Combinatorial proofs and generalizations on conjectures related with Euler's partition theorem

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Abstract

Being aware of the recent work of Andrews, we notice two conjectures concerning with some variations of odd partitions and distinct partitions posed by Beck, which are analytically proved by Andrews. Later, following the same method of Andrews, Chern presented the analytic proof of another Beck's conjecture related the gap-free partitions and distinct partitions with odd length. However, the combinatorial interpretations of these conjectures are still unclear and required. In this paper, motivated by Glaisher's bijection, we give the combinatorial proofs of these three conjectures directly or by proving more generalized results.

Keywords: Glaisher's bijection, distinct partition, odd partition, gap-free partition

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1 Introduction

A *partition* [1] of n is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $n = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$. We write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and call λ_i's the *parts* of λ. If a part i appears m_i times for $i \geq 1$, we also write λ as $(1^{m_1}, 2^{m_2}, \ldots)$, where the superscript m_i is neglected provided $m_i = 1$. The *size* of λ is the sum of all parts, which is denoted by $|\lambda|$, and the *length* of λ is the number of parts, which is denoted by $\ell(\lambda)$. The *conjugate* of λ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{\lambda_1})$, where $\lambda'_i = |\{\lambda_j : \lambda_j \geq i, 1 \leq j \leq m\}|$ for $1 \leq i \leq \lambda_1$, or λ' can be equivalently expressed as $(1^{\lambda 1 - \lambda 2}, 2^{\lambda 2 - \lambda 3}, \ldots, \ell - 1^{\lambda_{\ell-1} - \lambda_\ell}, \ell^{\lambda_\ell})$. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is called a distinct partition if $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell$, and an odd partition if λ_i is odd for all $1 \leq i \leq \ell$, respectively. In 1748, by using generating functions, Euler [6] gave the celebrated partition theorem as follows.

Theorem 1.1 (Euler's partition theorem) *The number of distinct partitions of n equals the number of odd partitions of n.*

After Euler's partition theorem proposed, there are many extensions and refinements, the famous ones of which are Glaisher's theorem and Franklin's theorem, and the reader can refer [11, 12, 14] for more details. Glaisher [10] bijectively proved the following extension.

Theorem 1.2 (Glaisher's theorem) *For any positive integer $k \geq 1$, the number of partitions of n with no part occurring k or more times equals the number of partitions of n with no part divisible by k.*

In 1882, Franklin [8, 13] acquired a more generalized result by giving constructive proof of the following theorem. Franklin [13, p. 268] also asserted that the generating function is easily obtained.

Theorem 1.3 (Franklin's theorem) *For any positive integer $k \geq 1$ and nonnegative integer $m \geq 0$, the number partitions of n with m distinct parts each occurring k or more times equals the number of partitions of n with exactly m distinct parts divisible by k.*

Thus by taking $m = 0$, Franklin's theorem degenerates to Glaisher's theorem, then by taking $k = 2$, Glaisher's theorem gives Euler's partition theorem.

From the works of Andrews and Chern, we notice three conjectures posed by Beck concerning with some variations of odd partitions and distinct partitions, which are only analytically proved by Andrews [4] and Chern [5] via differentiation technique in q-series introduced by Andrews [4]. In this paper, by extending Glaisher's bijection, we give the combinatorial proofs of the three conjectures directly or by proving more generalized results. For the consistency of notations, we utilize the same notations in [4] and [5] in the rest of paper as far as possible.

Let $a(n)$ denote the number of partitions of n with only one even part which is possible repeated. Beck [15] proposed the following conjecture:

Conjecture 1.1 *$a(n)$ is also the difference between the number of parts in the odd partitions of n and the number of parts in the distinct partitions of n.*

Example 1.1 *For $n = 6$, the set of partitions of 6 such that the set of even parts has only one element is $\{(6), (4, 1, 1), (3, 2, 1), (2, 2, 2), (2, 2, 1, 1), (2, 1, 1, 1, 1)\}$, which is consist of 6 partitions. The odd partitions of 6 are $\{(5, 1), (3, 3), (3, 1, 1, 1), (1, 1, 1, 1, 1, 1)\}$ with the sum of lengths is 14, and the distinct partitions of 6 are $\{(6), (5, 1), (4, 2), (3, 2, 1)\}$ with the sum of lengths is 8. Thus the difference is $14 - 8 = 6$.*

Let $c(n)$ denote the number of partitions of n in which exactly one part is repeated. Let $b(n)$ be the difference between the number of parts in the odd partitions of n and the number of parts in the distinct partitions of n. Andrews [4] analytically proved the following theorem by differentiation technique in q-series, which confirms the conjecture posed by Beck [15]:

Theorem 1.4 *[4, Theorem 1.] For all $n \geq 1, a(n) = b(n) = c(n)$.*

Later Fu and Tang [9, Theorem 1.5] extended the the result of Andrews and gave the analytic proof.

Note that setting $k = 2$ and $m = 1$ in Theorem 1.3 already gives the proof of $a(n) = c(n)$. Hence, in order to complete the bijective proof of Conjecture 1.1 or Theorem 1.4, we give the combinatorial proof of $a(n) = b(n)$ by proving a more generalized theorem. For $k \geq 1$, let $\mathcal{O}_k(n)$ be the set of partitions of n with no part divisible by k and $\mathcal{D}_k(n)$ be the set of partitions of n with no part occurring k or more times, respectively. Let $\mathcal{O}_{1,k}(n)$ be the partitions of n with exact one part (possible repeated) divisible by k. We study the numerical relationship between the cardinality of $\mathcal{O}_{1,k}(n)$ and the difference between the number of parts in $\mathcal{O}_k(n)$ and the number of parts in $\mathcal{D}_k(n)$, where the case of $k = 2$ gives the combinatorial interpretation of $a(n) = b(n)$.

Let $a_1(n)$ denote the number of partitions of n such that there is exactly one part occurring three times while all other parts occur only once. Beck [15] made the following conjecture:

Conjecture 1.2 *$a_1(n)$ is also the difference between the number of parts in the distinct partitions of n and the number of distinct parts in the odd partitions of n.*

Example 1.2 *Let $n = 6$, then the corresponding set counted by $a_1(6)$ is $\{(3, 1, 1, 1,), (2, 2, 2)\}$. By Example 1.1, the number of parts in the distinct partitions of 6 is 8 and the number of distinct parts in the odd partitions of 6 is 6, which implies the difference is $8 - 6 = 2$.*

Let $b_1(n)$ be the difference between the total number of parts in the partitions of n into distinct parts and the total number of distinct parts in the partitions of n into odd parts. This conjecture was also proved by Andrews [4] with analytic method.

Theorem 1.5 *[4, Theorem 2] $a_1(n) = b_1(n)$.*

To verify Conjecture 1.2 combinatorially, we prove a extension of Theorem 1.5. Let $\mathcal{T}_k(n)$ be the set of partitions of n such that there is exactly one part occurring more than k times and less than $2k$ times while all other parts occur less than k times. We prove that the cardinality of set $\mathcal{T}_k(n)$ is equal to the excess of the number of distinct parts of partitions in $\mathcal{D}_k(n)$, over the number of distinct parts of partitions in $\mathcal{O}_k(n)$. Hence taking $k = 2$ confirms Conjecture 1.2.

A partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is called a *gap-free*, or *compact*, partition if $0 \leq \lambda_i - \lambda_{i+1} \leq 1$ for all $1 \leq i \leq \ell - 1$. Let $a_2(n)$ be the number of gap-free partitions of n and Andrews [3] gave the generating function of $a_2(n)$. Beck [16] proposed the following conjecture of $a_2(n)$:

Conjecture 1.3 *$a_2(n)$ is also the sum of the smallest parts in the distinct partitions of n with an odd number of parts.*

Example 1.3 *For $n = 6$, there are 7 gag-free partitions, which are $\{(6), (3, 3), (3, 2, 1), (2, 2, 2), (2, 2, 1, 1),$ $(2, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1)\}$. The set of distinct partitions of 6 with odd length is $\{(6), (3, 2, 1)\}$, where the sum of the smallest parts is $6 + 1 = 7$.*

Let $b_2(n)$ denote the sum of the smallest parts in the distinct partitions of n with an odd number of parts. Chern [5] proved this conjecture analytically by q-series based on method used by Andrews in [4].

