

Graph-indexed random walks on special classes of graphs

Jan Bok

Computer Science Institute, Charles University in Prague, Malostranské náměstí 25,
11800, Prague, Czech Republic bok@iuuk.mff.cuni.cz

Abstract. We investigate the parameter of the average range of M -Lipschitz mapping of a given graph. We focus on well-known classes such as paths, complete graphs, complete bipartite graphs and cycles and show closed formulas for computing this parameter and also we conclude asymptotics of this parameter on these aforementioned classes.

1 Introduction

Graph-indexed random walks (or equivalently also M -Lipschitz mapping of graphs) are a generalization of standard random walk on \mathbb{Z} . Also, this concept has an important connections to statistical physics, namely to gas models (as is described by Zhao [22] and Cohen et al. [4]). Understanding the structure of all M -Lipschitz mappings of a given graph and corresponding parameters is also a point of interest because it can describe the expected behavior of a random homomorphism to a suitable graph.

This paper aims on examining specific classes graphs and its average range. This parameter can be described, without going into technical details from the very beginning, as the expected size of the homomorphic image of an uniformly picked random M -Lipschitz mapping of G .

Graph-indexed random walks and average range were studied for example in [1,9,2,21,14].

In the following text we will not mix the terms M -Lipschitz mapping and graph-indexed random walk and we will use only the first term.

1.1 Preliminaries

In this text we use the standard notation as for example in Diestel's monograph [7]. To avoid cumbersome notation, we will often write uv for undirected edge.

A *graph homomorphism* between digraphs G and H is a mapping $f : V(G) \rightarrow V(H)$ such that for every edge $uv \in E(G)$, $f(u)f(v) \in E(H)$. That means that graph homomorphism is an adjacency-preserving mapping between the vertex sets of two digraphs. The set $I := \{w \in V(H) \mid \exists v \in V(G) : f(v) = w\}$ for a graph homomorphism f is called the *homomorphic image* of f .

For a comprehensive and more complete source on graph homomorphisms, the reader is invited to see [12]. A quick introduction is given in [11] as well.

Definition 1. For $M \in \mathbb{N}$, an M -Lipschitz mapping of a connected graph $G = (V, E)$ with root $v_0 \in V$ is a mapping $f : V \rightarrow \mathbb{Z}$ such that $f(v_0) = 0$ and for every edge $(u, v) \in E$ it holds that $|f(u) - f(v)| \leq M$. The set of all M -Lipschitz mappings of a graph G is denoted by $\mathcal{L}_M(G)$.

By the term *Lipschitz mappings of graph* we mean the union of sets of M -Lipschitz mappings for every $M \in \mathbb{N}$.

The importance of having rooted graphs is the following. We want to have finitely many Lipschitz mappings for a fixed graph G . Mappings with $f(v_0) \neq 0$ are just linear shifts of some mapping with $f(v_0) = 0$. Formally, consider a mapping f' with $f'(v_0) = a$. Then we can define a linear transformation T_a as $T_a(f) \mapsto f - a$. Applying T_a to f' yields a Lipschitz mapping of G with v_0 as its root.

We note that we are interested in connected graphs only. Components without the root would also allow infinitely many new M -Lipschitz mappings.

In literature, we will often meet a slightly different definition of 1-Lipschitz mappings. In it the restriction $|f(u) - f(v)| \leq 1$, for all $uv \in E$, is removed and instead, the restriction $|f(u) - f(v)| = 1$, for all $uv \in E$, is added. In [14] authors call these mappings *strong Lipschitz mappings*. We generalize this in the following definition.

Definition 2. For $M \in \mathbb{N}$, a strong M -Lipschitz mapping of a connected graph $G = (V, E)$ with root $v_0 \in V$ is a mapping $f : V \rightarrow \mathbb{Z}$ such that $f(v_0) = 0$ and for every edge $(u, v) \in E$ it holds that $|f(u) - f(v)| = M$. The set of all M -Lipschitz mappings of a graph G is denoted by $\mathcal{L}_{\pm M}(G)$.

Note that strong M -Lipschitz mappings are a special case of M -Lipschitz mappings of graph. Also, B -Lipschitz mappings are a superset of A -Lipschitz mapping whenever $B \geq A$. See Figure 1 for the Hasse diagram of various types of Lipschitz mappings.

Analogously, by the term *strong Lipschitz mappings of graph* we mean the union of sets of strong M -Lipschitz mappings for every $M \in \mathbb{N}$.

First of all, let us define the *range* of a mapping.

Definition 3. The range of a Lipschitz mapping f of G is the size of the homomorphic image of f . Formally:

$$r_G(f) := |\{z \in \mathbb{Z} \mid z = f(v) \text{ for some } v \in V(G)\}|.$$

We define the *average range* of graph G as follows.

Definition 4. (Average range) The average range of graph G over all M -Lipschitz mappings is defined as

$$\bar{r}_M(G) := \frac{\sum_{f \in \mathcal{L}_M(G)} r(f)}{|\mathcal{L}_M(G)|}.$$

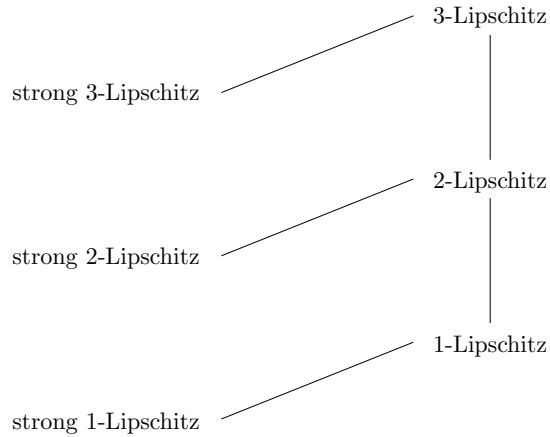


Fig. 1: The Hasse diagram of different types of Lipschitz mappings of graphs.

