COUNTING INVERSIONS AND DESCENTS OF RANDOM ELEMENTS IN FINITE COXETER GROUPS

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ABSTRACT. We investigate Mahonian and Eulerian probability distributions given by inversions and descents in general finite Coxeter groups. We provide uniform formulas for the mean values and variances in terms of Coxeter group data in both cases. We also provide uniform formulas for the double-Eulerian probability distribution of the sum of descents and inverse descents. We finally establish necessary and sufficient conditions for general sequences of Coxeter groups of increasing rank for which Mahonian and Eulerian probability distributions satisfy central and local limit theorems.

CONTENTS

1. INTRODUCTION

Properties of random permutations are important in many areas of applied mathematics, for example in statistical ranking where the collected data consists of permutations. Instead of studying the actual permutations, applications often work with permutation statistics. The most common include the numbers of cycles of various sizes, or the numbers of inversions and descents. When permutations in the symmetric group are drawn uniformly at random, the asymptotics of the resulting random variables (as the size of the symmetric group tends to infinity) are well-studied. Exact formulas for the moments and limit theorems for the corresponding distributions are known. In this paper we extend the study of counting inversions and descents of random permutations to random elements of finite Coxeter groups. We illustrate in detail how to compute mean values and variances, and follow the product formula approach by Bender [1] to give necessary and sufficient conditions on sequences of finite Coxeter groups of increasing rank such that the numbers of inversions and descents satisfy central and local limit theorems. For permutations these are well-known phenomena. We refer to [4, 5, 16] for these and further applications of Bender's approach. Limit theorems for permutation statistics are a topic of continuing interest, we refer to [8] for a recent consideration of the statistic given by

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the number of descents of a permutation plus the number of descents of its inverse. We also provide uniform formulas for mean values and variances of this statistic in general finite Coxeter groups.

Section 2 contains relevant notions for finite Coxeter groups and the associated random variables. In Sections 3, 4 and 5, we compute means and variances of the *W-Mahonian distribution* given by the number of inversions of a random Coxeter group element, the *W-Eulerian distribution* given by the number of descents, and the *W-double-Eulerian distribution* given by the number of descents plus the number of inverse descents. In the final Section 6, we exhibit necessary and sufficient conditions for central and local limit theorems to hold for the W-Mahonian and the W-Eulerian distributions. Curiously, these conditions only depend on the sizes of the dihedral parabolic subgroups in the sequence of Coxeter groups. At the moment such necessary and sufficient conditions for limit theorems remain open for the W-double Eulerian distribution of an arbitrary finite Coxeter group.

This project began with an experimental investigation of the asymptotics of permutation statistics. We present these investigations in Appendix A. In particular, we found the variances for the Mahonian, the Eulerian and the double-Eulerian distributions. The first two are classical, while the latter was computed recently in [8]. Using the same procedure, we also found conjectured formulas for the other classical types B_n and D_n. These are now Theorems 3.1, 4.1 and 5.1.

Next to mean and variance of distributions of permutation statistics, one might as well try to guess formulas for higher moments and cumulants. These computations can then suggest central limit theorems. For Mahonian, Eulerian and double-Eulerian distributions in the symmetric group, the central limit theorems are known. The first two have many different proofs, but the central limit theorem for the double-Eulerian distribution required some advanced techniques [8]. Our computations in the other classical types resulted in Theorem 6.1.

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2. Probability distributions from Coxeter group statistics

A polynomial $f = \sum_i a_i z^i \in \mathbb{N}[z]$ with $\mathbb{N} = \{0, 1, 2, \ldots\}$ gives rise to a random variable X_f on $\mathbb{N}$ via

$$\mathrm{Prob}(X_f = k) = \frac{a_k}{\sum_i a_i} = [z^k]f/f(1).$$

This is, the probability for X_f to have value k is the coefficient of z^k in f divided by $f(1)$. A *permutation statistics* is, in its simplest form, a map $\mathrm{st} : \mathfrak{S}_n \longrightarrow \mathbb{N}$, where $\mathfrak{S}_n$ is the group of permutations of $\{1, \ldots, n\}$. Each such permutation statistic yields a random variable X_{st} on $\mathbb{N}$ when evaluated on permutation that is chosen uniformly at random. These two concepts are linked via the generating function of a statistic

$$\mathcal{G}_{\mathrm{st}}(z) = \sum_{\pi \in \mathfrak{S}_n} z^{\mathrm{st}(\pi)}$$

since $\mathrm{Prob}(X_{\mathrm{st}} = k) = \mathrm{Prob}(X_{\mathcal{G}_{\mathrm{st}}} = k)$. In particular, the random variable X_{st} only depends on the generating function of the statistic st.

Two basic and important examples of permutation statistics are the number of *inversions* $\mathrm{inv}(\pi) = \#\mathrm{Inv}(\pi)$ (findstat.org/St000018) and of *descents* $\mathrm{des}(\pi) = \#\mathrm{Des}(\pi)$ of $\pi \in \mathfrak{S}_n$ (findstat.org/St000021), where

$$\mathrm{Inv}(\pi) = \big\{1 \leq i < j \leq n \mid \pi(i) > \pi(j)\big\}, \quad \mathrm{Des}(\pi) = \big\{1 \leq i < n \mid \pi(i) > \pi(i+1)\big\}.$$

The *Mahonian number* (oeis.org/A000302) is the number of permutations in $\mathfrak{S}_n$ with k inversions and the *Eulerian number* (oeis.org/A008292) is the number of permutations in $\mathfrak{S}_n$ with k descents. The Eulerian numbers have a long history. Euler encountered them in the context of the evaluation of the sum of alternating powers $(1^n - 2^n + 3^n - \cdots)$. The combinatorial definition that we use now became popular only the 20th century. See [15] for everything on Eulerian numbers. The probability distributions for the random variables X_{inv} and X_{des} are thus respectively called *Mahonian probability distribution* and the *Eulerian probability distribution*. Both are well studied, see [1] for a unified treatment. Many extensions of this result are known. Two examples are a central limit theorem for the Mahonian probability distribution on multiset permutations [7], and a central limit theorem for Mahonian and Eulerian distribution on colored permutations [9].

In this paper, we generalize and extend results about inversions and descents to general finite Coxeter groups. Let $(W, \mathcal{S})$ be a finite Coxeter group of rank $n = |\mathcal{S}|$. The elements in $\mathcal{S}$ are the *simple reflections*. Let $\Delta \subseteq \Phi^+ \subset \Phi = \Phi^+ \sqcup \Phi^-$ be a root system for $(W, \mathcal{S})$ with simple roots Δ and positive roots Φ^+. We refer to [3, Part 1] for background on finite Coxeter groups. Slightly abusing notation, we always think of a Coxeter group as coming with a fixed system of simple roots. As usual, let $m_{s,t}$ denote the order of the product $st \in W$ for two simple reflections $s \neq t$. All elements of the form $c = s_1 \cdots s_n$ for $\mathcal{S} = \{s_1, \ldots, s_n\}$ are conjugate in W and thus have the same order h which is called *Coxeter number* of W. The eigenvalues of these elements are $\{e^{2\pi i(d_k - 1)/h}\}$ where $\{d_1, \ldots, d_n\}$ are the *degrees* of W.

Inversions and descents can be defined in the above setting using the (*weak*) *length function* $\mathrm{inv} : W \to \mathbb{N}$ which assigns the length of a shortest word in $\mathcal{S}$ to any $w \in W$. Equivalently, $\mathrm{inv}(w)$ is the number of positive roots sent to negative roots by w. For $w \in W$, one thus defines *W-inversions* and *W-descents* by

$$\mathrm{Inv}(w) = \big\{\beta \in \Phi^+ \mid w(\beta) \in \Phi^-\big\}, \quad \mathrm{Des}(w) = \big\{\beta \in \Delta \mid w(\beta) \in \Phi^-\big\}.$$

We continue to write $\mathrm{inv}(w) = \#\mathrm{Inv}(w)$ and $\mathrm{des}(w) = \#\mathrm{Des}(w)$. These definitions specialize to the known definitions in the permutation group. Positive roots in $A_n = \mathfrak{S}_{n+1}$ can be realized as $\Phi^+ = \{e_i - e_j \mid 1 \leq i < j \leq n+1\}$ and simple roots as $\{e_i - e_{i+1} \mid 1 \leq i \leq n\}$. Therefore inversions and descents in the one-line notation for $\mathfrak{S}_{n+1}$ correspond to A_n-inversions and, respectively, to A_n-descents. Consider for example the permutation

$$\pi = [2, 5, 1, 3, 6, 4] = (12)(45)(34)(23)(56).$$

In this case, we have

$$\begin{aligned} \mathrm{Inv}(\pi) &= \{13, 23, 24, 26, 56\} \leftrightarrow \{e_1 - e_3, e_2 - e_3, e_2 - e_4, e_2 - e_4, e_2 - e_6, e_5 - e_6\}, \\ \mathrm{Des}(\pi) &= \{\quad 23, \qquad 56\} \leftrightarrow \{\qquad e_2 - e_3, \qquad\qquad e_5 - e_6\}. \end{aligned}$$

As above, the *W-Mahonian numbers* and *W-Eulerian numbers* are numbers of elements in W with exactly k W-inversions, and, respectively, W-descents. Finally, the

W-Mahonian distribution X_{inv} and *W-Eulerian distribution* X_{des} on $\mathbb{N}$ are defined using their generating functions

$$\mathcal{G}_{\mathrm{inv}}(W;z) = \sum_{w\in W} z^{\mathrm{inv}(w)} \quad \text{and} \quad \mathcal{G}_{\mathrm{des}}(W;z) = \sum_{w\in W} z^{\mathrm{des}(w)}.$$

Remark 2.1. One could also study more general statistics interpolating between W-descents and W-inversions by defining $\mathrm{st}_I(w) = \{\beta \in I \mid w(\beta) \in \Phi^-\}$ where I is any subset of positive roots. At the end of Section 3, we discuss how to analyze mean value and variance of the distribution of any such statistic. However, the arguments for limit theorems depend on the concrete product structure of the generating functions, and do not apply to interpolating distributions in general.

