THE ASCENT-PLATEAU STATISTICS ON STIRLING PERMUTATIONS

SHI-MEI MA, JUN MA, AND YEONG-NAN YEH

Abstract. In this paper, several variants of the ascent-plateau statistic are introduced, including flag ascent-plateau, double ascent and descent-plateau. We first study the flag ascent-plateau statistic on Stirling permutations by using context-free grammars. We then present a unified refinement of the ascent polynomials and the ascent-plateau polynomials. In particular, by using Foata and Strehl's group action, we prove two bistatistics over the set of Stirling permutations of order n are equidistributed.

Keywords: Stirling permutations; Context-free grammars; Ascents; Plateaus; Ascent-plateaus

1. Introduction

A *Stirling permutation* of order n is a permutation of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ such that for each i, $1 \le i \le n$, all entries between the two occurrences of i are larger than i. Denote by $\mathcal{Q}_n$ the set of *Stirling permutations* of order n. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{Q}_n$. For $1 \le i \le 2n$, we say that an index i is a *descent* of σ if $\sigma_i > \sigma_{i+1}$ or $i = 2n$, and we say that an index i is an *ascent* of σ if $\sigma_i < \sigma_{i+1}$ or $i = 1$. Hence the index $i = 1$ is always an ascent and $i = 2n$ is always a descent. Moreover, a *plateau* of σ is an index i such that $\sigma_i = \sigma_{i+1}$, where $1 \le i \le 2n-1$. Let $\operatorname{des}(\sigma), \operatorname{asc}(\sigma)$ and $\operatorname{plat}(\sigma)$ be the numbers of descents, ascents and plateaus of σ, respectively.

Stirling permutations were defined by Gessel and Stanley [8], and they proved that

$$(1-x)^{2k+1}\sum_{n=0}^{\infty}\left\{ {n+k \atop n} \right\}x^n = \sum_{\sigma\in\mathcal{Q}_k} x^{\operatorname{des}\sigma},$$

where $\left\{ {n \atop k} \right\}$ is the *Stirling number of the second kind*, i.e., the number of ways to partition a set of n objects into k non-empty subsets. A classical result of Bóna [2] says that descents, ascents and plateaus have the same distribution over $\mathcal{Q}_n$, i.e.,

$$\sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{des}\sigma} = \sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{asc}\sigma} = \sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{plat}\sigma}.$$

This equidistributed result and associated multivariate polynomials have been extensively studied by Janson, Kuba, Panholzer, Haglund, Chen et al., see [5, 9, 10, 11] and references therein.

Recently, Ma and Toufik [15] introduced the definition of ascent-plateau statistic and presented a combinatorial interpretation of the $1/k$-Eulerian polynomials. The purpose of this paper is to explore variants of the ascent-plateau statistic. In the following, we collect some definitions, notation and results that will be needed throughout this paper.

2010 *Mathematics Subject Classification.* Primary 05A05; Secondary 05A15.

Definition 1. *An occurrence of an ascent-plateau of $\sigma \in \mathcal{Q}_n$ is an index i such that $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$, where $i \in \{2, 3, \ldots, 2n-1\}$. An occurrence of a left ascent-plateau is an index i such that $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$, where $i \in \{1, 2, \ldots, 2n-1\}$ and $\sigma_0 = 0$.*

Let $\mathrm{ap}\,(\sigma)$ and $\mathrm{lap}\,(\sigma)$ be the numbers of ascent-plateaus and left ascent-plateaus of σ, respectively. For example, $\mathrm{ap}\,(442\mathbf{3}32115\mathbf{6}65) = 2$ and $\mathrm{lap}\,(\mathbf{4}42\mathbf{3}32115\mathbf{6}65) = 3$.

Define

$$M_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{ap}\,(\sigma)},\ N_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma)}.$$

According to [15, Theorem 2, Theorem 3], we have

$$M(x,t) = \sum_{n \geq 0} M_n(x) \frac{t^n}{n!} = \sqrt{\frac{x-1}{x - e^{2t(x-1)}}}, \tag{1}$$

$$N(x,t =) \sum_{n \geq 0} N_n(x) \frac{t^n}{n!} = \sqrt{\frac{1-x}{1 - xe^{2t(1-x)}}}. \tag{2}$$

It should be noted that the polynomials $M_n(x)$ and $N_n(x)$ are also enumerative polynomials of perfect matchings. A *perfect matching* of $[2n]$ is a partition of $[2n]$ into n blocks of size 2. Let $\mathcal{M}_{2n}$ be the set of perfect matchings of $[2n]$. Let $\mathrm{el}\,(\mathrm{M})$ (resp. $\mathrm{ol}\,(\mathrm{M})$) be the number of blocks of $\mathrm{M} \in \mathcal{M}_{2n}$ with even (resp. odd) larger entries. According to [17], we have

$$M_n(x) = \sum_{\mathrm{M} \in \mathcal{M}_{2n}} x^{\mathrm{ol}\,(\mathrm{M})},\ N_n(x) = \sum_{\mathrm{M} \in \mathcal{M}_{2n}} x^{\mathrm{el}\,(\mathrm{M})}.$$

Let $\#C$ denote the cardinality of a set C. Let $\mathfrak{S}_n$ denote the symmetric group of all permutations $\pi = \pi(1)\pi(2)\ldots\pi(n)$ of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. A *descent* of π is an index $i \in [n-1]$ such that $\pi(i) > \pi(i+1)$. For $\pi \in \mathfrak{S}_n$, let $\mathrm{des}\,(\pi)$ be the number of descents of π. The classical Eulerian polynomials are defined by

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{des}\,(\pi)}.$$

The *hyperoctahedral group* B_n is the group of signed permutations of the set $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all i, where $\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. Throughout this paper, we always identify a signed permutation $\pi = \pi(1) \cdots \pi(n)$ with the word $\pi(0)\pi(1) \cdots \pi(n)$, where $\pi(0) = 0$. For each $\pi \in B_n$, we define

$$\begin{aligned} \mathrm{des}_A(\pi) &= \#\{i \in [n-1] : \pi(i) > \pi(i+1)\}, \\ \mathrm{des}_B(\pi) &= \#\{i \in \{0, 1, 2 \ldots, n-1\} : \pi(i) > \pi(i+1)\}. \end{aligned}$$

It is clear that

$$\sum_{\pi \in B_n} x^{\mathrm{des}_A(\pi)} = 2^n A_n(x).$$

Following [1], the *flag descents* of $\pi \in B_n$ is defined by

$$\mathrm{fdes}\,(\pi) = \begin{cases} 2\mathrm{des}_A(\pi) + 1, & \text{if } \pi(1) < 0; \\ 2\mathrm{des}_A(\pi), & \text{otherwise.} \end{cases}$$

The Eulerian polynomial of type B and the flag descent polynomial are respectively defined by

$$B_n(x) = \sum_{\pi \in B_n} x^{\operatorname{des}_B(\pi)},\ F_n(x) = \sum_{\pi \in B_n} x^{\operatorname{fdes}(\pi)}.$$

Very recently, we studied the following combinatorial expansions (see [18, Section 4]):

$$2^n x A_n(x) = \sum_{k=0}^{n} \binom{n}{k} N_k(x) N_{n-k}(x),\ B_n(x) = \sum_{k=0}^{n} \binom{n}{k} N_k(x) M_{n-k}(x).$$

It is now well known that $F_n(x) = (1+x)^n A_n(x)$ (see [1, Theorem 4.4]). This paper is motivated by exploring an expansion of $F_n(x)$ in terms of some enumerative polynomials of Stirling permutations.

