w & w & x & z & y & w & w & z
\end{array} .
$$

If $n+1$ is inserted immediately before $\pi_{k}$, where $2 \leq k \leq n$, we label $n+1$ by $x$ and relabel $\pi_{k-1}$ by $w$ as shown below:

$$
\begin{array}{cccccc}
\pi_{k-1} & >\pi_{k} \\
x & z & \Longrightarrow & \pi_{k-1} & < & n+1 \\
w & > & \pi_{k} \\
z
\end{array} .
$$

For example, if we insert 7 before 4 in the above example, we get

$$
\begin{array}{cccccccc}
3 & 5 & 6 & 7 & 4 & 1 & 2 & 0 \\
w & w & w & x & z & y & w & z
\end{array} .
$$

In summary, the insertion in this case always corresponds to the rule $z \rightarrow z w$ and produces $i+1$ permutations on $[n+1]$ with $i$ exterior peaks and $j$ proper double descents. So we obtain a total weight $(i+1) x^{i} y^{j} z^{i+1} w^{n-2 i-j+1}$.

Case 4: $n+1$ is inserted immediately before $\pi_{k}$ with label $w$, where $1 \leq k \leq n$. Then we label $n+1$ by $x$ and relabel $\pi_{k}$ by $z$ :

$$
\begin{array}{cccc}
\pi_{k} \\
w & \Longrightarrow & n+1 & > \\
x & & \pi_{k} \\
z
\end{array} .
$$

For example, if we insert 7 before 2 , we get

$$
\begin{array}{cccccccc}
3 & 7 & 5 & 6 & 4 & 1 & 2 & 0 \\
w & x & z & x & z & y & w & z
\end{array} .
$$

This insertion corresponds to the rule $w \rightarrow x z$ and produces $n-2 i-j$ permutations on [ $n+1$ ] with $i+1$ exterior peaks and $j$ proper double descents. So we get a total weight $(n-2 i-j) x^{i+1} y^{j} z^{i+2} w^{n-2 i-j-1}$. Combining the above cases, we see that

$$
\begin{aligned}
P_{n+1}(x, y, z, w)= & \sum_{i, j=0}^{n} P_{n}(i, j)\left(i x^{i} y^{j+1} z^{i+1} w^{n-2 i-j}+j x^{i+1} y^{j-1} z^{i+2} w^{n-2 i-j}\right. \\
& \left.+(i+1) x^{i} y^{j} z^{i} w^{n-2 i-j+1}+(n-2 i-j) x^{i+1} y^{j} z^{i+2} w^{n-2 i-j-1}\right)
\end{aligned}
$$

But, by the grammar $G$ we obtain that

$$
\begin{aligned}
D\left(x^{i} y^{j} z^{i+1} w^{n-2 i-j}\right)= & i x^{i} y^{j+1} z^{i+1} w^{n-2 i-j}+j x^{i+1} y^{j-1} z^{i+2} w^{n-2 i-j} \\
& +(i+1) x^{i} y^{j} z^{i} w^{n-2 i-j+1}+(n-2 i-j) x^{i+1} y^{j} z^{i+2} w^{n-2 i-j-1} .
\end{aligned}
$$

Hence, by the induction hypothesis, we find that

$$
\begin{aligned}
D^{n+1}(z)= & D\left(P_{n}(x, y, z, w)\right) \\
= & D\left(\sum_{i, j=0}^{n} P_{n}(i, j) x^{i} y^{j} z^{i+1} w^{n-2 i-j}\right) \\
= & \sum_{i, j=0}^{n} P_{n}(i, j) D\left(x^{i} y^{j} z^{i+1} w^{n-2 i-j}\right) \\
= & \sum_{i, j=0}^{n} P_{n}(i, j)\left(i x^{i} y^{j+1} z^{i+1} w^{n-2 i-j}+j x^{i+1} y^{j-1} z^{i+2} w^{n-2 i-j}\right. \\
& \left.\quad+(i+1) x^{i} y^{j} z^{i} w^{n-2 i-j+1}+(n-2 i-j) x^{i+1} y^{j} z^{i+2} w^{n-2 i-j-1}\right) \\
= & P_{n+1}(x, y, z, w)
\end{aligned}
$$