Using the factorization of generating functions (which corresponds to the independence of the associated random variables) the computation of mean values and variances for finite Coxeter groups reduces to the irreducible Coxeter groups.

Lemma 2.2. *Let $W = W' \times W''$ be a product of two Coxeter groups W' and W'' and denote by X_{st} either the number of inversions of a random element in W or the number of descents. Define X'_{st} and X''_{st} analogously. Then*

$$\mathbb{E}(X_{\mathrm{st}}) = \mathbb{E}(X'_{\mathrm{st}}) + \mathbb{E}(X''_{\mathrm{st}}), \qquad \mathbb{V}(X_{\mathrm{st}}) = \mathbb{V}(X'_{\mathrm{st}}) + \mathbb{V}(X''_{\mathrm{st}}).$$

The main ingredients in the subsequent constructions from general finite Coxeter groups are the following properties of inversions and descents. Following [19], a polynomial $f = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_i z + a_0 \in \mathbb{N}[z]$ is

- *unimodal* if $a_0 \le \cdots \le a_{i-1} \le a_i \ge a_{i+1} \ge \cdots \ge a_n$ for some $1 \le i \le n$,
- *log-concave* if $a_i^2 \ge a_{i-1}a_{i+1}$ for all $1 \le i < n$.

If the sequence $a_0, \ldots, a_n$ has no internal zeroes, then log-concavity implies unimodality. A stronger condition implying log-concavity is that f has only real nonpositive roots, this is $f = \prod_k (z + q_i)$ with $q_i \in \mathbb{R}_{\ge 0}[z]$, see [19, Theorem 2].

Let $[d]_z$ denote the *z-integer* $\frac{1-z^d}{1-z} = 1 + z + z^2 + \cdots + z^{d-1}$ (often used as *q-integer*). The following statement can be found for example in [3, Chapter 7].

Theorem 2.3. *Let W be a finite Coxeter group of rank n with degrees $d_1, \ldots, d_n$. The generating function for the number of inversions satisfies*

$$\mathcal{G}_{\mathrm{inv}}(W;z) = \prod_{i=1}^{n} [d_i]_z. \tag{2.1}$$

In particular, the sequence of coefficients of $\mathcal{G}_{\mathrm{inv}}$ is log-concave and unimodal.

The next statement was proven in all irreducible types except type D in [6] while type D was only recently settled in [18].

Theorem 2.4. *Let $(W, \mathcal{S})$ be a finite Coxeter group of rank n. Then $\mathcal{G}_{\mathrm{des}}$ has only real negative roots,*

$$\mathcal{G}_{\mathrm{des}}(W;z) = \prod_{i=1}^{n} (z + q_i) \tag{2.2}$$

for some $q_1, \ldots, q_n \in \mathbb{R}_{\ge 0}$. In particular, the sequence of coefficients of $\mathcal{G}_{\mathrm{des}}$ is log-concave and unimodal.

2.1. **Inversions and descents in classical types.** The Coxeter group of type B_n can be realized as the group of *signed permutations*, that is antisymmetric bijections on $\{\pm 1, \dots, \pm n\}$. In symbols,

$$B_n = \big\{\pi : \{\pm 1, \dots, \pm n\} \xrightarrow{\sim} \{\pm 1, \dots, \pm n\} \mid \pi(-i) = -\pi(i)\big\}.$$

We represent signed permutations in their one-line notation $\pi = [\pi(1), \dots, \pi(n)]$ where $\pi(i) \in \{\pm 1, \dots, \pm n\}$ and $\{|\pi(1)|, |\pi(2)|, \dots, |\pi(n)|\} = \{1, \dots, n\}$. The Coxeter group of type D_n can be realized as the group of *even signed permutations*, the subgroup of B_n of index 2 containing all signed permutations whose one-line notation contains an even number of negative entries. This is,

$$D_n = \big\{\pi \in B_n \mid \pi(1) \cdot \pi(2) \cdot \ \cdots \ \cdot \pi(n) > 0\big\}.$$

Following [3, Prop. 8.1.1] in type B_n and [3, Prop. 8.2.1] in type D_n, we set

$$\begin{aligned} \mathrm{Inv}^+(\pi) &= \big\{1 \le i < j \le n \mid \pi(i) > \pi(i+1)\big\} \\ \mathrm{Inv}^-(\pi) &= \big\{1 \le i < j \le n \mid -\pi(i) > \pi(i+1)\big\} \\ \mathrm{Inv}^\circ(\pi) &= \big\{1 \le i \le n \mid \pi(i) < 0\big\} \end{aligned}$$

and obtain

$$\mathrm{Inv}(\pi) = \begin{cases} \mathrm{Inv}^+(\pi) & \text{for } \pi \in A_{n-1}, \\ \mathrm{Inv}^+(\pi) \cup \mathrm{Inv}^-(\pi) \cup \mathrm{Inv}^\circ(\pi) & \text{for } \pi \in B_n, \\ \mathrm{Inv}^+(\pi) \cup \mathrm{Inv}^-(\pi) & \text{for } \pi \in D_n. \end{cases} \tag{2.3}$$

Similarly, following [3, Prop. 8.1.2] in type B_n and [3, Prop. 8.2.2] in type D_n, we set

$$\pi(0) = \begin{cases} 0 & \text{for } \pi \in A_{n-1}, \\ 0 & \text{for } \pi \in B_n, \\ -\pi(2) & \text{for } \pi \in D_n. \end{cases}$$

and define descents as

$$\mathrm{Des}(\pi) = \big\{0 \le i < n \mid \pi(i) > \pi(i+1)\big\}. \tag{2.4}$$

3. The Mahonian distribution

Theorem 3.1. *Let W be a finite Coxeter group. The W-Mahonian distribution X_{inv} has mean and variance*

$$\mathbb{E}(X_{\mathrm{inv}}) = \frac{1}{2}\sum_{k=1}^{n}(d_k - 1), \quad \mathbb{V}(X_{\mathrm{inv}}) = \frac{1}{12}\sum_{k=1}^{n}(d_k^2 - 1),$$

where n is the rank of W and $d_1, \dots, d_n$ are the degrees of W.

The theorem can be written explicitly as follows.

Corollary 3.2. *In the situation of the previous theorem, the W-Mahonian distribution has variances*

(type A_n)	$\mathbb{E}(X_{\text{inv}}) = n(n+1)/4$	$\mathbb{V}(X_{\text{inv}}) = (2n^3+9n^2+7n)/72$
(type B_n)	$\mathbb{E}(X_{\text{inv}}) = n^2/2$	$\mathbb{V}(X_{\text{inv}}) = (4n^3+6n^2-n)/36$
(type D_n)	$\mathbb{E}(X_{\text{inv}}) = n(n-1)/2$	$\mathbb{V}(X_{\text{inv}}) = (4n^3-3n^2-n)/36$
(type E_6)	$\mathbb{E}(X_{\text{inv}}) = 18$	$\mathbb{V}(X_{\text{inv}}) = 29$
(type E_7)	$\mathbb{E}(X_{\text{inv}}) = 63/2$	$\mathbb{V}(X_{\text{inv}}) = 287/4$
(type E_8)	$\mathbb{E}(X_{\text{inv}}) = 60$	$\mathbb{V}(X_{\text{inv}}) = 650/3$
(type F_4)	$\mathbb{E}(X_{\text{inv}}) = 12$	$\mathbb{V}(X_{\text{inv}}) = 61/3$
(type H_3)	$\mathbb{E}(X_{\text{inv}}) = 15/2$	$\mathbb{V}(X_{\text{inv}}) = 137/12$
(type H_4)	$\mathbb{E}(X_{\text{inv}}) = 30$	$\mathbb{V}(X_{\text{inv}}) = 361/3$
(type $I_2(m)$)	$\mathbb{E}(X_{\text{inv}}) = m/2$	$\mathbb{V}(X_{\text{inv}}) = (m^2+2)/12$

We prove Theorem 3.1 using a well-known description of the generating function of the number of inversions in general finite Coxeter groups. Corollary 3.2 follows from this description but we also provide an explicit proof in the classical types.

