Some algebraic and arithmetic properties of Feynman diagrams

Yajun Zhou

Abstract This article reports on some recent progresses in Bessel moments, which represent a class of Feynman diagrams in 2-dimensional quantum field theory. Many challenging mathematical problems on these Bessel moments have been formulated as a vast set of conjectures, by David Broadhurst and collaborators, who work at the intersection of high energy physics, number theory and algebraic geometry. We present the main ideas behind our verifications of several such conjectures, which revolve around linear and non-linear sum rules of Bessel moments, as well as relations between individual Feynman diagrams and critical values of modular L-functions.

1 Introduction

1.1 Bessel moments and Feynman diagrams

In perturbative quantum field theory (pQFT), we use *Feynman diagrams* to quantify the interactions among elementary particles [31, 1, 11, 37]. In this survey, we will focus on 2-dimensional pQFT, where the propagator of a free particle with proper mass m_0 takes the following form:

$$\frac{1}{(2\pi)^2} \lim_{\varepsilon \to 0^+} \iint_{\mathbb{R}^2} \frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x} - \varepsilon|\boldsymbol{p}|^2}\mathrm{d}^2\boldsymbol{p}}{|\boldsymbol{p}|^2 + m_0^2} = \frac{K_0(m_0|\boldsymbol{x}|)}{2\pi} \tag{1}$$

for $\boldsymbol{x} \in \mathbb{R}^2 \smallsetminus \{\boldsymbol{0}\}$. Here, $K_0(t) := \int_0^\infty e^{-t\cosh u}\mathrm{d}u, t > 0$ is the modified Bessel function of the second kind and zeroth order.

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Some results in 2-dimensional pQFT also find their way into the finite part of renormalized perturbative expansions of $(4 - \varepsilon)$-dimensional quantum electrodynamics [36]. For example, in Stefano Laporta's recent computation of the 4-loop contribution to electron's magnetic moment [37], one of the master integrals is the 4-loop sunrise diagram for 2-dimensional pQFT:

$$
\text{diagram} := 2^4 \int_0^\infty I_0(t)[K_0(t)]^5 t\,\mathrm{d}t
$$

$$
= \int_0^\infty \frac{\mathrm{d}x_1}{x_1} \int_0^\infty \frac{\mathrm{d}x_2}{x_2} \int_0^\infty \frac{\mathrm{d}x_3}{x_3} \int_0^\infty \frac{\mathrm{d}x_4}{x_4} \frac{1}{\left(1 + \sum_{k=1}^4 x_k\right)\left(1 + \sum_{k=1}^4 \frac{1}{x_k}\right) - 1}. \tag{2}
$$

Here, the single integral over the variable t represents the Feynman diagram in configuration space (see [1, §1], [10, §9.2] or [11, (84)]), and $I_0(t) = \frac{1}{\pi} \int_0^\pi e^{t\cos\theta}\,\mathrm{d}\theta$ is the modified Bessel function of the first kind and zeroth order; alternatively, a quadruple integral over a rational function in the variables x_1, x_2, x_3 and x_4 represents the same Feynman diagram in the Schwinger parameter space (see [10, §9.1] or [44, §8]).

On one hand, Feynman diagrams provide us with many physically meaningful multiple integrals over rational functions, which are special cases of motivic integrals [44, 3], playing prominent rôles in the arena for algebraic geometers. On the other hand, certain Feynman diagrams are (conjecturally or provably) related to arithmetically interesting objects [42, 11, 50], such as special values of modular L-functions inside their critical strips, inviting pilgrims to the shrine for number theorists.

After high-precision computations of Feynman diagrams, Bailey–Borwein–Broadhurst–Glasser [1], Broadhurst [10, 11], Broadhurst–Schnetz [18] and Broadhurst–Mellit [17] had formulated various conjectures on *Bessel moments*

$$
\mathbf{IKM}(a,b;n) := \int_0^\infty [I_0(t)]^a [K_0(t)]^b t^n\,\mathrm{d}t \tag{3}
$$

with $a, b, n \in \mathbb{Z}_{\geq 0}$. The last few years had witnessed rapid progress in these conjectures proposed by David Broadhurst and coworkers. In §§1.2–1.3 below, we give precise statements of some recently proven conjectures about Bessel moments, before presenting in §1.4 a road map for their mathematical understanding.

1.2 Some algebraic relations involving Bessel moments

The following theorem about linear sum rules for Bessel moments grew out of numerical conjectures by Bailey–Borwein–Broadhurst–Glasser [1, (220)], Broadhurst–Mellit [17, (7.10)] and Broadhurst–Roberts [12, Conjecture 2]. The first proof appeared in [49].

Theorem 1 (Generalized Bailey–Borwein–Broadhurst–Glasser sum rules and generalized Crandall numbers).

(a) *We have*

$$\int_0^\infty \frac{[\pi I_0(t) + iK_0(t)]^m + [\pi I_0(t) - iK_0(t)]^m}{i} [K_0(t)]^m t^n \mathrm{d}t = 0 \qquad (4)$$

for $m \in \mathbb{Z}_{>1}, n \in \mathbb{Z}_{\geq 0}, \frac{m-n}{2} \in \mathbb{Z}_{>0}$, and

$$\int_0^\infty \frac{[\pi I_0(t) + iK_0(t)]^m - [\pi I_0(t) - iK_0(t)]^m}{i} [K_0(t)]^m t^n \mathrm{d}t = 0 \qquad (5)$$

for $m \in \mathbb{Z}_{>0}, n \in \mathbb{Z}_{\geq 0}, \frac{m-n-1}{2} \in \mathbb{Z}_{>0}$, which generalize the Bailey–Borwein–Broadhurst–Glasser sum rule [1, (220)].

(b) *The Crandall numbers (OEIS `A262961` [43])*

$$A(n) := \left(\frac{2}{\pi}\right)^4 \int_0^\infty \left\{[\pi I_0(t)]^2 - [K_0(t)]^2\right\} I_0(t)[K_0(t)]^5 (2t)^{2n-1} \mathrm{d}t \qquad (6)$$

are integers for all $n \in \mathbb{Z}_{>0}$. More generally, the integral

$$C_{m,n} = \frac{2^{1+2(n-1)[1-(-1)^m]}}{\pi^{m+1}} \int_0^\infty \frac{[\pi I_0(t) + iK_0(t)]^m - [\pi I_0(t) - iK_0(t)]^m}{i} \times$$
$$\times [K_0(t)]^m (2t)^{2n+m-3} \mathrm{d}t \qquad (7)$$

evaluates to a positive integer for each $m, n \in \mathbb{Z}_{>0}$.

The next theorem includes two sets of non-linear sum rules, which were originally discovered by Broadhurst–Mellit [17, (6.12) and (7.13)] through numerical experiments on moderately-sized determinants. An analytic proof has recently been found [52] for Broadhurst–Mellit determinants that come in arbitrary sizes.

Theorem 2 (Broadhurst–Mellit determinant formulae). *Define $\mathbf{M}_k$ and $\mathbf{N}_k$ as $k \times k$ matrices with elements*

$$(\mathbf{M}_k)_{a,b} := \int_0^\infty [I_0(t)]^a [K_0(t)]^{2k+1-a} t^{2b-1} \mathrm{d}t, \qquad (8)$$

$$(\mathbf{N}_k)_{a,b} := \int_0^\infty [I_0(t)]^a [K_0(t)]^{2k+2-a} t^{2b-1} \mathrm{d}t. \qquad (9)$$

Then we have the following determinant formulae:

$$\det \mathbf{M}_k = \prod_{j=1}^k \frac{(2j)^{k-j} \pi^j}{\sqrt{(2j+1)^{2j+1}}}, \qquad (10)$$

$$\det \mathbf{N}_k = \frac{2\pi^{(k+1)^2/2}}{\Gamma((k+1)/2)} \prod_{j=1}^{k+1} \frac{(2j-1)^{k+1-j}}{(2j)^j}, \qquad (11)$$

where Euler's gamma function[1] is defined by $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$ for $x > 0$.