Theorem 1.6 *[5, Theorem 1.2] For all $n \geq 1$, $a_2(n) = b_2(n)$.*

In this paper, we not only give the combinatorial proof of Conjecture 1.3, but also study the relationship between the gap-free partitions and the distinct partitions with even length, which leads us to rediscover a classical combinatorial identity found by Fokkink-Fokkink-Wang [7] combinatorially and proved by Andrews [2] analytically.

The rest of this paper is organized as follows. As our main tool to prove Conjecture 1.1 and 1.2, we introduce Glaisher's bijection detailedly in Section 2. In section 3, we give the combinatorial proof of Theorem 3.1, which is a extension of Conjecture 1.1. By proving a more generalized result (Theorem 4.1), we confirm Conjecture 1.2 in Section 4. The combinatorial proof of Conjecture 1.3 (Theorem 5.4) and a similar theorem (Theorem 5.5) connecting the gap-free partitions with the distinct partitions of even length are contained in Section 5.

2 Glaisher's bijection

In this section, we mainly recall Glaisher's bijection since it will be used frequently through this paper. Recall that for any positive integer $k \geq 1$, $\mathcal{O}_k(n)$ is the set of partitions of n with no part divisible by k, and $\mathcal{D}_k(n)$ is the set of partitions of n with no part occurring k or more times. Glaisher [10] gave the bijective proof of $|\mathcal{O}_k(n)| = |\mathcal{D}_k(n)|$.

Glaisher's bijection is defined as follows. and one can refer [10, 11, 12, 14] for more details. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition in $\mathcal{D}_k(n)$, then for $1 \leq i \leq \ell$, each part λ_i can be uniquely written as $k^{m_i} f_i$ with $k \nmid f_i$. Thus $\phi(\lambda)$ is established from λ by replacing the part λ_i by k^{m_i} parts f_i for $1 \leq i \leq \ell$. Since each f_i is not divisible by k, then we have $\phi(\lambda) \in \mathcal{O}_k(n)$.

Example 2.1 *If $k = 3$ and $\lambda = (2^2, 6, 8^2, 9, 12) \in \mathcal{D}_3(47)$, then $2 = 3^0 \cdot 2$, $6 = 3^1 \cdot 2$, $8 = 3^0 \cdot 8$, $9 = 3^2 \cdot 1$ and $12 = 3^1 \cdot 4$. Hence $\phi(\lambda) = (1^9, 2^5, 4^3, 8^2) \in \mathcal{O}_3(47)$.*

In another direction, given a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \in \mathcal{O}_k(n)$, assume that there are m_i parts i in μ for $1 \leq i \leq n$. Then for each $m_i \geq 1$, m_i can be uniquely expressed as

$$m_i = b_{i_1} k^{a_{i_1}} + b_{i_2} k^{a_{i_2}} + \cdots + b_{i_{p_i}} k^{a_{i_{p_i}}},$$

where $a_{i_1} > a_{i_2} > \cdots > a_{i_{p_i}} \geq 0$ and $1 \leq b_{i_j} \leq k-1$ for $1 \leq j \leq p_i$. Hence we can construct $\varphi(\mu)$ by μ via substituting m_i parts i by b_{i_1} parts $i \cdot k^{a_{i_1}}$, b_{i_2} parts $i \cdot k^{a_{i_2}}, \ldots, b_{i_{p_i}}$ parts $i \cdot k^{a_{i_{p_i}}}$ whenever $m_i \geq 1$ for $1 \leq i \leq n$. It is clear that $\varphi(\mu) \in \mathcal{O}_k(n)$ because $1 \leq b_{i_j} \leq k-1$ for $1 \leq i \leq n$ and

$1 \le j \le p_i$, and $i \cdot k^r = j \cdot k^s$ if and only if both $i = j$ and $r = s$ hold since i and j are not divisible by k. Actually, we can describe the process of producing $\varphi(\mu)$ more simply. It is easy to check that $\varphi(\mu)$ is obtained from μ by keeping gluing k same parts into one part until there is no part occurring at least k times.

Example 2.2 *If $k = 3$ and $\mu = (1^9, 2^5, 4^3, 8^2) \in \mathcal{O}_3(47)$, then $9 = 1 \cdot 3^2$, $5 = 1 \cdot 3^1 + 2 \cdot 3^0$, $3 = 1 \cdot 3^1$ and $2 = 2 \cdot 3^0$. Thus $\varphi(\mu) = (2^2, 6, 8^2, 9, 12) \in \mathcal{D}_3(47)$.*

Therefore, the maps $\phi\colon \mathcal{D}_k \to \mathcal{O}_k$ and $\varphi\colon \mathcal{O}_k \to \mathcal{D}_k$ are well defined bijections inverse to each other: $\varphi = \phi^{-1}$, which gives Glaisher's bijection. In the rest of paper, according to the specific circumstances, we are free to choose Glaisher's bijection as ϕ mapping $\mathcal{D}_k(n)$ to $\mathcal{O}_k(n)$ or φ mapping $\mathcal{O}_k(n)$ to $\mathcal{D}_k(n)$.

3 Combinatorial proof of Conjecture 1.1

Recall that $\mathcal{O}_{1,k}(n)$ is the set of partitions of n containing exactly one part divisible by k which is possible repeated. To verify Conjecture 1.1, we prove the following generalized theorem combinatorially.

Theorem 3.1 *For $k \ge 2$ and $n \ge 0$, we have*

$$\left|\mathcal{O}_{1,k}(n)\right| = \frac{1}{k-1} \cdot \left(\sum_{\lambda \in \mathcal{O}_k(n)} \ell(\lambda) - \sum_{\lambda \in \mathcal{D}_k(n)} \ell(\lambda) \right),$$

where $\ell(\lambda)$ is the number of parts in λ.

Thus, letting $k = 2$ in Theorem 3.1 reduces the set $\mathcal{O}_{1,k}(n)$ to the set of partition of n with only one even part which is possible repeated, and the set $\mathcal{O}_k(n)$ (resp. $\mathcal{D}_k(n)$) to the set of odd (resp. distinct) partitions of n, which gives the combinatorial proof of Conjecture 1.1.

In this section, we use Glaisher's bijection $\varphi\colon \mathcal{O}_k(n) \to \mathcal{D}_k(n)$. First we introduce some notations. Let m be a positive integer, then denote by $p(m)$ the sum of nonzero digits in the k-adic representation of m and $a(m)$ the highest digit in the k-adic representation of m. Precisely, m can be uniquely written as $b_1 k^{a_1} + b_2 k^{a_2} + \cdots + b_p k^{a_p}$ with $a_1 > a_2 > \cdots > a_p \ge 0$ and $1 \le b_i \le k-1$ for $1 \le i \le p$, then $p(m) = \sum_{j=1}^{p} b_j$ and $a(m) = a_1$.

Lemma 3.2 *Let $\lambda = (1^{m_1}, 2^{m_2}, \ldots) \in \mathcal{O}_k(n)$, we have*

$$\ell(\lambda) - \ell(\varphi(\lambda)) = \sum_{m_i \neq 0} (m_i - p(m_i)). \tag{3.1}$$

Proof. Assume that λ contains m_i parts i with $m_i \neq 0$, then by the construction of Glaisher's bijection $\varphi\colon \mathcal{O}_k(n) \to \mathcal{D}_k(n)$, for each $m_i \neq 0$, we rewrite m_i as $b_{i_1}k^{a_{i_1}} + b_{i_2}k^{a_{i_2}} + \cdots + b_{i_{p_i}}k^{a_{ip_i}}$ and replace m_i parts i by b_{i_1} parts $i \cdot k^{a_{i_1}}$, b_{i_2} parts $i \cdot k^{a_{i_2}}$, ..., $b_{i_{p_i}}$ parts $i \cdot k^{a_{ip_i}}$. Therefore the number of parts of $\varphi(\lambda)$ is decreased to $\sum_{m_i \neq 0} p(m_i)$, which implies (3.1) since the number of parts of λ is $\sum_{m_i \neq 0} m_i$. ∎

Next we will construct a series of subsets of $\mathcal{O}_{1,k}(n)$ by $\lambda \in \mathcal{O}_k(n)$. Let $\lambda = (1^{m_1}, 2^{m_2}, \ldots) \in \mathcal{O}_k(n)$, for each $m_i \geq k$, we establish a set of partitions $\pi^i_{j,r}$ from λ by replacing $r \cdot k^j$ parts i by r parts $i \cdot k^j$, where $1 \leq j \leq a(m_i)$ and $1 \leq r \leq \lfloor m_i/k^j \rfloor$. Define

$$\mathcal{O}_{\lambda,k,i} = \left\{ \pi^i_{j,r}\colon\ 1 \leq r \leq \lfloor m_i/k^j \rfloor \text{ and } 1 \leq j \leq a(m_i) \right\}.$$

