Some Ulam's reconstruction problems for quantum states

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Abstract

Provided by a complete set of putative k-body reductions of a multipartite quantum state, can one determine if a joint state exists? We derive necessary conditions for this to be true. In contrast to what is known as the quantum marginal problem, we consider a setting where the labeling of the subsystems is unknown. The problem can be seen in analogy to Ulam's reconstruction conjecture in graph theory. The conjecture – still unsolved – claims that every graph can uniquely be reconstructed from the set of its vertex-deleted subgraphs. When considering quantum states, we demonstrate that the non-existence of joint states can, in some cases, already be inferred from a set of marginals having the size of just more than half of the parties. We apply these methods to graph states, where many constraints can be evaluated by knowing the number of stabilizer elements of certain weights that appear in the reductions. This perspective links with constraints that were derived in the context of quantum error-correcting codes and polynomial invariants. Some of these constraints can be interpreted as monogamy-like relations that limit the correlations arising from quantum states.

1 Introduction

The relationship of the whole to its parts lies at the heart of the theory of quantum entanglement. A pure quantum state is said to be *entangled* if it can not be written as the tensor product of its reductions. A particularly intriguing and important consequence of this mathematical definition is, that given a set of quantum marginals, it is not clear from the outset if and how they can be assembled into a pure joint state. Understanding this problem is not only important in the theory of entanglement, but also for applications in solid-state physics and quantum chemistry, such as calculating the energies of ground states [1, 2].

These kind of reconstruction questions have a long tradition in the mathematics literature. From our side, a particularly interesting context is the 1960's *Ulam graph reconstruction conjecture* [3–6]. Indeed, since graphs as well as quantum states can both be represented by positive semi-definite matrices – the Laplacian and the density matrix respectively – this highlights a common theme. Moreover, it may be valuable to notice that in the quantum mechanical setting, relational information is associated with correlations between subsystems. The family of graphs states allows to directly encode such relational information in pure quantum states. Using approaches from quantum mechanics, this may provide new insights into certain aspects of graph theory.

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Informally, the Ulam reconstruction conjecture is as follows: given a complete set of vertex-deleted subgraphs, is the original graph (without vertex labels) the only possible joint graph? Over the last decade a substantial amount of research focused on this problem. However, this remains one of the outstanding unsolved questions in graph theory.

In this work, we start by providing background to the so-called *quantum marginal problem* (QMP), which asks analogous questions for the case that the labels of the individual subsystems are known [1, 7, 8]. This is in contrast to the question which we will investigate later, namely a setting in which the labels are unknown to us. Originally coined the *N-representability problem* by Coleman, its first formulation asks how to recognize when a putative two-party reduced density matrix is in fact the reduction of an N-particle system of indistinguishable Fermions [1]. In fact, the N-representability problem has been highlighted as one of the most prominent research challenges in theoretical and computational chemistry [2]. This question was subsequently expanded to the case of distinguishable particles, and in particular, to qubits. In the case of the marginals being disjoint, the conditions for the existence of a joint state have been completely characterized: considering the existence of a pure joint qubit state, the characterization is given by the so-called Polygon inequalities, which constrain the spectra of pure state reductions [9]. Constraints for the existence of a mixed joint state on two qubits have been subsequently been obtained by Bravyi [10]. Solving the QMP in case of disjoint marginals completely, Klyachko extended the spectral conditions to the existence to a mixed joint state on n parties of arbitrary local dimensions [7].

The QMP problem in the case of overlapping marginals has turned out to be an even harder problem. Only few necessary conditions for the general case are known [11–14], of which many are based on entropic inequalities such as the strong subadditivity. Other constraints are posed by monogamy (in)equalities [15–17], some of which will also be used in our work. Interestingly, the special case of the symmetric extension of two qubits, where a two-party density matrix ϱ_{AB} is extended to a tripartite state $\varrho_{ABB'}$ with $\varrho_{AB} = \varrho_{AB'}$, has completely been characterized [18]. Despite many efforts, a general necessary and sufficient condition for the QMP with overlapping marginals is still lacking.

A question related to the QMP are the conditions uniqueness of the joint state, given its marginals. This is motivated by a naturally arising physical question: Considering a Hamiltonian with local interactions only, its groundstate is non-degenerate only if no other states with the same local reductions exist. In this context, Linden et al. showed that almost every pure state of three qubits is completely determined by its two-particle reduced density matrices [19, 20]. This result has been subsequently been expanded to systems of n qubits, where having access to a certain subset of all marginals of size $\lfloor n/2 \rfloor + 1$ is almost always sufficient to uniquely specify a joint pure state [21]. Finally, it is useful to remark that, while the QMP can in principle be stated as a semidefinite program [22], its formulation scales exponentially in system size. In fact, the QMP has been shown to be QMA-complete [23].

In contrast to previous work on the QMP, we consider in this work only *unlabeled* marginals, that is, marginals whose corresponding subsystems are unknown to us. Thus, one is free to arrange them as necessary in order to obtain a joint state. Should the reductions to one party be all different (e.g. when considering reductions of random states), such labels can naturally be restored by comparing the one-body reductions. However, we are here mainly considering a special type of quantum states called graph states. These have proven to be useful for certain tasks in quantum information such as quantum error correction [24, 25] and measurement-based quantum computation [26–28]. The few-body reductions of this type of states are typically maximally mixed, so the strategy of comparing one-body reductions does not find an immediate application. Thus the quantum marginal problem amounts to a kind of jigsaw puzzle: we are given overlapping parts, the task being to determine whether or not they indeed can be assembled to one or many different puzzles.

Here, we address similar questions in the case of unlabeled marginals and derive necessary constraints for the Ulam reconstruction problem for quantum graph states. These are based solely on the number of so-called stabilizer elements present in the complete set of reductions having a given size. Our results connect with constraints that were derived in the context of quantum error-correcting codes that involve polynomial invariants. These can be interpreted as monogamy-like relations that limit the correlations that can arise from quantum states.

The remainder of this paper is organized as follows. In Sec. 2, we introduce the classical Ulam conjecture. In Secs. 3 and 4, we introduce the basic notions of many-qubit systems and graph states that will be useful in our context. The main tool of this paper, the so-called weight distribution, is introduced in Sec. 5. We derive constraints on the weight distribution in Sec. 6. These are then applied in Secs. 7 and 8 to state legitimacy conditions on marginals to originate from a putative joint state. We conclude and provide an outlook in Sec. 9.

2 Realizability and uniqueness in graphs

Consider a simple graph $G = (V, E)$ on n vertices. Denote by $N(i)$ the neighborhood of vertex i, that is, the vertices adjacent to i. By deleting a single vertex $j \in V$ and deleting each edge e incident with j, one obtains the vertex-deleted subgraph $G_j = (V \backslash \{j\}, E \backslash \{e\} | j \in e)$ on $(n-1)$ vertices. By forming all vertex-deleted subgraphs G_j induced by G, the so-called *cards*, we obtain its unordered *deck*, the multi-set $D(G) = \{G_1, \dots, G_n\}$.

The Ulam *reconstruction problem* can be stated as follows: given a deck $D(G)$, is there, up to graph isomorphisms, a unique graph corresponding to it? The Ulam graph reconstruction conjecture states that this must indeed be the case for all graphs.