1.3 Some arithmetic properties of Bessel moments

In what follows, we write $f_{k,N}$ for a *modular form* (see §2.3 for technical details) of weight k and level N, and define its L-function through a Mellin transform:

$$L(f_{k,N}, s) := \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f_{k,N}(iy)y^{s-1}\mathrm{d}y. \tag{12}$$

A special L-value $L(f_{k,N}, s)$ is said to be *critical*, if $s \in \mathbb{Z} \cap (0, k)$. In this survey, we will be interested in the following three special modular forms:

$$f_{3,15}(z) = [\eta(3z)\eta(5z)]^3 + [\eta(z)\eta(15z)]^3, \tag{13}$$

$$f_{4,6}(z) = [\eta(z)\eta(2z)\eta(3z)\eta(6z)]^2, \tag{14}$$

$$f_{6,6}(z) = \frac{[\eta(2z)\eta(3z)]^9}{[\eta(z)\eta(6z)]^3} + \frac{[\eta(z)\eta(6z)]^9}{[\eta(2z)\eta(3z)]^3}, \tag{15}$$

where the Dedekind eta function is defined as $\eta(z) := e^{\pi i z/12}\prod_{n=1}^\infty (1 - e^{2\pi i n z})$ for complex numbers z satisfying $\mathrm{Im}\, z > 0$. For $y > 0$, one can deduce

$$f_{k,N}\left(\frac{i}{Ny}\right) = (\sqrt{N}y)^k f_{k,N}(iy) \tag{16}$$

from the modular transformation $\eta(-1/\tau) = \sqrt{\tau/i}\eta(\tau)$ for $\tau/i > 0$. Consequently, the L-functions attached to these three modular forms satisfy the following reflection formulae [11, (95), (106), (138)]:

$$\Lambda(f_{k,N}, s) := \left(\frac{\sqrt{N}}{\pi}\right)^s \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)L(f_{k,N}, s) = \Lambda(f_{k,N}, k-s). \tag{17}$$

The studies of the Bessel moments **IKM**$(1,4;1)$ and **IKM**$(2,3;1)$ had been initiated by Bailey–Borwein–Broadhurst–Glasser [1, §5]. Back in 2008, it was analytically confirmed that

$$\mathbf{IKM}(2,3;1) = \frac{\sqrt{15}\pi}{2}C \tag{18}$$

where

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[1] Throughout this survey, we reserve the upright Γ for Euler's gamma function, and write Γ in slanted typeface for congruence subgroups (to be introduced in §2.3).

$$C = \frac{1}{240\sqrt{5}\pi^2} \Gamma\left(\frac{1}{15}\right) \Gamma\left(\frac{2}{15}\right) \Gamma\left(\frac{4}{15}\right) \Gamma\left(\frac{8}{15}\right) \tag{19}$$

is the *Bologna constant* attributed to Broadhurst [9, 1] and Laporta [36]. Later on, it was realized that (18) can be rewritten as $\mathbf{IKM}(2,3;1) = \frac{3}{4}L(f_{3,15},2) = \frac{3\pi}{2\sqrt{15}}L(f_{3,15},1)$ [11, (96)–(97)], thanks to the work of Rogers–Wan–Zucker [40, Theorem 5]. An innocent-looking conjecture $\mathbf{IKM}(1,4;1) = \frac{2\pi}{\sqrt{15}}\mathbf{IKM}(2,3;1)$ was proposed in 2008 [1, (95)], but was not resolved until Bloch–Kerr–Vanhove carried out a *tour de force* in motivic cohomology during 2015 [3], and Samart elucidated the computations of special gamma values in 2016 [42]. We have recently simplified [50, Theorem 2.2.2] the result of Bloch–Kerr–Vanhove and Samart, as stated in the theorem below.

Theorem 3 (3-loop sunrise via Bologna constant). *We have*

$$\mathbf{IKM}(1,4;1) = \pi^2 C = \frac{\pi^2}{5}L(f_{3,15},1) = \frac{3\pi}{2\sqrt{15}}L(f_{3,15},2). \tag{20}$$

Based on a discussion with Francis Brown at Les Houches in 2010, and encouraged by a result of Zhiwei Yun published in 2015 [47], David Broadhurst discovered some relations between $\mathbf{IKM}(a,6-a;1)$ and $L(f_{4,6},s)$ [11, §7.3], as well as between $\mathbf{IKM}(a,8-a;1)$ and $L(f_{6,6},s)$ [11, §7.6]. All these conjectures have been verified recently [50, §§4–5], so they are included in the theorem below.

Theorem 4 (Critical *L*-values for 6-Bessel and 8-Bessel problems).

(a) *We have*

$$\frac{3}{\pi^2}\mathbf{IKM}(1,5;1) = \mathbf{IKM}(3,3;1) = \frac{3}{2}L(f_{4,6},2), \tag{21}$$

$$\mathbf{IKM}(2,4;1) = \frac{\pi^2}{2}L(f_{4,6},1) = \frac{3}{2}L(f_{4,6},3), \tag{22}$$

where the first equality in (21) *comes from Theorem 1(a), and the last equality in* (22) *descends from* (17).

(b) *We have*

$$\mathbf{IKM}(4,4;1) = L(f_{6,6},3), \tag{23}$$

$$\frac{1}{\pi^2}\mathbf{IKM}(1,7;1) = \mathbf{IKM}(3,5;1) = \frac{9}{4}L(f_{6,6},4), \tag{24}$$
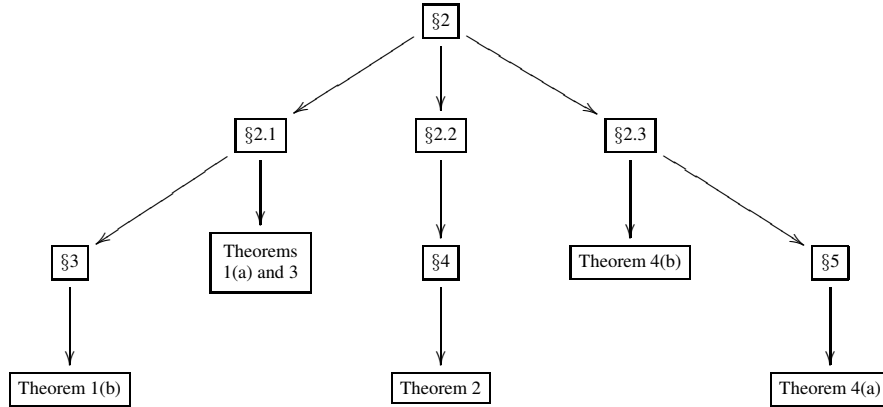
$$\mathbf{IKM}(2,6;1) = \frac{27}{4}L(f_{6,6},5), \tag{25}$$

where the first equality in (24) *follows from Theorem 1(a),*

1.4 Plan of proofs

To help our readers navigate through this survey, we present the *Leitfaden* in Table 1.

Table 1 Organizational chart

In §2.1, we begin with a summary of useful analytic properties for Bessel functions, which result in a proof of Theorem 1(a). We then present Wick rotations, which are special contour deformations connecting moment problems for **IKM**$(a,b;n)$ to those for

$$\mathbf{JYM}(\alpha,\beta;n) := \int_0^\infty [J_0(t)]^\alpha [Y_0(t)]^\beta t^n \mathrm{d}t, \tag{26}$$

where $a+b = \alpha+\beta$, and $J_0(x) := \frac{2}{\pi} \int_0^{\pi/2} \cos(x\cos\varphi)\mathrm{d}\varphi$, $Y_0(x) := -\frac{2}{\pi} \int_0^\infty \cos(x\cosh u)\mathrm{d}u$ are Bessel functions of the zeroth order, defined for $x > 0$. The **JYM** problems have some desirable properties [8, 48], which lead us to a quick proof of Theorem 3. Further applications of Wick rotations are given in §3, in the context of Theorem 1.

In §2.2, we give a brief overview of Vanhove's differential equations [44, §9], and compute certain Wrońskian determinants involving Bessel moments. These preparations allow us to present the main ideas behind the proof of Theorem 2, in §4.

In §2.3, we describe how to obtain critical *L*-values via integrations over products of certain modular forms, illustrating our general procedures with the proof of Theorem 4(b). Some extensions in §5 then lead to a sketched proof of all the statements in Theorem 4.

In §6, we wrap up this survey with some open questions on Bessel moments.

2 Toolkit

2.1 Wick rotations of Bessel moments

As we may recall, for $v \in \mathbb{C}, -\pi < \arg z < \pi$, the Bessel functions J_v and Y_v are defined by

$$J_v(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+v+1)} \left(\frac{z}{2}\right)^{2k+v}, \quad Y_v(z) := \lim_{\mu \to v} \frac{J_\mu(z)\cos(\mu\pi) - J_{-\mu}(z)}{\sin(\mu\pi)}, \tag{27}$$

which may be compared to the modified Bessel functions I_v and K_v:

$$I_v(z) := \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+v+1)} \left(\frac{z}{2}\right)^{2k+v}, \quad K_v(z) := \frac{\pi}{2} \lim_{\mu \to v} \frac{I_{-\mu}(z) - I_\mu(z)}{\sin(\mu\pi)}. \tag{28}$$

Here, the fractional powers of complex numbers are defined through $w^\beta = \exp(\beta \log w)$ for $\log w = \log|w| + i\arg w$, where $|\arg w| < \pi$.

The cylindrical Hankel functions $H_0^{(1)}(z) = J_0(z) + iY_0(z)$ and $H_0^{(2)}(z) = J_0(z) - iY_0(z)$ are both well defined for $-\pi < \arg z < \pi$. In view of (27) and (28), we can verify

$$J_0(ix) = I_0(x) \quad \text{and} \quad \frac{\pi i}{2} H_0^{(1)}(ix) = K_0(x) \tag{29}$$

along with

$$H_0^{(1)}(\pm x + i0^+) = \pm J_0(x) + iY_0(x) \tag{30}$$

for $x > 0$.