We can view this quantity as the expected size of the homomorphic image of an uniformly picked random M -Lipschitz mapping of G .

Whenever we want to talk about the counterparts of these definitions for strong Lipschitz mappings, we denote it with \pm in subscript. For example, $\bar{r}_{\pm M}$ is the average range of strong M -Lipschitz mapping of graph.

Whenever we write average range without saying which M -Lipschitz mappings we use, it should be clear from the context what M do we mean.

It is worth noting that for computing the average range, the choice of root does not matter. That is why we often omit the details of picking the root. For better analysis in proofs we occasionally pick the root in some convenient way.

1.2 Conjectures on the average range

Fundamental conjectures on the average range of (strong) 1-Lipschitz mappings say that paths P_n are extremal with regard to this parameter on the n -vertex graphs.

The first one is from Benjamini, Häggström and Mossel.

Conjecture 1. [1] (Benjamini-Häggström-Mossel) *For any connected bipartite graph $G \in \mathcal{G}_n$, $\bar{r}_{\pm 1}(G) \leq \bar{r}_{\pm 1}(P_n)$ holds.*

Newer version which generalizes the previous one is the following conjecture by Loeb, Nešetřil and Reed.

Conjecture 2. [14] (Loeb-Nešetřil-Reed) *For any connected graph $G \in \mathcal{G}_n$, $\bar{r}_1(G) \leq \bar{r}_1(P_n)$ holds.*

We will occasionally abbreviate Conjecture 1 to BHM Conjecture and Conjecture 2 to LNR Conjecture.

1.3 Structure of this paper

Each of the following sections deals with a different graph class.

2 Complete graphs

For completeness of the picture we will show the formula for $\bar{\tau}_1(K_n)$.

Theorem 1. *For a complete graph K_n we have $\bar{\tau}_1(K_n) = 2 - (2^n - 1)^{-1}$.*

Proof. Let us count the number of 1-Lipschitz mappings of K_n . We cannot choose the image of the root r but we can do it for other vertices. Namely, we must choose integers from interval $[-1, 1]$ due to the fact that every vertex $v \neq r$ is a neighbor of the root. Furthermore, 1-Lipschitz mapping f with vertices $u, v \in V(K_n)$ such that $f(u) = -1$ and $f(v) = 1$ cannot exist since $uv \in E(K_n)$. Thus, apart from the trivial case of setting image of all vertices to 0, we can choose to map vertices other than r either to -1 and 0, or to 1 and 0 – exclusively. For each of this choice we have $2^{n-1} - 1$ of such 1-Lipschitz mappings and each of them has the range equal to 2. We conclude:

$$\bar{\tau}_1(K_n) = \frac{2 \cdot 2 \cdot (2^{n-1} - 1) + 1}{2 \cdot (2^{n-1} - 1) + 1} = 2 - \frac{1}{2^n - 1}.$$

□

Theorem 1 implies the limiting behavior of $\bar{\tau}_1(K_n)$.

Corollary 1. *It holds that $\lim_{n \rightarrow \infty} \bar{\tau}_1(K_n) = 2$.*

3 Complete bipartite graphs

We prove an exact formula for another well-known class of graphs, *complete bipartite graphs*.

Theorem 2. *For every $p, q \in \mathbb{N}$, a complete bipartite graph $K_{p,q}$ satisfies*

$$|\mathcal{L}_1(K_{p,q})| = 3^p + 3^q + 2^{p+q} - 2^{p+1} - 2^{q+1} + 1,$$

and

$$\bar{\tau}_1(K_{p,q}) = 3 - 2^{p+q} \cdot (3^p + 3^q + 2^{p+q} - 2^{p+1} - 2^{q+1} + 1)^{-1}.$$

Proof. We use Theorem ?? that implies that the possible ranges of $K_{p,q}$ form a subset of $\{1, 2, 3\}$. We analyze separate cases of possible ranges and count how many such mappings exist. Let us denote the part of size p by P and the other one, with the size q , as Q . Without loss of generality, assume that all 1-Lipschitz mappings are rooted in some fixed vertex of P .

- *Range equal to 1:* Clearly, there is exactly one such mapping, sending everything to zero.

- *Range equal to 2:* A homomorphic image of a 1-Lipschitz mapping is some closed interval, as we observed earlier in preliminary chapter. Thus the possibilities for the homomorphic image of the range 2 are $\{0, 1\}$ and $\{-1, 0\}$. These cases are symmetric to each other, so let us analyze, without loss of generality, the case $\{0, 1\}$.

There are 2^{p+q-1} possibilities how to assign 0 and 1 to the vertices excluding the root. However, one of these possibilities is the trivial mapping of the range 1 (everything mapped to zero). The result is that there are $2^{p+q-1} - 1$ mappings with the homomorphic image $\{0, 1\}$.

- *Range equal to 3:* Again we have multiple cases. The cases $\{0, 1, 2\}$ and $\{-2, -1, 0\}$ are symmetric, the third is $\{-1, 0, 1\}$.

Let us solve the case $\{0, 1, 2\}$ first. Clearly, 0 and 2 cannot be in different parts, otherwise there would exist an edge with endpoints mapped to 0 and to 2, violating the definition of 1-Lipschitz mapping. That further implies the impossibility of $v_q \in Q$ mapped to 2. By a similar argument we get that only in the case that all vertices of Q are mapped to one we can get the homomorphic image $\{0, 1, 2\}$. We can then place any of the numbers from $\{0, 1, 2\}$ on the part P . However, we must exclude assignments with no 2 on the part P . That yields $3^{p-1} - 2^{p-1}$ possibilities.

The remaining case is $\{-1, 0, 1\}$. Again, we see that 1 and -1 cannot be in different parts. Thus, either the part P has all vertices mapped to zero and on Q we can choose for every vertex an image from the set $\{-1, 0, 1\}$, or vice versa. That gives us $3^{p-1} + 3^q$ choices from which we must exclude those that use only some proper subset of $\{-1, 0, 1\}$. Finally, we get the formula

$$3^{p-1} + 3^q - 2^p - 2^{q+1} + 2$$

for this case.