Example 3.1 *Let $k = 3$ and $\lambda = (1^{11}, 4^2, 5^7) \in \mathcal{O}_3(47)$, then it follows that $m_1 = 11$ and $m_5 = 7$. For $m_1 = 11$ and $a(m_1) = 2$, we have $\pi^1_{1,1} = (1^8, 3, 4^2, 5^7)$, $\pi^1_{1,2} = (1^5, 3^2, 4^2, 5^7)$, $\pi^1_{1,3} = (1^2, 3^3, 4^2, 5^7)$, and $\pi^1_{2,1} = (1^2, 4^2, 5^7, 9)$. For $m_5 = 7$ and $a(m_5) = 1$, we have $\pi^5_{1,1} = (1^{11}, 4^2, 5^4, 15)$, and $\pi^5_{1,2} = (1^{11}, 4^2, 5, 15^2)$. Therefore, we conclude that $\mathcal{O}_{\lambda,3,1} = \{(1^8, 3, 4^2, 5^7), (1^5, 3^2, 4^2, 5^7), (1^2, 3^3, 4^2, 5^7), (1^2, 4^2, 5^7, 9)\}$ and $\mathcal{O}_{\lambda,3,5} = \{(1^{11}, 4^2, 5^4, 15), (1^{11}, 4^2, 5, 15^2)\}$.*

From the above example, we find that $|\mathcal{O}_{\lambda,3,1}| = 4$ and $|\mathcal{O}_{\lambda,3,5}| = 2$, which are equal to $(m_1 - p(m_1))/2 = (11-3)/2 = 4$, $(m_5 - p(m_5))/2 = (7-3)/2 = 2$, respectively.

Lemma 3.3 *Let $\lambda = (1^{m_1}, 2^{m_2}, \ldots) \in \mathcal{O}_k(n)$, then for each $m_i \geq k$, we have $\mathcal{O}_{\lambda,k,i} \subseteq \mathcal{O}_{1,k}(n)$ and*

$$\left|\mathcal{O}_{\lambda,k,i}\right| = \frac{m_i - p(m_i)}{k-1}.$$

Proof. By the construction of $\mathcal{O}_{\lambda,k,i}$ for $1 \leq j \leq a(m_i)$ and $1 \leq r \leq \lfloor m_i/k^j \rfloor$, we know that there exists only one part $i \cdot k^j$ divisible by k which occurs r times in $\pi^i_{j,r}$. Moreover, substituting $r \cdot k^j$ parts i by r parts $i \cdot k^j$ preserves the size of partition. Thus we have $\pi^i_{j,r} \in \mathcal{O}_{1,k}(n)$, which leads to $\mathcal{O}_{\lambda,k,i} \subseteq \mathcal{O}_{1,k}(n)$ for each $m_i \geq k$.

For the cardinality of $\mathcal{O}_{\lambda,k,i}$, we may compute directly as follows. Assume that

$$m_i = b_{i_1}k^{a_{i_1}} + b_{i_2}k^{a_{i_2}} + \cdots + b_{i_{p_i}}k^{a_{ip_i}},$$

where $a_{i_1} > a_{i_2} > \cdots > a_{i_{p_i}} \geq 0$, then $p(m_i) = \sum_{j=1}^{p_i} b_{i_j}$ and $a(m_i) = a_{i_1}$. Thus we see that

$$\begin{aligned}
\left|\mathcal{O}_{\lambda,k,i}\right| &= \left\lfloor \frac{m_i}{k} \right\rfloor + \left\lfloor \frac{m_i}{k^2} \right\rfloor + \cdots + \left\lfloor \frac{m_i}{k^{a_{i_1}}} \right\rfloor \\
&= \left\lfloor b_{i_1}k^{a_{i_1}-1} + b_{i_2}k^{a_{i_2}-1} + \cdots + b_{i_{p_i}}k^{a_{ip_i}-1} \right\rfloor + \left\lfloor b_{i_1}k^{a_{i_1}-2} + b_{i_2}k^{a_{i_2}-2} + \cdots + b_{i_p}k^{a_{ip_i}-2} \right\rfloor + \\
&\qquad \cdots + \left\lfloor b_{i_1} + b_{i_2}k^{a_{i_2}-a_{i_1}} + \cdots + b_{i_{p_i}}k^{a_{ip_i}-a_{i_1}} \right\rfloor \\
&= \left(b_{i_1}k^{a_{i_1}-1} + b_{i_1}k^{a_{i_1}-2} + \cdots + b_{i_1} \right) + \left(b_{i_2}k^{a_{i_2}-1} + b_{i_2}k^{a_{i_2}-2} + \cdots + b_{i_2} \right) +
\end{aligned}$$

$$
\begin{aligned}
&\cdots + \left(b_{i_{p_i}} k^{a_{i p_i}-1} + b_{i_{p_i}} k^{a_{i p_i}-2} + \cdots + b_{i_{p_i}}\right) \\
&= b_{i_1} \cdot \frac{k^{a_{i_1}} - 1}{k-1} + b_{i_2} \cdot \frac{k^{a_{i_2}} - 1}{k-1} + \cdots + b_{i_{p_i}} \cdot \frac{k^{a_{i p_i}} - 1}{k-1} \\
&= \frac{1}{k-1} \cdot \left(b_{i_1} k^{a_{i_1}} + b_{i_2} k^{a_{i_2}} + \cdots + b_{i_{p_i}} k^{a_{i p_i}} - (b_{i_1} + b_{i_2} + \cdots + b_{i_{p_i}})\right) \\
&= \frac{m_i - p(m_i)}{k-1},
\end{aligned}
$$

which completes the proof. ∎

Theorem 3.4 *For $n \geq 1$, we have*

$$
\mathcal{O}_{1,k}(n) = \bigcup_{\lambda \in \mathcal{O}_k(n)} \left(\bigcup_{m_i \geq k} \mathcal{O}_{\lambda,k,i} \right),
$$

where the sets $\mathcal{O}_{\lambda,k,i}$'s are pairwise disjoint.

By Theorem 3.4 with Lemma 3.2 and Lemma 3.3, we can easily give the proof of Theorem 3.1.

Proof of Theorem 3.1. We have

$$
\begin{aligned}
\left|\mathcal{O}_{1,k}(n)\right| &= \left| \bigcup_{\lambda \in \mathcal{O}_k(n)} \left(\bigcup_{m_i \geq k} \mathcal{O}_{\lambda,k,i} \right) \right| = \sum_{\lambda \in \mathcal{O}_k(n)} \left(\sum_{m_i \geq k} \left|\mathcal{O}_{\lambda,k,i}\right| \right) \\
&= \sum_{\lambda \in \mathcal{O}_k(n)} \sum_{m_i \geq k} \frac{m_i - p(m_i)}{k-1} = \frac{1}{k-1} \cdot \left(\sum_{\lambda \in \mathcal{O}_k(n)} \ell(\lambda) - \ell(\varphi(\lambda)) \right) \\
&= \frac{1}{k-1} \cdot \left(\sum_{\lambda \in \mathcal{O}(n)} \ell(\lambda) - \sum_{\lambda \in \mathcal{D}(n)} \ell(\lambda) \right),
\end{aligned}
$$

where the last but one equation is due to the fact that $p(m_i) = m_i$ if $m_i < k$, and the last equation is due to Glaisher's bijection. ∎

Proof of Theorem 3.4. By Lemma 3.3, it is obvious that $\bigcup_{\lambda \in \mathcal{O}_k(n)} (\bigcup_{m_i \geq k} \mathcal{O}_{\lambda,k,i}) \subseteq \mathcal{O}_{1,k}(n)$. Hence to prove the theorem, we only need to show that for each $\pi \in \mathcal{O}_{1,k}(n)$, there exists an unique pair (λ, i) satisfying $\pi \in \mathcal{O}_{\lambda,k,i}$, where $\lambda \in \mathcal{O}_k(n)$ contains m_i parts i with $m_i \geq k$. Fix a partition $\pi \in \mathcal{O}_{1,k}(n)$ and suppose that the only part divisible by k in π is a, we write a as $a = k^j \cdot i$, where $i \nmid k$ and $j \geq 1$. Let λ be the partition constructed from π by replacing every part a by k^j parts i, thus we obtain $\lambda \in \mathcal{O}_k(n)$. Noting that $k^j \geq k$ since $j \geq 1$, hence λ contains at least k parts i, implying $m_i \geq k$. Therefore we find the unique required pair (λ, i) such that $\pi \in \mathcal{O}_{\lambda,k,i}$, which completes the proof. ∎