This paper is organized as follows. In Section 2, we present a combinatorial expansion of $F_n(x)$. In Section 3, we study a multivariate enumerative polynomials of Stirling permutations. In particular, we consider Foata and Strehl's group action on Stirling permutations.

2. The flag descent polynomials and flag ascent-plateau polynomials

Context-free grammar is a powerful tool to study exponential structures (see [5, 18] for instance). In this section, we first present a grammatical description of the flag descent polynomials by using grammatical labeling introduced by Chen and Fu [5]. And then, we study the flag ascent-plateau statistics over Stirling permutations.

2.1. **Context-free grammars.**

For an alphabet A, let $\mathbb{Q}[[A]]$ be the rational commutative ring of formal power series in monomials formed from letters in A. Following [4], a context-free grammar over A is a function $G : A \rightarrow \mathbb{Q}[[A]]$ that replaces a letter in A by a formal function over A. The formal derivative D is a linear operator defined with respect to a context-free grammar G. More precisely, the derivative $D = D_G$: $\mathbb{Q}[[A]] \rightarrow \mathbb{Q}[[A]]$ is defined as follows: for $x \in A$, we have $D(x) = G(x)$; for a monomial u in $\mathbb{Q}[[A]]$, $D(u)$ is defined so that D is a derivation, and for a general element $q \in \mathbb{Q}[[A]]$, $D(q)$ is defined by linearity.

Let us now recall a result on context-free grammars.

Proposition 2 ([13, Theorem 10])**.** *Let $A = \{x, y, z\}$ and*

$$G = \{x \rightarrow xyz, y \rightarrow yz^2, z \rightarrow y^2 z\}. \tag{3}$$

For $n \geq 0$, we have

$$D^n(xy) = xy \sum_{\pi \in B_n} y^{\operatorname{fdes}(\pi)} z^{2n - \operatorname{fdes}(\pi)}. \tag{4}$$

Moreover,

$$D^n(y^2)=y^2\sum_{\pi\in B_n}y^{2\mathrm{des}_A(\pi)}z^{2n-2\mathrm{des}_A(\pi)},$$
$$D^n(yz)=yz\sum_{\pi\in B_n}y^{2\mathrm{des}_B(\pi)}z^{2n-2\mathrm{des}_B(\pi)},$$
$$D^n(y)=y\sum_{\pi\in \mathcal{Q}_n}y^{2\mathrm{ap}\,(\sigma)}z^{2n-2\mathrm{ap}\,(\sigma)},$$
$$D^n(z)=z\sum_{\pi\in \mathcal{Q}_n}y^{2\mathrm{lap}\,(\sigma)}z^{2n-2\mathrm{lap}\,(\sigma)}.$$

The grammatical labeling is illustrated in the following proof of (4). Let $\pi\in B_n$. As usual, denote by $\bar{i}$ the negative element $-i$. We define an *ascent* (resp. a descent) of π to be a position $i\in\{0,1,2\ldots,n-1\}$ such that $\pi(i)<\pi(i+1)$ (resp. $\pi(i)>\pi(i+1)$). Now we give a labeling of $\pi\in B_n$ as follows:

(L_1) If $i\in[n-1]$ is an ascent, then put a superscript label z and a subscript label z right after $\pi(i)$;
(L_2) If $i\in[n-1]$ is a descent, then put a superscript label y and a subscript label y right after $\pi(i)$;
(L_3) If $\pi(1)>0$, then put a superscript label z and a subscript label x right after $\pi(0)$;
(L_4) If $\pi(1)<0$, then put a superscript label x and a subscript label y right after $\pi(0)$;
(L_5) Put a superscript label y and a subscript label z at the end of π.

Note that the weight of π is given by $w(\pi)=xy^{\mathrm{fdes}\,(\pi)+1}z^{\mathrm{fasc}\,(\pi)+1}$.

Let

$$F_n(i,j)=\{\pi\in B_n:\mathrm{fdes}\,(\pi)=i,\mathrm{fasc}\,(\pi)=j\}.$$

When $n=1$, we have $F_1(0,1)=\{0_x^z1_z^y\}$ and $F_1(1,0)=\{0_y^x\overline{1}_z^y\}$. Note that $D(xy)=xyz^2+xy^2z$. Thus the sum of weights of the elements of B_1 is given by $D(xy)$.

Suppose we get all labeled permutations in $F_n(i,j)$ for all i,j,k, where $n\geq 1$. Let $\pi'\in B_{n+1}$ be obtained from $\pi\in F_n(i,j)$ by inserting the entry $n+1$ or $\overline{n+1}$. We distinguish the following five cases:

(c_1) Let $i\in[n-1]$ be an ascent. If we insert $n+1$ (resp. $\overline{n+1}$) right after $\pi(i)$, then $\pi'\in F_{n+1}(i+2,j)$, and the insertion of $n+1$ (resp. $\overline{n+1}$) corresponds to applying the rule $z\to y^2z$ to the superscript (resp. subscript) label z associated with $\pi(i)$.
(c_2) Let $i\in[n-1]$ be a descent. If we insert $n+1$ (resp. $\overline{n+1}$) right after $\pi(i)$, then $\pi'\in F_{n+1}(i,j+2)$, and the insertion of $n+1$ (resp. $\overline{n+1}$) corresponds to applying the rule $y\to yz^2$ to the superscript (resp. subscript) label y associated with $\pi(i)$.
(c_3) If we insert $n+1$ (resp. $\overline{n+1}$) at the end of π, then $\pi'\in F_{n+1}(i,j+2)$ (resp. $\pi'\in F_{n+1}(i+2,j)$), and the insertion of $n+1$ (resp. $\overline{n+1}$) corresponds to applying the rule $y\to yz^2$ (resp. $z\to y^2z$) to the label y (resp. z) at the end of π;
(c_4) If $\pi(1)>0$ and we insert $n+1$ (resp. $\overline{n+1}$) immediately before $\pi(1)$, then $\pi'\in F_{n+1}(i+2,j)$ (resp. $\pi'\in F_{n+1}(i+1,j+1)$), and the insertion of $n+1$ (resp. $\overline{n+1}$

corresponds to applying the rule $z \to y^2 z$ (resp. $x \to xyz$) to the label z (resp. x) right after $\pi(0)$;

(c_5) If $\pi(1) < 0$ and we insert $n+1$ (resp. $\overline{n+1}$) immediately before $\pi(1)$, then $\pi' \in F_{n+1}(i+1, j+1)$ (resp. $\pi' \in F_{n+1}(i, j+2)$), and the insertion of $n+1$ (resp. $\overline{n+1}$ corresponds to applying the rule $x \to xyz$ (resp. $y \to yz^2$) to the label x (resp. y) right after $\pi(0)$.