Therefore, the theorem holds for $n+1$. This completes the proof.
We note that via a different grammatical labeling, the grammar $G$ can also be used to deal with the joint distribution of the number of peaks and the number of double descents of permutations on $[n]$. For a permutation $\pi=\pi_{1} \pi_{2} \cdots \pi_{n}$ on [ $n$ ], set $\pi_{0}=\pi_{n+1}=0$. For $1 \leq i \leq n$, an index $i$ is called an peak if $\pi_{i-1}<\pi_{i}>\pi_{i+1}$, see [18]. It is also called a maxima in [4], or a modified maximum in [14]. An index $i$ is called a double descent or a double fall if $\pi_{i-1}>\pi_{i}>\pi_{i+1}$, see $[14,18]$. For example, $\pi=4356721$ has two peaks: 1 and 5 , and two double descents: 6 and 7 .

Let $Q_{n}(i, j)$ denote the number of permutations on $[n]$ with $i$ peaks and $j$ double descents. Let

$$
Q_{n}(x, y, z, w)=\sum_{i, j} Q_{n}(i, j) x^{i} y^{j} z^{i} w^{n+1-2 i-j}
$$

where $0 \leq j \leq n$ and $2 i+j \leq n+1$. The first few values of $Q_{n}(x, y, z, w)$ are given below:

$$
\begin{aligned}
& Q_{1}(x, y, z, w)=x z \\
& Q_{2}(x, y, z, w)=x y z+x z w \\
& Q_{3}(x, y, z, w)=x y^{2} z+2 x^{2} z^{2}+2 x y z w+x z w^{2} \\
& Q_{4}(x, y, z, w)=x z w^{3}+3 x y z w^{2}+8 x^{2} z^{2} w+3 x y^{2} z w+8 x^{2} y z^{2}+x y^{3} z
\end{aligned}
$$

The following theorem shows that the polynomials $Q_{n}(x, y, z, w)$ can also be generated by the formal derivative $D$ with respect to the grammar $G$.

Theorem 2.2. For $n \geq 1$,

$$
\begin{equation*}
D^{n}(y)=Q_{n}(x, y, z, w) \tag{2.9}
\end{equation*}
$$

For example, it can be checked that $D^{4}(y)=Q_{4}(x, y, z, w)$. The coefficient of $x^{2} y z^{2}$ in $D^{4}(y)$ equals eight, and there are eight permutations on $\{1,2,3,4\}$ with two peaks and one double descent: $2143,3142,3214,3241,4132,4213,4231$ and 4312.

The proof of the above theorem will be omitted. Here we just provide a grammatical labeling. Let $\pi=\pi_{1} \pi_{2} \cdots \pi_{n}$ be a permutation on [ $n$ ]. First, set $\pi_{0}=0$ and $\pi_{n+1}=0$. For $1 \leq i \leq n+1$, if $i$ is a peak, then we label $\pi_{i}$ by $x$ and $\pi_{i+1}$ by $z$; if $i$ is a double descent, then we label $\pi_{i+1}$ by $y$. All other elements are labeled by $w$. For example, the permutation $\pi=4356721$ has the following labeling:

$$
\begin{array}{ccccccccc}
0 & 4 & 3 & 5 & 6 & 7 & 2 & 1 & 0 \\
& x & z & w & w & x & z & y & y
\end{array},
$$

and the weight of $\pi$ equals $x^{2} y^{2} z^{2} w^{2}$.
It should be noticed that Carliltz and Scoville defined the generating function on the joint distribution of the number of peaks, the number of valleys, the number of double descents and the number of double rises as:

$$
F_{n}(x, y, z, w)=\sum_{\pi} x^{\# \text { peaks }-1} y^{\# \text { double descents }} z^{\# \text { valleys }} w^{\# \text { double rises }}
$$

where $\pi$ runs over the permutations on $[n]$. It turns out that the polynomials $Q_{n}(x, y, z, w)$ are essentially the polynomials $F_{n}(x, y, z, w)$ defined by Carlitz and Scoville. More precisely, it can be easily seen that for $n \geq 1$,

$$
\begin{equation*}
Q_{n}(x, y, z, w)=x z F_{n}(x, y, z, w) \tag{2.10}
\end{equation*}
$$