Conjecture 1 (Ulam [29–31])**.** *We have $D(G) = D(H)$ if and only if G is isomorphic to H.*

Let us remark that one can also consider the situation where a given deck does not necessarily need to originate from an existing graph. Suppose we are given a putative deck containing n cards of size $(n-1)$ each, whose origin is unknown to us. A naturally arising question is: can this deck indeed be obtained from a graph on exactly n vertices? This is also called the *legitimate deck* problem, and is a type of realizability problem [3]. It is useful to state a legitimacy condition introduced by Bondy for this to be the case, called Kelly's condition [6]:

Theorem 1 (Bondy [6], Kelly's condition)**.** *Let $D = \{G_i\}$ be a complete deck of a putative graph of n vertices. For any graph F with less than n vertices, denote by $s(F, G_i)$ the number of induced subgraphs of G_i that are isomorphic to F. Then the following expression must be an integer,*

$$\frac{\sum_{i=1}^{n} s(F, G_i)}{n - |V(F)|} \,. \tag{1}$$

Clearly, whenever the above expression is not an integer, the deck cannot originate from a graph. This seems to detect already a majority of illegitimate decks.

Interestingly, a specific set of three cards is sufficient to uniquely reconstruct the original graph for most decks [32]. This type of asymptotic result shows that most graphs are easy to specify, in analogy with the WL-method for graph isomorphism. In the following, we aim to treat the Ulam graph reconstruction and the legitimate deck problem for a special type of quantum states called *graph states*. Given a collection of graph state marginals, we ask for a corresponding realization as a joint state. Conversely, if a joint state exists, we are interested in its uniqueness.

3 Set up

A few definitions concerning multipartite quantum states are in order. Denote by I, X, Y, and Z the Pauli matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

The single qubit Pauli group is defined as $\mathcal{G}_1 = \langle i, X, Y, Z \rangle$, from which the n-qubit Pauli group is constructed by its n-fold tensor product $\mathcal{G}_n = \mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_1$ (n times). By forming tensor products of Pauli matrices, we obtain an orthonormal basis $\mathcal{P} = \{P\}$ of Hermitian operators acting on $(\mathbb{C}^2)^{\otimes n}$, with $\mathrm{tr}(P_\alpha P_\beta) = \delta_{\alpha\beta} 2^n$. We will write X_j, Y_j, Z_j for the Pauli matrices acting on particle j alone. Denote by $\mathrm{supp}(P)$ the support of an operator $P \in \mathcal{P}$, that is, the parties on which P acts non-trivially with X, Y, or Z. The weight of an operator is then the size of its support, $\mathrm{wt}(E) = |\,\mathrm{supp}(E)|$.

Pure quantum states $|\psi\rangle$ of n qubits are represented by unit vectors in $(\mathbb{C}^2)^{\otimes n}$. Their corresponding density matrix $\varrho = |\psi\rangle\langle\psi|$ on n qubits can be expanded in terms of Pauli matrices as

$$\varrho = 2^{-n} \sum_{P \in \mathcal{P}} \mathrm{tr}[P^\dagger \varrho] P\,. \tag{3}$$

Given a quantum state ϱ, we obtain its marginal (also called reduction) on subset A by acting with the partial trace on its complement A^c, $\varrho_A = \mathrm{tr}_{A^c}(\varrho)$. In the Bloch decomposition [Eq. (3)], the reduction onto subsystem A tensored by the identity on A^c can also be written as

$$\varrho_A \otimes \mathbb{1}_{A^c} \;=\; \sum_{\mathrm{supp}(P) \subseteq A} \mathrm{tr}[P^\dagger \varrho] P\,. \tag{4}$$

This follows from $\mathrm{tr}_{A^c}(E) = 0$ if $\mathrm{supp}(E) \nsubseteq A$.

A pure multipartite state is called *entangled*, if it cannot be written as the tensor product of single-party states,

$$|\psi\rangle^{\mathrm{ent}} \neq |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle\,. \tag{5}$$

For a mixed state, entanglement is present if it cannot be written as a convex combination of product states,

$$\varrho^{\mathrm{ent}} \neq \sum_i p_i \varrho_1 \otimes \varrho_2 \otimes \cdots \otimes \varrho_n\,, \tag{6}$$

for all single-party states $\varrho_1 \dots \varrho_n$ and probabilities p_i with $\sum_i p_i = 1$, $p_i \geq 0$. Note that for a pure product state, all reductions are pure too, having the purity $\mathrm{tr}(\varrho_A^2) = 1$.

Lastly, we note that any pure state can, for every bipartition $A|B$, be written as

$$|\psi_{AB}\rangle = \sum_i \sqrt{\lambda_i} |i_A\rangle \otimes |i_B\rangle\,, \tag{7}$$

where $\{|i_A\rangle\}$ and $\{|i_B\rangle\}$ are orthonormal bases for subsystems A and B, and $\sqrt{\lambda_i} \geq 0$ with $\sum_i \lambda_i = 1$. This is called the *Schmidt decomposition*.

We are now able to introduce the analogue of a graph deck for quantum states.

Definition 1. A *quantum k-deck* is a collection of quantum marginals, also called quantum cards, of size k each. The marginals are unlabeled and thus not associated to any specific subsystems. A deck is called *complete* if it contains $\binom{n}{k}$ cards, and *legitimate* if it originates from a joint state.

Thus, given a quantum state $|\psi\rangle$, its corresponding k-deck is given by the collection of all its marginals of size k,

$$D(|\psi\rangle) = \{\varrho_A \mid |A| = k, A \subseteq \{1 \ldots n\}\} . \tag{8}$$

4 Graph states

To approach Ulam type problems in the quantum setting, let us introduce *graph states*. These are a type of pure quantum states which are completely characterized by corresponding graphs. Of course we could also approach a more general setting by considering generic pure states. Our choice is suggested by the immediate connection between graph states and graphs. Our hope is that the mathematical richness of graph states could highlight some new perspective on Ulam's problem, even when we restrict our attention to graph states only.

Definition 2 ([28])**.** Given a simple graph $G = (V, E)$ having n vertices, its corresponding *graph state* $|G\rangle$ is defined as the common $(+1)$-eigenstate of the n commuting operators $\{g_i\}$,

$$g_i = X_i \bigotimes_{j \in N(i)} Z_j . \tag{9}$$

Thus $g_i|G\rangle = |G\rangle$ for all g_i. The set $\{g_i\}$ is called the *generator* of the graph state.

To obtain $|G\rangle\langle G|$ explicitly, the notion of its *stabilizer* is helpful. The stabilizer S is the Abelian group obtained by the multiplication of generator elements,

$$S = \left\{ s = \prod_{i \in I} g_i \mid I \subseteq \{1, \ldots, n\} \right\} . \tag{10}$$

Each of its 2^n elements stabilize the state, $g_i|G\rangle = |G\rangle$ for all g_i. Naturally, the stabilizer forms a subgroup of the n-party Pauli-group which consists of all elements in $\mathcal{P}$ in addition to a complex phase $\{\pm 1, \pm i\}$. With this, the graph state can be written as [28]

$$|G\rangle\langle G| = \frac{1}{2^n} \sum_{s \in S} s . \tag{11}$$

On the other hand, it can be shown that the graph state can also be written as

$$|G\rangle = \prod_{e \in E} C_e |+\rangle_V , \tag{12}$$

where $|+\rangle_V = \bigotimes_{j \in V} (|0\rangle_j + |1\rangle_j)/\sqrt{2}$. The controlled-$Z$ gate acting on parties i and j of edge $e = (i, j)$ reads $C_e = \mathrm{diag}(1, 1, 1 - 1)$.