Table 1 summarize all the cases. The number of 1-Lipschitz mappings of $K_{p,q}$ is equal to

$$3^p + 3^q + 2^{p+q} - 2^{p+1} - 2^{q+1} + 1,$$

i.e. the sum of the third column of Table 1. By straightforward calculations we get

$$\bar{r}_1(K_{p,q}) = 3 - 2^{p+q} \cdot (3^p + 3^q + 2^{p+q} - 2^{p+1} - 2^{q+1} + 1)^{-1}.$$

□

We conclude this section with the observation on the limiting behavior of $\bar{r}_1(K_{p,q})$ as $(p+q) \rightarrow \infty$. Clearly, the average range is 3 in limit.

4 Stars

Definition 5. A star graph S_n is a tree with n vertices; one vertex of degree $n-1$ and $n-1$ leaves (vertices of degree one). Or, alternatively, it is a complete bipartite graph $K_{1,n-1}$.

range	homomorphic image	number of such mappings
1	{0}	1
2	{0, 1}	$2^{p+q-1} - 1$
	{0, -1}	dtto
3	{0, 1, 2}	$3^{p-1} - 2^{p-1}$
	{-2, -1, 0}	dtto
	{-1, 0, 1}	$3^{p-1} + 3^q - 2^p - 2^{q+1} + 2$

Table 1: Table for Theorem 2.

Theorem 3. *A star S_n satisfies*

$$\bar{r}_1(S_n) = 3 - \frac{2^n}{3^{n-1}},$$

and

$$\bar{r}_{\pm 1}(S_n) = 3 - 2^{2-n}.$$

Proof. We will use the definition of stars as a special case of complete bipartite graphs. We can then use Theorem 2 for the case of 1-Lipschitz mappings with $p := 1$ and $q := n - 1$. The desired claim follows.

We will now prove the second formula. Without loss of generality, we will root our graph in the central vertex. Observe that all leaves will get either +1 or -1 and only cases in which range is equal to 2 are the cases of either all leaves mapped to 1 or to -1. The rest of the cases have the range equal to 3. Totally, there are 2^{n-1} of strong 1-Lipschitz mappings. That concludes our claim. \square

5 Paths

In [21], authors compute several values of $\bar{r}_1(P_n)$ (see Table 2) and claim that no explicit formula for an average range of a path is known. We fill this gap and present such formula, exploiting the tool used in the random walk analysis called *reflection principle*.

n	2	3	4	5	6	7	8	9	10	11	12
$\bar{r}_1(P_n)$	$\frac{5}{3}$	$\frac{19}{9}$	$\frac{67}{27}$	$\frac{227}{81}$	$\frac{751}{243}$	$\frac{2445}{729}$	$\frac{7869}{2187}$	$\frac{25107}{6561}$	$\frac{78767}{19683}$	$\frac{250793}{59049}$	$\frac{786985}{177147}$

Table 2: Table of values of $\bar{r}_1(P_n)$ for $2 \leq n \leq 12$.

We will define auxiliary random variables and we will speak for a while also in the language of standard random walks which are naturally encoded in 1-Lipschitz mapping of P_n . We refer reader to [15] for a general treatment of random walks.

Definition 6. For a given 1-Lipschitz mapping f :

- $M_n^+(f)$ is a random variable corresponding to the maximum non-negative number in the image of a 1-Lipschitz mapping f .
- $M_n^-(f)$ is a random variable corresponding to the minimum non-positive number in the image of a 1-Lipschitz mapping f .
- $X_n(f)$ denotes the number $f(v_n)$, i.e. image of the second endpoint of P_n .

We omit (f) if f is clear from the context.

Theorem 4. For a path P_n we have

$$\bar{r}_1(P_n) = 1 + 3^{-n+1} \cdot 2 \cdot \sum_{k=0}^{n-1} k \cdot \sum_{i=0}^{\lfloor \frac{n-1-k}{2} \rfloor} \left(\binom{n-1}{k+i} \binom{n-k-i-1}{i} + \binom{n-1}{k+1+i} \binom{n-k-i-2}{i} \right).$$

Proof. The average range of path P_n can be formulated as:

$$\bar{r}_1(P_n) = E[M_n^+ - M_n^- + 1].$$

From the symmetry of M_n^+ and M_n^- and from the linearity of expectation, one gets:

$$\bar{r}_1(P_n) = E[M_n^+ + M_n^+ + 1] = E[M_n^+] + E[M_n^+] + 1 = 2E[M_n^+] + 1.$$

Set $M_n := M_n^+$. Now let us prove that $P(M_n \geq r) = P(X_n \geq r) + P(X_n \geq r + 1)$.

The walks with $M_n \geq r$ fit into two groups. Either such walks end in $s \geq r$ or in $s < r$. In the second case, we can reflect the section of the path after the first time we get to r and we get a new walk which now ends in $s' > r$. See Figure 2 for an illustration. Since this process is invertible and every path that reaches $s \geq r$ must have $M_n \geq r$, we get:

$$P(M_n \geq r) = P(X_n \geq r) + P(X_n \geq r + 1).$$

Next we will prove that: $P(M_n = r) = P(X_n = r) + P(X_n = r + 1)$.

$$\begin{aligned} P(M_n = r) &= P(M_n \geq r) - P(M_n \geq r + 1) \\ &= P(X_n \geq r) + P(X_n \geq r + 1) - P(X_n \geq r + 1) - P(X_n \geq r + 2) \\ &= P(X_n = r) + P(X_n = r + 1). \end{aligned}$$

Now we need to determine $P(X_n = r)$. Recall the aforementioned bijection between $\{1, -1, 0\}$ -sequences and walks from Section ??.