The first few values of $F_{n}(x, y, z, w)$ are as follows:

$$
\begin{aligned}
& F_{1}(x, y, z, w)=1 \\
& F_{2}(x, y, z, w)=y+w \\
& F_{3}(x, y, z, w)=y^{2}+2 x z+2 y w+w^{2} \\
& F_{4}(x, y, z, w)=w^{3}+3 y w^{2}+8 x z w+3 y^{2} w+8 x y z+y^{3}
\end{aligned}
$$

Let $F(x, y, z, w ; t)$ denote the generating function of $F_{n}(x, y, z, w)$, that is,

$$
F(x, y, z, w ; t)=\sum_{n=1}^{\infty} F_{n}(x, y, z, w) \frac{t^{n}}{n!}
$$

Carliltz and Scoville showed that

$$
\begin{equation*}
F(x, y, z, w ; t)=\frac{e^{v t}-e^{u t}}{v e^{u t}-u e^{v t}} \tag{2.11}
\end{equation*}
$$

where $u v=x z$ and $u+v=y+w$, see [14, Exercise 3.3.46] and [18, Exercise 1.61].
To conclude this section, we use the grammar $G$ to produce a recurrence relation of $P_{n}(x, y, z, w)$.

Theorem 2.3. For $n \geq 0$,

$$
\begin{equation*}
P_{n+1}(x, y, z, w)=w P_{n}(x, y, z, w)+\sum_{k=0}^{n-1}\binom{n}{k} P_{k}(x, y, z, w) Q_{n-k}(x, y, z, w) \tag{2.12}
\end{equation*}
$$

Proof. Since

$$
\begin{equation*}
D^{n+1}(z)=D^{n}(z w) \tag{2.13}
\end{equation*}
$$

by the Leibniz formula, we obtain that

$$
D^{n+1}(z)=\sum_{k=0}^{n}\binom{n}{k} D^{k}(z) D^{n-k}(w) .
$$

But, for $k \geq 1$,

$$
\begin{equation*}
D^{k}(w)=D^{k}(y) \tag{2.14}
\end{equation*}
$$

hence, by (2.13) and (2.14), we deduce that

$$
\begin{equation*}
D^{n+1}(z)=w D^{n}(z)+\sum_{k=0}^{n-1}\binom{n}{k} D^{k}(z) D^{n-k}(y) . \tag{2.15}
\end{equation*}
$$

Combining (2.15) with Theorem 2.1 and Theorem 2.2, we arrive at (2.12).
Setting $z=w=1$ in (2.12) and letting $Q_{n}(x, y)=Q_{n}(x, y, 1,1)$, we get

$$
\begin{equation*}
P_{n+1}(x, y)=P_{n}(x, y)+\sum_{k=0}^{n-1}\binom{n}{k} P_{k}(x, y) Q_{n-k}(x, y) . \tag{2.16}
\end{equation*}
$$

Here we mention some special cases of (2.16).
Recall that $T_{n}(x)$ is defined by (1.1). Let $R(n, k)$ be the number of permutations on $[n]$ with $k$ peaks and let

$$
R_{n}(x)=\sum_{k=0}^{n} R(n, k) x^{k} .
$$

Taking $y=z=w=1$, (2.12) becomes

$$
\begin{equation*}
T_{n+1}(x)=T_{n}(x)+\sum_{k=0}^{n-1}\binom{n}{k} T_{k}(x) R_{n-k}(x) . \tag{2.17}
\end{equation*}
$$

Let $W(n, k)$ be the number of permutations on $[n]$ with $k$ double descents and let

$$
W_{n}(y)=\sum_{k=0}^{n} W(n, k) y^{k} .
$$

Taking $x=z=w=1$ in (2.12) yields

$$
\begin{equation*}
U_{n+1}(y)=U_{n}(y)+\sum_{k=0}^{n-1}\binom{n}{k} U_{k}(y) W_{n-k}(y), \tag{2.18}
\end{equation*}
$$

where $U_{n}(y)$ is defined by (1.3).