Proposition 3.3. *Let $d_1, \dots, d_n$ be* any *sequence of positive integers and X_f the random variable for the polynomial $f = \prod_{k=1}^n [d_k]_z$. Then the mean and variance of X_f are*

$$\mathbb{E}(X_f) = \frac{1}{2}\sum (d_k - 1), \quad \mathbb{V}(X_f) = \frac{1}{12}\sum_{k=1}^{n}(d_k^2 - 1).$$

Proof. For $d \geq 2$, let X_d be the random variable for the polynomial $[d]_z$. That is, X_d is distributed uniformly on the integers $\{0, \dots, d-1\}$. A simple count yields that

$$X_f = X_{d_1} + \cdots + X_{d_n}$$

for independent random variables $X_{d_1}, \dots, X_{d_n}$. Therefore, the mean and variance of X_f are, respectively, the sums of the means and variances of the individual X_{d_k}. These are well-known to be $\mathbb{E}(X_d) = (d-1)/2$ and $\mathbb{V}(X_d) = \frac{1}{12}(d^2-1)$. □

Proof of Theorem 3.1. This is a direct application of Proposition 3.3 given (2.1). □

For the proof of Corollary 3.2 it is now sufficient to look up the degrees of the classical finite Coxeter groups. We also discuss an instructive direct proof below, using combinatorial interpretations of inversions in the classical types.

Proof of Corollary 3.2. In type A_n, we have $(d_1, \dots, d_n) = (2, 3, \dots, n+1)$ and thus

$$\begin{aligned}\mathbb{V}(X_{\text{inv}}) &= \frac{1}{12}\sum_{k=2}^{n+1}(k^2-1) \\ &= \frac{1}{12}\left(\tfrac{1}{6}(n+1)(n+2)(2n+3) \ - \ (n+1)\right) \\ &= \frac{1}{72}(2n^3+9n^2+7n).\end{aligned}$$

In type B_n, we have $(d_1, \dots, d_n) = (2, 4, \dots, 2n)$ and thus obtain

$$\begin{aligned}\mathbb{V}(X_{\mathrm{inv}}) &= \frac{1}{12}\sum_{k=1}^{n}(4k^2-1)\\ &= \frac{1}{12}\left(\tfrac{4}{6}n(n+1)(2n+1) \;-\; n\right)\\ &= \frac{1}{36}(4n^3+6n^2-n).\end{aligned}$$

In type D_n, we have $(d_1, \dots, d_n) = (2, 4, \dots, 2n-2,\ n)$ and thus obtain

$$\begin{aligned}\mathbb{V}(X_{\mathrm{inv}}) &= \frac{1}{12}\sum_{k=1}^{n-1}(4k^2-1) + \frac{1}{12}(n^2-1)\\ &= \frac{1}{12}\left(\tfrac{4}{6}n(n-1)(2n-1) \;-\; (n-1)+n^2-1\right)\\ &= \frac{1}{36}(4n^3-3n^2-n).\end{aligned}$$

□

We next provide a sum decomposition to obtain Theorem 3.1 in the classical types. We then describe how to use such sum decompositions to analyze the variance of any statistic st_I for $I \subseteq \Phi^+$ as in Remark 2.1.

To this end, define indicator random variables corresponding to the three sets in (2.3).

$$Y_{ij}^{+} = \begin{cases} 1 & \text{if } \pi(i) > \pi(j) \\ 0 & \text{otherwise} \end{cases}$$

$$Y_{ij}^{-} = \begin{cases} 1 & \text{if } -\pi(i) > \pi(j) \\ 0 & \text{otherwise} \end{cases}$$

$$Y_{i}^{\circ} = \begin{cases} 1 & \text{if } \pi(i) < 0 \\ 0 & \text{otherwise} \end{cases}$$

These random variables can be interpreted as indicating how π acts on the positive roots if one identifies

$$Y_{ij}^{+} \leftrightarrow e_i - e_j, \qquad Y_{ij}^{-} \leftrightarrow e_i + e_j, \qquad Y_{i}^{\circ} \leftrightarrow e_i \tag{3.1}$$

With these definitions and Propositions 8.1.1 and 8.2.1 in [3] we have

(type A_{n-1}) $$X_{\mathrm{inv}} = \sum_{i<j} Y_{ij}^{+}$$

(type B_n) $$X_{\mathrm{inv}} = \sum_{i<j} Y_{ij}^{+} + \sum_{i<j} Y_{ij}^{-} + \sum_{i} Y_{i}^{\circ}$$

(type D_n) $$X_{\mathrm{inv}} = \sum_{i<j} Y_{ij}^{+} + \sum_{i<j} Y_{ij}^{-}.$$

For a direct proof of Corollary 3.2 using $\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$ one needs to control the covariances among the random variables. The mean values of X_{inv} are easily confirmed as a warm-up to the following computation recalculating $\mathbb{V}(X_{\text{inv}})$ in type B_n:

$$
\begin{aligned}
&\mathbb{E}(X_{\text{inv}}^2) = \mathbb{E}\Big(\sum_{i<j} Y_{ij}^+ + \sum_{i<j} Y_{ij}^- + \sum_i Y_i^\circ\Big)^2 \\
&(Y^+ \text{ with } Y^+) \quad = \binom{n}{2}\frac{1}{2} + \binom{n}{2}\binom{n-2}{2}\frac{1}{4} + 2\binom{n}{3}\frac{1}{6} + 4\binom{n}{3}\frac{1}{3} \\
&(Y^- \text{ with } Y^-) \quad + \binom{n}{2}\frac{1}{2} + \binom{n}{2}\binom{n-2}{2}\frac{1}{4} + 2\binom{n}{3}\frac{1}{3} + 4\binom{n}{3}\frac{1}{3} \\
&(Y^\circ \text{ with } Y^\circ) \quad + n\frac{1}{2} + 2\binom{n}{2}\frac{1}{4} \\
&(Y^+ \text{ with } Y^-) \quad + 2\Bigg[\binom{n}{2}\frac{1}{4} + \binom{n}{2}\binom{n-2}{2}\frac{1}{4} + \binom{n}{3}\frac{1}{3} + \binom{n}{3}\frac{1}{6} + 2\binom{n}{3}\frac{1}{3} + 2\binom{n}{3}\frac{1}{6} \\
&(Y^+ \text{ with } Y^\circ) \quad\quad + 3\binom{n}{3}\frac{1}{4} + \binom{n}{2}\frac{1}{8} + \binom{n}{2}\frac{3}{8} \\
&(Y^- \text{ with } Y^\circ) \quad\quad + 3\binom{n}{3}\frac{1}{4} + \binom{n}{2}\frac{3}{8} + \binom{n}{2}\frac{3}{8}\Bigg] \\
&\quad = \frac{1}{4}n^4 + \frac{1}{36}(4n^3 + 6n^2 - n)
\end{aligned}
$$

The formula is written so that each summand is a product of a number of patterns of indices ij, kl or ij, k and a probability of the configuration that yields products $Y_{ij}Y_{kl} = 1$. Working out all the coefficients is an instructive exercise. After subtracting $\mathbb{E}(X)^2 = \frac{1}{4}n^4$ from the result above we find Corollary 3.2 in type B_n. The variance formulas for types A_{n-1} and D_n can be deduced from above, omitting all terms that contain Y^- or Y° in type A_{n-1} and those that contain Y° in type D_n.

The same argument can also be used to analyze the distribution X_{st} of any statistic $\text{st} = \text{st}_I = \{\beta \in I \mid w(\beta) \in \Phi^-\}$ where I is any subset of positive roots as in Remark 2.1. First, there is the obvious uniform argument to compute the mean value.

Proposition 3.4. *Let W be a finite Coxeter group and let $I \subseteq \Phi^+$ be a subset of positive roots. Then*

$$\mathbb{E}(X_{\text{st}_I}) = \tfrac{1}{2}|I|.$$

Proof. Let $w_\circ \in W$ be the unique element of longest length. Then for any element $w \in W$, there is a unique $w' \in W$ such that $ww' = w_\circ$ and

$$\text{Inv}(w) \cup \text{Inv}(w') = \Phi^+, \quad \text{Inv}(w) \cap \text{Inv}(w') = \emptyset.$$

Since $\text{st}_I(w) = |\text{Inv}(w) \cap I|$, we obtain that $\text{st}_I(w) + \text{st}_I(w') = |I|$. Because $w \neq w'$, the statement follows. □

To obtain the variance of st_I as well, one proceeds as in the direct proof of Corollary 3.2, this time using only the variables $Y_{ij}^+, Y_{ij}^-, Y_i^\circ$ corresponding to positive roots in I. The matching is as in (3.1) and

$$X_{\text{st}_I} = \sum_{\beta \in I} X_\beta.$$

where X_β is the random variable corresponding to the positive root $\beta \in I$.

4. The Eulerian distribution

Theorem 4.1. *Let $(W, \mathcal{S})$ be an irreducible finite Coxeter system and denote by m the maximal order of the product of two simple reflections in $\mathcal{S}$, $m = \max\{m_{s,t} \mid s, t \in \mathcal{S}\}$. The W-Eulerian distribution X_{des} has mean and variance*

$$\mathbb{E}(X_{\mathrm{des}}) = n/2, \quad \mathbb{V}(X_{\mathrm{des}}) = (n-2)/12 + 1/m.$$

where n is the rank of W.

The theorem can be written explicitly as follows.