We have $n - 1$ edges so if we want to attain some fixed k , we need to sum up our sequence to k . Thus we need to pick k additional 1's over -1 's. Summing up through the all possible values of the number of -1 's we get:

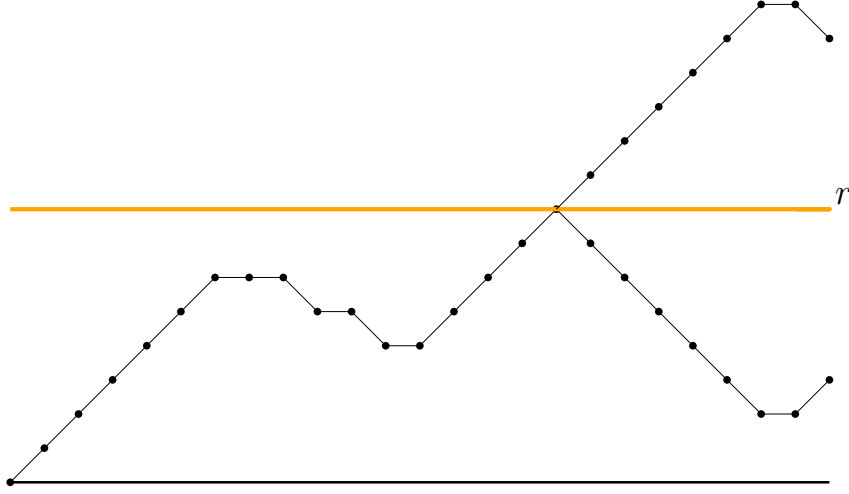


Fig. 2: An illustration of the reflection principle.

$$P(X_n = k) = 3^{-n+1} \cdot \sum_{i=0}^{\lfloor \frac{n-1-k}{2} \rfloor} \binom{n-1}{k+i} \binom{n-k-i-1}{i}. \quad (1)$$

And for $P(X_n = k + 1)$ analogously:

$$P(X_n = k + 1) = 3^{-n+1} \cdot \sum_{i=0}^{\lfloor \frac{n-1-k}{2} \rfloor} \binom{n-1}{k+1+i} \binom{n-k-i-2}{i}. \quad (2)$$

We are now ready to combine all of this together and we get:

$$\bar{r}_1(P_n) = 1 + 2 \cdot \sum_{k=0}^{n-1} k \cdot (P(X_n = k) + P(X_n = k + 1)).$$

We note that $\binom{a}{b}$ is defined as zero if $b > a$. Substituting $P(X_n = k)$ by (1) and $P(X_n = k + 1)$ by (2) we get the desired claim. \square

Besides the exact formula for the average range of a path, we prove the following relation between the \bar{r}_1 of paths P_n and P_{n+1} .

Lemma 1. For every $n \in \mathbb{N}$, $\bar{r}_1(P_{n+1}) - \bar{r}_1(P_n) \leq 2/3$.

Proof. Let us write v_1, v_2, \dots, v_n for vertices of P_n consecutively and let

$$E(P_n) := \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}.$$

For P_{n+1} , set $V(P_{n+1}) := V(P_n) \cup \{v_{n+1}\}$ and $E(P_{n+1}) = E(P_n) \cup \{v_nv_{n+1}\}$.

Pick v_1 as the root of P_n and P_{n+1} as well and consider all 1-Lipschitz mappings $\mathcal{L}(P_n)$ and $\mathcal{L}(P_{n+1})$. Choose an arbitrary f from $\mathcal{L}(P_n)$. Now $f(v_n) = r$ for some $r \in \mathbb{Z}$. If we want to extend this f to a 1-Lipschitz mapping f' of P_{n+1} , we see that we can set $f'(v_{n+1})$ to either r , $r + 1$ or $r - 1$. Choosing r does not increase the range. Since we want to do an upper estimate, let us presume that choosing $r + 1$ or $r - 1$ always increases the range. Thus we get:

$$\bar{r}(P_{n+1}) \leq \bar{r}(P_n) + 2/3.$$

Which is only a different form of the desired claim. □

This simple upper bound has two corollaries.

Corollary 2. *For every $r, q \in \mathbb{N}$, $r > q$, $\bar{r}_1(P_r) \leq \bar{r}_1(P_q) + (r - q) \cdot \frac{2}{3}$ holds.*

Proof. Use Lemma 1 ($r - q$) times. □

Corollary 3. *For every P_n , $\bar{r}_1(P_n) \leq \frac{2n+1}{3}$ holds.*

Proof. Choose $r := n$ and $q := 1$. Then use the previous lemma and observe that $\bar{r}_1(P_1)$ is equal to one. □

We remark that for all paths in general we cannot get a better upper bound by a constant than in Lemma 1 since $\bar{r}_1(P_2) - \bar{r}_1(P_1) = \frac{2}{3}$.

6 Trees

The most recent result in the area of graph-indexed random walks is the result of Wu, Xhu and Zhu from 2016 [21]. The authors tried to attack the LNR and BHM conjecture and got the following partial result.

Theorem 5. [21] *For any tree T_n on n vertices holds the following,*

1. $\bar{r}_1(T_n) \leq \bar{r}_1(P_n)$,
2. $\bar{r}_{\pm 1}(T_n) \leq \bar{r}_{\pm 1}(P_n)$.

Their approach is to use a special transformation called KC-transformation, named by Kelmans [13], which we already mentioned in Section ???. Csikvári [5,6] proved that this transformation induces a partially ordered set on the class of all n -vertex trees with the path P_n as the maximum element and the star S_n as the minimum element. By carefully choosing a right chain in this poset they prove Theorem 5.

We note that by proving Theorem 3 and Theorem 4 we showed the precise formulas for the minimum and the maximum possible average range of trees on n vertices for the case of 1-Lipschitz mappings.

n	3	4	5	6	7	8	9	10	11	12
$\bar{r}_1(C_n)$	$\frac{13}{7}$	$\frac{41}{19}$	$\frac{121}{51}$	$\frac{365}{141}$	$\frac{1093}{393}$	$\frac{3281}{1107}$	$\frac{9841}{3139}$	$\frac{29525}{8953}$	$\frac{88573}{25653}$	$\frac{265721}{73789}$

Table 3: Table of values of $\bar{r}_1(C_n)$ for $3 \leq n \leq 12$.