In general, the insertion of $n+1$ (resp. $\overline{n+1}$) into π corresponds to the action of the formal derivative D on a superscript label (resp. subscript label). By induction, we get a grammatical proof of (4).

Example 3. *For example, let* $\pi = 04\overline{3}152$. *Then* π *can be generated as follows:*

$$0_x^z 1_z^{\boldsymbol{y}} \mapsto 0_x^z 1_z^z 2_z^y;$$
$$0_{\boldsymbol{x}}^z 1_z^z 2_z^y \mapsto 0_y^x \overline{3}_z^z 1_z^z 2_z^y;$$
$$0_y^{\boldsymbol{x}} \overline{3}_z^z 1_z^z 2_z^y \mapsto 0_x^z 4_y^y \overline{3}_z^z 1_z^z 2_z^y;$$
$$0_x^z 4_y^y \overline{3}_z^z 1_z^{\boldsymbol{z}} 2_z^y \mapsto 0_x^z 4_y^y \overline{3}_z^z 1_z^z 5_y^y 2_z^y.$$

2.2. **The flag ascent-plateau statistic.**

Definition 4. *Let* $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{Q}_n$. *The number of flag ascent-plateau of* σ *is defined by*

$$\mathrm{fap}\,(\sigma) = \begin{cases} 2\mathrm{ap}\,(\sigma) + 1, & \text{if } \sigma_1 = \sigma_2; \\ 2\mathrm{ap}\,(\sigma), & \text{otherwise.} \end{cases}$$

We can now present the first main result of this paper.

Theorem 5. *Let* D *be the formal derivative with respect to the grammar* (3). *For* $n \geq 1$, *we have*

$$D^n(x) = x \sum_{\sigma \in \mathcal{Q}_n} y^{\mathrm{fap}\,(\sigma)} z^{2n - \mathrm{fap}\,(\sigma)}. \tag{5}$$

Therefore,

$$\sum_{\pi \in B_n} x^{\mathrm{fdes}\,(\pi)} = \sum_{k=0}^{n} \binom{n}{k} \sum_{\sigma \in \mathcal{Q}_k} x^{\mathrm{fap}\,(\sigma)} \sum_{\sigma \in \mathcal{Q}_{n-k}} x^{2\mathrm{ap}\,(\sigma)}. \tag{6}$$

Proof. We first introduce a grammatical labeling of $\sigma \in \mathcal{Q}_n$ as follows:

(L_1) If $i \in \{2, 3, \ldots, 2n-1\}$ is an ascent-plateau, then put a superscript label y immediately before σ_i and a superscript label y right after σ_i;

(L_2) If $\sigma_1 = \sigma_2$, then put a superscript label y immediately before σ_1 and a superscript x right after σ_1;

(L_3) If $\sigma_1 < \sigma_2$, then put a superscript label x immediately before σ_1;

(L_4) The rest of positions in σ are labeled by a superscript label z.

Note that the weight of σ is given by

$$w(\sigma) = xy^{\mathrm{fap}\,(\sigma)} z^{2n - \mathrm{fap}\,(\sigma)}.$$

For example, The labeling of 1223314554 and 661223314554 are respectively given as follows:

$$^x1^y2^y2^y3^y3^z1^z4^y5^y5^z4^z,\ ^y6^x6^z1^y2^y2^y3^y3^z1^z4^y5^y5^z4^z.$$

We then prove (5) by induction. Let $S_n(i)=\{\sigma\in\mathcal{Q}_n : \mathrm{fap}\,(\sigma)=i\}$. For $n=1$, we have $S_1(1)=\{^y1^x1^z\}$. For $n=2$, the elements of $\mathcal{Q}_2$ are labeled as follows:

$$S_2(1)=\{^y2^x2^z1^z1^z\},\ S_2(2)=\{^x1^y2^y2^z1^z\},\ S_2(3)=\{^y1^x1^y2^y2^z\}.$$

Note that $D(x)=xyz$ and $D^2(x)=xyz(y^2+yz+z^2)$. Hence the result holds for $n=1,2$. Suppose we get all labeled Stirling permutations of $S_n(i)$ for all i, where $n\geq 2$. Let $\sigma'\in\mathcal{Q}_{n+1}$ be obtained from $\sigma\in S_n(i)$ by inserting the pair $(n+1)(n+1)$ into σ. We distinguish the following three cases:

(c_1) If $\sigma_1=\sigma_2$ and the pair $(n+1)(n+1)$ is inserted at the front of σ, then the change of labeling is illustrated as follows:

$$^y\sigma_1^x\sigma_2\cdots\mapsto^y(n+1)^x(n+1)^z\sigma_1^z\sigma_2\cdots.$$

In this case, the insertion corresponds to the rule $y\mapsto yz^2$ and $\sigma'\in S_{n+1}(i)$;

(c_2) If $\sigma_1<\sigma_2$ and the pair $(n+1)(n+1)$ is inserted at the front of σ, then the change of labeling is illustrated as follows:

$$^x\sigma_1\cdots\mapsto^y(n+1)^x(n+1)^z\sigma_1\cdots.$$

In this case, the insertion corresponds to the rule $x\mapsto xyz$ and $\sigma'\in S_{n+1}(i+1)$;

(c_3) If i is an ascent plateau of σ, and the pair $(n+1)(n+1)$ is inserted immediately before or right after σ_i, then the change of labeling are illustrated as follows:

$$\cdots\sigma_{i-1}^y\sigma_i^y\sigma_{i+1}\cdots\mapsto\cdots\sigma_{i-1}^y(n+1)^y(n+1)^z\sigma_i^z\sigma_{i+1}\cdots,$$

$$\cdots\sigma_{i-1}^y\sigma_i^y\sigma_{i+1}\cdots\mapsto\cdots\sigma_{i-1}^z\sigma_i^y(n+1)^y(n+1)^z\sigma_{i+1}\cdots.$$

In this case, the insertion corresponds to the rule $y\mapsto yz^2$ and $\sigma'\in S_{n+1}(i)$;

(c_4) If the pair $(n+1)(n+1)$ is inserted to a position with the label z, then the change of labeling are illustrated as follows:

$$\cdots\sigma_i^z\cdots\mapsto\cdots\cdots\sigma_i^y(n+1)^y(n+1)^z\cdots.$$

In this case, the insertion corresponds to the rule $z\mapsto y^2z$ and $\sigma'\in S_{n+1}(i+2)$.