Corollary 4.2. *In the situation of the previous theorem, the W-Eulerian distribution has variances*

(type A_n)	$\mathbb{V}(X_{\mathrm{des}}) = (n+2)/12$
(type B_n)	$\mathbb{V}(X_{\mathrm{des}}) = (n+1)/12$
(type D_n)	$\mathbb{V}(X_{\mathrm{des}}) = (n+2)/12$
(type E_n)	$\mathbb{V}(X_{\mathrm{des}}) = (n+2)/12$
(type F_4)	$\mathbb{V}(X_{\mathrm{des}}) = 5/12$
(type H_3)	$\mathbb{V}(X_{\mathrm{des}}) = 17/60$
(type H_4)	$\mathbb{V}(X_{\mathrm{des}}) = 11/30$
(type $I_2(m)$)	$\mathbb{V}(X_{\mathrm{des}}) = 1/m$

Remark 4.3. The groups of types A_{n-1} and B_n are also wreath products $\mathfrak{S}_n \wr \mathcal{C}_r$ where $\mathcal{C}_r$ is the cyclic group on r letters. Chow and Mansour computed in [9] the means and variances for all such groups using a different approach.

Remark 4.4. Theorem 4.1 can be used to obtain information about the ominous negative roots of $\mathcal{G}_{\mathrm{des}}(W; z) = \prod_i (z + q_i)$, since by [1, Theorem 2],

$$\mathbb{E}(X_{\mathrm{des}}) = \sum_{i=1}^{n} \frac{1}{1+q_i}, \qquad \mathbb{V}(X_{\mathrm{des}}) = \sum_{i=1}^{n} \frac{q_i}{(1+q_i)^2}.$$

Observe that the palindromicity $\mathcal{G}_{\mathrm{des}}(W; z) = z^n \cdot \mathcal{G}_{\mathrm{des}}(W; z^{-1})$ implies that the left equation for the mean is trivially satisfied because the roots come in inverse pairs q and q^{-1}. On the other hand, we are not aware of any previously known property of the roots q_i which imply the right equation for the variance.

We prove Theorem 4.1 using combinatorial interpretations of descents in the classical types. These can be found in [3] (we refer to the exact sections below). For later use we define the following indicator random variables

$$Y^{(i)} = \begin{cases} 1 & \pi(i) > \pi(i+1) \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

The $Y^{(i)}$ depend on the exact definition of descent (which differs from type to type). In any case, the number of descents of a random element $\pi \in W$ is the sum of such random variables and mean and variance can be computed from this sum since (2.4) implies that

$$X_{\mathrm{des}} = \sum_{i=0}^{n-1} Y^{(i)} \tag{4.2}$$

in types B_n and D_n, while the sum is from 1 to n in type A_n. The A_n-case is well-known.

Proposition 4.5. *The mean and variance of the Eulerian distribution on A_n are*

$$\mathbb{E}(X_{\mathrm{des}}) = \frac{n}{2}, \qquad \mathbb{V}(X_{\mathrm{des}}) = \frac{n+2}{12}$$

Proof. The mean is clear from linearity of the mean value and $\mathbb{E}(Y^{(i)}) = 1/2$. To compute $\mathbb{E}(X_{\mathrm{des}}^2) = \sum_{i,j} \mathbb{E}(Y^{(i)}Y^{(j)})$ we distinguish three types of summands:

- The n summands with $i = j$ give $\mathbb{E}(Y^{(i)}Y^{(j)}) = 1/2$.
- The $2(n-1)$ summands with $|i-j| = 1$ give $\mathbb{E}(Y^{(i)}Y^{(j)}) = 1/6$, since $\pi(a) > \pi(a+1) > \pi(a+2)$ for $1 \leq a < n-1$ occurs exactly once among the six equally likely possibilities.
- For the summands with $|i-j| > 1$ we have $\mathbb{E}(Y^{(i)}Y^{(j)}) = \mathbb{E}(Y^{(i)})\mathbb{E}(Y^{(j)}) = 1/4$.

We thus find

$$\begin{aligned}\mathbb{V}(X_{\mathrm{des}}) &= \mathbb{E}(X_{\mathrm{des}}^2) - \mathbb{E}(X_{\mathrm{des}})^2 \\ &= \frac{n}{2} + \frac{2(n-1)}{6} + \frac{n^2 - n - 2(n-1)}{4} - \frac{n^2}{4} \\ &= \frac{n+1}{12}.\end{aligned}$$

□

Proposition 4.6. *The mean and variance of the B_n-Eulerian distribution are*

$$\mathbb{E}(X_{\mathrm{des}}) = \frac{n}{2}, \qquad \mathbb{V}(X_{\mathrm{des}}) = \frac{n+1}{12}$$

Proof. Again, $\mathbb{E}(X_{\mathrm{des}}) = n/2$ is clear from linearity of $\mathbb{E}$. To compute $\mathbb{E}(X_{\mathrm{des}}^2)$ we split the sum over pairs $i, j \in \{0, \ldots, n-1\}$ into four types of summands.

- The n summands with $i = j$ give $\mathbb{E}(Y^{(i)}Y^{(j)}) = 1/2$.
- The $2(n-2)$ summands with $|i-j| = 1$ and $i, j > 0$ give $\mathbb{E}(Y^{(i)}Y^{(j)}) = 1/6$ for the same reason as in Proposition 4.5.
- The 2 summands with $\{i, j\} = \{0, 1\}$ give $\mathbb{E}(Y^{(i)}Y^{(j)}) = 1/8$. This is because $0 > w_1 > w_2$ occurs in exactly one of eight equally likely possibilities $w_1, w_2 \lesseqgtr 0$ and $w_1 > w_2$.
- Finally, the $n^2 - n - 2(n-2) - 2$ summands with $|i-j| > 1$ give $\mathbb{E}(Y^{(i)}Y^{(j)}) = \mathbb{E}(Y^{(i)})\mathbb{E}(Y^{(j)}) = 1/4$.

We thus find

$$\begin{aligned}\mathbb{V}(X_{\mathrm{des}}) &= \mathbb{E}(X_{\mathrm{des}}^2) - \mathbb{E}(X_{\mathrm{des}})^2 \\ &= \frac{n}{2} + \frac{2(n-2)}{6} + \frac{2}{8} + \frac{n^2 - n - 2(n-2) - 2}{4} - \frac{n^2}{4} \\ &= \frac{n+1}{12}.\end{aligned}$$

□

Proposition 4.7. *The mean and variance of the D_n-Eulerian distribution are*

$$\mathbb{E}(X_{\mathrm{des}}) = \frac{n}{2}, \qquad \mathbb{V}(X_{\mathrm{des}}) = \frac{n+2}{12}.$$

Proof. By linearity of $\mathbb{E}$ again $\mathbb{E}(X_{\mathrm{des}}) = n/2$. To compute $\mathbb{E}(X_{\mathrm{des}}^2)$ we here consider five types of pairs $i, j \in \{0, \ldots, n-1\}$.

- The n summands with $i = j$ give $\mathbb{E}(Y^{(i)}Y^{(j)}) = 1/2$.
- The $2(n-2)$ summands with $|i-j| = 1$ and $i, j > 0$ give $\mathbb{E}(Y^{(i)}Y^{(j)}) = 1/6$ for the same reason as in Proposition 4.5.

- The 2 summands with $\{i,j\} = \{0,1\}$ yield $\mathbb{E}(Y^{(i)}Y^{(j)}) = 1/4$ since one quarter of the elements of D_n satisfies $-\pi(2) > \pi(1) > \pi(2)$.
- The 2 summands with $\{i,j\} = \{0,2\}$ yield $\mathbb{E}(Y^{(i)}Y^{(j)}) = 1/6$. This is because one asks how often $-w_3 > -w_2 > w_1$.
- Finally, in all other summands $\mathbb{E}(Y^{(i)}Y^{(j)}) = \mathbb{E}(Y^{(i)})\mathbb{E}(Y^{(j)}) = 1/4$.

In total we have

$$\begin{aligned}\mathbb{V}(X_{\mathrm{des}}) &= \mathbb{E}(X_{\mathrm{des}}^2) - \mathbb{E}(X_{\mathrm{des}})^2 \\ &= \frac{n}{2} + \frac{2(n-2)}{6} + \frac{2}{4} + \frac{2}{6} + \frac{n^2 - n - 2(n-2) - 4}{4} - \frac{n^2}{4} \\ &= \frac{n+2}{12}. \end{aligned}$$ □

Proof of Theorem 4.1. The classical types were computed in Propositions 4.5, 4.6 and 4.7. The computation in the dihedral types $I_2(m)$ is obvious, and the remaining were computed using SAGE [11]. □

5. The double-Eulerian distribution

An *inverse descent* (also known as *recoil* or *ligne of route*) of a permutation π is a descent of π^{-1},

$$\mathrm{ides}(\pi) = \mathrm{des}(\pi^{-1}).$$

Permutations with k descents and ℓ inverse descents have been studied in various contexts. We refer to the unpublished manuscript by Foata and Han [13] for a detailed combinatorial treatment of this bi-statistic. To emphasize its bivariate nature, we refer to the numbers of permutations with k descents and ℓ inverse descents as the *bi-Eulerian numbers* and to the numbers of permutations such that $\mathrm{des}(\pi) + \mathrm{ides}(\pi)$ equals k as the *double-Eulerian numbers* (`oeis.org/A298248`). Several papers use the term double-Eulerian numbers already for the bivariate version. Others, such as [15], refer to the bi-statistic as the *two-sided Eulerian numbers*. We have chosen the present terms in order to clarify the distinction between the bivariate statistic $(\mathrm{des}(\pi), \mathrm{ides}(\pi))$ and the univariate statistic $\mathrm{des}(\pi) + \mathrm{ides}(\pi)$ (`findstat.org/St000824`). We thus call the probability distributions for the random variables $X_{\mathrm{des+ides}}$ *double-Eulerian probability distribution*.