As $|z| \to \infty, -\pi < \arg z < \pi$, we have the following asymptotic expansions [46, §7.2]:

$$\begin{cases} H_0^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{i\left(z-\frac{\pi}{4}\right)} \left\{ 1 + \sum_{n=1}^{N} \frac{\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2}{(2iz)^n \pi n!} + O\left(\frac{1}{|z|^{N+1}}\right) \right\}, \\ H_0^{(2)}(z) = \sqrt{\frac{2}{\pi z}} e^{i\left(\frac{\pi}{4}-z\right)} \left\{ 1 + \sum_{n=1}^{N} \frac{\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2}{(-2iz)^n \pi n!} + O\left(\frac{1}{|z|^{N+1}}\right) \right\}, \end{cases} \tag{31}$$

which allow us to establish a vanishing identity

$$\int_{-i\infty}^{i\infty} [H_0^{(1)}(z)H_0^{(2)}(z)]^m z^n \mathrm{d}z = 0, \quad n \in \mathbb{Z} \cap [0, m-1] \tag{32}$$

by closing the contour rightwards. One can transcribe the last vanishing integral into the statements in Theorem 1(a), bearing in mind that

$$H_0^{(1)}(it)H_0^{(2)}(it) = \frac{4K_0(|t|)}{\pi^2}\left[K_0(|t|) - \frac{\pi it}{|t|}I_0(|t|)\right], \quad \forall t \in (-\infty, 0) \cup (0, \infty). \quad (33)$$

Lemma 1 (An application of Wick rotation). *We have the following relation between* **IKM** *and* **JYM***:*

$$\left(\frac{2}{\pi}\right)^4 \mathbf{IKM}(1,4;1) = -\mathbf{JYM}(5,0;1) + 6\mathbf{JYM}(3,2;1) - \mathbf{JYM}(1,4;1). \quad (34)$$

Proof. From (29), we know that

$$\left(\frac{2}{\pi}\right)^4 \mathbf{IKM}(1,4;1) = -\operatorname{Re}\int_0^{i\infty} J_0(z)[H_0^{(1)}(z)]^4 z\mathrm{d}z, \quad (35)$$

where the contour runs along the positive $\operatorname{Im} z$-axis.

Noting that the asymptotic behavior of $J_0(z) = [H_0^{(1)}(z) + H_0^{(2)}(z)]/2$ can be inferred from (31), we can rotate the contour $90°$ clockwise, from the positive $\operatorname{Im} z$-axis to the positive $\operatorname{Re} z$-axis (see Fig. 1*a*), thereby equating (35) with

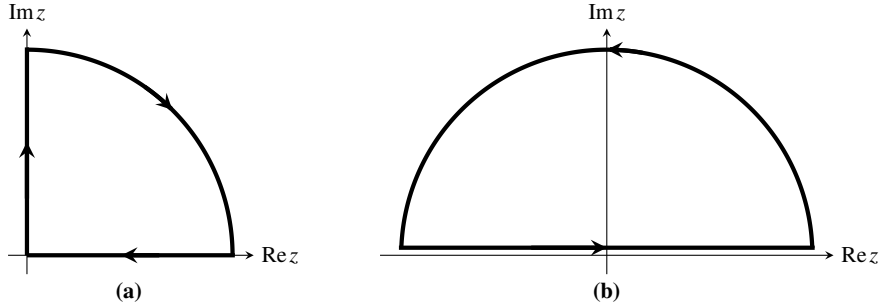$$-\operatorname{Re}\int_0^{\infty} J_0(x)[H_0^{(1)}(x)]^4 x\mathrm{d}x = -\operatorname{Re}\int_0^{\infty} J_0(x)[J_0(x) + iY_0(x)]^4 x\mathrm{d}x, \quad (36)$$

hence the right-hand side of (34). □

Proposition 1 (Evaluation of IKM$(1,4;1)$). *We have*

$$\mathbf{IKM}(1,4;1) = \frac{\pi^4}{30}\mathbf{JYM}(5,0;1) = \pi^2 C, \quad (37)$$

where C is the Bologna constant defined in (19)*.*

Fig. 1 a Wick rotation that turns an **IKM** to a sum of several **JYM**'s. Note that the contribution from the circular arc tends to zero as $|z| \to \infty$, thanks to Jordan's lemma being applicable to the asymptotic behavior of Hankel functions. **b** Contour of integration that leads to a cancelation formula for **JYM**'s.

Proof. For $\ell, m, n \in \mathbb{Z}_{\geq 0}$ satisfying either $\ell - (m+n)/2 < 0; m < n$ or $\ell - m = \ell - n < -1$, we can prove

$$\int_{i0^+ - \infty}^{i0^+ + \infty} [J_0(z)]^m [H_0^{(1)}(z)]^n z^\ell \mathrm{d}z := \lim_{\varepsilon \to 0^+} \lim_{R \to \infty} \int_{i\varepsilon - R}^{i\varepsilon + R} [J_0(z)]^m [H_0^{(1)}(z)]^n z^\ell \mathrm{d}z = 0, \quad (38)$$

by considering the contour in Fig. 1*b*. According to (30) and $J_0(-x) = J_0(x)$, we may reformulate (38) as

$$\int_0^\infty [J_0(x)]^m \left\{ [J_0(x) + iY_0(x)]^n + (-1)^\ell [-J_0(x) + iY_0(x)]^n \right\} x^\ell \mathrm{d}x = 0, \qquad (38')$$

which is a convenient cancelation formula for **JYM**'s.

With $J(J^4 - 6J^2 Y^2 + Y^4) - \frac{2J^2}{3}[(J+iY)^3 - (-J+iY)^3] - \frac{(J+iY)^5 - (-J+iY)^5}{10} = -\frac{8J^5}{15}$ in hand, we can identify the right-hand side of (34) with $\frac{8}{15}$ **JYM**$(5, 0; 1)$. This proves the first equality in (37). The second equality can be directly deduced from [8, (5.2)]. $\square$

So far, we have recapitulated an analytic proof of Theorem 3, as originally given in [50, §2]. It is worth pointing out that Kluyver's function [35]

$$p_n(x) = \int_0^\infty J_0(xt)[J_0(t)]^n xt \, \mathrm{d}t \qquad (39)$$

characterizes the probability density of the distance x traveled by a rambler, who walks in the Euclidean plane, taking n consecutive steps of unit lengths, aiming at uniformly distributed random directions. The analytic properties of such probability densities have been extensively studied [4, 8, 7, 6, 51]. Recently, we have shown [51, Theorem 5.1] that $p_n(x)$ is expressible through Feynman diagrams when n is odd, as stated in the theorem below.

Theorem 5 ($p_{2j+1}(x)$ **as Feynman diagrams**). *For each $j \in \mathbb{Z}_{>1}$, the function $p_{2j+1}(x), 0 \leq x \leq 1$ is a unique $\mathbb{Q}$-linear combination of*

$$\int_0^\infty I_0(xt)[I_0(t)]^{2m+1} \left[\frac{K_0(t)}{\pi} \right]^{2(j-m)} xt \, \mathrm{d}t, \quad where \ m \in \mathbb{Z} \cap \left[0, \frac{j-1}{2} \right]. \qquad (40)$$

(When $j = 1$, the same is true for $0 \leq x < 1$.)

2.2 Vanhove's differential equations and Wrońskians of Bessel moments

In [44, §9], Vanhove has constructed n-order differential operators $\widetilde{L}_n$ (written in the variable u in this survey) so that the relation

$$\widetilde{L}_n \int_0^\infty I_0(\sqrt{u}t)[K_0(t)]^{n+1}t\mathrm{d}t = \mathrm{const} \tag{41}$$

holds for all $n \in \mathbb{Z}_{>0}$ and $u \in (0, (n+1)^2)$. The first few Vanhove operators $\widetilde{L}_n$ are listed in Table 2, where $D^n = \partial^n/\partial u^n$ for $n \in \mathbb{Z}_{>0}$ and D^0 is the identity operator.

Table 2 The first few Vanhove differential operators (abridged from [44, §9, Table 1])

n	$\widetilde{L}_n$
1	$u(u-4)D^1 + (u-2)D^0$
2	$u(u-1)(u-9)D^2 + (3u^2 - 20u + 9)D^1 + (u-3)D^0$
3	$u^2(u-4)(u-16)D^3 + 6u(u^2 - 15u + 32)D^2 + (7u^2 - 68u + 64)D^1 + (u-4)D^0$
4	$u^2(u-1)(u-9)(u-25)D^4 + 2u(5u^3 - 140u^2 + 777u - 450)D^3 + (25u^3 - 518u^2 + 1839u - 450)D^2 + (3u-5)(5u-57)D^1 + (u-5)D^0$

In general, for each $n \in \mathbb{Z}_{\geq 1}$, Vanhove's operator $\widetilde{L}_n$ satisfies

$$\begin{cases} t\widetilde{L}_n I_0(\sqrt{u}t) = \frac{(-1)^n}{2^n} L_{n+2}^* \frac{I_0(\sqrt{u}t)}{t}, \\ t\widetilde{L}_n K_0(\sqrt{u}t) = \frac{(-1)^n}{2^n} L_{n+2}^* \frac{K_0(\sqrt{u}t)}{t}, \end{cases} \tag{42}$$

where L_{n+2}^* is the formal adjoint to the Borwein–Salvy operator L_{n+2} [5], the latter of which is the $(n+1)$-st symmetric power of the Bessel differential operator $(t\partial/\partial t)^2 - t^2$ that annihilates both $I_0(t)$ and $K_0(t)$. Using the Bronstein–Mulders–Weil algorithm [19] for symmetric powers, we have shown [52, Lemma 4.2] that the following homogeneous differential equations

$$\widetilde{L}_n \left[\int_0^\infty I_0(\sqrt{u}t)[K_0(t)]^{n+1}t\mathrm{d}t + (n+1) \int_0^\infty K_0(\sqrt{u}t)I_0(t)[K_0(t)]^n t\mathrm{d}t \right] = 0, \tag{43}$$

$$\widetilde{L}_n \int_0^\infty I_0(\sqrt{u}t)[I_0(t)]^{j-1}[K_0(t)]^{n+2-j}t\mathrm{d}t = 0, \quad \forall j \in \mathbb{Z} \cap \left[2, \frac{n}{2}+1\right], \tag{44}$$

$$\widetilde{L}_n \int_0^\infty K_0(\sqrt{u}t)[I_0(t)]^{j}[K_0(t)]^{n+1-j}t\mathrm{d}t = 0, \quad \forall j \in \mathbb{Z} \cap \left[2, \frac{n+1}{2}\right] \tag{45}$$

hold for $u \in (0,1)$.

