A Variation on Mills-Like Prime-Representing Functions

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Abstract

Mills showed that there exists a constant A such that $\lfloor A^{3^n} \rfloor$ is prime for every positive integer n. Kuipers and Ansari generalized this result to $\lfloor A^{c^n} \rfloor$ where $c \in \mathbb{R}$ and $c \geq 2.106$. The main contribution of this paper is a proof that the function $\lceil B^{c^n} \rceil$ is also a prime-representing function, where $\lceil X \rceil$ denotes the ceiling or least integer function. Moreover, the first 10 primes in the sequence generated in the case $c = 3$ are calculated. Lastly, the value of B is approximated to the first 5500 digits and is shown to begin with $1.2405547052\ldots$.

1 Introduction

Mills [6] showed in 1947 that there exists a constant A such that $\lfloor A^{3^n} \rfloor$ is prime for all positive integers n. Kuipers [5] and Ansari [1] generalized this result to all $\lfloor A^{c^n} \rfloor$ where $c \in \mathbb{R}, c \geq 2.106$, i.e., there exist infinitely many A's such that the above expression yields a prime for all positive integers n. Caldwell and Cheng [2] calculated the minimum constant A for the case $c = 3$ up to the first 6850 digits (A051021), and found it to be approximately equal to $1.3063778838\ldots$. This process involved computing the first 10 primes b_i in the sequence generated by the function (A051254), with b_{10} having 6854 decimal digits.

The main contribution of this paper is a proof that the function $\lceil B^{c^n} \rceil$ satisfies the same criteria, where $\lceil X \rceil$ denotes the ceiling function (the least integer greater than or equal to X). In other words, there exists a constant B such that for all positive integers n, the expression $\lceil B^{c^n} \rceil$ yields a prime for $c \geq 3, c \in \mathbb{N}$. Moreover, the sequence of primes generated by such functions is monotonically increasing. Lastly, analogously to [2] the case $c = 3$ is studied in more detail and the value of B is approximated up to the first 5500 decimal digits by calculating the first 10 primes b_i of the sequence.

In contrast to Mills' formula and given that here the floor function is replaced by a ceiling function, the process of generating the prime number sequence $P_0, P_1, P_2, \ldots$ involves taking the greatest prime smaller than P_n^c at each step instead of smallest prime greater than P_n^c, in order to find P_{n+1}. As a consequence, the sequence of primes generated by $\lceil B^{c^n} \rceil$ is different from the one generated by $\lfloor A^{c^n} \rfloor$ for the same value of c and the same starting prime (apart from the first element of course).

2 The prime-representing function

This paper begins with a proof of the case $c = 3$ and will proceed to a generalization of the function to all $c \geq 3, c \in \mathbb{N}$.

By using Ingham's result [4] on the difference of consecutive primes:

$$p_{n+1} - p_n < K p_n^{5/8},$$

and analogously to Mills' reasoning [6], we construct an infinite sequence of primes $P_0, P_1, P_2, \ldots$ such that $\forall n \in \mathbb{N} : (P_n - 1)^3 + 1 < P_{n+1} < P_n^3$ using the following lemma.

Lemma 1. $\forall N > K^8 + 1 \in \mathbb{N} : \exists p \in \mathbb{P} : (N-1)^3 + 1 < p < N^3$, *where* $\mathbb{P}$ *denotes the set of prime numbers.*

Proof. Let p_n be the greatest prime smaller than $(N-1)^3$.

$$\begin{aligned}
(N-1)^3 &< p_{n+1} \\
&< p_n + K p_n^{5/8} \\
&< (N-1)^3 + K\left((N-1)^3\right)^{5/8} && \text{(since } p_n < (N-1)^3\text{)} \\
&< (N-1)^3 + (N-1)^2 && \text{(since } N > K^8 + 1\text{)} \\
&< N^3 - 2N^2 + N \\
&< N^3.
\end{aligned}$$

Note that since $(N-1)^3 < p_{n+1}$, $(N-1)^3 + 1 < p_{n+1}$ since $(N-1)^3 + 1 = N(N^2 - 3N + 3)$ is not prime.

□

Given the above we can construct an infinite sequence of primes $P_0, P_1, P_2, \ldots$ such that for every positive integer n, we have: $(P_n - 1)^3 + 1 < P_{n+1} < P_n^3$.

We now define the following two functions:

$$\forall n \in \mathbb{Z}^+ : u_n = (P_n - 1)^{3^{-n}},$$
$$\forall n \in \mathbb{Z}^+ : v_n = P_n^{3^{-n}}.$$

The following statements can immediately be deduced:

- $u_n < v_n$,

- $u_{n+1} = (P_{n+1} - 1)^{3^{-n-1}} > ((P_n - 1)^3 + 1) - 1)^{3^{-n-1}} = (P_n - 1)^{3-n} = u_n,$
- $v_{n+1} = P_{n+1}^{3^{-n-1}} < (P_n^3)^{3^{-n-1}} = P_n^{3^{-n}} = v_n.$

It follows that u_n forms a bounded and monotone increasing sequence.