In type A_n, Chatterjee and Diaconis [8] computed the mean and variance of the double-Eulerian distribution as

$$\mathbb{E}(X_{\mathrm{des+ides}}) = n, \qquad \mathbb{V}(X_{\mathrm{des+ides}}) = \frac{n+8}{6} - \frac{1}{n+1}.$$

We generalize this result uniformly to all finite Coxeter groups.

Theorem 5.1. *Let W be an irreducible finite Coxeter group of rank n and Coxeter number h. Then*

$$\mathbb{E}(X_{\mathrm{des+ides}}) = n, \qquad \mathbb{V}(X_{\mathrm{des+ides}}) = 2\mathbb{V}(X_{\mathrm{des}}) + n/h. \tag{5.1}$$

The theorem can be written explicitly as follows.

Corollary 5.2. *In the situation of the previous theorem, the W-double-Eulerian distribution has variances*

(type A_n)	$\mathbb{V}(X_{\text{des+ides}}) = \dfrac{n+2}{6} + \dfrac{n}{n+1}$
(type B_n)	$\mathbb{V}(X_{\text{des+ides}}) = \dfrac{n+4}{6}$
(type D_n)	$\mathbb{V}(X_{\text{des+ides}}) = \dfrac{n+2}{6} + \dfrac{n}{2n-2}$
(type E_6)	$\mathbb{V}(X_{\text{des+ides}}) = 11/6$
(type E_7)	$\mathbb{V}(X_{\text{des+ides}}) = 17/9$
(type E_8)	$\mathbb{V}(X_{\text{des+ides}}) = 29/15$
(type F_4)	$\mathbb{V}(X_{\text{des+ides}}) = 7/6$
(type H_3)	$\mathbb{V}(X_{\text{des+ides}}) = 13/15$
(type H_4)	$\mathbb{V}(X_{\text{des+ides}}) = 13/15$
(type $I_2(m)$)	$\mathbb{V}(X_{\text{des+ides}}) = 4/m$

We divide the proof of Theorem 5.1 into three propositions, one for each type. In analogy to the random variables $Y^{(i)}$ from (4.2), define

$$\tilde{Y}^{(j)} = \begin{cases} 1 & \pi^{-1}(j) > \pi^{-1}(j+1), \\ 0 & \text{otherwise.} \end{cases}$$

Using the two sets of random variables we write

$$X_{\text{des+ides}} = \sum_{i=1}^{n} \big(Y^{(i)} + \tilde{Y}^{(i)}\big). \tag{5.2}$$

Remark 5.3. The locations of inverse descents of π can be read off the one-line notation. In type A, j is an inverse descent if the location of $j+1$ is to the left of the location of j. In types B and D the signs also play a role. Specifically, $\pi^{-1}(j) > \pi^{-1}(j+1)$ if one of the following four orderings occurs

$$\pm(j+1) \text{ left of } j \quad \text{or} \quad \pm j \text{ left of } -(j+1).$$

Proposition 5.4. *The mean and variance of the distribution $X_{\text{des+ides}}$ on A_n are*

$$\mathbb{E}(X_{\text{des+ides}}) = n, \qquad \mathbb{V}(X_{\text{des+ides}}) = 2\mathbb{V}(X_{\text{des}}) + n/(n+1).$$

Proof. The computation for the mean value is obvious. For the variance, we first record that (5.2) implies that

$$\begin{aligned} \mathbb{V}(X_{\text{des+ides}}) &= \mathbb{E}(X_{\text{des+ides}}^2) - \mathbb{E}(X_{\text{des+ides}})^2 \\ &= 2\mathbb{E}(X_{\text{des}}^2) + 2\sum_{i,j=1}^{n} \mathbb{E}(Y^{(i)}\tilde{Y}^{(j)}) - 2\mathbb{E}(X_{\text{des}})^2 - n^2/2 \\ &= 2\mathbb{V}(X_{\text{des}}) + 2\sum_{i,j=1}^{n} \mathbb{E}(Y^{(i)}\tilde{Y}^{(j)}) - n^2/2 \end{aligned}$$

where we used that $X_{\text{des}} = X_{\text{ides}}$ and that $n = \mathbb{E}(X_{\text{des+ides}}) = 2\mathbb{E}(X_{\text{des}})$. We thus aim to show that

$$2\sum_{i,j=1}^{n} \mathbb{E}(Y^{(i)}\tilde{Y}^{(j)}) - \frac{n^2}{2} = \frac{n}{n+1}.$$

For fixed $1 \le i, j \le n$, by Remark 5.3, $Y^{(i)}\tilde{Y}^{(j)} = 1$ if and only if $\pi(i) > \pi(i+1)$ and $j, j+1$ are out of order in the one-line notation of π. We claim the following expression for the mean:

$$\mathbb{E}(Y^{(i)}\tilde{Y}^{(j)}) = \frac{1}{(n+1)!}\Big[\tfrac{1}{4}(n-1)(n-2)(n-1)! + (n-1)! \\ + (n-2)!\big((i-1)(j-1) + (i-1)(n-j) + (n-i)(j-1) + (n-i)(n-j)\big)\Big].$$

Since $|A_n| = (n+1)!$ we show that the numerator counts the number of permutations for which $Y^{(i)}\tilde{Y}^{(j)} = 1$. We consider 6 different types of permutations $\pi \in A_n$. The following table lists a type of permutation together with the number of such permutations and the probability that $Y^{(i)}\tilde{Y}^{(j)} = 1$.

$\big\{\pi(i), \pi(i+1)\big\} \cap \big\{j, j+1\big\} = \emptyset$	:	$(n-1)(n-2)(n-1)!$	$\cdot$	$1/4$
$\big\{\pi(i), \pi(i+1)\big\} = \big\{j, j+1\big\}$	:	$2(n-1)!$	$\cdot$	$1/2$
$\pi(i) = j, \quad \pi(i+1) \neq j+1$	:	$(n-1)(n-1)!$	$\cdot$	$(i-1)(j-1)/(n-1)^2$
$\pi(i) = j+1, \quad \pi(i+1) \neq j$	:	$(n-1)(n-1)!$	$\cdot$	$(n-i)(j-1)/(n-1)^2$
$\pi(i) \neq j, \quad \pi(i+1) = j+1$	:	$(n-1)(n-1)!$	$\cdot$	$(n-i)(n-j)/(n-1)^2$
$\pi(i) \neq j+1, \quad \pi(i+1) = j$	:	$(n-1)(n-1)!$	$\cdot$	$(i-1)(n-j)/(n-1)^2$

Using that

$$\sum_{i,j=1}^{n}(i-1)(j-1) = \sum_{i,j=1}^{n}(i-1)(n-j) = \sum_{i,j=1}^{n}(n-i)(j-1) = \sum_{i,j=1}^{n}(n-i)(n-j) = \binom{n}{2}^2,$$

we obtain

$$\begin{aligned} 2\sum_{i,j=1}^{n}\mathbb{E}(Y^{(i)}\tilde{Y}^{(j)}) &= \frac{2}{(n+1)!}\left[\binom{n}{2}^2(n-2)(n-2)! + n\cdot n! + 4(n-2)!\binom{n}{2}^2\right] \\ &= \frac{2}{(n+1)!}\left[\binom{n}{2}^2(n+2)(n-2)! + n\cdot n!\right] \\ &= \frac{2n}{n+1}\big(\tfrac{1}{4}(n-1)(n+2) + 1\big) \\ &= \frac{n}{n+1} + \frac{n^2}{2}. \end{aligned}$$

□

Proposition 5.5. *The mean and variance of the distribution $X_{\text{des+ides}}$ on B_n are*

$$\mathbb{E}(X_{\text{des+ides}}) = n, \qquad \mathbb{V}(X_{\text{des+ides}}) = 2\mathbb{V}(X_{\text{des}}) + 1/2.$$

Proof. The computation for the mean value is obvious. For the variance, we follow the same argument as for A_n, except that we have to deal with more cases. The main step is again to analyze the mean value of a summand $\mathbb{E}(Y^{(i)}\tilde{Y}^{(j)})$, using in particular Remark 5.3. We organize the summands into different cases which are presented as tables containing numbers of occurrences and probabilities. The caption of each table is one of the 6 mutually exclusive situations as for the symmetric group. Now each table has (at most) four rows indicating the special cases that $i=0$ or $j=0$ as follows:

$i,j>0$	$i=0<j$	$i>0=j$	$i,j=0$
$++$	$0+$	$+0$	00

Rows for impossible situations are omitted. Every row contains in order the sign indicator, the number of signed permutations in this situation, and the probability that $Y_i\tilde{Y}_j=1$. In cases 3–6, these probabilities also depend on the signs of $\pi(i),\pi(i+1),\pi^{-1}(j),\pi^{-1}(j+1)$. In these tables there are four columns with probabilities, labeled by $\pm$-sequences.