For $N \in \mathbb{Z}_{>1}$, we write $W[f_1(u), \ldots, f_N(u)]$ for the Wrońskian determinant $\det(D^{i-1}f_j(u))_{1 \leq i,j \leq N}$. In [52, §4.1], we have constructed some Wrońskians as precursors to Broadhurst–Mellit determinants $\mathbf{M}_k$ and $\mathbf{N}_k$ (see Theorem 2). Concretely speaking, for each $k \in \mathbb{Z}_{\geq 2}$, we set

$$\begin{cases} \mu_{k,1}^\ell(u) = \frac{1}{2k+1} \int_0^\infty \{I_0(\sqrt{u}t)K_0(t) + 2kK_0(\sqrt{u}t)I_0(t)\}[K_0(t)]^{2k-1}t^{2\ell-1}\mathrm{d}t, \\ \mu_{k,j}^\ell(u) = \int_0^\infty I_0(\sqrt{u}t)[I_0(t)]^{j-1}[K_0(t)]^{2k+1-j}t^{2\ell-1}\mathrm{d}t, \forall j \in \mathbb{Z} \cap [2,k], \\ \mu_{k,j}^\ell(u) = \int_0^\infty K_0(\sqrt{u}t)[I_0(t)]^{j-k+1}[K_0(t)]^{3k-1-j}t^{2\ell-1}\mathrm{d}t, \forall j \in \mathbb{Z} \cap [k+1, 2k-1], \end{cases} \tag{46}$$

and

$$
\begin{cases}
v_{k,1}^\ell(u) = \frac{1}{2(k+1)} \int_0^\infty \{I_0(\sqrt{u}t)K_0(t) + (2k+1)K_0(\sqrt{u}t)I_0(t)\}[K_0(t)]^{2k}t^{2\ell-1}\mathrm{d}t, \\
v_{k,j}^\ell(u) = \int_0^\infty I_0(\sqrt{u}t)[I_0(t)]^{j-1}[K_0(t)]^{2k+2-j}t^{2\ell-1}\mathrm{d}t, \forall j \in \mathbb{Z} \cap [2,k+1], \\
v_{k,j}^\ell(u) = \int_0^\infty K_0(\sqrt{u}t)[I_0(t)]^{j-k}[K_0(t)]^{3k+1-j}t^{2\ell-1}\mathrm{d}t, \forall j \in \mathbb{Z} \cap [k+2,2k],
\end{cases}
\tag{47}
$$

and consider the Wrońskian determinants $\Omega_{2k-1}(u) := W[\mu_{k,1}^1(u), \ldots, \mu_{k,2k-1}^1(u)]$, $\omega_{2k}(u) := W[v_{k,1}^1(u), \ldots, v_{k,2k}^1(u)]$. For $k \in \mathbb{Z}_{\geq 2}$, Vanhove's operators $\widetilde{L}_{2k-1}$ and $\widetilde{L}_{2k}$ take the following forms [44, (9.11)–(9.12)]:

$$
\widetilde{L}_{2k-1} = \mathfrak{m}_{2k-1}(u)D^{2k-1} + \frac{2k-1}{2}\frac{\mathrm{d}\mathfrak{m}_{2k-1}(u)}{\mathrm{d}u}D^{2k-2} + L.O.T.,
\tag{48}
$$

$$
\widetilde{L}_{2k} = \mathfrak{n}_{2k}(u)D^{2k} + k\frac{\mathrm{d}\mathfrak{n}_{2k}(u)}{\mathrm{d}u}D^{2k-1} + L.O.T.,
\tag{49}
$$

where

$$
\mathfrak{m}_{2k-1}(u) = u^k \prod_{j=1}^k [u - (2j)^2], \quad \mathfrak{n}_{2k}(u) = u^k \prod_{j=1}^{k+1}[u - (2j-1)^2],
\tag{50}
$$

and "*L.O.T.*" stands for "lower order terms". Therefore, we have the following evolution equations for Wrońskians:

$$
D^1\Omega_{2k-1}(u) = \frac{2k-1}{2}\Omega_{2k-1}(u)D^1 \log \frac{1}{u^k \prod_{j=1}^k [(2j)^2 - u]},
\tag{51}
$$

$$
D^1\omega_{2k}(u) = k\omega_{2k}(u)D^1 \log \frac{1}{u^k \prod_{j=1}^{k+1}[(2j-1)^2 - u]},
\tag{52}
$$

where $0 < u < 1$. These differential equations will play crucial rôles in the proof of Theorem 2 in §4.

2.3 Modular forms and their integrations

Let

$$
\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z}; ad - bc = 1; c \equiv 0 \,(\mathrm{mod}\,N) \right\}
\tag{53}
$$

be the Hecke congruence group of level $N \in \mathbb{Z}_{>0}$. A modular form $M_{k,N}(z)$ of weight k and level N is a holomorphic[2] function that transforms like

---

[2] For technical requirements on holomorphy at $i\infty$, see [23, Definition 1.2.3].

$$M_{k,N}\left(\frac{az+b}{cz+d}\right) = (cz+d)^k M_{k,N}(z), \tag{54}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ runs over all the members of $\Gamma_0(N)$, and z is an arbitrary point in the upper half-plane $\mathfrak{H} := \{w \in \mathbb{C} \mid \operatorname{Im} w > 0\}$. Modular forms of weight 0 (relaxing the requirement on holomorphy at $i\infty$) are called modular functions. These $\Gamma_0(N)$-invariant modular functions are effectively defined on the moduli space $Y_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash\mathfrak{H}$ (see Fig. 2) for isomorphism classes of complex elliptic curves.

Following the notation of Chan–Zudilin [20], we write $\hat{W}_3 = \frac{1}{\sqrt{3}}\left(\begin{smallmatrix} 3 & -2 \\ 6 & -3 \end{smallmatrix}\right)$ and construct a group $\Gamma_0(6)_{+3} = \langle\Gamma_0(6), \hat{W}_3\rangle$ by adjoining $\hat{W}_3$ to $\Gamma_0(6)$. This group is of particular importance to the following motivic integral [3, §2]:
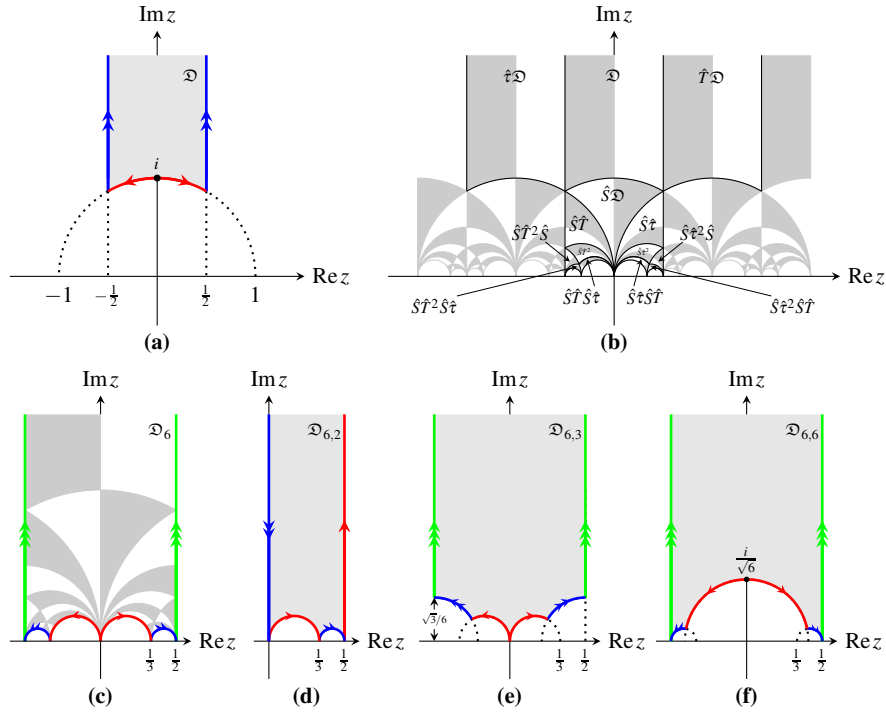
Fig. 2 (Adapted from [26, Fig. 61]) **a** Fundamental domain $\mathfrak{D}$ of $\Gamma_0(1) = SL(2,\mathbb{Z})$. The moduli space $Y_0(1)(\mathbb{C}) = SL(2,\mathbb{Z})\backslash\mathfrak{H}$ is a quotient space of $\mathfrak{D}$ that identifies the corresponding sides of the boundary $\partial\mathfrak{D}$ along the *arrows*. **b** Tessellation of the upper half-plane $\mathfrak{H}$ by successive translations [generator $\hat{T} = \hat{t}^{-1} : z \mapsto z+1$] and inversions [generator $\hat{S} = \hat{S}^{-1} : z \mapsto -1/z$] of the fundamental domain $\mathfrak{D}$. Each tile is subdivided and painted in *gray* or *white* according as the pre-image satisfies $\operatorname{Re} z < 0$ or $\operatorname{Re} z > 0$ in the fundamental domain $\mathfrak{D}$. **c** Fundamental domain $\mathfrak{D}_6$ of $\Gamma_0(6)$, dissected with $SL(2,\mathbb{Z})$-tiles (cf. panel **b**). Gluing the three pairs of boundary sides of $\mathfrak{D}_6$ along the *arrows*, one obtains the moduli space $Y_0(6)(\mathbb{C}) = \Gamma_0(6)\backslash\mathfrak{H}$. **d**–**f** Fundamental domains $\mathfrak{D}_{6,k}$ for the Chan–Zudilin groups $\Gamma_0(6)_{+k} = \langle\Gamma_0(6), \hat{W}_k\rangle$, where $\hat{W}_2 z = (2z-1)/(6z-2)$, $\hat{W}_3 z = (3z-2)/(6z-3)$, and $\hat{W}_6 z = -1/(6z)$.