$\lambda \in \mathcal{O}_3(9)$	$\varphi(\lambda) \in \mathcal{D}_3(9)$	$\mathcal{O}_{\lambda,3,i} \subseteq \mathcal{O}_{1,3}(9)$
(1^9)	(9)	$\{(1^6,3),(1^3,3^2,3^3),(9)\}$
$(1^7,2)$	$(1,2,3^2)$	$\{(1^4,2,3),(1,2,3^2)\}$
$(1^5,2^2)$	$(1^2,2^2,3)$	$\{(1^2,2^2,3)\}$
$(1^3,2^3)$	$(3,6)$	$\{(2^3,3)\}$, $\{(1^3,6)\}$
$(1,2^4)$	$(1,2,6)$	$\{(1,2,6)\}$
$(1^5,4)$	$(1^2,3,4)$	$\{(1^2,3,4)\}$
$(1^3,2,4)$	$(2,3,4)$	$\{(2,3,4)\}$
$(1^4,5)$	$(1,3,5)$	$\{(1,3,5)\}$

Table 3.1: the correspondence among the subset of $\mathcal{O}_3(9)$, the subset of $\mathcal{D}_3(9)$ and $\mathcal{O}_{1,3}(9)$

Example 3.2 *In Table 3.1, we give the correspondence among $\mathcal{O}_3(9)$, $\mathcal{D}_3(9)$ and $\mathcal{O}_{1,3}(9)$, where each row contains a partition $\lambda \in \mathcal{O}_3(9)$, the corresponding partition $\varphi(\lambda) \in \mathcal{D}_3(9)$, and the corresponding sets $\mathcal{O}_{\lambda,3,i}$'s arranged in the increasing order of i. Note that for the brevity, we just list the pair $(\lambda, \varphi(\lambda))$ whose the difference of the number of parts are nonzero.*

Remark 3.1 *Let $\mathcal{D}_{1,k}$ denote the set of partitions of n with exactly one part occurring at least k times. Let $E_k(n)$ denote the difference between the number of parts congruent to 1 modulo k in partitions of $\mathcal{O}_k(n)$ and the number of distinct parts of partitions in $\mathcal{D}_k(n)$. Fu and Tang [9, Theorem 1.5] analytically obtained that for $n \geq 0$ and $k \geq 2$, $\mathcal{O}_{1,k}(n) = \mathcal{D}_{1,k}(n) = E_k(n)$, which remains the combinatorial proof still unsolved. By letting $k = 2$, this theorem implicates the validity of Conjecture 1.1, where the part of $\mathcal{O}_{1,k}(n) = \mathcal{D}_{1,k}(n)$ is ensured by Franklin's theorem.*

4 Combinatorial proof of Conjecture 1.2

Denote by $\mathcal{T}_k(n)$ the set of partitions of n such that there is exactly one part occurring more than k times and less than $2k$ times while all other parts occur less than k times. Motivated by Glaisher's bijection, in stead of proving Conjecture 1.2 directly, we prove a more generalized theorem as follows.

Theorem 4.1 *Let $k \geq 2$ be any positive integer, then for $n \geq 0$, we have*

$$|\mathcal{T}_k(n)| = \sum_{\lambda \in \mathcal{D}_k(n)} \bar{\ell}(\lambda) - \sum_{\lambda \in \mathcal{O}_k(n)} \bar{\ell}(\lambda),$$

where $\bar{\ell}(\lambda)$ is the number of distinct parts in λ.

Thus by taking $k = 2$, $\mathcal{T}_2(n)$ becomes the set of partitions of n with exactly one part occurring three times and other parts occurring only once while $\mathcal{D}_2(n)$ and $\mathcal{O}_2(n)$ become the set of distinct and odd partitions of n, respectively, which gives a positive answer to Conjecture 1.2.

To prove Theorem 4.1, we first recall Glaisher's bijection $\phi\colon \mathcal{D}_k(n) \to \mathcal{O}_k(n)$ in Section 2. Let λ be a partition in $\mathcal{D}_k(n)$, then for $1 \le i \le \ell(\lambda)$, each part λ_i can be uniquely written as $k^{m_i} f_i$ with $k \nmid f_i$. Thus $\phi(\lambda) \in \mathcal{O}_k(n)$ is consist of k^{m_i} parts of f_i, where i ranges from 1 to $\ell(\lambda)$. Define $f_k(n)$ be the largest factor of n which is not divisible by k. For any positive integer d with $k \nmid d$, define $F_{k,d} = \{n \ge 1\colon\ f_k(n) = d\}$. Therefore, by the construction of ϕ, we easily obtain the following lemma.

Lemma 4.2 *For any $\lambda \in \mathcal{D}_k(n)$, we have*

$$\bar{\ell}(\lambda) - \bar{\ell}(\phi(\lambda)) = \sum_{\substack{k \nmid d \\ F_{k,d} \cap \lambda \neq \emptyset}} \left(\left|F_{k,d} \cap \lambda\right| - 1\right), \tag{4.1}$$

here we view λ as a multiset.

Proof. Let $\lambda \in \mathcal{D}_k(n)$, then we know that $\bar{\ell}(\lambda) = \sum_{k \nmid d} |F_{k,d} \cap \lambda|$ and $\bar{\ell}(\phi(\lambda)) = \sum_{k \nmid d} |F_{k,d} \cap \phi(\lambda)|$ since $\{F_{k,d}\colon\ k \nmid d\}$ is a partition of the positive integer set. By the definition of Glaisher's bijection ϕ, $|F_{k,d} \cap \phi(\lambda)| \neq \emptyset$ if and only if $|F_{k,d} \cap \lambda| \neq \emptyset$ for any $k \nmid d$. Thus, the difference between $\bar{\ell}(\lambda)$ and $\bar{\ell}(\phi(\lambda))$ is the sum of the differences between $|F_{k,d} \cap \lambda|$ and $|F_{k,d} \cap (\phi(\lambda))|$ for any $|F_{k,d} \cap \lambda| \neq \emptyset$. Therefore (4.1) holds since $F_{k,d} \cap \phi(\lambda) \in \{\{d\}, \emptyset\}$ for all $k \nmid d$ by Glaisher's bijection. ∎

Example 4.1 *Let $k = 3$ and $\lambda = (2, 3^2, 4^2, 6^2, 12, 18) \in \mathcal{D}_3(58)$, then we see that $|F_{3,1} \cap \lambda| = |\{3\}| = 1$, $|F_{3,2} \cap \lambda| = |\{2, 6, 18\}| = 3$, $|F_{3,4} \cap \lambda| = |\{4, 12\}| = 2$, Thus $\bar{\ell}(\lambda) - \bar{\ell}(\phi(\lambda)) = (1-1) + (3-1) + (2-1) = 3$.*

Let $\lambda \in \mathcal{D}_k(n)$, if $|F_{k,d} \cap \lambda| \ge 2$ for some $k \nmid d$, we construct $\mathcal{T}_{\lambda,k,d} \subseteq \mathcal{T}_k(n)$ of size $|F_{k,d} \cap \lambda| - 1$. Assuming $|F_{k,d} \cap \lambda| = p \ge 2$, then λ contains

$$\Big\{ \underbrace{k^{a_1} d, k^{a_1} d, \ldots, k^{a_1} d}_{\le k-1}, \underbrace{k^{a_2} d, k^{a_2} d, \ldots, k^{a_2} d}_{\le k-1}, \ldots, \underbrace{k^{a_p} d, k^{a_p} d, \ldots, k^{a_p} d}_{\le k-1} \Big\},$$

as parts with $0 \le a_1 < a_2 < \cdots < a_p$. For $1 \le i \le p-1$, we construct τ_d^i from λ by substituting one part $k^{a_{i+1}} d$ in λ by k parts $k^{a_i} d$, $k-1$ parts $k^{a_i+1} d$, $k-1$ parts $k^{a_i+2} d$, $\ldots$, $k-1$ parts $k^{a_{i+1}-1} d$. Since

$$k \cdot k^{a_i} d + (k-1) \cdot k^{a_i+1} d + (k-1) \cdot k^{a_i+2} + \cdots + (k-1) \cdot k^{a_{i+1}-1} d = k^{a_{i+1}} d,$$

it follows that $|\tau_d^i| = |\lambda|$. Moreover, the number of parts $k^{a_i} d$ in τ_d^i is between $k+1$ and $2k-1$ while other parts in τ_d^i occur less than k times, Therefore $\tau_d^i \in \mathcal{T}_k(n)$.