Case 1: $\{|\pi(i)|,|\pi(i+1)|\}\cap\{|\pi^{-1}(j)|,|\pi^{-1}(j+1)|\}=\emptyset$:

$++$	$2^n\cdot 2\binom{n-2}{2}(n-2)!$	$\frac{1}{4}$
$0+$	$2^n\cdot\binom{n-2}{1}(n-1)!$	$\frac{1}{4}$
$+0$	$2^n\cdot 2\binom{n-1}{2}(n-2)!$	$\frac{1}{4}$

Case 2: $\{|\pi(i)|,|\pi(i+1)|\}=\{|\pi^{-1}(j)|,|\pi^{-1}(j+1)|\}$:

$++$	$2^n\cdot 2(n-2)!$	$\frac{3}{8}$
00	$2^n\cdot(n-1)!$	$\frac{1}{2}$

Case 3: $|\pi(i)|=j,\quad |\pi(i+1)|\neq j+1$:

		$+++\pm$	$+-+\pm$	$-+-\pm$	$---\pm$
$++$	$2^{n-3}(n-2)(n-2)!$	$\frac{j-1}{n-2}\left(\frac{i-1}{n-2}+1\right)$	$1\cdot\left(\frac{i-1}{n-2}+1\right)$	0	$\frac{n-j-1}{n-2}\left(0+\frac{n-i-1}{n-2}\right)$
00	$2^{n-3}(n-1)(n-1)!$	0	$1\cdot(0+1)$	0	$1\cdot(0+1)$

Case 4: $|\pi(i)|=j+1,\quad |\pi(i+1)|\neq j$:

		$++\pm+$	$+-\pm+$	$-+\pm-$	$--\pm-$
$++$	$2^{n-3}(n-2)(n-2)!$	$\frac{j-1}{n-2}\left(\frac{n-i-1}{n-2}+0\right)$	$1\cdot\left(\frac{n-i-1}{n-2}+0\right)$	0	$\frac{n-j-1}{n-2}\left(1+\frac{i-1}{n-2}\right)$
$+0$	$2^{n-3}(n-1)!$	0	$1\cdot(0+0)$	0	$1\cdot(1+1)$

Case 5: $|\pi(i)|\neq j,\quad |\pi(i+1)|=j+1$:

		$++\pm+$	$-+\pm+$	$+-\pm-$	$--\pm-$
$++$	$2^{n-3}(n-2)(n-2)!$	$\frac{n-j-1}{n-2}\big(\frac{n-i-1}{n-2}+0\big)$	0	$1\cdot\big(1+\frac{i-1}{n-2}\big)$	$\frac{j-1}{n-2}\big(1+\frac{i-1}{n-2}\big)$
$0+$	$2^{n-3}(n-1)!$	0	0	$1\cdot(0+1)$	$1\cdot(0+1)$
$+0$	$2^{n-3}(n-1)!$	0	0	$1\cdot(1+1)$	0

Case 6: $|\pi(i)|\neq j, \quad |\pi(i+1)|=j+1$:

		$+++\pm$	$-++\pm$	$+--\pm$	$---\pm$
$++$	$2^{n-3}(n-2)(n-2)!$	$\frac{n-j-1}{n-2}\big(\frac{i-1}{n-2}+1\big)$	0	$1\cdot\big(0+\frac{n-i-1}{n-2}\big)$	$\frac{j-1}{n-2}\big(0+\frac{n-i-1}{n-2}\big)$
$0+$	$2^{n-3}(n-1)!$	0	0	$1\cdot\big(1+0\big)$	$1\cdot\big(1+0\big)$

We discuss one entry in detail to illustrate how to read these tables. Consider the highlighted situation $i,j>0$ with $\pi(i),\pi(i+1),\pi^{-1}(j+1)>0$ in **Case 4**. The two possible signs for $\pi^{-1}(j)$ are treated separately and correspond to the sum in the entry. That is, for $\pi^{-1}(j)>0$ the probability is $\frac{j-1}{n-2}\cdot\frac{n-i-1}{n-2}$, while for $\pi^{-1}(j)<0$ the probability is $\frac{j-1}{n-2}\cdot 0$.

First, we count signed permutations in this case, treating absolute value and signs individually. The value $|\pi(i)|=j+1$ is fixed, and $|\pi(i+1)|\neq j$ means that there are $n-2$ choices for the absolute value of $\pi(i+1)$ and $(n-2)!$ choices for the absolute values of $\{\pi(k)\mid k\neq i,i+1\}$. Four signs are fixed by the column label, but the signs of $\pi(i)$ and $\pi^{-1}(j+1)$ coincide, giving in total 2^{n-3} possible signs for the remaining entries.

Second, the probability that i is a descent is $\frac{j-1}{n-2}$ since $\pi(i)=j+1,\pi(i+1)>0$ and $\pi(i+1)\neq j$ leaving $j-1$ possible values for $\pi(i+1)$ out of $n-2$ in total.

Third, we consider the two possibilities for the sign of $\pi^{-1}(j)$. The probability that i is a descent is independent of this because $|\pi(i+1)|\neq j$. If $\pi^{-1}(j)>0$, we have, according to Remark 5.3, that $j+1$ must be to the left of j. Since $j+1$ is in position i, and j cannot be in position $i+1$, there are $n-i-1$ positions to the right, out of $n-2$ positions in total. If $\pi^{-1}(j)<0$, than j cannot be an inverse descent since this situation does not appear as a possibility in Remark 5.3.

In total, a random signed permutation in this situation has a descent in position i and an inverse descent in position j with probability

$$2^{n-3}(n-2)(n-2)!\ \tfrac{j-1}{n-2}\big(\tfrac{n-i-1}{n-2}+0\big).$$

Summing all 6 cases individually for $0\leq i,j<n$, and then summing the cases yields

$$\begin{aligned}2^{n-2}(n-1)!(n-1)\big((n-2)(n-3)+2(n-2)\big)+2^{n-2}(n-1)!(3n-1)+\\ 2^{n-4}(n-1)(n-1)!(5n-6)+2^{n-4}(n-1)!(n-1)(3n-2)+\\ 2^{n-4}(n-1)(n-1)!(5n-2)+2^{n-4}(n-1)!(n-1)(3n-2)=\\ 2^{n-2}\cdot n!\cdot(n^2+1),\end{aligned}$$

giving in total

$$2\sum_{i,j=1}^{n}\mathbb{E}(Y^{(i)}\tilde{Y}^{(j)}) = \frac{1}{2^{n-1}\cdot n!}\cdot 2^{n-2}\cdot n!\cdot(n^2+1) = \frac{n^2+1}{2} = \frac{n^2}{2}+\frac{1}{2}. \qquad \square$$

Proposition 5.6. *The mean and variance of the distribution $X_{\text{des+ides}}$ on D_n are*

$$\mathbb{E}(X_{\text{des+ides}}) = n, \qquad \mathbb{V}(X_{\text{des+ides}}) = 2\mathbb{V}(X_{\text{des}}) + n/(2n-2).$$

Proof. The computation for the mean value is obvious. This time, we have to show that

$$2\sum_{i,j=1}^{n}\mathbb{E}(Y^{(i)}\tilde{Y}^{(j)}) - \frac{n^2}{2} = \frac{n}{2n-2}.$$

This can be obtained from the variance in type B as follows. Even though we follow the convention $\pi(0) = -\pi(2)$ for computing descents in type D, we follow the type B convention to distinguish the cases. That is, we let $\pi(0) = 0$ in the case distinction. One can check that except for three situations listed below, one obtains the same probabilities, but half the counts compared to type B (since D_n is an index 2 subgroup of B_n). The three exceptions are the following replacements

$$\begin{aligned} \text{situation } 0+ \text{ in Case 6:} \quad & 2^{n-2}(n-1)! \;\rightsquigarrow\; 2^{n-3}\cdot n(n-2)! \\ \text{situation } +0 \text{ in Case 4:} \quad & 2^{n-2}(n-1)! \;\rightsquigarrow\; 2^{n-3}\cdot n(n-2)! \\ \text{situation } 00 \text{ in Case 2:} \quad & 2^{n-1}(n-1)! \;\rightsquigarrow\; 2^{n-3}\cdot n(n-2)!\ . \end{aligned}$$

Here, each situation is meant as the total contribution of this complete row in the above table. This is,

$$\begin{aligned} 2^{n-2}(n-1)! &= 2^{n-3}(n-1)!\cdot(0+0+1(1+0)+1(1+0) \\ &= 2^{n-3}(n-1)!\cdot(0+1(0+0)+0+1(1+1) \\ 2^{n-1}(n-1)! &= 2^{n}(n-1)!\cdot\tfrac{1}{2} \end{aligned}$$