$$\int_0^\infty I_0(\sqrt{u}t)[K_0(t)]^4 t\mathrm{d}t$$

$$= \frac{1}{8}\int_0^\infty \frac{\mathrm{d}X}{X}\int_0^\infty \frac{\mathrm{d}Y}{Y}\int_0^\infty \frac{\mathrm{d}Z}{Z}\frac{1}{(1+X+Y+Z)\left(1+\frac{1}{X}+\frac{1}{Y}+\frac{1}{Z}\right)-u}. \tag{55}$$

As pointed out in Verrill's thesis [45, Theorems 1 and 2], the differential equation $\widetilde{L}_3 f(u) = 0$ (cf. Table 2) is the Picard–Fuchs equation for a pencil of $K3$ surfaces:

$$\mathscr{X}_{A_3} : (1+X+Y+Z)\left(1+\frac{1}{X}+\frac{1}{Y}+\frac{1}{Z}\right) = u, \tag{56}$$

whose monodromy group is isomorphic to $\overline{\Gamma_0(6)_{+3}}$, the image of $\Gamma_0(6)_{+3}$ after quotienting out by scalars. As a consequence, the general solutions to $\widetilde{L}_3 f(u) = 0$ admit a modular parametrization

$$f(u) = Z_{6,3}(z)(c_0 + c_1 z + c_2 z^2), \tag{57}$$

where $c_0, c_1, c_2 \in \mathbb{C}$ are constants, and

$$u = -64X_{6,3}(z) := -\left[\frac{2\eta(2z)\eta(6z)}{\eta(z)\eta(3z)}\right]^6, \tag{58}$$

$$Z_{6,3}(z) := \frac{[\eta(z)\eta(3z)]^4}{[\eta(2z)\eta(6z)]^2}. \tag{59}$$

Here, $X_{6,3}(z)$ is a modular function on $\Gamma_0(6)_{+3}$ [20, (2.2)], while $Z_{6,3}(z)$ is a modular form of weight 2 and level 6 [20, (2.5)].

Since $\int_0^\infty J_0(\sqrt{-u}t)I_0(t)[K_0(t)]^3 t\mathrm{d}t$ is annihilated by Vanhove's operator $\widetilde{L}_3$, we can establish the following modular parametrization

$$\int_0^\infty J_0\left(8\sqrt{X_{6,3}(z)}t\right)I_0(t)[K_0(t)]^3 t\mathrm{d}t = \frac{\pi^2}{16}Z_{6,3}(z) \tag{60}$$

through asymptotic analysis of both sides near the infinite cusp $z \to i\infty$ [whereupon the left-hand side tends to $\int_0^\infty I_0(t)[K_0(t)]^3 t\mathrm{d}t = \mathbf{IKM}(1,3;1)$ and the right-hand side tends to $\frac{\pi^2}{16} = \mathbf{IKM}(1,3;1)$]. Here, the positive $\mathrm{Im}\,z$-axis corresponds to $\sqrt{-u} = 8\sqrt{X_{6,3}(z)} \in (0,\infty)$. In a similar fashion, one can show that

$$\int_0^\infty J_0\left(8\sqrt{X_{6,3}(z)}t\right)[I_0(t)]^2[K_0(t)]^2 t\mathrm{d}t = \frac{\pi z}{4i}Z_{6,3}(z) \tag{61}$$

holds for $z/i > 0$. Now, we can prove Theorem 4(b) by throwing (60)–(61) into the Parseval–Plancherel identity for Hankel transforms [1, (16)]

$$\int_0^\infty f(t)g(t)t\mathrm{d}t = \int_0^\infty \left[\int_0^\infty J_0(xt)f(t)t\mathrm{d}t\right]\left[\int_0^\infty J_0(x\tau)g(\tau)\tau\mathrm{d}\tau\right]x\mathrm{d}x, \tag{62}$$

and noting that [50, Theorem 5.1.1]

$$[Z_{6,3}(z)]^2 \frac{\mathrm{d}X_{6,3}(z)}{\mathrm{d}z} = 2\pi i f_{6,6}(z). \tag{63}$$

Actually, we can say a little more about the 8-Bessel problem than what has been stated in Theorem 4(b). With heavy use of Wick rotations and integrations of $f_{6,6}(z)z^n, n \in \{0,1,2,3,4\}$ over the boundary $\partial \mathfrak{D}_{6,3}$ of the fundamental domain $\mathfrak{D}_{6,3}$ (Fig. 2*e*), one may show that [50, §5]

$$\frac{L(f_{6,6},5)}{L(f_{6,6},3)} = \frac{2\pi^2}{21}. \tag{64}$$

Comparing this to Theorem 4(b), one confirms a sum rule $9\pi^2 \mathbf{IKM}(4,4;1) = 14\mathbf{IKM}(2,6;1)$, which was originally proposed in 2008 [1, at the end of §6.3, between (228) and (229)].

3 Some linear sum rules of Feynman diagrams

The contour integral in (32) is no longer convergent when $n \in \mathbb{Z} \cap [m,\infty)$, so the methods in §2.1 do not directly apply to Theorem 1(b), which involves Bessel moments $\mathbf{IKM}(a,b;n)$ with high orders $n \geq (a+b-2)/2$. In [49, §3], I used a real-analytic approach (based on Hilbert transforms), to circumvent divergent contour integrals while handling Theorem 1(b). After email exchanges with Mark van Hoeij on Oct. 24, 2017, about the asymptotic expansion of $[I_0(x)K_0(x)]^4$ for large and positive x (see van Hoeij's update on [43], dated Oct. 23, 2017), I realized that the divergence problem in the complex-analytic approach can be amended by subtracting Laurent polynomials from $[H_0^{(1)}(z)H_0^{(2)}(z)]^m$. This amendment is described in the lemma below.

Lemma 2 (Asymptotic expansions and Bessel moments). *We have the following asymptotic expansion as* $|z| \to \infty, -\pi < \arg z < \pi$:

$$\begin{aligned}
&\left(\frac{\pi}{2}\right)^{2m} [H_0^{(1)}(z)H_0^{(2)}(z)]^m \\
&= \sum_{n=1}^{N} \frac{(-1)^{n+1}}{z^{2n+m-2}} \int_0^\infty \frac{[\pi I_0(t)+iK_0(t)]^m - [\pi I_0(t)-iK_0(t)]^m}{\pi i}[K_0(t)]^m t^{2n+m-3}\mathrm{d}t \\
&\quad + O\left(\frac{1}{|z|^{2N+m}}\right),
\end{aligned} \tag{65}$$

where $m,N \in \mathbb{Z}_{>0}$.