Example 4.2 *From Example 4.1, we have* $|F_{3,2}\cap\lambda| = 3$ *and* $|F_{3,4}\cap\lambda| = 2$*, then* $\tau_2^1 = (2^4, 3^2, 4^2, 6, 12, 18)$, $\tau_2^2 = (2, 3^2, 4^2, 6^5, 12)$ *and* $\tau_4^1 = (2, 3^2, 4^5, 6^2, 18)$ *belong to* $\mathcal{T}_k(58)$.

Theorem 4.3 *For* $k \geq 2$ *and* $\lambda \in \mathcal{D}_k(n)$ *with* $|F_{k,d} \cap \lambda| = p \geq 2$*, define*

$$\mathcal{T}_{\lambda,k,d} = \left\{\tau_d^1, \tau_d^2, \ldots, \tau_d^{p-1}\right\}.$$

Then the sets $\mathcal{T}_{\lambda,k,d}$*'s are pairwise disjoint and*

$$\mathcal{T}_k(n) = \bigcup_{\lambda\in\mathcal{D}_k(n)} \Bigg(\bigcup_{\substack{k\nmid d \\ |F_{k,d}\cap\lambda|\geq 2}} \mathcal{T}_{\lambda,k,d}\Bigg).$$

Therefore by the combination of Theorem 4.3 and Lemma 4.2, we can give the proof of Theorem 4.1.

Proof of Theorem 4.1. We obtain that

$$\begin{aligned}|\mathcal{T}_k(n)| &= \sum_{\lambda\in\mathcal{D}_k(n)} \sum_{\substack{k\nmid d \\ |F_{k,d}\cap\lambda|\geq 2}} \left|\mathcal{T}_{\lambda,k,d}(n)\right| = \sum_{\lambda\in\mathcal{D}_k(n)} \sum_{\substack{k\nmid d \\ F_{k,d}\cap\lambda\neq\emptyset}} \left(\left|F_{k,d}\cap\lambda\right| - 1\right) \\ &= \sum_{\lambda\in\mathcal{D}_k(n)} \left(\bar{\ell}(\lambda) - \bar{\ell}(\phi(\lambda))\right) = \sum_{\lambda\in\mathcal{D}_k(n)} \bar{\ell}(\lambda) - \sum_{\lambda\in\mathcal{O}_k(n)} \bar{\ell}(\lambda),\end{aligned}$$

where the last equality is due to Glaisher's bijection. ∎

Proof of Theorem 4.3. By the construction of $\mathcal{T}_{\lambda,k,d}$, it is clear that fix any $\lambda \in \mathcal{D}_k(n)$, then $\mathcal{T}_{\lambda,k,d_1} \cap \mathcal{T}_{\lambda,k,d_2} = \emptyset$ if $d_1 \neq d_2$ with $|F_{k,d_1} \cap \lambda|, |F_{k,d_1} \cap \lambda| \geq 2$ since $F_{k,d_1} \cap F_{k,d_2} = \emptyset$. Hence to complete the proof, we only need to show that for arbitrary partition $\tau \in \mathcal{T}_k(n)$, there must exists one and only one partition $\lambda \in \mathcal{D}_k(n)$ such that $\tau \in \mathcal{T}_{\lambda,k,d}$ for some $k \nmid d$.

Assume $\tau \in \mathcal{T}_k(n)$ and there is only one part $k^a d$ in λ for some $a \geq 0$ and $k \nmid d$, the time of occurrence of which is more than k and less than $2k$. We replace k parts $k^a d$ by one part $k^{a+1}d$, which preserves the size of partition and reduces the number of parts $k^a d$ to at most $k-1$. If it cause the number of parts $k^{a+1}d$ increased to k, which is possible, then we continue to replace k parts $k^{a+1}d$ by one part $k^{a+2}d$. Thus as long as the number of parts $k^{a+i}d$ is k, we replace these k parts $k^{a+i}d$ by one part $k^{a+i+1}d$. This process would stop at $a+m$ for some $m \geq 1$ since the number of parts of λ is finite. Then we obtain the desired $\lambda \in \mathcal{D}_k(n)$ since the number of parts $k^{a+i}d$ is less than k for all $0 \leq i \leq m$ and other parts of τ whose time of occurrence is less than k originally are unchanged. Note that $\{k^a d, k^{a+1}d, \ldots, k^{a+m}d\} \subseteq F_{k,d} \cap \lambda$, which implies $|F_{k,d} \cap \lambda| \geq m+1 \geq 2$. Therefore we deduce that $\tau \in \mathcal{T}_{\lambda,k,d}$. For example, let $k = 3$ and $\tau = (1^2, 2^4, 4) \in \mathcal{T}_3(14)$, then we find that the part 2 occurring four times, which gives us $\lambda = (1^2, 2, 4, 6) \in \mathcal{D}_3(14)$ by substituting three parts 2 by one part 6, hence $\tau \in \mathcal{T}_{\lambda,3,2}$. This completes the proof. ∎

$\lambda \in \mathcal{D}_3(12)$	$\phi(\lambda) \in \mathcal{O}_3(12)$	$\mathcal{T}_{\lambda,3,d} \subseteq \mathcal{T}_3(12)$
$(1^2, 2^2, 3^2)$	$(1^8, 2^2)$	$\{(1^5, 2, 3)\}$
$(1, 2^2, 3, 4)$	$(1^4, 2^2, 4)$	$\{(1^4, 2^2, 4)\}$
$(1^2, 3^2, 4)$	$(1^8, 4)$	$\{(1^5, 3, 4)\}$
$(1, 3, 4^2)$	$(1^4, 4^2)$	$\{(1^4, 4^2)\}$
$(1^2, 2, 3, 5)$	$(1^5, 2, 5)$	$\{(1^5, 2, 5)\}$
$(1, 3^2, 5)$	$(1^7, 5)$	$\{(1^4, 3, 5)\}$
$(1^2, 2^2, 6)$	$(1^2, 2^5)$	$\{(1^2, 2^5)\}$
$(1, 2, 3, 6)$	$(1^4, 2^4)$	$\{(1^4, 2, 6)\}, \{(1, 2^4, 3)\}$
$(2, 4, 6)$	$(2^4, 4)$	$\{(2^4, 4)\}$
$(1^2, 3, 7)$	$(1^5, 7)$	$\{(1^5, 7)\}$
$(1, 3, 8)$	$(1^4, 8)$	$\{(1^4, 8)\}$
$(1, 2, 9)$	$(1^{10}, 2)$	$\{(1^4, 2, 3^2)\}$
$(3, 9)$	(1^{12})	$\{(3^4)\}$

Table 4.1: the correspondence among the subset of $\mathcal{O}_3(12)$, the subset of $\mathcal{D}_3(12)$ and $\mathcal{T}_3(12)$

Example 4.3 *We give the correspondence among $\mathcal{O}_3(12)$, $\mathcal{D}_3(12)$ and $\mathcal{T}_k(12)$ in Table 4.1, where each row contains a partition $\lambda \in \mathcal{D}_3(12)$, the corresponding $\phi(\lambda) \in \mathcal{O}_3(12)$, and the sets $\mathcal{T}_{\lambda,3,d}$ arranged in the increasing order of d. For the sake of conciseness, we just list the corresponding pairs $(\lambda, \phi(\lambda))$ whose differences of the number of distinct parts are nonzero.*

5 Combinatorial proof of Conjecture 1.3

In this section we are going to prove Conjecture 1.3 as follows. Let $\mathcal{D}_O(n)$ be the set of distinct partitions of n with odd length, and $\mathcal{G}(n)$ be the set of gap-free partitions of size n. First we build a one-to-one correspondence between the set $\mathcal{D}_O(n)$ and one special kind of gap-free partitions of size n denoted by $\mathcal{G}_{\mathcal{I}}(n)$. Then by using $\lambda \in \mathcal{G}_{\mathcal{I}}(n)$ as indices, we partition $\mathcal{G}(n)$ into $|\mathcal{G}_{\mathcal{I}}(n)|$ pairwise disjoint subsets denoted by $\mathcal{G}_\lambda$. For any $\lambda \in \mathcal{G}_{\mathcal{I}}$, we can prove that the number of partitions in $\mathcal{G}_\lambda$ equals the smallest part of the partition in $\mathcal{D}_O(n)$ corresponding to λ. Hence, the conjecture 1.3 is true.