In this picture, it is evident that graph states can be described as real equally weighted states: by initializing in $|+\rangle_V$, an equal superposition of all computational basis states is created. The subsequent application of C_e gates then only changes certain signs in this superposition.

To understand when two non-isomorphic graphs give different but comparable quantum states, let us take a small detour. The first step is to clarify the meaning of *comparable*. When looking at similarities between quantum states, the equivalence up to local unitaries (LU) is often considered. Two n-qubit states σ and ϱ are said to be *LU-equivalent*, if there exist unitaries $U_1, \ldots, U_n \in SU(2)$, such that $\sigma = U_1 \otimes \cdots \otimes U_n \; \varrho \; U_1^\dagger \otimes \cdots \otimes U_n^\dagger$. If no such matrices exist the states are said to be *LU-inequivalent*. An interesting subset of unitaries to consider is the so-called local Clifford group $\mathcal{C}_n$. It is obtained by the n-fold tensor product of the one-qubit Clifford group $\mathcal{C}_1$,

$$\mathcal{C}_1 = \left\langle \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \right\rangle . \tag{13}$$

The group $\mathcal{C}_1$ maps the one-qubit Pauli group $\mathcal{G}_1 = \langle i, X, Y, Z \rangle$ to itself under conjugation. The n-qubit local Clifford group $\mathcal{C}_n = \mathcal{C}_1 \otimes \ldots \otimes \mathcal{C}_1$ (n times) then similarly maps the n-qubit Pauli group $\mathcal{G}_n$ to itself under conjugation. Interestingly, it was shown that the action of local Clifford operations on a graph state can be understood as a sequence of local complementations on the corresponding graph [33]. This works in the following way: given a graph G, its local complementation with respect to vertex j is defined as the complementation of the subgraph in G consisting of all vertices in its neighborhood $N(j)$ and their common edges. We conclude that if two graphs are in the same local complementation orbit, then their corresponding graph states must be equivalent under the action of local Clifford operations, and vice versa. Thus they must also be LU-equivalent. The contrary however is not necessarily true. Indeed, it has been shown that there exist LU-equivalent graph states which are not local Clifford equivalent [34, 35].

We begin our analysis with an observation concerning the reductions of graph states onto $n-1$ parties [36]. For this, let us define a *vertex-shrunken* graph: the vertex-shrunken graph S_i is obtained by deleting vertex i and by shrinking all of its incident edges (i, j) to so-called one-edges (j). For a simple graph, the operation simply marks all vertices adjacent to i. To clarify the meaning of this notion, it is useful to look at a generalization to hypergraphs. Hypergraphs can have edges containing more than two vertices. Consider now deleting a single vertex i from a hypergraph: an incident hyperedge e can, instead of simply being discarded, be *shrunken* such as to still contain all remaining vertices. The shrunken edge then reads $e \backslash \{i\}$. In that way, a k-edge, initially connected k vertices, becomes a $(k-1)$-edge. Consequently, shrinking a 2-edge yields a one-edge. One has

$$S_i = \left(V \backslash i, \{e \backslash i | i \in e\} \cup \{e | i \notin e\}\right), \tag{14}$$

and the notion of a one-edge is well-motivated. The following Proposition is concerned with expressing reductions of graph states in terms of vertex-deleted and -shrunken graphs.

Proposition 1 (Lyons et al. [28])**.** *Consider the quantum $(n-1)$-deck of a graph state $|G\rangle$. Then, each of its cards can be represented by two graphs: a vertex-deleted graph G_j and a vertex-shrunken graph S_j, each having $(n-1)$ vertices.*

Proof. In Eq. (12), let us single out vertex j to be traced over.

$$|G\rangle = \prod_{e \in E} C_e |+\rangle_V = \left(|0\rangle_j + \prod_{e \in E \,|\, j \in e} C_{e \backslash \{j\}} |1\rangle_j \right) \otimes \prod_{e' \in E \,|\, j \notin e'} C_{e'} |+\rangle_{V \backslash \{j\}} . \tag{15}$$

Note that if C_e is a controlled Z-gate acting on parties i and j, then $C_{e\setminus\{j\}}$ is the local Z_i gate acting on party i alone. Thus performing a partial trace over subsystem j yields

$$\begin{aligned}
\mathrm{tr}_j[|G\rangle\langle G|] &= \langle 0_j|G\rangle\langle G|0_j\rangle + \langle 1_j|G\rangle\langle G|1_j\rangle \\
&= \frac{1}{2}\Big(\underbrace{\prod_{e\in E\,|\,j\notin e} C_e \,(|+\rangle\langle+|)_{V\setminus\{j\}} \prod_{e'\in E\,|\,j\notin e'} C_{e'}}_{\text{delete}} \\
&\quad + \underbrace{\prod_{i\in N(j)} Z_i \prod_{e\in E\,|\,j\notin e} C_e \,(|+\rangle\langle+|)_{V\setminus\{j\}} \prod_{e'\in e\,|\,j\notin e'} C_{e'} \prod_{i\in N(j)} Z_i \Big)}_{\text{shrink}} .
\end{aligned} \tag{16}$$

The reduction of a graph state onto $(n-1)$ parties is thus given by the equal mixture of two graph states: a vertex-deleted graph state $|G_i\rangle$, whose graph is the vertex-deleted subgraph of G, and a vertex-shrunken graph state $|S_j\rangle$, whose graph is a vertex-deleted subgraph with additional one-edges on $N(j)$ caused by shrinking all edges adjacent to j. These one-edges correspond to local Z_j-gates. One obtains

$$|G_j\rangle = \prod_{e\in E\,|\,j\notin e} C_e|+\rangle_{V\setminus\{j\}}\,, \tag{17}$$

$$|S_j\rangle = \prod_{i\in N(j)} Z_i \prod_{e\in E\,|\,j\notin e} C_e|+\rangle_{V\setminus\{j\}}\,, \tag{18}$$

and we can write

$$\mathrm{tr}_j(|G\rangle\langle G|) = \frac{1}{2}(|G_j\rangle\langle G_j| + |S_j\rangle\langle S_j|)\,. \tag{19}$$

This ends the proof. □

If the graph G is fully connected, then $\langle G_j|S_j\rangle = 0$ for all j. This follows from the fact that all stabilizer elements corresponding to a fully connected graph must have weights larger or equal than two. Thus the one-body reductions are maximally mixed, and the complementary $(n-1)$-body reductions must be proportional to projectors of rank two. When tracing out more than one party, this procedure of substituting each graph by the equal mixture of its vertex-deleted and vertex-shrunken subgraphs is iteratively repeated. Thus the reduction of a graph state of size $n-k$ is represented by a collection of 2^k graphs.

Let us now consider a specific formulation of the Ulam graph problem in the quantum setting where all $(n-1)$-body reductions of a graph state are given in the computational basis. What can one say about the joint state?

Proposition 2 (Lyons et al. [36]). *Given a legitimate $(n-1)$-deck of a graph state $|G\rangle$ in the computational basis, the joint state $|G\rangle$ can be reconstructed up to local Z_j gates from any single card.*

Proof. Let us expand the graph states $|G_j\rangle$ and $|S_j\rangle$ as appearing in (19) in the computational basis. Due to our ignorance about the joint state, denote them by $|\alpha\rangle$ and $|\beta\rangle$, where either one could be the vertex-deleted graph state, with the other one being the vertex-shrunken graph state.