## 3 The Generating Functions

In this section, we use the grammar

$$
G: x \rightarrow x y, \quad y \rightarrow x z, \quad z \rightarrow z w, \quad w \rightarrow x z
$$

to derive the generating function of $P_{n}(x, y)$ without solving differential equations. In fact, we shall consider the generating function $\operatorname{Gen}(z, t)$ of the polynomials $D^{n}(z)$ in four variables $x, y, z, w$, that is,

$$
\operatorname{Gen}(z, t)=\sum_{n=0}^{\infty} D^{n}(z) \frac{t^{n}}{n!}
$$

Furthermore, we show that the generating function for the joint distribution of the number of peaks and the number of double descents can also be determined by $\operatorname{Gen}(y, t)$. This leads to a grammatical approach to the generating function of Carliltz and Scoville.

Theorem 3.1. We have

$$
\operatorname{Gen}(z, t)=\frac{2 z \sqrt{(w+y)^{2}-4 x z} e^{t / 2 \cdot\left(w-y+\sqrt{(w+y)^{2}-4 x z}\right)}}{w+y+\sqrt{(w+y)^{2}-4 x z}-\left(w+y-\sqrt{(w+y)^{2}-4 x z}\right) e^{t \sqrt{(w+y)^{2}-4 x z}}} .
$$

Combining Theorem 2.1 and Theorem 3.1, we readily deduce (1.9) by setting $w=z=1$.
To prove Theorem 3.1, we need the following relation between $\operatorname{Gen}(z, t)$ and $\operatorname{Gen}(x, t)$.
Lemma 3.2. We have

$$
\begin{equation*}
\operatorname{Gen}(z, t)=z x^{-1} \operatorname{Gen}(x, t) e^{(w-y) t} . \tag{3.1}
\end{equation*}
$$

Proof. For the grammar $G$ and its formal derivative $D$, it is evident that

$$
\begin{equation*}
D(w-y)=0 . \tag{3.2}
\end{equation*}
$$

By the Leibniz rule (2.1), we get

$$
D\left(x^{-1}\right)=-x^{-2} D(x)=-x^{-2} x y=-x^{-1} y .
$$

It follows that

$$
\begin{equation*}
D\left(z x^{-1}\right)=z x^{-1}(w-y) . \tag{3.3}
\end{equation*}
$$

Combining (3.2) and (3.3), we deduce that

$$
D^{n}\left(z x^{-1}\right)=z x^{-1}(w-y)^{n} .
$$

Thus,

$$
\begin{equation*}
\operatorname{Gen}\left(z x^{-1}, t\right)=\sum_{n=0}^{\infty} D^{n}\left(z x^{-1}\right) \frac{t^{n}}{n!}=z x^{-1} \sum_{n=0}^{\infty}(w-y)^{n} \frac{t^{n}}{n!}=z x^{-1} e^{(w-y) t} \tag{3.4}
\end{equation*}
$$

By (2.3), we see that

$$
\begin{equation*}
\operatorname{Gen}(z, t)=\operatorname{Gen}\left(z x^{-1}, t\right) \operatorname{Gen}(x, t) . \tag{3.5}
\end{equation*}
$$

Substituting (3.4) into (3.5), we find that

$$
\operatorname{Gen}(z, t)=z x^{-1} \operatorname{Gen}(x, t) e^{(w-y) t} .
$$

This completes the proof.
Proof of Theorem 3.1. By (2.3), we see that

$$
\begin{equation*}
\operatorname{Gen}(z x, t)=\frac{1}{\operatorname{Gen}\left(z^{-1} x^{-1}, t\right)} . \tag{3.6}
\end{equation*}
$$

Applying Lemma 3.2, we find that

$$
\begin{equation*}
\operatorname{Gen}(z x, t)=\operatorname{Gen}(z, t) \operatorname{Gen}(x, t)=x z^{-1} e^{(y-w) t} \operatorname{Gen}^{2}(z, t) \tag{3.7}
\end{equation*}
$$