Proof. From (31), we know that as $|z| \to \infty, -\pi < \arg z < \pi$, there exist certain constant coefficients $a_{m,n}$ such that the following relation holds:

$$F_{m,N}(z) := [H_0^{(1)}(z)H_0^{(2)}(z)]^m - \sum_{n=1}^{N} \frac{a_{m,n}}{z^{2n+m-2}} = O\left(\frac{1}{|z|^{2N+m}}\right). \qquad (66)$$

To determine $a_{m,N}$, we consider

$$\lim_{\varepsilon \to 0^+} \lim_{T \to \infty} \left(\int_{-iT}^{-i\varepsilon} + \int_{C_\varepsilon} + \int_{i\varepsilon}^{iT} \right) F_{m,N}(z) z^{2N+m-3} \mathrm{d}z, \qquad (67)$$

where C_ε is a semi-circular arc in the right half-plane, joining $-i\varepsilon$ to $i\varepsilon$. For each fixed $\varepsilon > 0$, the contour integral in question tends to zero, as $T \to \infty$, because we can close the contour to the right. Recalling (33), and integrating the Laurent polynomial over C_ε, we arrive at the claimed result. $\quad\square$

Before moving onto the proof of Theorem 1(b) in the next proposition, we point out that one can also generalize the method in the last lemma into other cancelation formulae. For example, in [53, Lemma 3.3.1], we used a vanishing contour integral

$$\lim_{T \to \infty} \int_{-iT}^{iT} H_0^{(1)}(z)H_0^{(2)}(z) \left\{ [H_0^{(1)}(z)H_0^{(2)}(z)]^2 - \frac{4}{\pi^2 z^2} \right\} z^3 \mathrm{d}z = 0 \qquad (68)$$

to prove

$$\int_0^\infty I_0(t)[K_0(t)]^5 t^3 \mathrm{d}t = \frac{\pi^2}{3} \int_0^\infty I_0(t)K_0(t) \left\{ [I_0(t)]^2[K_0(t)]^2 - \frac{1}{4t^2} \right\} t^3 \mathrm{d}t, \qquad (69)$$

which paved way for the verification of a conjecture [53, (1.2.4)] due to Laporta [37, (29)] and Broadhurst (private communication on Nov. 10, 2017).

Thanks to van Hoeij's observation that led to Lemma 2, we see that the expression $C_{m,n}$ in (7) evaluates to a rational number [cf. (31)], and these sequences of rational numbers satisfy a discrete convolution relation with respect to the power m. To show that $C_{m,n}$ is in fact a positive integer, it now suffices to prove that, for each $\ell \in \mathbb{Z}_{>0}$,

$$C_{1,\ell} = \frac{[(2\ell-2)!]^3}{2^{2\ell-2}[(\ell-1)!]^4} = [(2\ell-3)!!]^2 \binom{2\ell-2}{\ell-1} \qquad (70)$$

and

$$C_{2,\ell} = \frac{1}{2^{4(\ell-1)}} \sum_{k=1}^{\ell} \frac{[(2\ell-2k)!]^3}{[(\ell-k)!]^4} \frac{[(2k-2)!]^3}{[(k-1)!]^4} \qquad (71)$$

are both integers. Here, we have $\binom{n}{k} = \frac{n!}{k!(n-k)!} \in \mathbb{Z}$ for $n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{Z} \cap [0,n]$, and $(2n-1)!! := (2n)!/(n!2^n) \in \mathbb{Z}$ for $n \in \mathbb{Z}_{\geq 0}$, so the statement $C_{1,\ell} \in \mathbb{Z}$ holds true. The integrality of $C_{2,\ell}$ will be explained below.

Proposition 2 (An integer sequence). *For each $\ell \in \mathbb{Z}_{>0}$, the number $\alpha_\ell := C_{2,\ell}$ is a positive integer.*

Proof. In [41, Theorem 3.1], Mathew Rogers has effectively shown that the following identity holds for $|u|$ sufficiently small:

$$\sum_{\ell=1}^{\infty} \frac{\alpha_{\ell+1} - \ell^2 \alpha_\ell}{(\ell!)^2} u^\ell = 3 \sum_{n=1}^{\infty} [(2n-1)!!]^2 \binom{3n-1}{2n} \frac{1}{(n!2^n)^2} \frac{u^{2n}}{(1-u)^{3n}}. \tag{72}$$

Comparing the coefficients of u^n on both sides, we see that, for each $n \in \mathbb{Z}_{>0}$, the expression $\alpha_{n+1} - n^2 \alpha_n$ equals a sum of finitely many terms, each of which is an integer multiple of $(k!!)^2 \in \mathbb{Z}$ for a certain odd positive integer k less than n. Therefore, we have $\alpha_1 = 1, \alpha_{\ell+1} - \ell^2 \alpha_\ell \in \mathbb{Z}$ for $\ell \in \mathbb{Z}_{>0}$, which entails the claimed result. $\square$

4 Some non-linear sum rules of Feynman diagrams

As we did in §3, we will build non-linear sum rules of Feynman diagrams without evaluating individual Bessel moments in closed form. In what follows, we describe a key step towards the proof of Broadhurst–Mellit determinant formulae (Theorem 2), namely, the asymptotic analysis of the Wrońskians $\Omega_{2k-1}(u)$ and $\omega_{2k}(u)$ introduced in §2.2.

As in [52, §4], we differentiate (46) with respect to u and define

$$\acute{\mu}_{k,j}^\ell(u) := 2\sqrt{u} D^1 \mu_{k,j}^\ell(u), \quad \forall j \in \mathbb{Z} \cap [1, 2k-1]. \tag{73}$$

Through iterated applications of the Bessel differential equations $(uD^2 + D^1)I_0(\sqrt{u}t) = \frac{t^2}{4}I_0(\sqrt{u}t)$ and $(uD^2 + D^1)K_0(\sqrt{u}t) = \frac{t^2}{4}K_0(\sqrt{u}t)$, we can verify

$$(2\sqrt{u})^{(k-1)(2k-1)} \Omega_{2k-1}(u) = \det \begin{pmatrix} \mu_{k,1}^1(u) & \cdots & \mu_{k,2k-1}^1(u) \\ \acute{\mu}_{k,1}^1(u) & \cdots & \acute{\mu}_{k,2k-1}^1(u) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \mu_{k,1}^k(u) & \cdots & \mu_{k,2k-1}^k(u) \end{pmatrix}, \tag{74}$$

where the μ (resp. $\acute{\mu}$) terms occupy the odd-numbered (resp. even-numbered) rows. Since $W[I_0(u), K_0(u)] = -I_0(u)K_1(u) - K_0(u)I_1(u) = -1/u$, we can show that

$$\begin{cases} \mu_{k,j}^\ell(1) = \mu_{k,k+j-1}^\ell(1), \\ \acute{\mu}_{k,k+j-1}^\ell(1) - \acute{\mu}_{k,j}^\ell(1) = -\mu_{k-1,j-1}^\ell(1) \end{cases} \tag{75}$$

for all $j \in \mathbb{Z} \cap [2, k]$. Thus, we obtain, after column eliminations and row bubble sorts,

$$2^{(k-1)(2k-1)}\Omega_{2k-1}(1)$$

$$= \det \begin{pmatrix} \mu_{k,1}^1(1) & \cdots & \mu_{k,k}^1(1) & 0 & \cdots & 0 \\ \acute{\mu}_{k,1}^1(1) & \cdots & \acute{\mu}_{k,k}^1(1) & -\mu_{k-1,1}^1(1) & \cdots & -\mu_{k-1,k-1}^1(1) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mu_{k,1}^k(1) & \cdots & \mu_{k,k}^k(1) & 0 & \cdots & 0 \end{pmatrix}$$

$$= (-1)^{\frac{k(k-1)}{2}} \det \left(\begin{array}{c|c} \mathbf{M}_k^{\mathrm{T}} & \mathbf{O} \\ \hline \begin{array}{ccc} \acute{\mu}_{k,1}^1(1) & \cdots & \acute{\mu}_{k,k}^1(1) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \acute{\mu}_{k,1}^{k-1}(1) & \cdots & \acute{\mu}_{k,k}^{k-1}(1) \end{array} & -\mathbf{M}_{k-1}^{\mathrm{T}} \end{array} \right), \tag{76}$$

which factorizes into

$$\Omega_{2k-1}(1) = (-1)^{\frac{(k-1)(k-2)}{2}} \frac{\det \mathbf{M}_{k-1}}{2^{(k-1)(2k-1)}} \det \mathbf{M}_k \tag{77}$$

for each $k \in \mathbb{Z}_{\geq 2}$. By a similar procedure (see [52, Proposition 4.4] for detailed asymptotic analysis), one can show that

$$\lim_{u\to 0^+} u^{k(2k-1)/2}\Omega_{2k-1}(u) = (-1)^{\frac{(k-1)(k-2)}{2}} \frac{k[\Gamma(k/2)]^2}{(2k+1)} \frac{(\det \mathbf{N}_{k-1})^2}{2^{(k-1)(2k-1)+1}}. \tag{78}$$

Consequently, the evolution equation in (51) admits a solution

$$\Omega_{2k-1}(u) = \frac{(-1)^{\frac{(k-1)(k-2)}{2}} k[\Gamma(k/2)]^2}{u^{k(2k-1)/2}(2k+1)} \frac{(\det \mathbf{N}_{k-1})^2}{2^{(k-1)(2k-1)+1}} \prod_{j=1}^{k} \left[\frac{(2j)^2}{(2j)^2 - u} \right]^{k-\frac{1}{2}} \tag{79}$$

for $u \in (0,1]$.