It is routine to check that each element of $\mathcal{Q}_{n+1}$ can be obtained exactly once. By induction, we present a constructive proof of (5). Using the Leibniz's formula, we have $D^n(xy)=\sum_{k=0}^n D^k(x)D^{n-1}(y)$. Combining (5) and Proposition 2, we get the desired formula (6). □

Let

$$T_n(x)=\sum_{\sigma\in\mathcal{Q}_n}x^{\mathrm{fap}\,(\sigma)}=\sum_{k\geq 1}T(n,k)x^k.$$

From the proof of (5), we see that the numbers $T(n,k)$ satisfy the recurrence relation

$$T(n+1,k)=kT(n,k)+T(n,k-1)+(2n-k+2)T(n,k-2).$$

with the initial conditions $T(0,0)=1,\ T_(1,1)=1$ and $T(1,k)=0$ for $k\neq 1$. It should be noted that $T(n,k)$ is also the number of dual Stirling permutations of order n with k alternating runs (see [16]). Recall that (see [20, A008292]):

$$A(x,t)=\sum_{n\geq 0}A_n(x)\frac{t^n}{n!}=\frac{x-1}{x-e^{t(x-1)}}.$$

Hence

$$F(x,t)=\frac{x-1}{x-e^{t(x^2-1)}}.$$

Let $T(x,t)=\sum_{n\geq 0}T_n(x)\frac{t^n}{n!}$. Write the formula (6) as follows:

$$F_n(x)=\sum_{k=0}^{n}\binom{n}{k}T_k(x)M_{n-k}(x^2).$$

Thus, $F(x,t)=T(x,t)M(x^2,t)$. Combining (1), we get

$$T(x,t)=\frac{F(x,t)}{M(x^2,t)}=\frac{x-1}{x-e^{t(x^2-1)}}\sqrt{\frac{x^2-e^{2t(x^2-1)}}{x^2-1}}. \tag{7}$$

Combining (2) and (7), we have

$$T(x,t)N(x^2,t)=1-x+xA(x,t(1+x)).$$

Therefore, a dual formula of (6) is given as follows:

$$\sum_{\pi\in B_n}x^{\mathrm{fdes}\,(\pi)+1}=\sum_{k=0}^{n}\binom{n}{k}\sum_{\sigma\in\mathcal{Q}_k}x^{\mathrm{fap}\,(\sigma)}\sum_{\sigma\in\mathcal{Q}_{n-k}}x^{2\mathrm{lap}\,(\sigma)}.$$

for $n\geq 1$.

Let $\delta_{i,j}$ be the Kronecker delta, i.e., $\delta_{i,j}=1$ if $i=j$ and $\delta_{i,j}=0$ if $i\neq j$. It is not hard to verify that $T(x,t)T(-x,t)=1$. In other words,

$$\sum_{k=0}^{n}\binom{n}{k}T_k(x)T_{n-k}(-x)=\delta_{0,n}.$$

3. Multivariate polynomials over Stirling polynomials

Let

$$C_n(x)=\sum_{\sigma\in\mathcal{Q}_n}x^{\mathrm{asc}\,(\sigma)}.$$

The polynomials $C_n(x)$ and $N_n(x)$ respectively satisfy the following recurrence relation

$$C_{n+1}(x)=(2n+1)xC_n(x)+x(1-x)C_n'(x),$$

$$N_{n+1}(x)=(2n+1)xN_n(x)+2x(1-x)N_n'(x),$$

with the initial conditions $C_0(x)=N_0(x)=1$ (see [2, 8, 14] for instance). In this section, we shall present a unified refinement of the polynomials $C_n(x)$ and $N_n(x)$.

In the sequel, we always assume that Stirling permutations are prepended by 0. That is, we identify an n-Stirling permutation $\sigma_1\sigma_2\cdots\sigma_{2n}$ with the word $\sigma_0\sigma_1\sigma_2\cdots\sigma_{2n}$, where $\sigma_0=0$.

3.1. **A grammatical labeling of Stirling permutations.**

Definition 6. *Let $\sigma=\sigma_1\sigma_2\cdots\sigma_{2n}\in\mathcal{Q}_n$. For $1\leq i\leq 2n$, a double ascent of σ is an index i such that $\sigma_{i-1}<\sigma_i<\sigma_{i+1}$, a descent-plateau of σ is an index i such that $\sigma_{i-1}>\sigma_i=\sigma_{i+1}$.*

Let $\operatorname{dasc}(\sigma)$ and $\operatorname{dp}(\sigma)$ denote the numbers of double ascents and descent-plateaus of σ, respectively. For example, $\operatorname{dasc}(\mathbf{2}4433211\mathbf{5}665)=2$ and $\operatorname{dp}(244\mathbf{3}32\mathbf{1}15665)=2$. It is clear that

$$\operatorname{asc}(\sigma)=\operatorname{lap}(\sigma)+\operatorname{dasc}(\sigma),\operatorname{plat}(\sigma)=\operatorname{lap}(\sigma)+\operatorname{dp}(\sigma). \tag{8}$$

Define

$$P_n(x,y,z)=\sum_{\sigma\in\mathcal{Q}_n}x^{\operatorname{lap}(\sigma)}y^{\operatorname{dasc}(\sigma)}z^{\operatorname{dp}(\sigma)}=\sum_{i,j,k}P_n(i,j,k)x^iy^jz^k,$$

where $1\leq i\leq n, 0\leq j\leq n-1, 0\leq k\leq n-1$. In particular,

$$P_n(x,x,1)=P_n(x,1,x)=C_n(x),\ P_n(x,1,1)=N_n(x).$$

The first few of the polynomials $P_n(x,y,z)$ are given as follows:

$$\begin{aligned}
&P_1(x,y,z)=x,\\
&P_2(x,y,z)=xy+xz+x^2,\\
&P_3(x,y,z)=x(y^2+z^2)+4x^2(y+z)+2xyz+2x^2+x^3.
\end{aligned}$$

Now we present the second main result of this paper.

Theorem 7. *Let $A=\{x,y,z,p,q\}$ and*

$$G=\{x\rightarrow xzq, y\rightarrow yzp, z\rightarrow xyz, p\rightarrow xyz, q\rightarrow xyz\}. \tag{9}$$

Then

$$D^n(z)=z\sum_{i,j,k}P_n(i,j,k)(xy)^iq^jp^kz^{2n-2i-j-k},$$

where $1\leq i\leq n, 0\leq j\leq n-1, 0\leq k\leq n-1$ and $2i+j+k\leq 2n$. Set $P_n=P_n(x,y,z)$. Then the polynomials $P_n(x,y,z)$ satisfy the recurrence relation

$$P_{n+1}=(2n+1)xP_n+(xy+xz-2x^2)\frac{\partial}{\partial x}P_n+x(1-y)\frac{\partial}{\partial y}P_n+x(1-z)\frac{\partial}{\partial z}P_n, \tag{10}$$

with the initial condition $P_0(x,y,z)=1$.

