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Accurate estimates of $(1+x)^{1/x}$ Involved in Carleman Inequality and Keller Limit

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Abstract. In this paper, using the Maclaurin series of the functions $(1+x)^{1/x}$, some inequalities from papers [2] and [11] are generalized. For arbitrary Maclaurin series some general limits of Keller's type are defined and applying for generalization of some well known results.

Keyword. Constant e; Carleman's inequality; Keller's limit; approximations

1. Introduction

Let us start from the function

$$e(x) = \begin{cases} (1+x)^{\frac{1}{x}} & : \quad x>-1 \,\wedge\, x\neq 0\,, \\ \mathtt{e} & : \quad x=0\,; \end{cases} \tag{1}$$

where $\mathtt{e}$ is the base of the natural logarithm. In view of its importance in study Carleman inequality and Keller's Limit, many interesting and important results have been obtained for this function and related functions.

In [3], the following result was established:

$$e(x) = \mathtt{e}\left(1 + \sum_{k=1}^{\infty} e_k x^k\right), \tag{2}$$

for $x \in (-1,1)$ and where sequence e_n $(n \in N_0 = \{0,1,2,\ldots\})$ is determined by

$$\begin{aligned} e_0 &= 1\,, \\ e_n &= (-1)^n \sum_{k=0}^{n} \frac{(-1)^{n+k} S_1(n+k,k)}{(n+k)!} \sum_{m=0}^{n} \frac{(-1)^m}{(m-k)!}\,; \end{aligned} \tag{3}$$

$S_1(p,q)$ is the Stirling's number of the first kind. For the Stirling's numbers of the first kind the following recurrence relation is true

$$S_1(p+1,q) = -\,pS_1(p,q) + S_1(p,q-1), \tag{4}$$

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for $p>q$; $S_1(p,p)=1$ and $S_1(p,q)=0$ for $p<q$ $(p,q\in N=N_0\backslash\{0\})$, see for example [1], [3], [9]. Let us notice that sign of the STIRLING's number of the first kind $S_1(p,q)$ is equal to $(-1)^{p-q}$. Initial values of the sequences of fractions

$$e_0=1,\ e_1=-\frac{1}{2},\ e_2=\frac{11}{24},\ e_3=-\frac{7}{16},\ e_4=\frac{2447}{5760},\ e_5=-\frac{959}{2304},\ e_6=\frac{238043}{580608},\ \ldots \tag{5}$$

are tabled as the sequence A055505 in [18]. Based on the representation (3) we obtained the following representation

$$e_n=(-1)^n\sum_{k=1}^{\infty}\frac{S_1(n+k,k)}{(n+k)!}, \tag{6}$$

$n\in N$, as cited in [18].

In [6], the authors give the following form of the sequence

$$e_n\ =\ (-1)^n\sum_{(j_1,j_2,\ldots,j_n)\in\mathcal{A}_n}\frac{1}{j_1!\,j_2!\cdot\ldots\cdot j_n!}\left(\frac{1}{2}\right)^{j_1}\left(\frac{1}{3}\right)^{j_2}\cdot\ldots\cdot\left(\frac{1}{j+1}\right)^{j_n}; \tag{7}$$

where

$$\mathcal{A}_n:=\left\{(j_1,j_2,\ldots,j_n)\in N_0^n\ \middle|\ j_1+2j_2+\ldots+nj_n=n\right\} \tag{8}$$

is the set of the partition of the number $n\in N$. Representations (3) and (7) are equal based on uniqueness of the coefficients of the MACLAURIN series of a real analytic function $e(x)$.

Consider the following sequence of partial sums:

$$e_n(x)=\mathsf{e}\left(1+\sum_{k=1}^{n}(-1)^k f_k x^k\right), \tag{9}$$

where $x\in(-1,1)$, $f_n=(-1)^n e_n>0$ $(n\in N_0)$.

It is the first aim of the present paper to establish the monotoneity properties for the $e_n(x)$.

In [11] and [14], the authors proved that for any $c\in R$ the following KELLER's type limit holds:

$$\lim_{n\to\infty}\left((n+1)\left(1+\frac{1}{n+c}\right)^{n+c}-n\left(1+\frac{1}{n+c-1}\right)^{n+c-1}\right)=\mathsf{e}. \tag{10}$$

It is the second aim of the present paper to define some general limits of KELLER's type.