Comparing (77) and (79), we arrive at

$$\det \mathbf{M}_{k-1} \det \mathbf{M}_k = \frac{k[\Gamma(k/2)]^2 (\det \mathbf{N}_{k-1})^2}{2(2k+1)} \prod_{j=1}^{k} \left[\frac{(2j)^2}{(2j)^2 - 1} \right]^{k-\frac{1}{2}}, \tag{80}$$

for all $k \in \mathbb{Z}_{\geq 2}$. A similar service [52, §4.3] on $\omega_{2k}(u)$ then brings us

$$\det \mathbf{N}_{k-1} \det \mathbf{N}_k = \frac{2k+1}{k+1} \frac{(\det \mathbf{M}_k)^2}{(k-1)!} \prod_{j=2}^{k+1} \left[\frac{(2j-1)^2}{(2j-1)^2 - 1} \right]^{k}. \tag{81}$$

The last pair of equations, together with the initial conditions $\det \mathbf{M}_1 = \mathbf{IKM}(1,2;1) = \frac{\pi}{3\sqrt{3}}$ [1, (23)] and $\det \mathbf{N}_1 = \mathbf{IKM}(1,3;1) = \frac{\pi^2}{2^4}$ [1, (55)], allow us to prove Theorem 2 by induction.

As a by-product, we see from (79) that $\Omega_{2k-1}(u) = W[\mu^1_{k,1}(u),\ldots,\mu^1_{k,2k-1}(u)]$ is non-vanishing for $u \in (0,1]$. Therefore, the functions $\mu^1_{k,1}(u),\ldots,\mu^1_{k,2k-1}(u)$ (restricted to the interval $(0,1]$) form a basis for the kernel space of $\widetilde{L}_{2k-1}$. Consequently, for each $k \in \mathbb{Z}_{\geq 2}$, the function $p_{2k}(\sqrt{u})/\sqrt{u}, 0 < u \leq 1$ (where $p_{2k}(x) = \int_0^\infty J_0(xt)[J_0(t)]^{2k}xt\mathrm{d}t$ is Kluyver's probability density) is an $\mathbb{R}$-linear combination of the functions $\mu^1_{k,1}(u),\ldots,\mu^1_{k,2k-1}(u)$. Unlike our statement in Theorem 5, where the Bessel moment representation of $p_{2j+1}(x), j \in \mathbb{Z}_{\geq 2}$ leaves a convergent Taylor expansion for $0 \leq x \leq 1$, the representation of $p_{2k}(x), 0 \leq x \leq 1$ through a linear combination of Bessel moments may involve $O(x\log x)$ singularities in the $x \to 0^+$ regime, attributable to the Bessel function K_0. Such logarithmic singularities had been previously studied by Borwein–Straub–Wan–Zudilin [8].

5 Critical values of modular L-functions and multi-loop Feynman diagrams

As in the proof of Theorem 4(b) in §2.3, we need to fuse Hankel transforms in the Parseval–Plancherel identity to prove (22).

Fusing the following Hankel transform (cf. [50, (4.1.16)])

$$\int_0^\infty J_0\left(\frac{3[\eta(w)]^2[\eta(6w)]^4}{[\eta(3w)]^2[\eta(2w)]^4}it\right) I_0(t)[K_0(t)]^2 t\mathrm{d}t = \frac{\pi}{3\sqrt{3}}\frac{\eta(3w)[\eta(2w)]^6}{[\eta(w)]^3[\eta(6w)]^2} \quad (82)$$

$\left(\text{where } w = \frac{1}{2} + iy, y > 0 \text{ corresponds to } 0 < \frac{3[\eta(w)]^2[\eta(6w)]^4}{[\eta(3w)]^2[\eta(2w)]^4}i < \infty\right)$ with itself, we obtain (cf. [50, Proposition 4.2.1])

$$\mathbf{IKM}(2,4;1) = \frac{\pi^3 i}{3}\int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} f_{4,6}(w)\mathrm{d}w. \quad (83)$$

This is not quite the statement in (22) yet, as the integration path still sits on the "wrong" portion of $\partial\mathfrak{D}_{6,2}$ (Fig. 2d). To compensate for this, we need another Hankel fusion, together with some modular transforms on the Chan–Zudilin group $\Gamma_0(6)_{+2}$, to construct an identity [50, Proposition 4.2.2]:

$$\mathbf{JYM}(6,0;1) = \frac{12}{\pi i}\int_0^{i\infty} f_{4,6}(w)\mathrm{d}w - \frac{6}{\pi i}\int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} f_{4,6}(w)\mathrm{d}w. \quad (84)$$

Now that Wick rotation brings us $\mathbf{IKM}(2,4;1) = \frac{\pi^4}{30}\mathbf{JYM}(6,0;1)$ [50, (4.1.1)], we can deduce (22) from the last two displayed equations.

It takes slightly more effort to verify (21). Towards this end, we need a "Hilbert cancelation formula" [50, Lemma 4.2.4]

$$\int_0^\infty \left[\int_0^\infty J_0(xt)F(t)t\mathrm{d}t \right] \left[\int_0^\infty Y_0(x\tau)F(\tau)\tau\mathrm{d}\tau \right] x\mathrm{d}x = 0 \qquad (85)$$

for functions $F(t), t > 0$ satisfying certain growth bounds, along with modular parametrizations of some generalized Hankel transforms, such as (cf. [50, (4.1.31)])

$$
\begin{aligned}
&\int_0^\infty J_0 \left(\frac{3[\eta(w)]^2[\eta(6w)]^4}{[\eta(3w)]^2[\eta(2w)]^4} it \right) [K_0(t)]^3 t\mathrm{d}t \\
&\quad - \frac{3\pi}{2} \int_0^\infty Y_0 \left(\frac{3[\eta(w)]^2[\eta(6w)]^4}{[\eta(3w)]^2[\eta(2w)]^4} it \right) I_0(t)[K_0(t)]^2 t\mathrm{d}t \\
&= \frac{\pi^2(2w-1)}{2\sqrt{3}i} \frac{\eta(3w)[\eta(2w)]^6}{[\eta(w)]^3[\eta(6w)]^2} \qquad (86)
\end{aligned}
$$

for $w = \frac{1}{2} + iy, y > 0$. We refer our readers to [50, Proposition 4.1.3 and Theorem 4.2.5] for detailed computations that lead to (21).

6 Outlook

6.1 Broadhurst's p-adic heuristics

In §1.3, the modular forms $f_{3,15}$, $f_{4,6}$ and $f_{6,6}$ were not picked randomly, but were discovered by Broadhurst via some deep insights into p-adic analysis and étale cohomology [22, 33]. In short, Broadhurst's computations of Bessel moments at finite fields led him to local factors in the Hasse–Weil zeta functions, which piece together into the modular L-functions, namely, $L(f_{3,15}, s)$ for the 5-Bessel problem, $L(f_{4,6}, s)$ for the 6-Bessel problem, and $L(f_{6,6}, s)$ for the 8-Bessel problem.

On the arithmetic side, Broadhurst investigated Kloosterman moments ("Bessel moments at finite fields"), with extensive numerical experiments [11, §§2–6]. A Bessel function in a finite field [39], with respect to the variable $a \in \mathbb{F}_q = \mathbb{F}_{p^k}$, is defined by the following Kloosterman sum:

$$\mathrm{Kl}_2(\mathbb{F}_{p^k}, a) := \sum_{x_1, x_2 \in \mathbb{F}_q^\times, x_1 x_2 = a} e^{\frac{2\pi i}{p} \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x_1 + x_2)} = \sum_{x \in \mathbb{F}_q^\times} e^{\frac{2\pi i}{p} \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}\left(x + \frac{a}{x}\right)} \qquad (87)$$

where the Frobenius trace $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$ acts on an element $z \in \mathbb{F}_q$ as $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(z) := \sum_{j=0}^{k-1} z^{p^j}$. Writing $\mathrm{Kl}_2(\mathbb{F}_{p^k}, a) = -\alpha_a - \beta_a$ where $\alpha_a \beta_a = q$, and introducing the n-th symmetric power $\mathrm{Kl}_2^n := \mathrm{Sym}^n(\mathrm{Kl}_2)$ as $\mathrm{Kl}_2^n(\mathbb{F}_{p^k}, a) := \sum_{j=0}^n \alpha_a^j \beta_a^{n-j}$, we may further define Bessel moments in a finite field as the following Kloosterman moments:

$$S_n(q) := \sum_{a \in \mathbb{F}_q^\times} \mathrm{Kl}_2^n(\mathbb{F}_{p^k}, a) = \sum_{a \in \mathbb{F}_q^\times} \sum_{j=0}^n \alpha_a^j \beta_a^{n-j}. \qquad (88)$$

With $c_n(q) = -\frac{1+S_n(q)}{q^2}$ for a prime power $q = p^k$, one defines the Hasse–Weil local factor by a formula

$$Z_n(p,T) := \exp\left(-\sum_{k=0}^{\infty} \frac{c_n(p^k)}{k} T^k\right). \tag{89}$$

Following the notations of Fu–Wan [28], we set $L_p(\mathbb{P}^1_{\mathbb{F}_p} \smallsetminus \{0,\infty\}, \mathrm{Sym}^n(\mathrm{Kl}_2), s) = 1/Z_n(p,p^{-s})$, and define the Hasse–Weil zeta function

$$\zeta_{n,1}(s) := \prod_p L_p(\mathbb{P}^1_{\mathbb{F}_p} \smallsetminus \{0,\infty\}, \mathrm{Sym}^n(\mathrm{Kl}_2), s) = \prod_p \frac{1}{Z_n(p,p^{-s})}, \tag{90}$$

where the product runs over all the primes. It is known that $\zeta_{5,1}(s) = L(f_{3,15},s)$ [38] and $\zeta_{6,1}(s) = L(f_{4,6},s)$ [34]. The structure of $\zeta_{7,1}(s)$, which involves a Hecke eigenform of weight 3 and level 525, had been conjectured by Evans [25, Conjecture 1.1], before being completely verified by Yun [47, §4.7.7]. The story for the 8-Bessel problem is much more convoluted (see [47, Theorem 4.6.1 and Appendix B] as well as [11, §7.6]).