Theorem 2. *There exists a positive real constant B such that $\lceil B^{3^n} \rceil$ is a prime-representing function for all positive integers n.*

Proof. Since u_n is bounded and strictly monotone, there exists a number B such that

$$B := \lim_{n\to\infty} u_n.$$

From the above deduced properties of u_n and v_n, we have

$$\begin{aligned} u_n < B &< v_n, \\ (P_n - 1)^{3^{-n}} < B &< P_n^{3^{-n}}, \\ P_n - 1 < B^{3^n} &< P_n. \end{aligned}$$

□

Theorem 3. *There exists a positive real constant B such that $\lceil B^{c^n} \rceil$ is a prime-representing function for $c \geq 3, c \in \mathbb{N}$ and all positive integers n.*

Proof. We can use the generalizations to Mills' function as shown by Kuipers [5] and Dudley [3] in order to show that $\lceil B^{c^n} \rceil$ is also a prime-representing function for $c \geq 3, c \in \mathbb{N}$. This proof is short as it is essentially identical to the one presented above, with the following modifications.

As shown by Kuipers [5] for Mills' function, we first define $a = 3c - 4, b = 3c - 1$. Therefore $a/b \geq 5/8$. This means that in Ingham's equation there exists a constant K' such that

$$p_{n+1} - p_n < K' p_n^{a/b}.$$

Lemma 1 can then be modified by taking $N > K'^b + 1$, defining p_n as the greatest prime smaller than $(N-1)^c$ and noticing that $ca + 1 = b(c-1)$. Analogously to the proof in Lemma 1, we quickly obtain the bounds $(N-1)^c + 1 < p < N^c$. This means that we can construct a sequence of primes $P_0, P_1, P_2, \ldots$ such that for every positive integer n, $(P_n - 1)^c + 1 < P_{n+1} < P_n^c$.

This is then concluded with a similar reasoning as in the proof of Theorem 2.

□

3 Numerical calculation of B

In this section, a numerical approximation of B is presented for the case $c = 3$. Mills [6] suggested using the lower bound $K = 8$ for the first prime in the classical Mills function $\lfloor A^{3^n} \rfloor$, where K is the constant defined in Ingham's paper [4]. Other authors, including Caldwell and Cheng [2], decided to begin with the prime 2 and then choose the least possible prime at each step. In this case, since the ceiling function replaces the floor function, we choose the greatest possible prime smaller than P_n^3 as the next element P_{n+1}.

If p_i denotes the i^{th} prime in the sequence, we obtain

- $p_1 = 2$
- $p_2 = 7$
- $p_3 = 337$
- $p_4 = 38272739$
- $p_5 = 56062005704198360319209$
- $p_6 =$ 17619999581432728735667120910458586439705503907211069\
  6028654438846269
- $p_7 =$ 54703823381492990628407924713718713957740513297193414\
  21259587335767096542227048457036456872683352033529421007878\
  29141860830768725102385452609882503551811073140339908096068\
  8125590506176016285837338837682469

The primes p_8, p_9 and p_{10} are far too large to show in this paper — for instance p_{10} has 5528 decimal digits. The primes p_i for $i \leq 8$ were verified using a deterministic primality test in Wolfram Mathematica 11 with the `ProvablePrimeQ` function in the PrimalityProving package, while p_9 and p_{10} were certified prime by the Primo software [7]. The certification of p_{10} took 14 hours and 23 minutes on an Intel i7-4770 CPU and 4GB RAM. The prime certificates for p_9 and p_{10} as well as the primes themselves can be found alongside this paper as auxiliary files.

The value of B was calculated up to its first 5500 decimal digits. The first 600 are presented below:

1.2405547052	5201424067	4695153379	0034521235	3396725255
9232034386	1886622104	9111642316	9209174137	7064313608
3109555650	9480848158	9481662421	8378961303	7426392535
6658242301	8524802142	1960037621	1464734105	8229918628
4182439221	9437396337	9442594273	8936874985	9158491115
7886891108	4262398559	2731605607	5719554304	2915944781
6278755834	4774412491	8125993063	4590081972	8945860313
1303247244	0907981721	7119324606	1009855753	6063847008
6985820925	6038920740	0817313213	1691077511	3322609476
3239264899	5703729933	8452155290	5152647430	8960522935
3735771869	0936560934	8000430515	4856069064	6309177739
2832001365	6550953673	1549789328	9032942357	7708168137

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(Concerned with sequences A051021 and A051254.)

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