Based on the following lemma, we will give the definition of $\mathcal{G}_{\mathcal{I}}(n)$, and the one-to-one correspondence between $\mathcal{G}_\mathcal{I}(n)$ and $\mathcal{D}_O(n)$.

Lemma 5.1 *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathcal{D}(n)$, then the conjugate of λ, $\lambda' = (1^{m_1}, 2^{m_2}, \ldots, \ell^{m_\ell})$ satisfies that $m_i \geq 1$ for all $1 \leq i \leq \ell$ and $m_\ell = \lambda_\ell$. Thus $\lambda' \in \mathcal{G}(n)$.*

Proof. By the definition of the conjugate of partitions, we see that $m_i = \lambda_i - \lambda_{i+1}$ for $1 \leq i \leq \ell - 1$ and $m_\ell = \lambda_\ell$. Since $\lambda \in \mathcal{D}(n)$ is a distinct partition satisfying $\lambda_i - \lambda_{i+1} \geq 1$ for $1 \leq i \leq \ell - 1$, it is obvious that $m_i \geq 1$ and $\lambda' \in \mathcal{G}(n)$. ∎

Hence if we take λ from $\mathcal{D}_O(n)$ then take its conjugate, we obtain a gap-free partition λ' with the smallest part is 1 and the largest part is odd. Denote by $\mathcal{G}_\mathcal{I}$ the set of gap-free partitions satisfying that the smallest part is 1 and the largest part is odd, and $\mathcal{G}_\mathcal{I}(n)$ the set of such partitions of size n. Then Lemma 5.1 gives a one-to-one correspondence between $\mathcal{D}_O(n)$ and $\mathcal{G}_\mathcal{I}(n)$. For any partition λ, let $s(\lambda)$ be the smallest part of λ and $m(\lambda)$ be the times of the largest part of λ repeats. Then we have the following lemma.

Lemma 5.2 *There exist a bijection $\alpha\colon \mathcal{G}_\mathcal{I}(n) \to \mathcal{D}_O(n)$ such that for $\lambda \in \mathcal{G}_\mathcal{I}(n)$, $m(\lambda) = s(\alpha(\lambda))$.*

Example 5.1 *Let* $(\lambda) = (1, 2^3, 3, 4^2, 5^4) \in \mathcal{G}_\mathcal{I}(38)$*, then* $\alpha(\lambda) = (11, 10, 7, 6, 4) \in \mathcal{D}_O(38)$ *and* $m(\lambda) = s(\alpha(\lambda)) = 4$.

For convenience, we need to introduce some notations and operators on partitions. Let $\lambda = (1^{m_1}, 2^{m_2}, \ldots, k^{m_k})$ be a partition with $m_i \geq 0$ for $1 \leq i \leq k$. We call every i^{m_i} the *block* of i if $m_i \geq 1$ and denoted by B_i. Define $B_i < B_j$ if and only if $i < j$, and B_i to be odd (resp. even) if and only if i is odd (resp. even). The number of blocks of λ is exactly the number of distinct parts in λ denoted by $\bar{\ell}(\lambda)$ as used in Section 4. Hence if a partition λ is gap-free, then λ contains $\bar{\ell}(\lambda)$ continuous block(s), and $\mathcal{G}_\mathcal{I}$ is consist of the gap-free partitions whose smallest block is B_1 and the largest block is odd. Let λ be a gap-free partition of form

$$\Big(\underbrace{k, \ldots, \boldsymbol{k}}_{B_k}, \underbrace{k+1, \ldots, \boldsymbol{k+1}}_{B_{k+1}}, \ldots, \underbrace{k+r-1, \ldots, \boldsymbol{k+r-1}}_{B_{k+r-1}}, \ldots, \underbrace{k+\ell-1, \ldots, \boldsymbol{k+\ell-1}}_{B_{k+\ell-1}} \Big),$$

then for any $0 \leq r \leq \ell$, we can define increasing operator ξ_r which can increase the size of λ by r and preserve its property of gap-free. To be more specific, notice that $B_k, B_{k+1}, \ldots, B_{k+r-1}$ are the r smallest blocks of λ, then for each $1 \leq j \leq r$, add 1 on the last $k + j - 1$ part in the block B_{k+j-1}, which is in bold type. Let $\xi_r(\lambda)$ be the resulting partition, then it is also gap-free and of size $|\lambda| + r$. For example, if $\lambda = (3^2, 4, 5^3, 6) \in \mathcal{G}(31)$, then $\xi_2(\lambda) = (3, 4, 5^4, 6) \in \mathcal{G}(33)$. Symmetrically we can define decreasing operator ξ_r^- , where $0 \leq r \leq \ell$. Note that $B_{\ell-r+1}, B_{\ell-r+2}, \ldots, B_\ell$ are the r largest blocks of λ. $\xi_r^-(\lambda)$ is obtained from λ by subtracting 1 from the first $\ell - j + 1$ part in the block $B_{\ell-j+1}$ for each $1 \leq j \leq r$. Thus $\xi_r^{-1}(\lambda)$ is a gap-free partition of size $|\lambda| - r$. As the above partition λ, we have $\xi_2^-(\lambda) = (3^2, 4^2, 5^3) \in \mathcal{G}(29)$.

Lemma 5.3 *Let λ be a gap-free partition with $s(\lambda) > 1$, then the following hold:*

(1). $\bar{\ell}(\lambda) \le \bar{\ell}(\xi_r(\lambda)), \bar{\ell}(\xi_r^-(\lambda)) \le \bar{\ell}(\lambda)+1$ if $r=\bar{\ell}(\lambda)$;

(2). $\bar{\ell}(\lambda)-1 \le \bar{\ell}(\xi_r(\lambda)), \bar{\ell}(\xi_r^-(\lambda)) \le \bar{\ell}(\lambda)$ if $r=\bar{\ell}(\lambda)-1$;

(3). $\xi_r^{-1}=\xi_r^-$ if $r \in \{\bar{\ell}(\lambda)-1, \bar{\ell}(\lambda)\}$, where ξ_r^{-1} is the inverse operator of ξ_r.

Proof. Let $\lambda=(k^{m_k},(k+1)^{m_{k+1}},\ldots,(k+\ell-1)^{m_{k+\ell-1}})$ be the gap-free partition with $k>1$ and $\bar{\ell}(\lambda)=\ell$. Set $r=\ell$, then $\xi_\ell(\lambda)=\mu=(k^{m_k-1},(k+1)^{m_{k+1}},\ldots,(k+\ell-1)^{m_{k+\ell-1}},k+\ell)$. Note that m_k-1 can be zero, which means that $\ell \le \bar{\ell}(\mu) \le \ell+1$. Hence the acting of ξ_ℓ^- is valid and $\xi_\ell^-(\mu)=(k^{m_k},(k+1)^{m_{k+1}},\ldots,(k+\ell-1)^{m_{k+\ell-1}})=\lambda$. On the other hand, $\xi_\ell^-(\lambda)=\nu=(k-1,k^{m_k},\ldots,(k+\ell-1)^{m_{k+\ell-1}-1})$, implying that $\ell \le \bar{\ell}(\nu) \le \ell+1$. It follows that $\xi_\ell(\nu)=(k^{m_k},(k+1)^{m_{k+1}},\ldots,(k+\ell-1)^{m_{k+\ell-1}})=\lambda$. Thus we have verified (1) and the case of $r=\bar{\ell}(\lambda)$ in (3). The proof of (2) and the case of $r=\bar{\ell}(\lambda)-1$ in (3) are completely the same so we leave them to the reader. ∎

Let $\lambda=(1^{m_1},2^{m_2},\ldots,\ell^{m_\ell}) \in \mathcal{G}(n)$ be a gap-free partition with the smallest block B_1 and $\bar{\ell}(\lambda)=\ell$. Note that here we do not require the constraint that B_ℓ is odd. Then we shall define a series of operators ϱ_i for $1 \le i \le m_\ell$ as follows. First We delete $i-1$ parts ℓ from λ then λ becomes $\lambda_1^i=(1^{m_1},2^{m_2},\ldots,\ell^{m_\ell-i+1})$. It is clear that λ_1^i is also gap-free and of $\bar{\ell}(\lambda_1^i)=\ell$. For $2 \le j \le i$, let $\lambda_j^i=\xi_\ell(\lambda_{j-1}^i)$ iteratively and finally let $\varrho_i(\lambda)=\lambda_i^i$. It is easy to see that $\varrho_i(\lambda)$ and $\varrho_j(\lambda)$ are different whenever $i \ne j$. We will use the following example to illustrate how these sophisticated operators work.