On the geometric side, Broadhurst's L-functions $L(f_{3,15},s)$ and $L(f_{4,6},s)$ are closely related to the étale cohomologies of certain Calabi–Yau manifolds. Concretely speaking, one may regard the 4-loop sunrise [quadruple integral in (2)] as a motivic integral over the Barth–Nieto quintic variety [2, 30, 34], which is defined through a complete intersection

$$N := \left\{[u_0 : u_1 : u_2 : u_3 : u_4 : u_5] \in \mathbb{P}^5 \,\middle|\, \sum_{k=0}^{5} u_k = \sum_{k=0}^{5} \frac{1}{u_k} = 0\right\}. \tag{91}$$

The projective variety N has a smooth Calabi–Yau model Y. Its third étale cohomology group $H^3_{\text{ét}}(Y)$ is related to 2-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ [34, §3], so that for each prime $p \geq 5$, one has $L_p(H^3_{\text{ét}}(Y),s) = [1 - a_p(Y)p^{-s} + p^{3-2s}]^{-1}$ for

$$a_p(Y) = \mathrm{tr}(\mathrm{Frob}_p^*, H^3_{\text{ét}}(Y)) = 1 + 50p + 50p^2 + p^3 - \#Y(\mathbb{F}_p), \tag{92}$$

where $\#Y(\mathbb{F}_p)$ counts the number of points within Y in the finite field $\mathbb{F}_p$. The modular L-function $L(f_{4,6},s)$ coincides with $L(H^3_{\text{ét}}(Y),s) = \prod_p L_p(H^3_{\text{ét}}(Y),s)$ for all the local factors $L_p(\cdot,s)$ corresponding to primes $p \geq 5$ and $\mathrm{Re}\, s$ sufficiently large. A similar p-adic reinterpretation for $L(f_{3,15},s)$ also exists. Let A_n be the Fourier coefficient in $f_{3,15}(z) = \sum_{n=1}^{\infty} A_n e^{2\pi i n z}$, and $\left(\frac{p}{3}\right)$ be the Legendre symbol for a prime p other than 3 and 5, then [38, Theorem 5.3]

$$1 + p^2 + p\left(16 + 4\left(\frac{p}{3}\right)\right) + A_p \tag{93}$$

counts the number of $\mathbb{F}_p$-rational points of a $K3$ surface that is the minimal resolution of singularities of

$$\left\{ [u_0 : u_1 : u_2 : u_3 : u_4] \in \mathbb{P}^4 \,\middle|\, \sum_{k=0}^{4} u_k = \sum_{k=0}^{4} \frac{1}{u_k} = 0 \right\}. \tag{94}$$

Behind the aforementioned results on p-adic Bessel moments is a long and heroic tradition of algebraic geometry. Back in the 1970s, building upon the theories of Dwork [24] and Grothendieck [32, 33], Deligne interpreted Hasse–Weil L-functions as Fredholm determinants of Frobenius maps [21, (1.5.4)]. This tradition has been continued by Robba [39], Fu–Wan [27, 29, 28] and Yun [47], in their studies of p-adic Bessel functions and Kloosterman sheaves.

While Broadhurst's p-adic heuristics give strong hints that $L(f_{3,15}, s)$, $L(f_{4,6}, s)$ and $L(f_{6,6}, s)$ are appropriate mathematical models for 5-, 6- and 8-Bessel problems, our proofs of Theorems 3 and 4 described in this survey do not touch upon the p-adic structure. It is perhaps worthwhile to rework these proofs from the Hasse–Weil perspective, using local-global correspondence. We call for this effort because there are still many conjectures of Broadhurst (see §6.2 for a partial list) that go beyond the reach of this survey, but might appear tractable to specialists in p-adic analysis and étale cohomology.

6.2 Open questions

There are three outstanding problems involving 5-, 6- and 8-Bessel factors, originally formulated by Broadhurst–Mellit [17, (4.3), (5.8), (7.15)] and Broadhurst [11, (101), (114), (160)].

Conjecture 1 (Broadhurst–Mellit). The following determinant formulae hold:

$$\det \begin{pmatrix} \mathbf{IKM}(0,5;1) & \mathbf{IKM}(0,5;3) \\ \mathbf{IKM}(2,3;1) & \mathbf{IKM}(2,3;3) \end{pmatrix} \overset{?}{=} \frac{45}{8\pi^2} L(f_{3,15}, 4), \tag{95}$$

$$\det \begin{pmatrix} \mathbf{IKM}(0,6;1) & \mathbf{IKM}(0,6;3) \\ \mathbf{IKM}(2,4;1) & \mathbf{IKM}(2,4;3) \end{pmatrix} \overset{?}{=} \frac{27}{4\pi^2} L(f_{4,6}, 5), \tag{96}$$

$$\det \begin{pmatrix} \mathbf{IKM}(0,8;1) & \mathbf{IKM}(0,8;3) - 2\,\mathbf{IKM}(0,8;5) \\ \mathbf{IKM}(2,6;1) & \mathbf{IKM}(2,6;3) - 2\,\mathbf{IKM}(2,6;5) \end{pmatrix} \overset{?}{=} \frac{6075}{128\pi^2} L(f_{6,6}, 7). \tag{97}$$

Here, one might wish to compare the last conjectural determinant evaluation to the following proven result:

$$\frac{5\pi^8}{2^{19}3} = \det \mathbf{N}_3 = \det \begin{pmatrix} \mathbf{IKM}(1,7;1) & \mathbf{IKM}(1,7;3) & \mathbf{IKM}(1,7;5) \\ \mathbf{IKM}(2,6;1) & \mathbf{IKM}(2,6;3) & \mathbf{IKM}(2,6;5) \\ \mathbf{IKM}(3,5;1) & \mathbf{IKM}(3,5;3) & \mathbf{IKM}(3,5;5) \end{pmatrix}$$

$$= \frac{\pi^2}{2^8} \det \begin{pmatrix} \mathbf{IKM}(1,7;1) & \mathbf{IKM}(1,7;3) - 2\,\mathbf{IKM}(1,7;5) \\ \mathbf{IKM}(2,6;1) & \mathbf{IKM}(2,6;3) - 2\,\mathbf{IKM}(2,6;5) \end{pmatrix}. \tag{98}$$

To arrive at the last step, we have used the Crandall number relations [Theorem 1(b)] $\mathbf{IKM}(3,5;1) - \frac{\mathbf{IKM}(1,7;1)}{\pi^2} = 0$, $\mathbf{IKM}(3,5;3) - \frac{\mathbf{IKM}(1,7;3)}{\pi^2} = \frac{\pi^2}{2^7}$, $\mathbf{IKM}(3,5;5) - \frac{\mathbf{IKM}(1,7;5)}{\pi^2} = \frac{\pi^2}{2^8}$, along with row and column eliminations.

The special L-values $L(f_{k,N}, s)$ in Conjecture 1 all lie outside the critical strip $0 \leq \operatorname{Re} s \leq k$, so they do not yield to the methods given in §2.3 or §5.

Working with Anton Mellit at Mainz, David Broadhurst has discovered a numerical connection (see [17, (6.8)] or [11, (129)]) between $\zeta_{7,1}(s) = \prod_p \frac{1}{Z_7(p,p^{-s})}$ and the 7-Bessel problem, which still awaits a proof.

Conjecture 2 (Broadhurst–Mellit). We have

$$\mathbf{IKM}(2,5;1) \overset{?}{=} \frac{5\pi^2}{24} \zeta_{7,1}(2). \tag{99}$$

In a recent collaboration with David Roberts [12, 13, 14, 15, 16], David Broadhurst has discovered a lot more empirical formulae relating determinants of Bessel moments to special values of Hasse–Weil L-functions, which are outside the scope of the current exposition. Nevertheless, we believe that one day such determinant formulae will reveal deep p-adic structures of Bessel moments, as foreshadowed by pioneering works on Hasse–Weil L-functions and Fredholm determinants for Frobenius maps [21, 39, 38, 34, 27, 29, 28, 47].

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