From Eq. (12) follows that all graph states are real equally weighted states. Thus it is possible to expand

$$|\alpha\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \alpha_i |i\rangle \,, \qquad\qquad |\beta\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \beta_i |i\rangle \,, \tag{20}$$

where $N = 2^n$, $\alpha_i, \beta_i \in \{-1, 1\}$, and $\{|i\rangle\}$ being the computational basis for $V \backslash \{j\}$. We can therefore write the card $C^k = \mathrm{tr}_k(|G\rangle\langle G|)$ as

$$C^k = \frac{1}{2N} \sum_{i,j=0}^{N-1} (\alpha_i \alpha_j + \beta_i \beta_j) |i\rangle\langle j| \,. \tag{21}$$

Because of $\alpha_i, \beta_i \in \{-1, 1\}$, $2NC^k_{ij}$ can only be 0 or ± 1. Because $|0 \dots 0\rangle$ remains unaffected by conditional phase gates, $\alpha_1 = \beta_1 = 1$. Furthermore, $\alpha_j = \beta_j = \mathrm{sign}(C^k_{1j})$ for all j where $C^k_{1j} \neq 0$. On the other hand, when $C^k_{1l} = 0$, then $\alpha_l = -\beta_l$. Without loss of generality, set $\alpha_m = -\beta_m = 1$ for the first instance of m where this happens. The remaining but yet undetermined coefficients $\alpha_l = -\beta_l \in \{\pm 1\}$ are given from the entries C^k_{ml},

$$\alpha_m \alpha_l + \beta_m \beta_l = \alpha_l - \beta_l = 2\alpha_l = 2NC^k_{ml} \,. \tag{22}$$

This completely determines the remaining coefficients of $|\alpha\rangle$ and $|\beta\rangle$. Now the task is to reconstruct the graphs corresponding to $|\alpha\rangle$ and $|\beta\rangle$. This can be done by iteratively erasing all minus signs in the computational basis expansion [37]: first, minus signs in front of terms having a single excitation only, e.g. $|0 \dots 010 \dots \dots 0\rangle$, are removed by local Z_j gates. Then, conditional phase gates are applied to erase minus signs in front of terms having two excitations, and so on. By this procedure, one obtains the state $|+\rangle^{\otimes n}$ and all the gates necessary to obtain the original graph state, thus determining the graph.

Note that the symmetric difference of the two graphs corresponding to $|\alpha\rangle$ and $|\beta\rangle$ yields all edges that were severed under the partial trace operation,

$$\underbrace{\prod_{j \in e} C_{e \in E \,|\, e \backslash \{j\}} \prod_{e' \in E \,|\, j \notin e'} C_{e'}}_{\text{shrink}} \; \underbrace{\prod_{e'' \in E \,|\, j \notin e''} C_{e''}}_{\text{delete}} = \underbrace{\prod_{e \in E \,|\, j \in e} C_{e \backslash \{j\}}}_{\text{edges connected to } j} = \prod_{i \in N(j)} Z_j \,. \tag{23}$$

The original graph state can then only be one of the following

$$\begin{aligned} |G\rangle\langle G| &= \prod_{e \in E \,|\, j \in e} C_e \, |\alpha\rangle \otimes |+\rangle_j \,, \qquad \text{or} \\ |G\rangle\langle G| &= \prod_{e \in E \,|\, j \in e} C_e \, |\beta\rangle \otimes |+\rangle_j \,. \end{aligned} \tag{24}$$

This proves the claim. □

5 Weight distribution

In order to determine whether or not, given a quantum k-deck, a joint graph state could possibly exist, we introduce the *weight distribution* of quantum states. This is a tool from the theory of quantum error-correcting codes, and can be used to characterize the number of errors a code can correct. A partial weight distribution can be obtained from a complete quantum k-deck already, and no knowledge of the labeling of the individual parties is needed. This makes this tool useful for legitimate deck type problems. With it it is possible to detect certain illegitimate decks, that is, collections of marginals that are incompatible with any pure joint state.

Definition 3 ([38–40])**.** The weight distribution of a multipartite qubit quantum state ϱ is given by

$$A_j(\varrho) = \sum_{\substack{P\in\mathcal{P}\\ \mathrm{wt}(P)=j}} \mathrm{tr}(P\varrho)\,\mathrm{tr}(P^\dagger\varrho)\,, \tag{25}$$

where the sum is over all elements P of weight j in the n-qubit Pauli basis $\mathcal{P}$.

Note that for higher dimensional quantum systems any appropriate orthonormal tensor-product basis can be chosen instead of the Pauli basis, e.g. the Heisenberg-Weyl or Gell-Mann basis. The weights A_j, being quadratic in the coefficients of the density matrix and invariant under local unitaries, are so-called polynomial invariants of degree two. They characterize the distance of quantum error-correcting codes [38, 41] and can be used to detect entanglement [42]. For graph states, the weight distribution is particularly simple: because $\mathrm{tr}[P|G\rangle\langle G|]$ can only be either 0 or ±1, the weight distribution of $|G\rangle$ is simply given by the number of its stabilizer elements having weight j,

$$A_j(|G\rangle) = |\{s\in S|\,\mathrm{wt}(s)=j\}|\,. \tag{26}$$

Let us give an example.

Example 1. The three-qubit graph state corresponding to the fully connected graph of three vertices has the generator $G=\{XZZ, ZXZ, ZZX\}$ [1]. Its stabilizer reads

$$S = \{III, IYY, YIY, YYI, XZZ, ZXZ, ZZX, -XXX\}\,. \tag{27}$$

Accordingly, its weight distribution is $A=[A_0,A_1,A_2,A_3]=[1,0,3,4]$. By normalization, $A_0=\mathrm{tr}(\varrho)=1$ must hold for all states. Because ϱ is pure, $\mathrm{tr}(\varrho^2)=1$, and thus $\sum_{j=0}^n A_j(|\psi\rangle)=2^n$.

As a warm-up, let us derive a result known from quantum error correction [43] by using properties of Pauli matrices only.

Proposition 3. *Given a graph state, the sum* $A_\mathrm{e}=\sum_{j=0}^{\lfloor n/2\rfloor} A_{2j}$ *can only take two possible values,*

$$A_\mathrm{e} = \begin{cases} 2^{n-1} & \textit{(type I)}\,,\\ 2^n & \textit{(type II)}\,. \end{cases} \tag{28}$$

Proof. Note that a graph state $\varrho=|G\rangle\langle G|$ can be decomposed into

$$\varrho = \frac{1}{2^n}\Big(\sum_{\substack{P\in\mathcal{P}\\ \mathrm{wt}(P)\text{ even}}} \mathrm{tr}[P^\dagger M]P + \sum_{\substack{P\in\mathcal{P}\\ \mathrm{wt}(P)\text{ odd}}} \mathrm{tr}[P^\dagger M]P\Big) = \frac{1}{2^n}(P_\mathrm{e}+P_\mathrm{o})\,, \tag{29}$$

where P_e and P_o are the sums of all stabilizer elements having even and odd weight respectively. Because of $s\varrho=\varrho$ for all $s\in S$, also P_e and P_o have ϱ as an eigenvector. We apply this decomposition

[1]This is a state that is LU-equivalent to the Greenberger-Horne-Zeilinger state $(|000\rangle+|111\rangle)/\sqrt{2}$.