7 Cycles

In this section, more specifically in Theorem 7, we will show a formula for the average range of cycle graphs C_n .

See Table 3 for values of $r_1(C_n)$ of smaller cycles computed with a help of our computer program.

First, let us introduce what the *trinomial triangle* is.

7.1 Trinomial triangle

The trinomial triangle is similar to the Pascal (binomial) triangle of binomial coefficients. One can similarly define trinomial coefficients in a recursive way.

Definition 7. (Trinomial triangle and central trinomial coefficient) *Trinomial numbers (coefficients) $\binom{n}{k}_2$ are defined as:*

$$\binom{0}{0}_2 = 1$$

$$\binom{n+1}{k}_2 = \binom{n}{k-1}_2 + \binom{n}{k}_2 + \binom{n}{k+1}_2 \text{ for } n \geq 0,$$

where $\binom{n}{k}_2 = 0$ for $k < -n$ and $k > n$.

Central trinomial coefficients are the numbers $\binom{n}{0}_2$, where $n \in \mathbb{N}_0$.

1
1 1 1
1 2 3 2 1
1 3 6 7 6 3 1
1 4 10 16 19 16 10 4 1

Fig. 3: The trinomial triangle with central trinomial coefficients in blue color.

The sequence for central trinomial coefficients in OEIS is A123456 [18]. See Figure 3 for a visualization of the trinomial triangle with highlighted central trinomial coefficients. Trinomial coefficients appear quite often in enumerative combinatorics. Let us show one particular example.

Example 1. Suppose you have a king on a chessboard (it does not have to be the usual 8×8 one). Each entry of the triangle corresponds to the number of paths using the minimum number of steps between some cells of the chessboard. See Figure 4.

Useful fact is that central trinomial coefficients satisfy the following identity (for its derivation, see for example [3]):

$$\binom{n}{0}_2 = \sum_{k=0}^n \frac{n(n-1)\cdots(n-2k+1)}{(k!)^2} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k}. \quad (3)$$


1	3	6	7	6	3	1
3	1	2	3	2	1	3
6	2	1	1	1	2	6
7	3	1		1	3	7
6	2	1	1	1	2	6
3	1	2	3	2	1	3
1	3	6	7	6	3	1

Fig. 4: Each number represents the number of ways how to get to that cell with the minimum number of step with the figure of king. [20]

7.2 Motzkin numbers

For the proof of the formula for $\bar{r}_1(C_n)$ we need to define *generalized Motzkin number and paths*. We will further write only Motzkin numbers and Motzkin paths.

Definition 8. Consider a lattice path, beginning at $(0, 0)$, ending at (n, k) and satisfying that *y*-coordinate of every point is non-negative. Furthermore, every two consecutive steps (i, a) and $(i + 1, b)$ must satisfy $|a - b| \leq 1$. Such lattice path are called *Motzkin paths*.

The set of all the possible paths ending in (n, k) is denoted by $m(n, k)$ and the cardinality of this set is denoted by $M(n, k)$. We call $M(n, k)$ the *Motzkin number*.

For more details we refer to the seminal paper [8], Motzkin numbers $M(n, 0)$ form the sequence A001006 in OEIS [16]. See Figure 5 for an example of a Motzkin path.

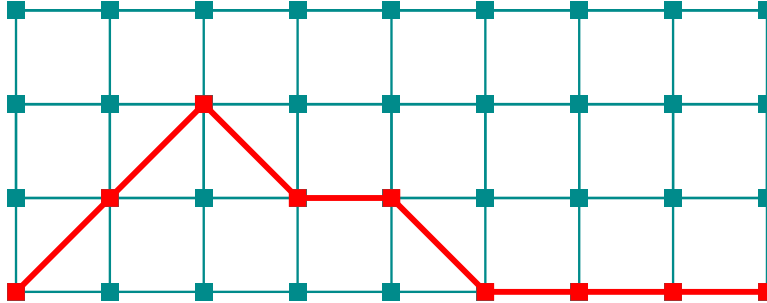


Fig. 5: A Motzkin path from $(0, 0)$ to $(8, 0)$.

7.3 Main theorem

Theorem 6. *For any cycle graph C_n , $n \geq 3$, we have*

$$\bar{\tau}_1(C_n) = \frac{3^n + (-1)^n}{2 \cdot \binom{n}{0}_2}.$$

We will prove Theorem 7 in series of lemmata that will be put together later.

Lemma 2. *There is a bijection between 1-Lipschitz mappings of C_n and the set of lattice paths starting at $(0, 0)$, ending at $(n, 0)$, and satisfying that for every two consecutive steps (i, a) and $(i + 1, b)$, $|a - b| \leq 1$.*

Proof. The proof is analogous to other bijections we made between 1-Lipschitz mappings of some type and some class of lattice walks. Take the sequence

$$v_1, v_2, \dots, v_n, v_1$$

of the vertices of C_n such that v_1 is the root and the vertices appear consecutively on the cycle precisely as in this sequence. For every 1-Lipschitz mapping f of C_n we can define another sequence

$$(v_1, f(v_1)), (v_2, f(v_2)), \dots, (v_n, f(v_n)), (v_1, f(v_1)).$$

The lemma follows easily. □

Let us prove the formula for $|\mathcal{L}(C_n)|$.