We explain this in Case 2, the others being similar. In type B_n in this situation and case, $\pi(1)$ is determined by $j+1$, so there are $(n-1)!$ permutations left, together with 2^{n-1} signs that yield a descents and an inverse descent at the same time. On the other hand, in type D_n, one has to check that both $\pi(2) < 0$ and $\pi^{-1}(2) < 0$. So one either has $|\pi(2)| = 2$ and obtains $(n-2)!$ permutations and 2^{n-2} possible signs, or one has $|\pi(2)| \neq 2$ and has $(n-2)(n-2)!$ permutation and 2^{n-3} possible signs. Summing these yields

$$2^{n-2}(n-2)! + 2^{n-3}(n-2)(n-2)! = 2^{n-3}\cdot n(n-2)!\ .$$

Observing that the situation 00 occurs once, while each of the situations $0+$ and $+0$ each occurs $n-1$ times, we obtain

$$\begin{aligned} 2^{n-1}(n-1)! + 2(n-1)\cdot 2^{n-2}(n-1)! &= 2^{n-1}\cdot n! \\ 2^{n-3}\cdot n(n-2)! + 2(n-1)\cdot 2^{n-3}\cdot n(n-2)! &= 2^{n-2}\cdot n! + 2^{n-3}\cdot n(n-2)!\ . \end{aligned} \tag{5.3}$$

We are thus ready to deduce the proposition. Let

$$S_B = 2^{n-2}\cdot n!\cdot(n^2+1)$$

be the formula from the proof in type B_n. Then the analogous formula in type D_n is

$$\begin{aligned}S_D &= \big(S_B - 2^{n-1}\cdot n!\big)/2 + 2^{n-2}\cdot n! + 2^{n-3}\cdot n(n-2)!\\ &= 2^{n-3}\cdot n!\cdot(n^2+1) + 2^{n-3}\cdot n(n-2)!\\ &= 2^{n-3}\cdot n(n-2)!\big((n-1)(n^2+1)+1\big)\\ &= 2^{n-3}\cdot n(n-2)!\big(n^2(n-1)+n\big).\end{aligned}$$

We finally compute

$$\begin{aligned}2\sum_{i,j=1}^{n}\mathbb{E}(Y^{(i)}\tilde{Y}^{(j)}) &= \frac{1}{2^{n-2}\cdot n!}\cdot 2^{n-3}\cdot n(n-2)!\big(n^2(n-1)+n\big)\\ &= \frac{1}{2(n-1)}\big(n^2(n-1)+n\big)\\ &= \frac{n^2}{2}+\frac{n}{2n-2}.\end{aligned}$$

□

6. Limit theorems

We finally turn to the limit theorems for Mahonian and Eulerian distributions of sequences of Coxeter groups of increasing rank. These depend only very mildly on the concrete sequence of finite Coxeter groups. Only the sizes and properties of the dihedral subgroups play a role.

Denote convergence in distribution by $\xrightarrow{\mathcal{D}}$ and the standard normal distribution by $N(0,1)$. A sequence X_n of random variables *satisfies the CLT* if, for $n\to\infty$,

$$\frac{X_n-\mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}}\xrightarrow{\mathcal{D}} N(0,1).$$

For functions $f,g:\mathbb{N}_+\to\mathbb{R}_{\geq 0}$, we use *big-O-notation* $f(n)\in O(g(n))$, if there exists $c>0$ and an $N\in\mathbb{N}$ such that for all $n\geq N$, we have $f(n)\leq cg(n)$, and we use *little-o-notation* $f(n)\in o(g(n))$, if for all $c>0$ there exists an $N\in\mathbb{N}$ with this property.

Theorem 6.1. *Let $W^{(1)}, W^{(2)},\ldots$ be an infinite sequence of finite Coxeter groups such that $W^{(n)}$ has rank n, and let $D^{(n)}$ be the dihedral parabolic subgroup of $W^{(n)}$ of maximal size. Denote by $X^{(n)}$ and by $Y^{(n)}$, for all n simultaneously, either the Mahonian or the Eulerian distribution for $W^{(n)}$ and for $D^{(n)}$, respectively. Then $X^{(n)}$ satisfies the CLT if and only if*

$$\mathbb{V}(Y^{(n)})/\mathbb{V}(X^{(n)})\longrightarrow 0 \quad \textit{for } n\to\infty.$$

Remark 6.2. For the Mahonian distribution, the condition $\mathbb{V}(Y^{(n)})/\mathbb{V}(X^{(n)})\longrightarrow 0$ can, by Theorem 3.1, be written in terms of the degrees $d_1^{(n)}\leq\ldots\leq d_n^{(n)}$ of $W^{(n)}$ as

$$d_n^{(n)}\in o(s_n),$$

where $s_n^2=\sum_i\big(d_i^{(n)}\big)^2$.

Remark 6.3. For the Eulerian distribution, $\mathbb{V}(Y^{(n)})$ is bounded, so the condition simplifies to $\mathbb{V}(X^{(n)})\longrightarrow\infty$. By Theorem 4.1, this is the case if and only if

- the non-dihedral component of $W^{(n)}$ is not globally bounded in rank, or
- the dihedral components $\{I_2(m_i(n))\}_{i\in I(n)}$ of $W^{(n)}$ satisfy

$$\sum_{i\in I(n)}\frac{1}{m_i(n)}\longrightarrow\infty.$$

The previous two remarks show that Theorem 6.1 holds in particular for sequences of groups of classical types A_n, B_n, or D_n.

Example 6.4. Given information about the dihedral subgroups, Theorem 6.1 can be used to work out whether a specific sequence of finite Coxeter groups satisfies a CLT or not.

(1) Fix $\alpha > 0$ and let $W^{(2n)} = \prod_{i=1}^{n} I_2(i^\alpha)$. Then $\mathbb{V}(Y_{\text{inv}}^{(2n)}) \sim n^{2\alpha}$ and $\mathbb{V}(X_{\text{inv}}^{(2n)}) \sim n^{2\alpha+1}$, implying that $X_{\text{inv}}^{(2n)}$ satisfies the CLT. On the other hand, $\mathbb{V}(X_{\text{des}}^{(2n)}) = \sum_i \frac{1}{i^\alpha}$, implying that $X_{\text{des}}^{(2n)}$ satisfies the CLT if and only if $\alpha \leq 1$.
(2) Let $W^{(2n)} = \prod_{i=1}^{n} I_2(2^i)$. Then $\mathbb{V}(Y_{\text{inv}}^{(n)}) \sim \mathbb{V}(X_{\text{inv}}^{(n)}) \sim 4^n$, implying that $X_{\text{inv}}^{(n)}$ does not satisfy the CLT, and $\mathbb{V}(X_{\text{des}}^{(n)}) = \sum_i \frac{1}{2^i} \longrightarrow 1$, implying that $X_{\text{des}}^{(n)}$ as well does not satisfy the CLT.
(3) Let $W^{(n)} = A_1^{n-2} \times I_2(n)$. Then $\mathbb{V}(Y_{\text{inv}}^{(n)}) \sim \mathbb{V}(X_{\text{inv}}^{(n)}) \sim n^2$, implying that $X_{\text{inv}}^{(n)}$ does not satisfy the CLT. On the other hand, $\mathbb{V}(X_{\text{des}}^{(n)}) \sim n \longrightarrow \infty$, implying that $X_{\text{des}}^{(n)}$ satisfies the CLT.

The central limit theorem gives a qualitative feel for the behavior of the distributions of X_{inv} and X_{des}. Following Bender [1] we can lift the central limit theorems to uniform convergence of the probabilities $\text{Prob}(X_{\text{inv}}^{(n)} = k)$ and $\text{Prob}(X_{\text{des}}^{(n)} = k)$ to the density of the normal distribution.

Theorem 6.5. *In the situation of Theorem 6.1,*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \sigma_n p_n(\lfloor \sigma_n x + \mu_n \rfloor) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| = 0.$$

where $p_n(k) = \text{Prob}(X^{(n)} = k)$, $\sigma_n^2 = \mathbb{V}(X^{(n)})$, *and* $\mu_n = \mathbb{E}(X^{(n)})$. *Furthermore the rate of convergence depends only on the rate of convergence in Theorem 6.1 and* σ_n.

Remark 6.6. One might be able to strengthen the above convergence to a mod-Gaussian convergence in the sense of [12]. For this one in particular needs to consider also the fourth cumulants of the Mahonian and the Eulerian distributions. For the W-Mahonian distribution one obtains a mod-Gaussian convergence in all classical types. With $\alpha_n = \beta_n = n$ in [12, Chapter 5.1] one computes

$$\kappa_2(X^{(n)}) = \sigma^2 n^3(1 + O(n^{-1})) \quad \kappa_4(X^{(n)}) = Ln^5(1 + O(n^{-1}))$$

for some constants σ, L, as needed for the mod-Gaussian convergence. Analogously, for the W-Eulerian distribution, one can use $\alpha_n = n$ and $\beta_n = 1$ and derive the needed property for $\kappa_2(X^{(n)})$. The computations for $\kappa_4(X^{(n)})$ might possibly be computed in the same way as the computation for $\kappa_2(X_n)$ in Section 4.