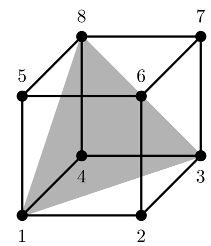

Figure 1: A hypergraph state that cannot be transformed into a graph state with local unitaries. All of its three-body marginals are maximally mixed.

to $\varrho = \varrho^2$, making use of Lemma 1 from Ref. [44] regarding the anti-commutators of elements from $\mathcal{P}$: the term $\{P_\mathrm{e}, P_\mathrm{o}\}$ appearing in $\varrho^2 = \{\varrho, \varrho\}/2$ can only contribute to terms of odd weight in ϱ, yielding $\{P_\mathrm{e}, P_\mathrm{o}\} = 2^n P_\mathrm{o}$. Consequently, one obtains

$$\mathrm{tr}(\{P_\mathrm{e}, P_\mathrm{o}\}\varrho) = \mathrm{tr}(2^n P_\mathrm{o}\varrho)\,. \tag{30}$$

Accordingly, $2A_\mathrm{e}A_\mathrm{o} = 2^n A_\mathrm{o}$, where A_e and A_o are the number of terms in the stabilizer that have even and odd weight respectively. Consider first $A_\mathrm{o} \neq 0$. Then $A_\mathrm{e} = 2^{n-1}$. Conversely, if $A_\mathrm{o} = 0$, then $A_\mathrm{e} = 2^n$, because ϱ is pure and must thus satisfy $\sum A_j = A_\mathrm{e} + A_\mathrm{o} = 2^n$. This ends the proof. ☐

The same argument can be done for reductions of graph states that happen to be proportional to projectors of rank 2^q. There, either $A_\mathrm{e} = 2^{n-q-1}$ or $A_\mathrm{e} = 2^{n-q}$ holds.

These two cases, that is, graph states of type I and type II, are also known from the theory of *classical* self-dual additive codes over $GF(4)$ [45, 43]. If only stabilizer elements of even weight are present the code is said to be of type II, while codes having both even and odd correlations in equal amount are of type I. It can be shown that all type II codes must have even length, and conversely, self-dual additive codes of odd length n are always of type I. This is also a direct consequence of the monogamy relation derived in Ref. [17] which is known to vanish for an odd number of parties, implying $A_\mathrm{e} = A_\mathrm{o} = 2^{n-1}$. Let us note that any graph states whose every vertex is connected to an odd number of other vertices is of type II [2]. For example, Greenberger-Horne-Zeilinger states of an even number of qubits are of this type, being LU-equivalent to fully connected graph states.

This result can be used to show that a particular state cannot be LU-equivalent to any graph state. Let us consider the state depicted in Fig. 5, which is a so-called *hypergraph state* [37]. It can be obtained by applying the additional gate $C_{138} = \mathrm{diag}(1, 1, 1, 1, 1, 1, 1, -1)$ between particles 1, 3, and 8 to the graph state of a cube, $|H\rangle = C_{138}|G_\mathrm{cube}\rangle$. Its weight distribution reads

$$A = [1, 0, 0, 0, 30, 48, 96, 48, 33]\,, \tag{31}$$

with $A_\mathrm{e} = \sum_{j \text{ even}} = 160$. This is incompatible with being a graph state of type I or type II, these having either $A_\mathrm{e} = 128$ or $A_\mathrm{e} = 256$ respectively. Because the weight distribution is invariant under LU-operations, the state must be LU-inequivalent to graph states. Let us add that all the three-body marginals of $|H\rangle$ are maximally mixed, and the state can thus be regarded as being highly entangled [43].

One could ask whether or not the presence of entanglement can be detected from the weight distribution of a state. This is indeed the case.

[2] See Theorem 15 in Ref. [46].

Proposition 4. *Let $|\psi\rangle$ be a pure product state on n qubits. Then $A_j(|\psi\rangle) = \binom{n}{j}$.*

Proof. Let us assume that we are given a product state on $m-1$ qubits with weights denoted by $A_j^{(m-1)}$. Tensoring it by a pure state on the last qubit, the weight $A_j^{(m)}$ of the resulting state on m qubits is

$$A_j^{(m)} = A_j^{(m-1)} A_0^{(1)} + A_{j-1}^{(m-1)} A_1^{(1)} = A_j^{(m-1)} + A_{j-1}^{(m-1)}\,, \tag{32}$$

because of $A_0 = A_1 = 1$ for a pure one-qubit state. But this is exactly the recurrence relation that is satisfied by the binomial coefficients, namely

$$\binom{m}{j} = \binom{m-1}{j} + \binom{m-1}{j-1}, \tag{33}$$

together with the initial condition $A_j^{(1)} = \binom{1}{j} = 1$. Thus a pure product state on n qubits has $A_j = \binom{n}{j}$. $\square$

If $|\psi\rangle$ is entangled across a partition of one party versus the rest, then above relation takes the form of a strict inequality

$$A_j^{(n)} > A_j^{(n-1)} + A_{j-1}^{(n-1)}\,. \tag{34}$$

This can be seen by considering the purity of the reductions and of the full state. Considering entanglement across a partition of $m \leq \lfloor n/2 \rfloor$ versus $(n-m)$ parties, one obtains in a similar fashion the inequality

$$A_j^{(n)} > A_j^{(n-m)} A_0^{(m)} + A_{j-1}^{(n-m)} A_1^{(m)} + \cdots + A_{j-m}^{(n-m)} A_m^{(m)}\,. \tag{35}$$

Further inequalities to detect entanglement from the weight distribution can be found in Ref. [42].

6 Constraints on the weight distribution

In the following, we derive further relations on the weight distribution of pure states. These are obtained from the Schmidt decomposition along bipartitions having fixes sizes and from monogamy-like relations. As we are dealing with Ulam type problems, not the complete weight distribution is given. Thus, let us define the *reduced* weight distribution, which is proportional to the average distribution that marginals of size m of a given quantum state ϱ show. This notion is useful for the Ulam type problems that we consider in this article, as the reduced weight distribution $A_j^m(\varrho)$ can already be obtained from a complete set of unlabeled marginals of size m. We refer to Sect. 7 for details how this can be achieved.

Definition 4. Let ϱ be a quantum state on n parties. Given its weight distribution $A_j(\varrho)$, define its associated *reduced weight distribution* $A_j^m(\varrho)$ for $0 \leq j \leq m$ as

$$A_j^m(\varrho) = \binom{n}{m}\binom{m}{j}\Big/\binom{n}{j} A_j(\varrho) = \binom{n-j}{n-m} A_j(\varrho)\,. \tag{36}$$

For the following proofs we also need the weight distribution on some subsystem $S \subseteq \{1 \dots n\}$,

$$A_j^S = \sum_{\substack{P \in P \\ \mathrm{supp}(P) \subseteq S \\ \mathrm{wt}(S) = j}} \mathrm{tr}(P\varrho)\,\mathrm{tr}(P^\dagger \varrho)\,. \tag{37}$$

We now state a first constraint on the weight distribution of pure states that arises from the Schmidt decomposition.