Combining (3.6) and (3.7), we deduce that

$$
\begin{equation*}
\operatorname{Gen}(z, t)=\sqrt{\frac{z x^{-1} e^{(w-y) t}}{\operatorname{Gen}\left(z^{-1} x^{-1}, t\right)}} . \tag{3.8}
\end{equation*}
$$

We continue to compute $\operatorname{Gen}\left(x^{-1} z^{-1}, t\right)$. It is easily checked that

$$
\begin{align*}
D\left(x^{-1} z^{-1}\right) & =-x^{-1} z^{-1}(w+y) \\
D^{2}\left(x^{-1} z^{-1}\right) & =x^{-1} z^{-1}\left((w+y)^{2}-2 x z\right) \\
D^{3}\left(x^{-1} z^{-1}\right) & =-x^{-1} z^{-1}(w+y)\left((w+y)^{2}-4 x z\right)  \tag{3.9}\\
D^{4}\left(x^{-1} z^{-1}\right) & =-x^{-1} z^{-1}(w+y)\left((w+y)^{2}-2 x z\right)\left((w+y)^{2}-4 x z\right) . \tag{3.10}
\end{align*}
$$

Observe that

$$
\begin{equation*}
D\left((w+y)^{2}-4 x z\right)=0 . \tag{3.11}
\end{equation*}
$$

In light of (3.11), it follows from (3.9) and (3.10) that for $k \geq 0$,

$$
\begin{aligned}
& D^{2 k+1}\left(x^{-1} z^{-1}\right)=-x^{-1} z^{-1}(w+y)\left((w+y)^{2}-4 x z\right)^{k} \\
& D^{2 k+2}\left(x^{-1} z^{-1}\right)=x^{-1} z^{-1}\left((w+y)^{2}-2 x z\right)\left((w+y)^{2}-4 x z\right)^{k} .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\operatorname{Gen}\left(x^{-1} z^{-1}, t\right)=\sum_{k=0}^{\infty} & \frac{D^{k}\left(z^{-1} x^{-1}\right) t^{k}}{k!}=\frac{1}{x z}\left(-\frac{2 x z}{(w+y)^{2}-4 x z}\right. \\
& +\left(\frac{w+y}{2 \sqrt{(w+y)^{2}-4 x z}}+\frac{(w+y)^{2}-2 x z}{2\left((w+y)^{2}-4 x z\right)}\right) e^{-t \sqrt{(w+y)^{2}-4 x z}} \\
& \left.-\left(\frac{w+y}{2 \sqrt{(w+y)^{2}-4 x z}}-\frac{(w+y)^{2}-2 x z}{2\left((w+y)^{2}-4 x z\right)}\right) e^{t \sqrt{(w+y)^{2}-4 x z}}\right) .
\end{aligned}
$$

Putting $\operatorname{Gen}\left(x^{-1} z^{-1}, t\right)$ into (3.8) gives the required formula for $\operatorname{Gen}(z, t)$. This completes the proof.

It can be seen that Theorem 3.1 serves as a unification of Theorems 1.1, 1.2 and 1.3. Taking $y=z=w=1$, Theorem 3.1 reduces to Theorem 1.1. Setting $x=z=w=1$ in Theorem 3.1 yields Theorem 1.2. Theorem 3.1 simplifies to Theorem 1.3 when $x=z=w=1$ and $y=0$.

We note that using the grammar $G$, it is easy to establish a connection between the generating functions of $P_{n}(x, y, z, w)$ and $Q_{n}(x, y, z, w)$.

Theorem 3.3. We have

$$
\begin{equation*}
\operatorname{Gen}(y, t)=\ln ^{\prime}(\operatorname{Gen}(z, t))-w+y . \tag{3.12}
\end{equation*}
$$

Proof. From the substitution rule $z \rightarrow z w$, we get

$$
\operatorname{Gen}^{\prime}(z, t)=\operatorname{Gen}(D(z), t)=\operatorname{Gen}(z w, t)=\operatorname{Gen}(z, t) \operatorname{Gen}(w, t),
$$

and hence

$$
\operatorname{Gen}(w, t)=\frac{\operatorname{Gen}^{\prime}(z, t)}{\operatorname{Gen}(z, t)}=\ln ^{\prime}(\operatorname{Gen}(z, t)) .
$$