Proof. Now we give a labeling of $\sigma\in\mathcal{Q}_n$ as follows:

- (L_1) If i is a left ascent-plateau, then put a superscript label y immediately before σ_i and a superscript label x right after σ_i;
- (L_2) If i is a double ascent, then put a superscript label q immediately before σ_i;
- (L_3) If i is a descent-plateau, then put a superscript label p right after σ_i;
- (L_4) The rest positions in σ are labeled by a superscript label z.

The weight of σ is defined by

$$w(\sigma)=z(xy)^{\mathrm{lap}\,(\sigma)}q^{\mathrm{dasc}\,(\sigma)}p^{\mathrm{dp}\,(\sigma)}z^{2n-2\mathrm{lap}\,(\sigma)-\mathrm{dasc}\,(\sigma)-\mathrm{dp}\,(\sigma)}.$$

For example, the labeling of 552442998813316776 is as follows:

$${}^y5^x5^z2^y4^x4^z2^y9^x9^z8^p8^z1^y3^x3^z1^q6^y7^x7^z6^z.$$

We proceed by induction on n. Note that $\mathcal{Q}_1=\{{}^y1^x1^z\}$ and

$$\mathcal{Q}_2=\{{}^y1^x1^y2^x2^z,\ {}^q1^y2^x2^z1^z,\ {}^y2^x2^z1^p1^z\}.$$

Thus the weight of ${}^y1^x1^z$ is given by $D(z)$ and the sum of weights of elements in $\mathcal{Q}_2$ is given by $D^2(z)$, since $D(z)=xyz$ and $D^2(x)=z(xyqz+xypz+x^2y^2)$.

Assume that the result holds for $n=m-1$, where $m\geq 3$. Let σ be an element counted by $P_{m-1}(i,j,k)$, and let σ' be an element of $\mathcal{Q}_m$ obtained by inserting the pair mm into σ. We distinguish the following five cases:

(c_1) If the pair mm is inserted at a position with label x, then the change of labeling is illustrated as follows:

$$\cdots\sigma_{i-1}^y\sigma_i^x\sigma_{i+1}\cdots\mapsto\cdots\sigma_{i-1}^q\sigma_i^y m^x m^z\sigma_{i+1}\cdots.$$

In this case, the insertion corresponds to the rule $x\mapsto xzq$ and produces i permutations in $\mathcal{Q}_m$ with i left ascent-plateaus, $j+1$ double ascents and k descent-plateaus;

(c_2) If the pair mm is inserted at a position with label y, then the change of labeling is illustrated as follows:

$$\cdots\sigma_{i-1}^y\sigma_i^x\sigma_{i+1}\cdots\mapsto\cdots\sigma_{i-1}^y m^x m^z\sigma_i^p\sigma_{i+1}\cdots.$$

In this case, the insertion corresponds to the rule $y\mapsto yzp$ and produces i permutations in $\mathcal{Q}_m$ with i left ascent-plateaus, j double ascents and $k+1$ descent-plateaus;

(c_3) If the pair mm is inserted at a position with label z, then the change of labeling is illustrated as follows:

$$\cdots\sigma_i^z\sigma_{i+1}\cdots\mapsto\cdots\sigma_i^y m^x m^z\sigma_{i+1}\cdots.$$

In this case, the insertion corresponds to the rule $z\mapsto xyz$ and produces $2m-2-2i-j-k$ permutations in $\mathcal{Q}_m$ with $i+1$ left ascent-plateaus, j double ascents and k descent-plateaus;

(c_4) If the pair mm is inserted at a position with label q, then the change of labeling is illustrated as follows:

$$\cdots\sigma_i^q\sigma_{i+1}\cdots\mapsto\cdots\sigma_i^y m^x m^z\sigma_{i+1}\cdots.$$

In this case, the insertion corresponds to the rule $q\mapsto xyz$ and produces j permutations in $\mathcal{Q}_m$ with $i+1$ left ascent-plateaus, $j-1$ double ascents and k descent-plateaus;

(c_5) If the pair mm is inserted at a position with label p, then the change of labeling is illustrated as follows:

$$\cdots\sigma_i^p\sigma_{i+1}\cdots\mapsto\cdots\sigma_i^y m^x m^z\sigma_{i+1}\cdots.$$

In this case, the insertion corresponds to the rule $p \mapsto xyz$ and produces k permutations in $\mathcal{Q}_m$ with $i+1$ left ascent-plateaus, j double ascents and $k-1$ descent-plateaus.

By induction, we see that grammar (9) generates all of the permutations in $\mathcal{Q}_m$.

Combining the above five cases, we see that

$$\begin{aligned} P_{n+1}(i,j,k) =& iP_n(i,j-1,k) + iP_n(i,j,k-1) + (j+1)P_n(i-1,j+1,k) + \\ & (k+1)P_n(i-1,j,k+1) + (2n+3-2i-j-k)P_n(i-1,j,k). \end{aligned}$$

Multiplying both sides of the above recurrence relation by $x^iy^jz^k$ for all i,j,k, we get (10) $\square$

3.2. **Equidistributed statistics.**

Let $i \in [2n]$ and let $\sigma = \sigma_1\sigma_2\ldots\sigma_{2n} \in \mathcal{Q}_n$. We define the action φ_i as follows:

- If i is a double ascent, then $\varphi_i(\sigma)$ is obtained by moving σ_i to the right of the second σ_i, which forms a new pleateau $\sigma_i\sigma_i$;
- If i is a descent-plateau, then $\varphi_i(\sigma)$ is obtained by moving σ_i to the right of σ_k, where $k = \max\{j \in \{0,1,2,\ldots,i-1\} : \sigma_j < \sigma_i\}$.

For instance, if $\sigma = 2447887332115665$, then

$$\varphi_1(\sigma) = 4478873322115665,\ \ \varphi_4(\sigma) = 2448877332115665,$$

and $\varphi_9(\varphi_1(\sigma)) = \varphi_6(\varphi_4(\sigma)) = \sigma$. In recent years, the Foata and Strehl's group action has been extensively studied (see [3, 12] for instance). We define the Foata-Strehl action on Stirling permutations by

$$\varphi_i'(\sigma) = \begin{cases} \varphi_i(\sigma), & \text{if } i \text{ is a double ascent or descent-plateau;} \\ \sigma, & \text{otherwise.} \end{cases}$$

It is clear that the φ_i''s are involutions and that they commute. Hence, for any subset $S \subseteq [2n]$, we may define the function $\varphi_S' : \mathcal{Q}_n \mapsto \mathcal{Q}_n$ by $\varphi_S'(\sigma) = \prod_{i\in S}\varphi_i'(\sigma)$. Hence the group $\mathbb{Z}_2^{2n}$ acts on $\mathcal{Q}_n$ via the function φ_S', where $S \subseteq [2n]$.