Proposition 5 (Cut-relations)**.** *Let $|\psi\rangle$ be a pure state of n qubits. For all $1 \leq m \leq n$, the reduced weight distributions $A_j^m(|\psi\rangle)$ satisfy*

$$2^{-m} \sum_{j=0}^{m} A_j^m(|\psi\rangle) = 2^{-(n-m)} \sum_{j=0}^{n-m} A_j^{n-m}(|\psi\rangle) \,. \tag{38}$$

Proof. In the following, let us write A_j for $A_j(|\psi\rangle)$. From the Schmidt decomposition of pure states, it follows that the purities of reductions on complementary subsystems must be equal,

$$\operatorname{tr}(\varrho_S^2) = \operatorname{tr}(\varrho_{S^c}^2) \,. \tag{39}$$

Summing Eq. (39) over all bipartitions S, S^c having fixed size $m \leq \lfloor n/2 \rfloor$ and $(n-m)$, one obtains

$$2^{-m} \sum_{|S|=m} \sum_{j=0}^{m} A_j^S = 2^{-(n-m)} \sum_{|S^c|=n-m} \sum_{j=0}^{n-m} A_j^{S^c} \,. \tag{40}$$

In the case of graph states, A_j^S is just the number of stabilizer elements of weight j having support in S. Note that in Eq. (40), the dimensional prefactor results from the difference in normalization of ϱ_S and ϱ_{S^c}. By summing over all subsystem pairs of fixed size, elements of weight j are overcounted by factors of $\binom{n}{m}\binom{m}{j}\binom{n}{j}^{-1} = \binom{n-j}{n-m}$ and $\binom{n}{n-m}\binom{n-m}{j}\binom{n}{j}^{-1} = \binom{n-j}{m}$ respectively. We arrive at

$$2^{-m} \sum_{j=0}^{m} \binom{n-j}{n-m} A_j = 2^{-(n-m)} \sum_{j=0}^{n-m} \binom{n-j}{m} A_j \,. \tag{41}$$

In terms of the reduced weight distribution, this simply reads as

$$2^{-m} \sum_{j=0}^{m} A_j^m = 2^{-(n-m)} \sum_{j=0}^{n-m} A_j^{n-m} \,. \tag{42}$$

This proves the claim. □

These are $\lfloor (n-1)/2 \rfloor$ independent linear equations that the weight distributions of pure states have to satisfy. We note that these relations can be seen as an alternate formulation of the so-called *quantum MacWilliams identity* for quantum codes in the special case of pure states [38, 41], and generalize naturally to states of higher local dimensions.

We now obtain further constraints on the reduced weight distributions that are obtained from the so-called *universal state inversion* and generalizations thereof [17, 41, 47, 48]. In the case of qubits it can simply be attained through a spin-flip, where every Pauli matrix in the Bloch decomposition changes sign, mapping $I \mapsto I$, $Y \mapsto -Y$, $X \mapsto -X$, and $Z \mapsto -Z$ [3]. When expanding a state ϱ in the Bloch representation as

$$\varrho = \frac{1}{2^n} \sum_{j=0}^{n} \sum_{\substack{P \in \mathcal{P} \\ \mathrm{wt}(P)=j}} \operatorname{tr}(P^\dagger \varrho) P \,, \tag{43}$$

[3] It is not hard to convince oneself that this spin-flip can be obtained by $\tilde{\varrho} = Y^{\otimes n} \varrho^T Y^{\otimes n}$, where ϱ^T is the transpose of the given state ϱ.

the spin-flipped state thus simply acquires a sign flip for all odd-body correlations,

$$\tilde{\varrho} = \frac{1}{2^n} \sum_{j=0}^{n} (-1)^j \sum_{\substack{P \in \mathcal{P} \\ \mathrm{wt}(P)=j}} \mathrm{tr}(P^\dagger \varrho) P \,. \tag{44}$$

Note that because $\tilde{\varrho}$ is positive semi-definite, the expression $\mathrm{tr}(\varrho\tilde{\varrho})$ must necessarily be non-negative. This leads to our next proposition.

Proposition 6. *Let ϱ be a state of n qubits. For all $1 \leq m \leq n$, the reduced weight distributions $A_j^m(\varrho)$ satisfy the inequality*

$$\sum_{j=0}^{m} (-1)^j A_j^m(\varrho) \geq 0 \,. \tag{45}$$

Proof. We evaluate $\mathrm{tr}(\varrho\tilde{\varrho}) \geq 0$ in the Bloch decomposition.

$$\begin{aligned}
\mathrm{tr}(\varrho\tilde{\varrho}) &= \frac{1}{2^{2n}} \mathrm{tr}\Big[\big(\sum_{j=0}^{n} (-1)^j \sum_{\substack{P \in \mathcal{P} \\ \mathrm{wt}(P)=j}} \mathrm{tr}(P\varrho) P^\dagger \big) \big(\sum_{j'=0}^{n} \sum_{\substack{P' \in \mathcal{P} \\ \mathrm{wt}(P)=j'}} \mathrm{tr}(P'^\dagger \varrho) P' \big) \Big] \\
&= \frac{1}{2^{2n}} \sum_{j=0}^{n} (-1)^j \sum_{\substack{P \in \mathcal{P} \\ \mathrm{wt}(P)=j}} \mathrm{tr}(P\varrho)\, \mathrm{tr}(P^\dagger \varrho)\, \mathrm{tr}(P^\dagger P) \\
&= \frac{1}{2^n} \sum_{j=0}^{n} (-1)^j A_j \; \geq 0 \,.
\end{aligned} \tag{46}$$

Applying the same method to all reductions ϱ_S of fixed size $|S| = m$, one obtains

$$\sum_{|S|=m} \mathrm{tr}[\varrho_S \tilde{\varrho}_S] = 2^{-m} \sum_{|S|=m} \sum_{j=0}^{m} (-1)^j A_j^S = 2^{-m} \sum_{j=0}^{m} (-1)^j \binom{n-j}{n-m} A_j \geq 0 \,. \tag{47}$$

Up to a dimensional constant, this can be rewritten as $\sum_{j=0}^{m} (-1)^j A_j^m \geq 0$. This ends the proof. □

For pure states, the expression $\mathrm{tr}(\varrho\tilde{\varrho})$ is an entanglement monotone called n-concurrence [49]. In light of Refs. [50, 41] on the shadow enumerator of quantum codes, Eq. (45) can also be restated as the requirement that the zeroth *shadow coefficient* $S_0(\varrho_S) = \mathrm{tr}(\varrho_S \tilde{\varrho}_S)$ be non-negative when averaged over all m-body marginals ϱ_S. In the case of graph states and stabilizer codes, this expression must necessarily be integer, as it is obtained by counting elements of the stabilizer with integer prefactors.

Let us point to the most general form of these inequalities, the so-called *shadow inequality*:

Theorem 2 (Rains, shadow inequality [38, 41])**.** *Let M and N be non-negative operators on n qubits. For any subset $T \subseteq \{1 \dots n\}$ it holds that*

$$\sum_{S \subseteq \{1 \dots n\}} (-1)^{|S \cap T|} \, \mathrm{tr}[\mathrm{tr}_{S^c}(M)\, \mathrm{tr}_{S^c}(N)] \geq 0 \,, \tag{48}$$

where the sum is taken over all subsets S in $\{1 \dots n\}$.