Example 5.2 *Let $\lambda=(1^2,2^3,3,4,5^4)$. Trivially $\varrho_1(\lambda)=\lambda=(1^2,2^3,3,4,5^4)$. For $i=2$, deleting one part 5 from λ then obtain $\lambda_1^2=(1^2,2^3,3,4,5^3)$, $\lambda_2^2=\xi_5(\lambda_1^2)=(1,2^3,3,4,5^3,6)$, hence we have $\varrho_2(\lambda)=(1,2^3,3,4,5^3,6)$. For $i=3$, by subtracting two parts 5, λ becomes $\lambda_1^3=(1^2,2^3,3,4,5^2)$. Thus we deduce that $\lambda_2^3=\xi_5(\lambda_1^3)=(1,2^3,3,4,5^2,6)$ and $\lambda_3^3=\xi_5(\lambda_2^3)=(2^3,3,4,5^2,6^2)$, which leads to $\varrho_3(\lambda)=(2^3,3,4,5^2,6^2)$. For $i=4$, we take three parts 5 out from λ, which change λ to $\lambda_1^4=(1^2,2^3,3,4,5)$. Then $\lambda_2^4=\xi_5(\lambda_1^4)=(1,2^3,3,4,5,6)$, $\lambda_3^4=\xi_5(\lambda_2^4)=(2^3,3,4,5,6^2)$ and $\lambda_4^4=\xi_5(\lambda_3^4)=(2^2,3,4,5,6^2,7)$. Hence we arrive at $\varrho_4(\lambda)=(2^2,3,4,5,6^2,7)$.*

Theorem 5.4 *For any $\lambda \in \mathcal{G}_{\mathcal{I}}(n)$, define*

$$\mathcal{G}_\lambda=\{\varrho_i(\lambda) : 1 \le i \le m(\lambda)\}.$$

Then the sets $\mathcal{G}_\lambda$'s are pairwise disjoint and

$$\mathcal{G}(n)=\bigcup_{\lambda \in \mathcal{G}_{\mathcal{I}}(n)} \mathcal{G}_\lambda.$$

Combining Theorem 5.4 and Lemma 5.2, we see that

$$|\mathcal{G}(n)|=\left|\bigcup_{\lambda \in \mathcal{G}_{\mathcal{I}}(n)} \mathcal{G}_\lambda\right|=\sum_{\lambda \in \mathcal{G}_{\mathcal{I}}(n)} |\mathcal{G}_\lambda|=\sum_{\lambda \in \mathcal{G}_{\mathcal{I}}(n)} m(\lambda)=\sum_{\lambda \in \mathcal{D}_O(n)} s(\lambda),$$

which confirms Conjecture 1.3.

Proof of Theorem 5.4. We first clarify that the operators ϱ_i's are well-defined. To this end, by the definition of increasing operator ξ_r for $r \geq 0$, we only need to show that when we act ξ_r on some partition λ, the number of the blocks of λ is no less than r. Actually, we can claim that for any $\lambda \in \mathcal{G}_{\mathcal{I}}(n)$ and $1 \leq i \leq m(\lambda)$, we have $\bar{\ell}(\lambda) \leq \bar{\ell}(\lambda_j^i) \leq \bar{\ell}(\lambda) + 1$ for $1 \leq j \leq i$. Let $\lambda \in \mathcal{G}_{\mathcal{I}}(n)$ be a gap-free partition with $\bar{\ell}(\lambda) = \ell$ and $m(\lambda) = m$, then for any $1 \leq i \leq m$, it is evident that λ_1^i is also of $\bar{\ell}(\lambda_1^i) = \ell$. By induction, we assume that $\ell \leq \bar{\ell}(\lambda_j^i) \leq \ell + 1$ for $1 \leq j \leq m-1$. By acting ξ_ℓ on λ_j^i, it produces $\xi_\ell(\lambda_j^i) = \lambda_{j+1}^i$. If $\bar{\ell}(\lambda_j^i) = \ell$, then by Lemma 5.3 (1), $\ell \leq \bar{\ell}(\lambda_{j+1}^i) \leq \ell + 1$. If $\bar{\ell}(\lambda_j^i) = \ell + 1$, then by Lemma 5.3 (2), we still have $\ell \leq \bar{\ell}(\lambda_{j+1}^i) \leq \ell + 1$, thus we derive $\ell \leq \bar{\ell}(\lambda_j^i) \leq \ell + 1$ for $1 \leq j \leq m$. Thus ϱ^i's are well-defined.

Next, for every $\lambda \in \mathcal{G}_{\mathcal{I}}(n)$, we shall show that $\mathcal{G}_\lambda \subseteq \mathcal{G}_n$, which implies that $\bigcup_{\lambda \in \mathcal{G}_{\mathcal{I}}(n)} \mathcal{G}_\lambda \subseteq \mathcal{G}(n)$. Since for any $\lambda \in \mathcal{G}_{\mathcal{I}}(n)$ and $1 \leq i \leq m(\lambda)$, λ_1^i is gap-free of size $n - (i-1)\bar{\ell}(\lambda)$ and each time acting $\xi_{\bar{\ell}(\lambda)}$ preserves the property of gap-free and increases the size by $\bar{\ell}(\lambda)$, then after $(i-1)$-time composition of $\xi_{\bar{\ell}(\lambda)}$, it is clear that $\varrho_i(\lambda)$ is gap-free and of size n. Thus we have $\mathcal{G}_\lambda \subseteq \mathcal{G}_n$ then $\bigcup_{\lambda \in \mathcal{G}_{\mathcal{I}}(n)} \mathcal{G}_\lambda \subseteq \mathcal{G}(n)$.

Finally, we prove that for any $\mu \in \mathcal{G}(n)$, there must exists only one $\lambda \in \mathcal{G}_{\mathcal{I}}(n)$ such that $\mu \in \mathcal{G}_\lambda$, which simultaneously demonstrates that $\mathcal{G}_\lambda$'s are pairwise disjoint and $\bigcup_{\lambda \in \mathcal{G}_{\mathcal{I}}(n)} \mathcal{G}_\lambda = \mathcal{G}(n)$. Given $\mu \in \mathcal{G}(n)$, if $\mu \in \mathcal{G}_{\mathcal{I}}(n)$, then we have done; otherwise let $\mu = (k^{m_k}, (k+1)^{m_{k+1}}, \ldots, (k+\ell-1)^{m_{k+\ell-1}})$ with $\bar{\ell}(\mu) = \ell$. Set $\mu_1 = \mu$ and $r = 2\lceil \ell/2 \rceil - 1$, i.e. r is the largest odd number that do not exceed ℓ, so that $r \leq \bar{\ell}(\mu_1) \leq r + 1$. For $i \geq 2$, we will keep constructing μ_i by letting $\mu_i = \xi_r^-(\mu_{i-1})$ until first getting $\mu_m \in \mathcal{G}_{\mathcal{I}}$ for some $m \geq 2$. This could be done since for $\ell \geq 1$, we have $r \geq 1$ and for $1 \leq i \leq m$, $r \leq \bar{\ell}(\mu_i) \leq r + 1$ by Lemma 5.3 (1) and (2). Since ξ_r^- is the inverse operator of ξ_r by Lemma 5.3 (3), it can be checked that the largest block of μ_m is B_r. Hence, by adding $m-1$ parts r back to μ_m, we obtain the unique $\lambda \in \mathcal{G}_{\mathcal{I}}(n)$. For example, if $\mu = (3^3, 4, 5^2, 6) \in \mathcal{G}(29) \setminus \mathcal{G}_{\mathcal{I}}(29)$, then $r = 3$ and we have $\mu_2 = (3^4, 4, 5^2)$, $\mu_3 = (2, 3^4, 4, 5)$, $\mu_4 = (2^2, 3^4, 4)$ and $\mu_5 = (1, 2^2, 3^4) \in \mathcal{G}_{\mathcal{I}}$. Thus $m = 5$ and $\lambda = (1, 2^2, 3^8) \in \mathcal{G}_{\mathcal{I}}(29)$, implying that $(3^3, 4, 5^2, 6) \in \mathcal{G}_{(1,2^2,3^8)}$. This completes the proof. ∎

Example 5.3 *We list $\mathcal{D}_{\mathcal{O}}(12)$, $\mathcal{G}_{\mathcal{I}}(12)$ and $\mathcal{G}(12)$ in Table 5.1, where each row contains a gap-free partition $\lambda \in \mathcal{G}_{\mathcal{I}}(12)$, the corresponding distinct partition $\alpha(\lambda) \in \mathcal{D}_{\mathcal{O}}(12)$, and the set $\mathcal{G}_\lambda$ of which partitions are arranged from $\varrho_1(\lambda)$ to $\varrho_{m(\lambda)}(\lambda)$.*

Let $\mathcal{D}_{\mathcal{E}}(n)$ be the set of distinct partitions of n with even length. We can also investigate the quantitative relationship between the cardinalities of $\mathcal{D}_{\mathcal{E}}(n)$ and $\mathcal{G}(n)$. Denote by $\mathcal{G}'_{\mathcal{I}}(n)$ the set of gap-free partitions of n whose the smallest block is B_1 and the largest block is even. By Lemma 5.1, we know that the conjugate also gives a bijection $\beta\colon \mathcal{G}'_{\mathcal{I}}(n) \to \mathcal{D}_{\mathcal{E}}(n)$ such that $m(\lambda) = s(\beta(\lambda))$ for any $\lambda \in \mathcal{G}'_{\mathcal{I}}(n)$.