The third main result of this paper is given as follows, which is implied by (10).

Theorem 8. *For any $n \geq 1$, we have*

$$P_n(x,y,z) = P_n(x,z,y). \tag{11}$$

Furthermore,

$$\sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{lap}(\sigma)}y^{\operatorname{asc}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{lap}(\sigma)}y^{\operatorname{plat}(\sigma)}. \tag{12}$$

Proof. For any $\sigma \in \mathcal{Q}_n$, we define

$$\begin{aligned} \operatorname{Dasc}(\sigma) &= \{i \in [2n] : \sigma_{i-1} < \sigma_i < \sigma_{i+1}\}, \\ \operatorname{DP}(\sigma) &= \{i \in [2n] : \sigma_{i-1} > \sigma_i = \sigma_{i+1}\}, \\ \operatorname{LAP}(\sigma) &= \{i \in [2n] : \sigma_{i-1} < \sigma_i = \sigma_{i+1}\}. \end{aligned}$$

Let $S = S(\sigma) = \operatorname{Dasc}(\sigma) \cup \operatorname{DP}(\sigma)$. Note that

$$\operatorname{Dasc}(\varphi_S'(\sigma)) = \operatorname{DP}(\sigma),\ \operatorname{DP}(\varphi_S'(\sigma)) = \operatorname{Dasc}(\sigma) \text{ and } \operatorname{LAP}(\varphi_S'(\sigma)) = \operatorname{LAP}(\sigma).$$

Therefore,

$$\begin{aligned} P_n(x,y,z) &= \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma)} y^{\mathrm{dasc}\,(\sigma)} z^{\mathrm{dp}\,(\sigma)} \\ &= \sum_{\sigma'\in\mathcal{Q}_n,} x^{\mathrm{lap}\,(\varphi'_{S(\sigma)}(\sigma))} y^{\mathrm{dp}\,(\varphi'_{S(\sigma)}(\sigma)))} z^{\mathrm{dasc}\,(\varphi'_{S(\sigma)}(\sigma)))} \\ &= \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma)} z^{\mathrm{dasc}\,(\sigma)} y^{\mathrm{dp}\,(\sigma)} \\ &= P_n(x,z,y). \end{aligned}$$

Combining (8) and (11), we see that $P_n(xy,y,1) = P_n(xy,1,y)$. This completes the proof. □

Theorem 9. *For $n \geq 1$, we have*

$$\sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma)} y^{\mathrm{dasc}\,(\sigma)} z^{\mathrm{dp}\,(\sigma)} = \sum_{\substack{1\leq i\leq n\\ 0\leq j\leq n-1}} \gamma_{n,i,j} x^i (y+z)^j,$$

where

$$\gamma_{n,i,j} = \#\{\sigma\in\mathcal{Q}_n : \mathrm{lap}\,(\sigma) = i, \mathrm{dasc}\,(\sigma) = j, \mathrm{dp}\,(\sigma) = 0\}.$$

Proof. Define

$$\mathrm{NDP}_{\,n,i,j} = \{\sigma\in\mathcal{Q}_n : \mathrm{lap}\,(\sigma) = i, \mathrm{dasc}\,(\sigma) = j, \mathrm{dp}\,(\sigma) = 0\}.$$

For any $\sigma \in \mathrm{NDP}_{\,n,i,j}$, let

$$[\sigma] = \{\varphi'_S(\sigma) \mid S \subseteq \mathrm{Dasc}\,(\sigma)\}.$$

For any $\sigma' \in [\sigma]$, suppose that $\sigma' = \varphi'_S(\sigma)$ for some $S \subseteq \mathrm{Dasc}\,(\sigma)$. Then

$$\mathrm{lap}\,(\sigma') = \mathrm{lap}\,(\sigma), \mathrm{dasc}\,(\sigma') = \mathrm{dasc}\,(\sigma) - |S| \text{ and } \mathrm{dp}\,(\sigma') = |S|.$$

Moreover, $\{[\sigma] \mid \sigma \in \mathrm{NDP}_{\,n,i,j}\}$ form a partition of $\mathcal{Q}_n$. Hence,

$$\begin{aligned} &\sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma)} y^{\mathrm{dasc}\,(\sigma)} z^{\mathrm{dp}\,(\sigma)} \\ =& \sum_{\sigma\in\mathrm{NDP}_{\,n}} \sum_{\sigma'\in[\sigma]} x^{\mathrm{lap}\,(\sigma')} y^{\mathrm{dasc}\,(\sigma')} z^{\mathrm{dp}\,(\sigma')} \\ =& \sum_{\sigma\in\mathrm{NDP}_{\,n}} \sum_{S\subseteq\mathrm{Dasc}\,(\sigma)} x^{\mathrm{lap}\,(\varphi'_S(\sigma))} y^{\mathrm{dasc}\,(\varphi'_S(\sigma))} z^{\mathrm{dp}\,(\varphi'_S(\sigma))} \\ =& \sum_{\sigma\in\mathrm{NDP}_{\,n}} \sum_{S\subseteq\mathrm{Dasc}\,(\sigma)} x^{\mathrm{lap}\,(\sigma)} y^{\mathrm{dasc}\,(\sigma)-|S|} z^{|S|} \\ =& \sum_{\sigma\in\mathrm{NDP}_{\,n}} x^{\mathrm{lap}\,(\sigma)} \sum_{S\subseteq\mathrm{Dasc}\,(\sigma)} y^{\mathrm{dasc}\,(\sigma)-|S|} z^{|S|} \\ =& \sum_{\sigma\in\mathrm{NDP}_{\,n}} x^{\mathrm{lap}\,(\sigma)} (y+z)^{\mathrm{dasc}\,(\sigma)} \\ =& \sum_{i,j} \gamma_{n,i,j} x^i (y+z)^j. \end{aligned}$$

□

Taking $y = z = 1$ in Theorem 9, we have

$$N_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma)} = \sum_{i=1}^{n} \left(\sum_{j=0}^{n-1} 2^j \gamma_{n,i,j} \right) x^i.$$

Let $N_n(x) = \sum_{k=1}^{n} N(n,k)x^k$. According to [14, Eq. (24)],

$$N_n(x) = \sum_{k=1}^{n} 2^{n-2k} \binom{2k}{k} k! \left\{ {n \atop k} \right\} x^k (1-x)^{n-k}.$$

Thus, for $n \geq 1$, we have

$$\sum_{j=0}^{n-1} 2^j \gamma_{n,i,j} = \sum_{j=1}^{i} (-1)^{i-j} 2^{n-2j} \binom{2j}{j} \binom{n-j}{i-j} j! \left\{ {n \atop j} \right\}.$$

Theorem 10. *Let $A = \{u, v, w\}$ and $G = \{u \to uvw, v \to 2uw, w \to uw\}$. Then*

$$D^n(w) = \sum_{\substack{1 \leq i \leq n \\ 0 \leq j \leq n-1}} \gamma_{n,i,j} u^i v^j w^{2n+1-2i-j}. \tag{13}$$

Furthermore, the numbers $\gamma_{n,i,j}$ satisfy the recurrence relation

$$\gamma_{n+1,i,j} = i\gamma_{n,i,j-1} + 2(j+1)\gamma_{n,i-1,j+1} + (2n+3-2i-j)\gamma_{n,i-1,j}, \tag{14}$$

with the initial conditions $\gamma_{1,1,0} = 1$ and $\gamma_{1,i,j} = 0$ for $i > 1$ and $j \geq 0$.