Theorem 7. *For any C_n , $n \geq 3$, $|\mathcal{L}(C_n)| = \binom{n}{0}_2$.*

Proof. We will encode all 1-Lipschitz mappings of the cycle into the sequences $\{-1, 0, 1\}^n$. Consider the lattice walks constructed in Lemma 2. For each sequence

$$(v_1, f(v_1)), (v_2, f(v_2)), \dots, (v_n, f(v_n)), (v_1, f(v_1)),$$

one can define the new sequence

$$f(v_2) - f(v_1), f(v_3) - f(v_2), \dots, f(v_1) - f(v_n).$$

We know that these sequences must add up to 0. Thus for any total number k of ones in this sequence we must have k times -1 in this sequence as well. Furthermore, we have $k \leq \lfloor n/2 \rfloor$.

Summing over all possible k 's we first pick $2k$ edges which have either $+1$ or -1 . Then from these $2k$ edges, we choose k edges for placing 1. The rest of $n - 2k$ edges gets 0's and the rest of $2k - k = k$ edges gets (-1) 's. Formally:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k}.$$

This coincides with identity (3) if we take into account that $\binom{a}{b}$ is defined to be equal to zero if $b > a$. \square

Definition 9. We denote by $\mathcal{L}(C_n, -d)$ the set of 1-Lipschitz mappings f of C_n satisfying

$$\min_{v \in V(C_n)} f(v) = -d.$$

In other words, $\mathcal{L}(C_n, -d)$ denotes the set of all 1-Lipschitz mappings of C_n with $-d$ as the minimum value in their homomorphic images.

Another ingredient we need is the following theorem of Van Leeuwen.

Theorem 8. [19] *Within the class of walks on Z starting at 0 and with steps advancing by $+1$, 0 or -1 , there is a bijection, conserving both the length of the walk and the number of steps 0, between on one hand the walks that end in 0, and on the other hand the walks that do not visit negative numbers. The bijection maps walks ending at 0 and whose minimal number visited is $-d$, to walks ending at $2d$, and is realized by reversing the direction of the d down-steps that first reach respectively the numbers $-1, -2, \dots, -d$.*

Now we need to show a bijection between $\mathcal{L}(C_n, -d)$ and the set $m(n, 2d)$.

Lemma 3. *There exist a bijection from the set of Motzkin paths $m(n, 2d)$ to the set $\mathcal{L}(C_n, -d)$.*

Proof. The existence of such sequence follows straightforwardly from combining Theorem 8 and Lemma 2. \square

For technical convenience, we will define the *irregular trinomial triangle* and *irregular trinomial coefficients*; see the sequence A027907 in OEIS [17]. See Figure 4, depicting a part of the irregular trinomial triangle.

Definition 10. The irregular trinomial coefficients are defined as

$$T^*(n, k) = \binom{n}{k+n}_2.$$

n/k	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	1	1	1										
2	1	2	3	2	1								
3	1	3	6	7	6	3	1						
4	1	4	10	16	19	16	10	4	1				
5	1	5	15	30	45	51	45	30	15	5	1		
6	1	6	21	50	90	126	141	126	90	50	21	6	1

Table 4: A part of the irregular trinomial triangle. The entries of the table are numbers $T^*(n, k)$.

The following lemmata, showing the relation of Motzkin paths and irregular trinomial coefficients will be crucial for the proof of the main theorem.

Remark 1. For every $n, k \in \mathbb{Z}$ the following identity holds:

$$T^*(n, k) = T^*(n-1, k) + T^*(n-1, k-1) + T^*(n-1, k-2).$$

Proof. This is easily verified from the definition of the trinomial coefficients. \square

Lemma 4. The following identity holds for every $n, k \in \mathbb{N}_0$, $k \leq n$,

$$M(n, k) = T^*(n, n-k) - T^*(n, n-k-2). \quad (4)$$

Proof. We prove this theorem by induction on n . For $n = 0, 1$, the identity holds.

We divide the rest of the proof into two cases (the first case is needed because in case of $n = k$, we would not be able to use induction hypothesis):

Case 1: $n = k$. Then $1 = M(n, n) = T^*(n, 0) + T^*(n, -2) = 1 + 0$, so this case is done.

Case 2: $n > k$. Now suppose the identity holds for all numbers up to $n-1$. By the definition of the generalized Motzkin numbers we have

$$M(n, k) = M(n-1, k) + M(n-1, k-1) + M(n-1, k+1). \quad (5)$$

And by induction hypothesis we can write

$$\begin{aligned} M(n, k) &= T^*(n-1, n-k) - T^*(n-1, n-k-2) \\ &\quad + T^*(n-1, n-k-1) - T^*(n-1, n-k-3) \\ &\quad + T^*(n-1, n-k-2) - T^*(n-1, n-k-4). \end{aligned}$$

From Remark 1 on the recurrence relation of irregular coefficients we get that the even summands and odd summands are equal to $T^*(n, n-k)$ and $-T^*(n, n-k-2)$, respectively. Our claim follows. \square

Now we need the last lemma, concerning the sum of irregular coefficients.

Lemma 5. *For every even $n \in N_0$ holds:*

$$\sum_{k=0}^n T^*(n, 2k) = (3^n + 1)/2, \quad (6)$$

and for odd $n \in N_0$ holds:

$$\sum_{k=1}^n T^*(n, 2k - 1) = (3^n - 1)/2. \quad (7)$$

Proof. We will prove these identities by induction. For $n = 0, 1$, the respective identities hold. Now assume that both identities hold for all $n' < n$. By parity of n we distinguish two cases. We will prove the lemma for the case of n even. Odd case is very similar.

$$\sum_{k=0}^n T^*(n, 2k) = \sum_{k=1}^{n-1} T^*(n, 2k - 1) + 2 \cdot \sum_{k=0}^{n-1} T^*(n, 2k) \quad (8)$$

$$= 2 \cdot 3^{n-1} - \sum_{k=1}^{n-1} T^*(n, 2k - 1) \quad (9)$$

$$= 2 \cdot 3^{n-1} - (3^{n-1} - 1)/2 \quad (10)$$

$$= (3^n + 1)/2. \quad (11)$$

- The first equation follows from Remark 1.
- The second equation follows from the fact that the sum of the n -th row of T^* is equal to 3^n . That can be easily proved by induction.
- The third equation follows from the induction hypothesis.
- The fourth equation is straightforward calculation.