Theorem 5.5 *For any $\lambda \in \mathcal{G}'_{\mathcal{I}}(n)$, define $\mathcal{G}_\lambda = \{\varrho_i(\lambda) : 1 \leq i \leq m(\lambda)\}$ as before. Then the sets $\mathcal{G}_\lambda$'s are*

$\alpha(\lambda) \in \mathcal{D}_{\mathcal{O}}(12)$	$\lambda \in \mathcal{G}_{\mathcal{I}}(12)$	$\mathcal{G}_\lambda \subseteq \mathcal{G}(12)$
(5,4,3)	$(1,2,3^3)$	$\{(1,2,3^3),(2,3^2,4),(3,4,5)\}$
(6,4,2)	$(1^2,2^2,3^2)$	$\{(1^2,2^2,3^2),(1,2^2,3,4)\}$
(6,5,1)	$(1,2^4,3)$	$\{(1,2^4,3)\}$
(7,3,2)	$(1^4,2,3^2)$	$\{(1^4,2,3^2),(1^3,2,3,4)\}$
(7,4,1)	$(1^3,2^3,3)$	$\{(1^3,2^3,3)\}$
(8,3,1)	$(1^5,2^2,3)$	$\{(1^5,2^2,3)\}$
(9,2,1)	$(1^7,2,3)$	$\{(1^7,2,3)\}$
(12)	(1^{12})	$\{(1^{12}),(1^{10},2),(1^8,2^2),(1^6,2^3),$ $(1^4,2^4),(1^2,2^5),(2^6),(2^3,3^2),$ $(3^4),(4^3),(6^2),(12)\}$

Table 5.1: the correspondence among $\mathcal{D}_{\mathcal{O}}(12)$, $\mathcal{G}_{\mathcal{I}}(12)$ and $\mathcal{G}(12)$

pairwise disjoint and

$$\mathcal{G}(n)\Big\backslash\mathcal{G}^0(n) = \bigcup_{\lambda\in\mathcal{G}'_{\mathcal{I}}(n)} \mathcal{G}_\lambda,$$

where $\mathcal{G}^0(n)$ is the set of gap-free partitions of n with only one block.

Proof. The proof is exactly the same as Theorem 5.4 except for setting $r = 2\lfloor \ell/2 \rfloor$, i.e., r is the largest even number that do not exceed ℓ. ∎

Example 5.4 *We list $\mathcal{D}_{\mathcal{E}}(12)$, $\mathcal{G}'_{\mathcal{I}}(12)$ and $\mathcal{G}(12)\Big\backslash\mathcal{G}^0(12)$ in Table 5.2, where each row contains a gap-free partition $\lambda \in \mathcal{G}'_{\mathcal{I}}(12)$, the corresponding distinct partition $\beta(\lambda) \in \mathcal{D}_{\mathcal{E}}(12)$, and the set $\mathcal{G}_\lambda$ of which partitions are arranged from $\varrho_1(\lambda)$ to $\varrho_{m(\lambda)}(\lambda)$.*

Remark 5.1 *Note that the number of gap-free partitions of n with only one block equals the number of divisors of n, which is usually denoted by d(n) in number theory and combinatorics. Thus by Theorem 5.4 and 5.5, we rediscover the following identity*

$$\sum_{\lambda\in\mathcal{D}_{\mathcal{O}}(n)} s(\lambda) - \sum_{\lambda\in\mathcal{D}_{\mathcal{E}}(n)} s(\lambda) = d(n), \tag{5.1}$$

which is first obtained by Fokkink, Fokkink and Wang [7] via the sum of polynomial quotients for n. Andrews [2] asserted that (5.1) is a corollary of the differentiation of the q-analog of Gauss's theorem [1, Corollary 2.4] then gave the analytic proof of (5.1) by q-series.

$\beta(\lambda) \in \mathcal{D}_{\mathcal{E}}(12)$	$\lambda \in \mathcal{G}'_{\mathcal{I}}(12)$	$\mathcal{G}_\lambda \subseteq \mathcal{G}(12) \setminus \mathcal{G}^0(12)$
(5,4,2,1)	$(1, 2^2, 3, 4)$	$(1, 2^2, 3, 4)$
(6,3,2,1)	$(1^3, 2, 3, 4)$	$(1^3, 2, 3, 4)$
(7,5)	$(1^2, 2^5)$	$(1^2, 2^5)$, $(1, 2^4, 3)$, $(2^3, 3^2)$, $(2, 3^2, 4)$, $(3, 4, 5)$
(8,4)	$(1^4, 2^4)$	$(1^4, 2^4)$, $(1^3, 2^3, 3)$, $(1^2, 2^2, 3^2)$, $(1, 2, 3^3)$
(9,3)	$(1^6, 2^3)$	$(1^6, 2^3)$, $(1^5, 2^2, 3)$, $(1^4, 2, 3^2)$
(10,2)	$(1^8, 2^2)$	$(1^8, 2^2)$, $(1^7, 2^2, 3)$
(11,1)	$(1^{10}, 2)$	$(1^{10}, 2)$

Table 5.2: the correspondence among $\mathcal{D}_{\mathcal{E}}(12)$, $\mathcal{G}'_{\mathcal{I}}(12)$ and $\mathcal{G}(12) \setminus \mathcal{G}^0(12)$

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References

[1] G.E. Andrews,*The theory of partitions*, Cambridge university press, 1998.

[2] G.E. Andrews, *The number of smallest parts in the partitions of n*, J. Reine Angew. Math., 624 (2008) 133–142.

[3] G.E. Andrews, *The Bhargava-Adiga Summation and Partitions*, J. Indian Math. Soc. (N.S.), 84 (2017), 151–160.

[4] G.E. Andrews, *Euler's partition identity and two problems of George Beck*, Math. Student, 86 (2017), 115–119

[5] S. Chern, *On a conjecture of George Beck*, Int. J. Number Theory, 2018.

[6] L. Euler, *Introduction in Analysin Infinitorum*, vol. 2. MM Bousquet, Hanson (1748).

[7] R. Fokkink, W. Fokkink and B. Wang, *A relation between partitions and the number of divisors*, Amer. Math. Monthly, 102 (1995), 345-347.

[8] F. Franklin, *On partitions*, Johns Hopkins Univ. Circular, 2 (1883), 72.

[9] S. Fu and D. Tang, *Generalizing a partition theorem of Andrews*, Math. Student, 86 (2017), 91–96.

[10] J.W.L. Glaisher, *A theorem in partitions*, Messenger of Math., 12 (1883), 158–170.

[11] I. Pak, *Partition bijections, a survey*, Ramanujan J., 12 (2006), 5–75.

[12] J.B. Remmel, *Bijective proofs of some classical partition identities*, J. Combin. Theory Ser. A, 33 (1982), 273–286.

[13] J.J. Sylvester and F. Franklin, *A constructive theory of partitions, arranged in three acts, an interact and an exodion*, Amer. J. Math., 5 (1882), 251–330.

[14] H.S. Wilf, *Lectures on integer partitions*, (unpublished), avaiable at: https://www.math.upenn.edu/~wilf, 2000.

[15] The On-Line Encyclopedia of Integer Sequences, Sequence A090867, https://oeis.org/A090867.

[16] The On-Line Encyclopedia of Integer Sequences, Sequence A034296, https://oeis.org/A034296.