Proof. From the grammar (9), we see that

$$D(xy) = xyz(p+q),$$
$$D(p+q) = 2xyz,$$
$$D(z) = xyz.$$

Set $u = xy, v = p+q$ and $w = z$. Then $D(u) = uvw, D(v) = 2uw$ and $D(w) = uw$. Combining Theorem 7 and Theorem 9, we get (13). Since $D^{n+1}(w) = D(D^n(w))$, we obtain that

$$\begin{aligned} D^{n+1}(w) &= D\left(\sum_{i,j} \gamma_{n,i,j} u^i v^j w^{2n+1-2i-j} \right) \\ &= \sum_{i,j} i\gamma_{n,i,j} u^i v^{j+1} w^{2n+2-2i-j} + 2\sum_{i,j} j\gamma_{n,i,j} u^{i+1} v^{j-1} w^{2n+2-2i-j} + \\ &\sum_{i,j} (2n+1-2i-j)\gamma_{n,i,j} u^{i+1} v^j w^{2n+1-2i-j}. \end{aligned}$$

Equating the coefficients of $u^i v^j w^{2n+1-2i-j}$ on both sides of the above equation, we obtain (14). □

Let $G_n(x,y) = \sum_{i,j} \gamma_{n,i,j} x^i y^j$. Multiplying both sides of the recurrence relation (14) by $x^i y^j$ for all i, j, we get that

$$G_{n+1}(x,y) = (2n+1)xG_n(x,y) + (xy - 2x^2)\frac{\partial}{\partial x} G_n(x,y) + (2x - xy)\frac{\partial}{\partial y} G_n(x,y). \tag{15}$$

The first few of the polynomials $G_n(x,y)$ are given as follows:

$$G_0(x,y)=1, G_1(x,y)=x, G_2(x,y)=xy+x^2, G_3(x,y)=xy^2+4x^2y+2x^2+x^3.$$

3.3. Connection with Eulerian numbers.

Recall that the *Eulerian numbers* are defined by

$$\left\langle {n \atop k} \right\rangle = \#\{\pi\in\mathfrak{S}_n : \operatorname{des}(\pi)=k\}.$$

The numbers $\left\langle {n \atop k} \right\rangle$ satisfy the recurrence relation

$$\left\langle {n+1 \atop k} \right\rangle = (k+1)\left\langle {n \atop k} \right\rangle + (n+1-k)\left\langle {n \atop k-1} \right\rangle,$$

with the initial conditions $\left\langle {1 \atop 0} \right\rangle=1$ and $\left\langle {1 \atop k} \right\rangle=0$ for $k\geq 1$.

Theorem 11. *For $n\geq 1$ and $0\leq k\leq n-1$, we have*

$$\gamma_{n,n-k,k} = \left\langle {n \atop k} \right\rangle.$$

Proof. Set $a(n,k)=\gamma_{n,n-k,k}$. Then $a(n,k-1)=\gamma_{n,n-k+1,k-1}$. Using (14), it is easy to verify that

$$\gamma_{n,i,j}=0 \qquad \text{for } i+j>n.$$

Hence $\gamma_{n,n-k,k+1}=0$. Therefore, the numbers $a(n,k)$ satisfy the recurrence relation

$$a(n+1,k)=(k+1)a(n,k)+(n+1-k)a(n,k-1).$$

Since the numbers $a(n,k)$ and $\left\langle {n \atop k} \right\rangle$ satisfy the same recurrence relation and initial conditions, so they agree. This completes the proof. □

A bijective proof of Theorem 11:

Proof. Let $\sigma\in\mathcal{Q}_n$. Note that every element of $[n]$ appears exactly two times in σ. Let $\alpha(\sigma)$ be the permutation of $\mathfrak{S}_n$ obtained from σ by deleting all of the first i from left to right, where $i\in[n]$. Then α is a map from $\mathcal{Q}_n$ to $\mathfrak{S}_n$. For example, $\alpha(\mathbf{34}43\mathbf{5}5\mathbf{6}6\mathbf{12}21)=435621$. Let

$$\mathcal{D}_n=\{\sigma\in\mathcal{Q}_n : \operatorname{lap}(\sigma)=i, \operatorname{dasc}(\sigma)=n-i, \operatorname{dp}(\sigma)=0\}.$$

Let x be a given element of $[n]$. For any $\sigma\in\mathcal{Q}_n$, we define the action β_x on $\mathcal{Q}_n$ as follows:

- Read σ from left to right and let i be the first index such that $\sigma_i=x$;
- Move σ_i to the right of σ_k, where $k=\max\{j\in\{0,1,2,\ldots,i-1\} : \sigma_j<\sigma_i\}$, where $\sigma_0=0$.

For example, if $\sigma=3443578876652211$, then

$$\beta_1(\sigma)=\mathbf{1}344357887665221,\ \beta_2(\sigma)=\mathbf{2}344357887665211,\ \beta_6(\sigma)=34435\mathbf{6}7887652211.$$

It is clear that $\beta_x(\beta_y(\sigma))=\beta_y(\beta_x(\sigma))$ for any $x,y\in[n]$. For any $S\subseteq[n]$, let $\beta_S:\mathcal{Q}_n\mapsto\mathcal{Q}_n$ be a function defined by

$$\beta_S(\sigma)=\prod_{x\in S}\beta_x(\sigma).$$

It is easy to verify that

$$\beta_{[n]}(\sigma) \in \mathcal{D}_n,\ \alpha(\sigma) = \alpha(\beta_{[n]}(\sigma)), \beta_{[n]}(\sigma) = \sigma \text{ if } \sigma \in \mathcal{D}_n.$$

Let $\alpha|_{\mathcal{D}_n}$ denote the restriction of the map α on the set $\mathcal{D}_n$. Then $\alpha|_{\mathcal{D}_n}$ is a map from $\mathcal{D}_n$ to $\mathfrak{S}_n$. Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. The inverse $\alpha|_{\mathcal{D}_n}^{-1}$ is defined as follows:

- let $\sigma = \sigma_1\sigma_2\ldots\sigma_{2n}$ be the Stirling permutation such that $\sigma_{2i-1} = \sigma_{2i} = \pi(i)$ for each $i = 1, 2, \ldots, n$;
- let $S(\pi) = \{\pi_i : \pi_{i-1} > \pi_i, 2 \le i \le n\}$;
- let $\alpha|_{\mathcal{D}_n}^{-1}(\pi) = \beta_{S(\pi)}(\sigma)$.