□

We can finally prove the main theorem of this section and one of the main results of this paper.

Proof (Proof of Theorem 7). We will first show the following identity for every $n \geq 3$.

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (2k+1)(T^*(n, n-2k) - T^*(n-1, n-2k-2)) = \begin{cases} \sum_{k=0}^n T^*(n, 2k), & n \text{ even} \\ \sum_{k=1}^n T^*(n, 2k - 1), & n \text{ odd} \end{cases}$$

This identity follows from the straightforward calculations and from the observation that $T^*(n, n - k) = T^*(n, n + k)$ for every $n, k \in \mathbb{Z}$.

For brevity, we will do the following calculation for n odd. The proof for n even is different in the last two equations but the only difference is the use of the different parts of lemma and identity 7.3, depending on the parity.

$$\begin{aligned}
\bar{r}_1(C_n) \cdot |\mathcal{L}(C_n)| &= \sum_{k=0}^{\lfloor n/2 \rfloor} (2k+1) \cdot M(n, 2k) \\
&\quad \text{(by Lemma 3 and linearity of expectation)} \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} (2k+1) \cdot (T^*(n, n-2k) - T^*(n-1, n-2k-2)) \\
&\quad \text{(by Lemma 4)} \\
&= \sum_{k=1}^n T^*(n, 2k-1) \\
&= \frac{3^n - 1}{2}. \quad \text{(by Lemma 5)}
\end{aligned}$$

Together with Theorem 7 taken into account we conclude the formula for $\bar{r}_1(C_n)$. \square

We present the following corollary regarding the asymptotics of $\bar{r}_1(C_n)$.

Corollary 4. *It holds that $\bar{r}_1(C_n) \sim 2\sqrt{\frac{\pi}{3}n}$.*

Proof. The asymptotics of central trinomial coefficients is known, see e.g. [10, p. 588]. Central trinomial coefficients satisfy

$$\binom{n}{0}_2 \sim \frac{3^{n+1/2}}{2\sqrt{\pi n}}.$$

The sign \sim denotes the relation of two sequences. Two sequences a_n and b_n are in relation $a_n \sim b_n$ if $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$. Using Theorem 7 and the mentioned asymptotics, we get: $\bar{r}_1(C_n) \sim 2\sqrt{\frac{\pi}{3}n}$. \square

8 Pseudotrees

We suspect that the following results might be the first step to prove LNR and BHM conjectures for the class of pseudotrees.

Definition 11. *We call a graph unicyclic if it contains exactly one cycle.*

Definition 12. *We call a graph pseudotree if it is a tree or a unicyclic graph. Equivalently, pseudotrees are graphs with at most one cycle.*

8.1 Counting the number of 1-Lipschitz mappings

Lemma 6. *The number of 1-Lipschitz mappings of unicyclic graphs with order n and cycle size c , $c \leq n$, is equal to*

$$\binom{c}{0}_2 \cdot 3^{n-c}.$$

Proof. Let us denote our unicyclic graph of order n and cycle size c by G and the subgraph induced by the vertices on its cycle by C . We use Theorem 7 to get the number of 1-Lipschitz mappings of the subgraph C . Now let us fix some f , a 1-Lipschitz mapping of C .

By deleting all the edges of the cycle C we get a forest \mathcal{T} of trees T_1, \dots, T_c . In this forest, exactly one vertex in each tree T_i has an image under the mapping f . Thus we obtain $3^{|V(T_i)|-1}$ different 1-Lipschitz mappings for each of the tree in \mathcal{T} . Because we can choose all these mapping independently on each other, we obtain, summing over all possible mappings f , the following identity.

$$\mathcal{L}_1(G) = \binom{c}{0}_2 \cdot 3^{\sum_{i=1}^c |V(T_i)|-1} = \binom{c}{0}_2 \cdot 3^{n-c}.$$

□

Observe that Lemma 6 implies that two same-order unicyclic graphs with the same-length cycle have the same number of Lipschitz mappings.

8.2 KC-transformation

In this section it will be useful for us to give a name to one special subset of unicyclic graphs. See Figure 6 for an example.

Definition 13. A corolla graph is a unicyclic graph obtained by taking a cycle graph and joining some path graphs to it by identifying their endpoints with some vertex of that cycle. Every path is joined to exactly one vertex of the cycle. And every vertex of the cycle has at most one path attached.

We note that cycles form a subset of corolla graphs.

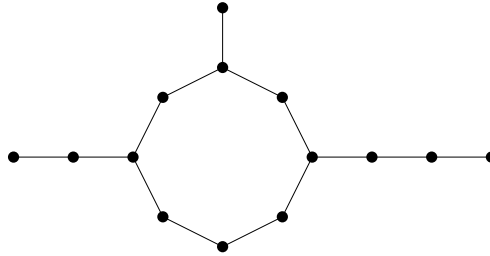


Fig. 6: An example of corolla graph.

Now we are ready to introduce the *generalized KC-transformation* and the main result of [21].

Definition 14 (Generalized KC-transformation). Take a connected graph G and pick $\{a, b\} \in \binom{V(G)}{2}$. Let $V_{a,b}(G)$ denote the set of those vertices which

cannot reach b without passing by a in G . If it is satisfied the following condition that

$$\min(|V_{a;b}(G)|, |V_{b;a}(G)|) > 1,$$

then we can get a new graph $G_{a \rightarrow b}$ by modifying G in the following way.

Remove the edges bb_1, \dots, bb_t , where b_1, \dots, b_t are all the neighbors of b in $V_{b;a}(G)$ and add new edges ab_1, \dots, ab_t .