Since $D(w)=D(y)$, we see that

$$
\operatorname{Gen}(y, t)=\operatorname{Gen}(w, t)-w+y=\ln ^{\prime}(\operatorname{Gen}(z, t))-w+y
$$

This completes the proof.
In view of Theorem 3.3, from the generating function of $P_{n}(x, y, z, w)$ we can deduce the generating function of $Q_{n}(x, y, z, w)$.
Theorem 3.4. We have

$$
\begin{equation*}
\operatorname{Gen}(y, t)=y+\frac{-2 x z+2 x z e^{t \sqrt{(y+w)^{2}-4 x z}}}{w+y+\sqrt{(w+y)^{2}-4 x z}-\left(w+y-\sqrt{(w+y)^{2}-4 x z}\right) e^{t \sqrt{(w+y)^{2}-4 x z}}} . \tag{3.13}
\end{equation*}
$$

Proof. By Theorem 3.1, we find that

$$
\begin{gathered}
\ln (\operatorname{Gen}(z, t))=\ln (2 z)+\frac{1}{2} \ln \left((w+y)^{2}-4 x z\right)+\frac{t}{2}\left(w-y+\sqrt{(w+y)^{2}-4 x z}\right) \\
-\ln \left(w+y+\sqrt{(w+y)^{2}-4 x z}\right. \\
\left.-\left(w+y-\sqrt{(w+y)^{2}-4 x z}\right) e^{t \sqrt{(w+y)^{2}-4 x z}}\right) .
\end{gathered}
$$

Hence

$$
\begin{align*}
\ln ^{\prime}(\operatorname{Gen}(z, t))= & \frac{1}{2}\left(w-y+\sqrt{(w+y)^{2}-4 x z}\right) \\
& +\frac{\left(w+y-\sqrt{(w+y)^{2}-4 x z}\right) \sqrt{(w+y)^{2}-4 x z} e^{t \sqrt{(w+y)^{2}-4 x z}}}{w+y+\sqrt{(w+y)^{2}-4 x z}-\left(w+y-\sqrt{(w+y)^{2}-4 x z}\right) e^{t \sqrt{(w+y)^{2}-4 x z}}} \tag{3.14}
\end{align*}
$$

Plugging (3.14) into (3.12), we obtain

$$
\begin{aligned}
\operatorname{Gen}(y, t)=y & -\frac{1}{2}\left(y+w-\sqrt{(w+y)^{2}-4 x z}\right) \\
& +\frac{\left(w+y-\sqrt{(w+y)^{2}-4 x z}\right) \sqrt{(w+y)^{2}-4 x z} e^{t \sqrt{(w+y)^{2}-4 x z}}}{w+y+\sqrt{(w+y)^{2}-4 x z}-\left(w+y-\sqrt{(w+y)^{2}-4 x z}\right) e^{t \sqrt{(w+y)^{2}-4 x z}}},
\end{aligned}
$$

which yields (3.16). This completes the proof.
To conclude this paper, we remark that the above formula for the generating function of $Q_{n}(x, y, z, w)$ can be recast as the formula (2.11) of Carlitz and Scoville for the generating function of $F_{n}(x, y, z, w)$.

Using (2.10), that is

$$
Q_{n}(x, y, z, w)=x z F_{n}(x, y, z, w)
$$

we get

$$
\begin{equation*}
\operatorname{Gen}(y, t)=y+x z F(x, y, z, w ; t) . \tag{3.15}
\end{equation*}
$$

Assume that $u+v=y+w$ and $u v=x z$, as in (2.11). Then (3.13) can be rewritten as

$$
\begin{align*}
\operatorname{Gen}(y, t) & =y+\frac{u v e^{t(u-v)}-u v}{u-v e^{t(u-v)}} \\
& =y+u v \frac{e^{t u}-e^{t v}}{u e^{t u}-v e^{t u}} . \tag{3.16}
\end{align*}
$$

Thus (2.11) follows from (3.16) and (3.15).

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