2. Monotoneity properties of the $e_n(x)$

The next statement is true:

Lemma 2.1. *The sequence f_n is strictly monotonic decreasing.*

Proof. Based on the recurrence relation for the STIRLING's numbers of first kind the following equalities are true

$$\begin{aligned} f_{n+1}\ &=\ (-1)^{n+1}\sum_{k=1}^{\infty}\frac{S_1(n+k+1,k)}{(n+k+1)!}\\ &=\ (-1)^{n+1}\sum_{k=1}^{\infty}\frac{-(n+k)S_1(n+k,k)+S_1(n+k-1,k-1)}{(n+k+1)!}\\ &=\ (-1)^{n+1}\sum_{k=1}^{\infty}\frac{-(n+k+1)S_1(n+k,k)+S_1(n+k,k)+S_1(n+k-1,k-1)}{(n+k+1)!}\\ &=\ (-1)^{n}\sum_{k=1}^{\infty}\frac{S_1(n+k,k)}{(n+k)!}+(-1)^{n+1}\sum_{k=1}^{\infty}\frac{S_1(n+k,k)+S_1(n+k-1,k-1)}{(n+k+1)!}. \end{aligned} \tag{11}$$

Therefore, based on sign of STIRLING's numbers of the first kind, we have the conclusion

$$f_n-f_{n+1}=(-1)^n\sum_{k=1}^{\infty}\frac{S_1(n+k,k)+S_1(n+k-1,k-1)}{(n+k+1)!}>0.\qquad\Box \tag{12}$$

Corollary 2.2. *The sequence f_n is convergent.*

Next, let us consider the BREDE's series representation

$$e(x) = \mathsf{e}\left(1 + \sum_{k=1}^{\infty}(-1)^k f_k x^k\right), \tag{13}$$

for $x \in (-1,1)$. For fixed $x \in (0,1)$ the following: $f_k x^k > 0$ and $f_k x^k \searrow 0$ are true $\Big(f_{k+1}x^{k+1} < f_k x^{k+1} = x(f_k x^k) \leq f_k x^k$ and $\lim\limits_{k\to\infty}(f_k\, x^k) = \lim\limits_{k\to\infty} f_k \cdot \lim\limits_{k\to\infty} x^k = 0\Big)$. Therefore, based on well-known properties of alternating series (see [8], Exercise 5.2.2) and previous Lemma, we can conclude that the following statement is true:

Theorem 2.3. *Let us form the sequence*

$$e_n(x) = \mathsf{e}\left(1 + \sum_{k=1}^{n}(-1)^k f_k x^k\right), \tag{14}$$

for $x \in (-1,1)$ $(n \in N_0)$. *For* $x \neq 0$ *is true:*

(i) if $x \in (0,1)$ *then*

$$e_1(x) < e_3(x) < \ldots < e_{2k-1}(x) < \ldots < e(x) < \ldots < e_{2k}(x) < \ldots < e_2(x) < e_0(x); \tag{15}$$

(ii) if $x \in (-1,0)$ *then*

$$e_0(x) < e_1(x) < e_2(x) < e_3(x) < \ldots < e_{2k}(x) < e_{2k+1}(x) < \ldots < e(x). \tag{16}$$

For $x = 0$ *is true that* $e_n(0) = e(0) = \mathsf{e}$ $(n \in N)$.

Corollary 2.4. *Based on previous theorem we obtained proofs of Lemma* 2.1 *from* [2] *and Theorem* 1 *from* [11] *over* $(0,1)$.

3. General limits of Keller's type

Let us consider an arbitrary MACLAURIN series

$$g(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{17}$$

for $x \in (-\varrho, \varrho)$ and some $\varrho \in (0,\infty)$, wherein a_k $(k \in N_0)$ is some real sequence. Therefore exist a convergent asymptotic expansion

$$G(y) = g\left(\frac{1}{y}\right) = \sum_{k=0}^{\infty} \frac{a_k}{y^k}, \tag{18}$$

for $y \in R\backslash\{0\}$ and $\left|\dfrac{1}{y}\right| < \min\{1, \varrho\}$. Then, the first general limit of KELLER's type of the function g we define by

$$L_1 = \lim_{y \longrightarrow \infty}\Big((y+1)G(y) - yG(y-1)\Big). \tag{19}$$

We consider the function $(y+1)G(y) - yG(y-1)$ for values $y > 1 + \max\left\{1, \dfrac{1}{\varrho}\right\}$. Using the binomial expansion $\dfrac{1}{(y-1)^k} = \dfrac{1}{y^k}\left(1 - \dfrac{1}{y}\right)^{-k} = \displaystyle\sum_{i=0}^{\infty} \dfrac{\binom{k+i-1}{k-1}}{y^{k+i}}$, for $\dfrac{1}{y} < 1$ and $k \in N$, we obtained for $y > 1 + \max\left\{1, \dfrac{1}{\varrho}\right\}$ the following convergent asym-

ptotic expansion

$$\begin{aligned}(y+1)G(y) - yG(y-1) &= (y+1)\left(a_0 + \sum_{k=1}^{\infty}\frac{a_k}{y^k}\right) - y\left(a_0 + \sum_{k=1}^{\infty}\frac{a_k}{(y-1)^k}\right)\\ &= a_0 + \sum_{k=1}^{\infty}\frac{(y+1)a_k}{y^k} - \sum_{k=1}^{\infty}\sum_{i=0}^{\infty}\frac{y\binom{k+i-1}{k-1}a_k}{y^{k+i}}.\end{aligned} \tag{20}$$