Note that

$$\operatorname{lap}(\alpha|_{\mathcal{D}_n}^{-1}(\pi)) + \operatorname{dasc}(\alpha|_{\mathcal{D}_n}^{-1}(\pi)) = n \text{ and } \operatorname{dasc}(\alpha|_{\mathcal{D}_n}^{-1}(\pi)) = \operatorname{des}(\pi).$$

Then $\alpha|_{\mathcal{D}_n}$ is a bijection from $\mathcal{D}_n$ to $\mathfrak{S}_n$. This completes the proof. □

Example 12. *The bijection between $\mathfrak{S}_3$ and $\mathcal{D}_3$ is demonstrated as follows:*

$$\begin{aligned}
&123 \leftrightarrow 112233\ (S = \emptyset) \leftrightarrow \beta_S(112233) = 112233;\\
&132 \leftrightarrow 113322\ (S = \{2\}) \leftrightarrow \beta_S(113322) = 112332;\\
&213 \leftrightarrow 221133\ (S = \{1\}) \leftrightarrow \beta_S(221133) = 122133;\\
&231 \leftrightarrow 223311\ (S = \{1\}) \leftrightarrow \beta_S(223311) = 122331;\\
&312 \leftrightarrow 331122\ (S = \{1\}) \leftrightarrow \beta_S(331122) = 133122;\\
&321 \leftrightarrow 332211\ (S = \{1,2\}) \leftrightarrow \beta_S(332211) = 123321.
\end{aligned}$$

4. Concluding remarks

In this paper, we introduce several variants of the ascent-plateau statistic on Stirling permutations. Recall that Park [19] studied the (p,q)-analogue of the descent polynomials of Stirling permutations:

$$C_n(x,p,q) = \sum_{\sigma \in \mathcal{Q}_n} x^{\operatorname{des}(\sigma)} p^{\operatorname{inv}(\sigma)} q^{\operatorname{maj}(\sigma)}.$$

It would be interesting to study the relationship between $C_n(x,p,q)$ and the following polynomials:

$$\sum_{\sigma \in \mathcal{Q}_n} x^{\operatorname{ap}(\sigma)} y^{\operatorname{lap}(\sigma)} p^{\operatorname{inv}(\sigma)} q^{\operatorname{maj}(\sigma)}.$$

In [6], Egge introduced the definition of Legendre-Stirling permutation, which shares similar properties with Stirling permutation. One may study the ascent-plateau statistic on Legendre-Stirling permutations.

References

[1] R.M. Adin, F. Brenti, Y. Roichman, Descent numbers and major indices for the hyperoctahedral group, *Adv. in Appl. Math.*, 27 (2001), 210–224.

[2] M. Bóna, Real zeros and normal distribution for statistics on Stirling permutations defined by Gessel and Stanley, *SIAM J. Discrete Math.*, 23 (2008/09), 401–406.

[3] P. Brändén, Actions on permutations and unimodality of descent polynomials *European J. Combin.*, 29 (2008), 514–531.

[4] W.Y.C. Chen, Context-free grammars, differential operators and formal power series, *Theoret. Comput. Sci.*, 117 (1993), 113–129.

[5] W.Y.C. Chen, A.M. Fu, Context-free grammars for permutations and increasing trees, *Adv. in Appl. Math.*, 82 (2017), 58–82.

[6] E.S. Egge, Legendre-Stirling permutations, *European J. Combin.*, 31 (7) (2010), 1735–1750.

[7] D. Foata, V. Strehl, Rearrangements of the symmetric group and enumerative properties of the tangent and secant numbers *Math. Z.*, 137 (1974), 257–264.

[8] I. Gessel and R.P. Stanley, Stirling polynomials, *J. Combin. Theory Ser. A*, 24 (1978), 25–33.

[9] J. Haglund, M. Visontai, Stable multivariate Eulerian polynomials and generalized Stirling permutations, *European J. Combin.*, 33 (2012), 477–487.

[10] S. Janson, Plane recursive trees, Stirling permutations and an urn model, Proceedings of Fifth Colloquium on Mathematics and Computer Science, *Discrete Math. Theor. Comput. Sci. Proc.*, vol. AI (2008), pp. 541–547.

[11] S. Janson, M. Kuba and A. Panholzer, Generalized Stirling permutations, families of increasing trees and urn models, *J. Combin. Theory Ser. A*, 118 (2011), 94–114.

[12] Z. Lin, Jiang Zeng, The γ-positivity of basic Eulerian polynomials via group actions, *J. Combin. Theory Ser. A,* 135 (2015), 112–129.

[13] S.-M. Ma, Some combinatorial arrays generated by context-free grammars, *European J. Combin.*, 34 (2013), 1081–1091.

[14] S.-M. Ma, A family of two-variable derivative polynomials for tangent and secant, *Electron. J. Combin.*, 20(1) (2013), #P11.

[15] S.-M. Ma, T. Mansour, The $1/k$-Eulerian polynomials and k-Stirling permutations, *Discrete Math.*, 338 (2015), 1468-1472.

[16] S.-M. Ma, H.-N. Wang, Enumeration of a dual set of Stirling permutations by their alternating runs, *Bull. Aust. Math. Soc.*, 94 (2016), 177–186.

[17] S.-M. Ma, Y.-N. Yeh, Stirling permutations, cycle structure of permutations and perfect matchings, *Electron. J. Combin.*, 22(4) (2015), #P4.42.

[18] S.-M. Ma, Y.-N. Yeh, Eulerian polynomials, Stirling permutations of the second kind and perfect matchings, *Electron. J. Combin.* 24(4) (2017), #P4.27.

[19] S.K. Park, The r-multipermutations, *J. Combin. Theory Ser. A*, 67(1) (1994), 44–71.

[20] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at http://oeis.org, 2010.

School of Mathematics and Statistics, Northeastern University at Qinhuangdao, Hebei 066004, P.R. China

E-mail address: shimeimapapers@163.com (S.-M. Ma)

Department of mathematics, Shanghai jiao tong university, Shanghai, P.R. China

E-mail address: majun904@sjtu.edu.cn(J. Ma)

Institute of Mathematics, Academia Sinica, Taipei, Taiwan

E-mail address: mayeh@math.sinica.edu.tw (Y.-N. Yeh)