Definition 15. Let G be a connected graph. Take two different cut vertices a and b of G . We write $V(G; a, b)$ for the set

$$(V(G) \setminus (V_{a;b}(G) \cup V_{b;a}(G))) \cup \{a, b\}.$$

Theorem 9. [21] Let G be a connected graph. Take two different cut vertices a and b of G . Let H be the subgraph of G induced by $V(G; a, b)$. Assume that H has an automorphism σ such that $\sigma(a) = b$ and $\sigma(b) = a$. Then $\bar{\tau}_1(G) \geq \bar{\tau}_1(G_{a \rightarrow b})$.

It is worth noting that one of the corollaries of Theorem 9 is the aforementioned Theorem 5. We will use Theorem 9 to show that for every unicyclic graph that is not a corolla graph there exists some corolla graph of the same order and cycle size that has higher or equal $\bar{\tau}_1$.

Theorem 10. For every unicyclic graph U on n vertices that is not a corolla graph there exist a corolla graph R on n vertices such that $\bar{\tau}_1(R) \geq \bar{\tau}_1(U)$.

Proof. Take an inclusion-wise maximal tree T rooted in r such that r is a vertex of the cycle of U and T is not isomorphic to a path graph. Furthermore, T must satisfy $V(U) \cap V(T) = \{r\}$. Since U is not a corolla graph, such tree must exist.

Consider a sequence T_1, T_2, \dots, T_s with $T_1 = T$ and T_s being a path graph such that for every $i \in \{2, \dots, s\}$, $\bar{\tau}_1(T_{i-1}) \leq \bar{\tau}_1(T_i)$ holds. The existence of such sequence directly follows from Theorem 9.

We can easily extend this argument and define the sequence U_1, U_2, \dots, U_s such that U_i is the graph in which T is replaced by T_i . Clearly, $\bar{\tau}_1(U_{i-1}) \leq \bar{\tau}_1(U_i)$.

We can repeatedly find another tree T' in U_s , satisfying the same conditions as T (except that the root has to be of different of course) in U and proceed similarly until we cannot find some next T' . We get a corolla graph and our claim follows. \square

9 Concluding remarks

We have showed closed formulas for several classes of graphs including paths, complete graphs, complete bipartite graphs (and specially stars) and most importantly we showed the formula for cycles by using properties of generalized Motzkin numbers. We also investigated pseudotrees in the effort of extending the results of [21].

Acknowledgments

This research was supported by the Charles University Grant Agency, project GA UK 1158216.

References

1. BENJAMINI, I., HÄGGSTRÖM, O., AND MOSSEL, E. On random graph homomorphisms into \mathbb{Z} . *Journal of Combinatorial Theory, Series B* 78, 1 (2000), 86–114.
2. BENJAMINI, I., AND SCHECHTMAN, G. Upper bounds on the height difference of the Gaussian random field and the range of random graph homomorphisms into \mathbb{Z} . *Random Structures and Algorithms* 17, 1 (2000), 20–25.
3. BLASIAK, P., DATTOLI, G., FRASCATI, D. F. A. C. R., AND HORZELA, A. Motzkin numbers, central trinomial coefficients and hybrid polynomials. *Journal of Integer Sequences* 11, 2 (2008), 3.
4. COHEN, E., PERKINS, W., AND TETALI, P. On the widom–rowlinson occupancy fraction in regular graphs. *Combinatorics, Probability and Computing* 26, 2 (2017), 183–194.
5. CSIKVÁRI, P. On a poset of trees. *Combinatorica* 30, 2 (2010), 125–137.
6. CSIKVÁRI, P. On a poset of trees II. *Journal of Graph Theory* 74, 1 (2013), 81–103.
7. DIESTEL, R. *Graph theory {Graduate texts in mathematics; 173}*. Springer-Verlag Berlin and Heidelberg GmbH, 2000.
8. DONAGHEY, R., AND SHAPIRO, L. W. Motzkin numbers. *Journal of Combinatorial Theory, Series A* 23, 3 (1977), 291–301.
9. ERSCHLER, A. Random mappings of scaled graphs. *Probability theory and related fields* 144, 3-4 (2009), 543–579.
10. FLAJOLET, P., AND SEDGEWICK, R. *Analytic Combinatorics*. Cambridge University Press, 2009.
11. GODSIL, C., AND ROYLE, G. F. *Algebraic graph theory*, vol. 207. Springer Science & Business Media, 2013.
12. HELL, P., AND NEŠETŘIL, J. Graphs and homomorphisms. *Oxford Lecture Series in Mathematics and its Applications* (2004).
13. KELMANS, A. On graphs with randomly deleted edges. *Acta Mathematica Hungarica* 37, 1-3 (1981), 77–88.
14. LOEBL, M., NEŠETŘIL, J., AND REED, B. A note on random homomorphism from arbitrary graphs to \mathbb{Z} . *Discrete mathematics* 273, 1 (2003), 173–181.
15. LOVÁSZ, L. Random walks on graphs. *Combinatorics, Paul Erdos is eighty 2* (1993), 1–46.
16. OEIS. A001006. <http://oeis.org/A001006>, 2016. [Online; accessed 23-June-2016].
17. OEIS. A027907. <http://oeis.org/A027907>, 2016. [Online; accessed 23-June-2016].
18. OEIS. A123456. <http://oeis.org/A123456>, 2016. [Online; accessed 23-June-2016].
19. VAN LEEUWEN, M. A. Some simple bijections involving lattice walks and ballot sequences. *arXiv preprint arXiv:1010.4847* (2010).
20. WIKIPEDIA. Number of ways to reach a cell with the minimum number of moves. https://upload.wikimedia.org/wikipedia/commons/9/92/King_walks.svg, 2017.

21. WU, Y., XU, Z., AND ZHU, Y. Average range of Lipschitz functions on trees. *Moscow Journal of Combinatorics and Number Theory* 1, 6 (2016), 96–116.
22. ZHAO, Y. Extremal regular graphs: independent sets and graph homomorphisms. *arXiv preprint arXiv:1610.09210* (2016).