For $M = N = \varrho$, the shadow inequality represents consistency conditions for quantum states in terms of purities of reductions, and for $T = \{1 \dots n\}$, it simply corresponds to $\operatorname{tr}(\varrho\tilde{\varrho}) \geq 0$. By symmetrizing the shadow inequality over all subsets T^c of some fixed size $0 \leq j \leq n$ one obtains the following constraints on the weight distribution of n-qubit states [4] [38, 45, 41],

$$S_j(\varrho) = \sum_{0 \leq k \leq n} (-1)^k K_j(k,n) A_k(\varrho) \geq 0\,. \tag{49}$$

The Krawtchouk polynomial above is given by

$$K_j(k,n) = \sum_{0 \leq \alpha \leq j} (-1)^\alpha 3^{j-\alpha} \binom{n-k}{j-\alpha}\binom{k}{\alpha}\,. \tag{50}$$

Naturally, Theorem 2 must also hold for all reductions of a joint state ϱ, these being quantum states themselves. Demanding this condition for all reductions ϱ_S of a fixed size m, we obtain following proposition for the reduced weight distribution:

Proposition 7. *Let ϱ be a state of n qubits. For all $1 \leq m \leq n$, the reduced weight distributions $A_k^m(\varrho)$ must satisfy*

$$S_j^m(\varrho) = \sum_{0 \leq k \leq m} (-1)^k K_j(k,m) A_k^m(\varrho) \geq 0\,. \tag{51}$$

Proof. Consider Eq. (49) for a m-body reduction of an n-qubit state. Summing over all marginals of size m, one obtains

$$\begin{aligned}\sum_{\substack{S \subseteq \{1 \dots n\} \\ \mathrm{wt}(S)=m}} \sum_{0 \leq k \leq m} (-1)^k K_j(k,m) A_k^S(\varrho_S) &= \sum_{0 \leq k \leq m} (-1)^k K_j(k,m) \binom{n-j}{n-m} A_k(\varrho) \\ &= \sum_{0 \leq k \leq m} (-1)^k K_j(k,m) A_k^m(\varrho) \geq 0\,. \end{aligned} \tag{52}$$

This ends the proof. ☐

Finally, let us note that constraints on weight distributions such as Eq. (48) can also be expressed in terms of *purities* or *linear entropies* of reductions, and vice versa. The linear entropy approximates the von Neumann entropy to its first order, and is defined as $S_L(\varrho_S) = 2[1 - \operatorname{tr}(\varrho_S^2)]$. The quantity $\operatorname{tr}(\varrho_S^2)$ is called the purity, measuring the pureness of a state.

As an example, let us show how the universal state inversion imposes constraints on the linear entropies of the two- and one-party reductions of a joint state.

Corollary 1. *Let ϱ be a multipartite quantum state, and denote by ϱ_i and ϱ_{ij} its one- and two-body reductions. The following inequality holds,*

$$(n-1)\sum_i S_L(\varrho_i) - \sum_{i<j} S_L(\varrho_{ij}) \geq 0\,. \tag{53}$$

[4] By convention, the shadow inequality is summed over the complement T^c of T, such that $S_0(\varrho) = \operatorname{tr}(\varrho\tilde{\varrho})$.

Proof. It has been shown in Ref. [48] that the universal state inversion can also be written as

$$\tilde{\varrho} = \sum_{S\subseteq\{1\dots n\}} (-1)^{|S|} \varrho_S \otimes \mathbb{1}_{S^c}\,. \tag{54}$$

Considering the state inversion on the two-party reductions, one obtains

$$\begin{aligned}\sum_{i<j} \mathrm{tr}[\varrho_{ij}\tilde{\varrho}_{ij}] &= \sum_{i<j} \mathrm{tr}[\varrho_{ij}(\mathbb{1} - \varrho_i \otimes \mathbb{1}_j - \mathbb{1}_i \otimes \varrho_j + \varrho_{ij})] \\ &= \sum_{i<j} (1 - \mathrm{tr}[\varrho_i^2] - \mathrm{tr}[\varrho_j^2] + \mathrm{tr}[\varrho_{ij}^2]) \\ &= (n-1)\sum_i S_L(\varrho_i) - \sum_{i<j} S_L(\varrho_{ij}) \geq 0\,. \end{aligned} \tag{55}$$

This ends the proof. □

Let us derive another relation involving three-body reductions.

Corollary 2. *Let ϱ be a multipartite quantum state. Denote by ϱ_i, ϱ_{ij}, and ϱ_{ijk} its one-, two-, and three-body reductions. The following inequality holds,*

$$\sum_i S_L(\varrho_i) + \sum_{i<j} S_L(\varrho_{ij}) - \sum_{i<j<k} S_L(\varrho_{ijk}) \geq 0\,. \tag{56}$$

Proof. Consider the shadow inequality [Eq. (48)] on a single three-body reduction ϱ_{ABC}. Choosing $T = \{AB\}$ and $M = N = \varrho_{ABC}$, one obtains

$$1 - \mathrm{tr}(\varrho_A^2) - \mathrm{tr}(\varrho_B^2) + \mathrm{tr}(\varrho_C^2) + \mathrm{tr}(\varrho_{AB}^2) - \mathrm{tr}(\varrho_{AC}^2) - \mathrm{tr}(\varrho_{BC}^2) + \mathrm{tr}(\varrho_{ABC}^2) \geq 0\,. \tag{57}$$

This can be rewritten in terms of linear entropies

$$S_L(\varrho_A) + S_L(\varrho_B) - S_L(\varrho_C) - S_L(\varrho_{AB}) + S_L(\varrho_{AC}) + S_L(\varrho_{BC}) - S_L(\varrho_{ABC}) \geq 0\,. \tag{58}$$

Summing this inequality over all three-body reductions of a multipartite quantum state ϱ proves the claim. □

Note that Corollary 1 simply corresponds to Propositon 7 for $m = 2$ and $j = 0$, expressed in terms of linear entropies, and Corollary 2 corresponds to the case of $m = 3$ and $j = 1$. Further relations can be obtained for other values of m and j, and in turn, these give consistency equations on decks of quantum marginals. Lastly, let us point out that the shadow inequality [Eq. (48)] also holds in operator form for any multipartite system having finite local dimensions [51]. For every $T \subseteq \{1\dots n\}$, the following expression is positive semidefinite,

$$\sum_{S\subseteq\{1\dots n\}} (-1)^{|S\cap T|} \varrho_S \otimes \mathbb{1}_{S^c} \geq 0\,. \tag{59}$$

For $T = \{1\dots n\}$, the expression reduces to the universal state inversion [Eq. (44)].

7 Detecting illegitimate decks

In the following, we use the relations derived in the previous section to detect illegitimate quantum decks. Let us first show how having access to all reduced states of size m directly yields the reduced weights A_j^m, and thus also all A_j with $j \leq m$.

Proposition 8. *Given a complete quantum m-deck $D = \{\varrho_S | \, |S| = m\}$, the weights $A_1, \ldots, A_m$ of the weight distribution of a putative joint state can be obtained.*

Proof. Given a complete deck D, we calculate

$$\sum_{\varrho_S \in D} \sum_{\substack{P \in \mathcal{P} \\ \mathrm{wt}(P)=j}} \mathrm{tr}(\varrho_S P) \,\mathrm{tr}(\varrho_S P^\dagger) = \sum_{S,\, |S|=m} A_j^S = \sum_{j=0}^{m} \binom{n-j}{n-m} A_j = \sum_{j=0}^{m} A_j^m \,. \tag{60}$$

From A_j^m, the weights A_j can be obtained for $0 \leq j \leq m$ from Eq. (36). This ends the proof. ☐

Note that for decks of putative joint graph states, A_j^m is exactly equal to the total number of stabilizer elements of weight j appearing in the quantum m-deck. To see how the cut-relations (Proposition 5) can help to decide compatibility of a quantum deck, let us provide some examples.