The above expansion is sufficient to conclude

$$L_1 = \lim_{y\longrightarrow\infty} \Big((y{+}1)G(y) - yG(y{-}1)\Big) = a_0\,. \tag{21}$$

Specially, if $g(x){=}e(x)$, i.e. $G(y){=}g\Big(\dfrac{1}{y}\Big){=}\Big(1{+}\dfrac{1}{y}\Big)^{y}$, then

$$L_1 = \lim_{y\longrightarrow\infty} \left(\frac{(y{+}1)^{y+1}}{y^y} - \frac{y^y}{(y{-}1)^{y-1}}\right) = \mathtt{e}\,. \tag{22}$$

For $y > 1{+}\max\left\{1, \dfrac{1}{\varrho}\right\}$ let us determine the representation

$$\begin{array}{rcl}(y{+}1)G(y){-}yG(y{-}1) &=& a_0{+}a_1{+}\dfrac{a_1{+}a_2}{y}{+}\dfrac{a_2{+}a_3}{y^2}{+}\dfrac{a_3{+}a_4}{y^3}{+}\dfrac{a_4{+}a_5}{y^4}{+}\ldots \\ && -\left(\dfrac{\binom{0}{0}a_1}{1} + \dfrac{\binom{1}{0}a_1}{y} + \dfrac{\binom{2}{0}a_1}{y^2} + \dfrac{\binom{3}{0}a_1}{y^3} + \ldots\right. \\ && + \dfrac{\binom{1}{1}a_2}{y} + \dfrac{\binom{2}{1}a_2}{y^2} + \dfrac{\binom{3}{1}a_2}{y^3} + \dfrac{\binom{4}{1}a_2}{y^4} + \ldots \\ && + \dfrac{\binom{2}{2}a_3}{y^2} + \dfrac{\binom{3}{2}a_3}{y^3} + \dfrac{\binom{4}{2}a_3}{y^4} + \dfrac{\binom{5}{2}a_3}{y^5} + \ldots \\ && \left. + \dfrac{\binom{3}{3}a_4}{y^3} + \dfrac{\binom{4}{3}a_4}{y^4} + \dfrac{\binom{5}{3}a_4}{y^5} + \dfrac{\binom{6}{3}a_4}{y^6} + \ldots\ldots\right),\end{array} \tag{23}$$

i.e.

$$\begin{array}{rcl}(y{+}1)G(y) - yG(y{-}1) &=& a_0 - \dfrac{a_1 + a_2}{y^2} \\ && - \dfrac{a_1 + 3a_2 + 2a_3}{y^3} \\ && - \dfrac{a_1 + 4a_2 + 6a_3 + 3a_4}{y^4} \\ && - \dfrac{a_1 + 5a_2 + 10a_3 + 10a_4 + 4a_5}{y^5} \\ && - \dfrac{a_1 + 6a_2 + 15a_3 + 20a_4 + 15a_5 + 5a_6}{y^6} \\ && \vdots \\ && - \dfrac{\sum\limits_{i=1}^{k-1}\binom{k}{i-1}a_i + \left(\binom{k}{k-1} - 1\right)a_k}{y^k} \\ && \vdots\end{array} \tag{24}$$

Finally for $y > 1 + \max\left\{1, \dfrac{1}{\varrho}\right\}$ we obtained the following convergent asymptotic expansion

$$(y{+}1)G(y) - yG(y{-}1) = a_0 - \sum_{k=2}^{\infty} \frac{\sum\limits_{i=1}^{k-1}\binom{k}{i-1}a_i + \left(\binom{k}{k-1} - 1\right)a_k}{y^k}\,. \tag{25}$$

Let us remark that coefficients $\binom{k}{0}, \binom{k}{1}, \binom{k}{2}, \ldots, \binom{k}{k-1}, \left(\binom{k}{k-1} - 1\right)$ respectively to $a_1, a_2, a_3, \ldots, a_{k-1}, a_k$ are tabled as the sequence A193815 in [18].

Next we use some expansion of type (25) for generalization of some results for limits of KELLER's type from paper [11]. The second general limit of KELLER's type of the function g we define by

$$L_2 = \lim_{y\longrightarrow\infty} \Big((y{+}1)G(y{+}c) - yG(y{+}c{-}1)\Big)\,, \tag{26}$$

for arbitrary $c\in R$. Similar to the above discussion, we can get

$$(y+1)G(y+c)-yG(y+c-1)=a_0+\sum_{k=1}^{\infty}\left(\frac{a_k}{(y+c)^k}-\sum_{i=1}^{k}\frac{\binom{k}{i-1}ya_i}{(y+c)^{k+1}}\right). \tag{27}$$

From (27), we can get immediately the limit (10).

4. Conclusions

In this paper, the monotoneity properties of the functions $e_n(x)$ were established and some general limits of KELLER's type are defined. We believe that the results will lead to a significant contribution toward the study of CARLEMAN inequality and KELLER's limit.

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