Example 2. Consider the case of a pure three qubit state. Setting $a = 1$ in Proposition 5 yields the condition $A_2 = 3$. From the normalization of the state, $\mathrm{tr}(\varrho) = 1$, it follows that $A_1 + A_3 = 4$. Thus it is not possible to join three Bell states together, as each one has the weights $A = [1, 0, 3]$ already.

Example 3. Let us consider a more elaborate example, the ring-cluster state of five qubits which is depicted in Fig. 7. Those of its three-body marginals that can be obtained by tracing out nearest neighbors are an equal mixture of the four graph states that are shown in Fig. 7, where the circles denote local Z-gates. Modifying the reductions to be the equal mixture of the states shown in the bottom row, it can be seen that no compatible joint state exists. This follows from their corresponding weight distribution: the ring-cluster state has $\binom{5}{3} = 10$ reductions on three qubits with $A = [1, 0, 0, 1]$. This is consistent with the cut-relations of Proposition 5, which read

$$-2A_1 + A_2 + A_3 = 10 \tag{61}$$

$$-4A_1 + 3A_2 + 2A_3 + A_4 = 35\,. \tag{62}$$

Slightly modifying some reductions to be an equal mixture of the four other states that are depicted in the lower row of Fig. 7, we obtain an illegitimate deck: these reductions have the weight distribution $A = [1, 0, 3/8, 11/8]$, and together with the rest of the deck, they do not satisfy Eq. (61). Thus no compatible joint state on five qubits exists.

Example 4. Let us ask whether or not a pure state ϱ on ten qubits could exist, that has all reductions on six qubits equal to

$$p|\mathrm{GHZ}_6\rangle\langle\mathrm{GHZ}_6| + (1-p)\mathbb{1}\,. \tag{63}$$

Above, the Greenberger-Horner-Zeilinger state on six qubits is defined as $|\mathrm{GHZ}_6\rangle = (|000000\rangle + |111111\rangle/\sqrt{2}$. Its weights are $A = [1, 0, 15, 0, 15, 0, 33]$. From it, we can obtain a part of the weight distribution of the putative joint state, namely

$$A_{j\leq 6}(\varrho) = \binom{10}{j}\binom{6}{j}^{-1} A_j(|\mathrm{GHZ}_6\rangle)\,. \tag{64}$$

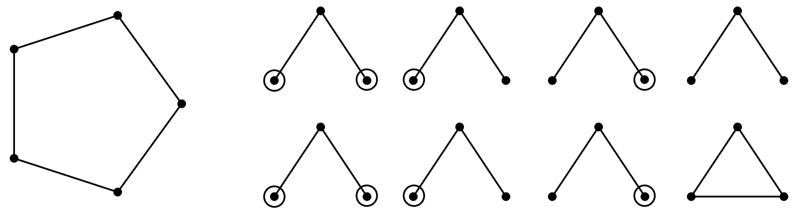

Figure 2: Left: the ring-cluster state on five qubits. Right, top row: the three-qubit reductions of the five qubit ring-cluster state that are obtained by tracing out nearest neighbors are the equal mixture of these graph states. Right, bottom row: modifying some reductions to be the equal mixture of the graph states shown in the bottom row, no compatible joint state on five qubits exists.

Thus the putative pure joint state must have $A = [1, 0, 45p, 0, 210p, 0, 6930p, \dots]$. Let us now see what value p can have, in order to satisfy Proposition 5. The cut-relation which involves the weights up to A_6 only requires that

$$-210A_1 - 42A_2 + 7A_3 + 11A_4 + 5A_5 + A_6 = 630\,. \tag{65}$$

This can only be fulfilled if $p = 3/35$.

Note that in above examples, one does not require to know the labeling of the parties. Despite that it is possible to make statements whether a joint state might exist, and to already detect illegitimate decks when provided by a deck whose cards have size $(\lfloor n/2 \rfloor + 1)$ only.

8 When is a weight distribution graphical?

Even when given the complete weight distribution $A_0, \dots, A_n$, one cannot always decide whether or not it can be realized by a graph state, that is, whether the weight distribution is *graphical*: the criteria derived in the previous sections are necessary but not sufficient for a realization. One can find weight distributions that satisfy all of the relations derived above, but for which it is known that no corresponding quantum state exists. As an example, consider a hypothetical pure state of seven qubits, having all three-body reductions maximally mixed [5]. Its weights distribution reads [43] $A = [1, 0, 0, 0, 35, 42, 28, 22]$. While it was known by exhaustive search that the distribution cannot be realized by a graph state, it was only recently shown that no state with such property exists [44]. Interestingly, weight distributions are known whose realizations as a graph states are unresolved. As an example, the existence of a graph state on 24 qubits, having all 9-body reductions maximally mixed, is a long-standing open problem [6]. Putative weights for such a state of type II, having even weights only, are [7]

$$\begin{aligned}&[A_{10}, A_{12}, A_{14}, \dots A_{24}] = \\ &[18216, 156492, 1147608, 3736557, 6248088, 4399164, 1038312, 32778]\,.\end{aligned} \tag{66}$$

[5] This a so-called absolutely maximally entangled state, having the code parameters $((7, 1, 4))_2$.

[6] Such state is equivalent to a self-dual additive code over GF_4, and corresponds to a quantum code having the parameters $[[24, 0, 10]]_2$. See also Research Problem 13.3.7 in Ref. [52] and the code tables of Ref. [53].

[7] This weight distribution can also be found in the On-Line Encyclopedia of Integer Sequences, see http://oeis.org/A030331.

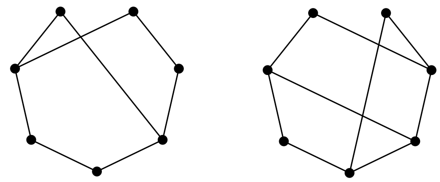

Figure 3: Two graph states on seven qubits that share the same weight distribution, but which can be shown to be inequivalent under local unitaries and graph isomorphism. These correspond to graphs No. 42 and 43 of Fig. 5 in Ref. [28].

Finally, we note that a weight distribution does not uniquely identify the corresponding graph state, as states inequivalent under LU-transformations and graph isomorphism can indeed have the same weight distribution. As an example, consider the two graph states on seven qubits that are depicted in Fig. 8. These can be shown to be inequivalent under local unitaries and graph isomorphism, but they share the same weight distribution of $A = [1, 0, 0, 7, 21, 42, 42, 15]$ [8]. We conclude that graph states are not uniquely identified by their weight distribution.

9 Conclusion

We have introduced the Ulam graph reconstruction problem to the case of quantum graph states. In contrast to classical graph decks, the full graph state can (up to local Z-gates) be reconstructed from a single card in the deck. As in the classical setting, the question of detecting illegitimate decks is of interest. Here, consistency equations can be derived which can detect some but not all illegitimate quantum decks from their weight distribution; in some cases it is already possible to detect the illegitimacy of decks containing marginals of size $\lfloor n/2 \rfloor + 1$. It would be interesting to see whether similar relations can also be obtained for classical decks of graphs.

The result by Bollobás [32], namely, that almost every graph can uniquely be reconstructed by a specific set of three cards, has an interesting counterpart in the quantum setting: It has been shown that almost all pure states are already uniquely determined amongst all pure states by three marginals of size $(n-2)$ [54]. It would be desirable to understand if similar results also hold for the special case of graph states.

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