We are now ready to prove Theorems 6.1 and 6.5, for which we combine arguments from [1] with the Lindeberg–Feller central limit theorem. For this, a *triangular array* is a set of random variables $X^{(n,i)}$ with $i = 1, \ldots, n$ for $n = 1, 2, \ldots$, such that for fixed n the random variables $X^{(n,i)}$ are independent with finite variances $0 < \mathbb{V}(X^{(n,i)}) < \infty$. Given a triangular array of random variables, it satisfies the *maximum condition* if

$$\max_i \{\mathbb{V}(X^{(n,i)})\}/\mathbb{V}(X^{(n)}) \longrightarrow 0,$$

and it satisfies the *Lindeberg condition* if, for any $\epsilon > 0$,

$$\frac{1}{s_n^2} \sum_{i=1}^{n} \mathbb{E}\big(X_{ni}^2 \cdot I\big\{|X_{ni}| \geq \epsilon s_n\big\}\big) \longrightarrow 0$$

where $s_n^2 = \sum_i \mathbb{V}(X^{(n,i)})$ is the variance of $\sum_i X^{(n,i)}$.

The following theorem goes back to the work of Lindeberg and Feller in the first half of the 20th century. See [14, Theorem 15.43] and [2, Theorem 27.4] for details.

Theorem 6.7 (Lindeberg–Feller theorem for triangular arrays)**.** *Let $X^{(n,i)}$ be a triangular array of random variables, and let $X^{(n)} = X^{(n,1)} + \cdots + X^{(n,n)}$. Then $X^{(n)}$ satisfies the Lindeberg condition if and only if it satisfies the CLT and the maximum condition.*

We moreover use the following well-known theorem.

Theorem 6.8 (Cramér's theorem [10])**.** *Let X, Y be independent random variables such that $X + Y$ has a normal distribution. Then X and Y both have normal distributions.*

To construct appropriate triangular arrays for the random variables X_{inv} and X_{des}, we make use of the factorizations (2.1) and (2.2). Let $W^{(n)}$ be a finite Coxeter group of rank n with degrees $d_1^{(n)} \leq \cdots \leq d_n^{(n)}$ and roots $q_1^{(n)}, \ldots, q_n^{(n)}$ of the descent generating function. We use the notation $[z^k]P$ for the coefficient of z^k of a polynomial $P \in \mathbb{N}[z]$.

For inversions define independent random variables $X_{\mathrm{inv}}^{(n,i)}$ with uniform distribution on $\{0, 1, \ldots, d_i^{(n)} - 1\}$. With $P_{\mathrm{inv}}^{(n)} = \mathcal{G}_{\mathrm{inv}}$ the $W^{(n)}$-Mahonian polynomial, we then have

$$\mathrm{Prob}\left(\sum_{i=1}^{n} X_{\mathrm{inv}}^{(n,i)} = k\right) = \frac{[x^k]P_{\mathrm{inv}}^{(n)}(x)}{P_{\mathrm{inv}}^{(n)}(1)} = \mathrm{Prob}(X_{\mathrm{inv}}^{(n)} = k),$$

implying

$$X_{\mathrm{inv}}^{(n)} = X_{\mathrm{inv}}^{(n,1)} + \cdots + X_{\mathrm{inv}}^{(n,n)}. \tag{6.1}$$

Similarly, define independent binary random variables

$$X_{\mathrm{des}}^{(n,i)} = \begin{cases} 0 & \text{with probability } \frac{q_i^{(n)}}{1+q_i^{(n)}}, \\ 1 & \text{with probability } \frac{1}{1+q_i^{(n)}}. \end{cases}$$

With $P_{\mathrm{des}}^{(n)} = \mathcal{G}_{\mathrm{des}}$ the $W^{(n)}$-Eulerian polynomial, we then have

$$\mathrm{Prob}\left(\sum_{i=1}^{n} X_{\mathrm{des}}^{(n,i)} = k\right) = \frac{[x^k]P_{\mathrm{des}}^{(n)}(x)}{P_{\mathrm{des}}^{(n)}(1)} = \mathrm{Prob}(X_{\mathrm{des}}^{(n)} = k),$$

implying

$$X_{\mathrm{des}}^{(n)} = X_{\mathrm{des}}^{(n,1)} + \cdots + X_{\mathrm{des}}^{(n,n)}. \tag{6.2}$$

Given these two decompositions of the $W^{(n)}$-Mahonian and Eulerian distributions, we deduce Theorem 6.1 from the following two propositions.

Proposition 6.9. *Fix $C > 0$ and for each $n \in \mathbb{N}_+$ and $1 \leq i \leq n$ let $X_{i,n}$ be a random variable with support contained in $[-C, C] \subset \mathbb{R}$ and nonzero variance $0 < s_{i,n}^2$. Set $X_n = \sum_{i=1}^{n} X_{i,n}$. Then X_n satisfies the CLT if and only if it satisfies the maximum condition if and only if $s_n^2 = \mathbb{V}(X_n) \to \infty$.*

Proof. First, the bounded support of $X_{i,n}$ implies that $\max_i\{s_{i,n}^2\}$ is bounded by C^2 and thus the maximum condition is equivalent to $s_n^2 \to \infty$.

Assume that $s_n \to \infty$ and let $\epsilon > 0$ be arbitrary. There exists an N such that for all $n > N$, $\epsilon s_n > C$ and thus the Lindeberg condition is trivially fulfilled.

Now assume $s_n \to S < \infty$. Then $\tilde{X} = \lim_{n\to\infty} \tilde{X}_n - \tilde{X}_1$ exists and has variance $\mathbb{V}(\tilde{X}) = S^2 - s_1^2$. Assume moreover for the sake of contradiction that $\tilde{X}_n/s_n \xrightarrow{\mathcal{D}} N(0,1)$, then by the decomposition $\lim_{n\to\infty} \tilde{X}_n = \tilde{X} + \tilde{X}_1$ and it follows from Cramér's theorem that $\tilde{X}_1/S$ has a normal distribution, contradicting the fact that it has bounded support. ☐

Proof of Theorem 6.1 for the Eulerian distribution. Remark 6.3 shows

$$\mathbb{V}(Y^{(n)})/\mathbb{V}(X^{(n)}) \longrightarrow 0 \iff \mathbb{V}(X^{(n)}) \longrightarrow \infty.$$

As in (6.2) the $W^{(n)}$-Eulerian distribution can be decomposed into a sum of Boolean independent random variables, which have finite support. Since $s_n^2 = \mathbb{V}(X^{(n)})$, the equivalence follows from Proposition 6.9. ☐

Proposition 6.10. *For each $n \in \mathbb{N}_+$, fix integers $2 \le d_{1,n} \le \cdots \le d_{n,n}$. Let $X_{i,n}$ be the discrete uniform distribution on $\{0, 1, \ldots, d_{i,n} - 1\}$. Set $X_n = \sum_{i=1}^n X_{i,n}$. Then X_n satisfies the CLT if and only if it satisfies the maximum condition.*

Proof. The maximum condition can be rephrased as

$$d_{n,n}/s_n \longrightarrow 0 \tag{6.3}$$

where $s_n^2 = \mathbb{V}(X_n)$. This implies the Lindeberg condition as follows. First,

$$\mathrm{Prob}(X_{i,n} = x) = 0 \text{ for } |x| \ge d_{n,n}. \tag{6.4}$$

By (6.3), we have that $d_{n,n} \in o(s_n)$ as functions in n. Together with (6.4) this implies

$$\mathbb{E}\big(X_{ni}^2 \cdot I\big\{|X_{ni}| \ge \epsilon s_n\big\}\big) = 0$$

for large enough n, implying the Lindeberg condition.

Vice versa, we aim to show that the fourth cumulant does not tend to zero if the maximum condition is violated. To this end, set $\mu_{i,n} = \mathbb{E}(X_{i,n}) = (d_{i,n} - 1)/2$ and

$$\tilde{X}_{i,n} = X_{i,n} - \mu_{i,n}, \quad \tilde{X}_n = \sum_{i=1}^n \tilde{X}_{i,n},$$

Computing the second and fourth central moments as

$$\mu_2(\tilde{X}_{i,n}) = \mathbb{E}(\tilde{X}_{i,n}^2) = (d_{i,n}^2 - 1)/12$$
$$\mu_4(\tilde{X}_{i,n}) = \mathbb{E}(\tilde{X}_{i,n}^4) = (3d_{i,n}^4 - 10d_{i,n}^2 + 7)/240,$$

we obtain the fourth cumulant as

$$\kappa_4(\tilde{X}_n) = \mu_4(\tilde{X}_n) - 3\mu_2(\tilde{X}_n)^2 = \frac{1}{120}\sum_{i=1}^n (1 - d_{i,n}^4).$$

If the maximum condition is violated, then $d_{n,n}/s_n \to C > 0$ for $n \to \infty$, implying that $d_{n,n}^4/s_n^4 \to C^4$. This bounds $\kappa_4(\tilde{X}_n/s_n) = \kappa_4(\tilde{X}_n)/s_n^4$ away from zero, as desired. ☐

Proof of Theorem 6.1 for the Mahonian distribution. As in (6.1) the Mahonian distribution on $W^{(n)}$ can be decomposed into a sum of discrete uniform distributions. The equivalence follows from Proposition 6.10 when $d_{1,n} \le \cdots \le d_{n,n}$ are the degrees of $W^{(n)}$. ☐

Proof of Theorem 6.5. Given Theorem 6.1 this is [1, Lemma 2] and the log-concavity from Theorems 2.3 and 2.4. ☐

We finish this section with the open problem of providing an analogue of Theorem 6.1 for the W-double-Eulerian distribution. Chatterjee and Diaconis have shown that the double-Eulerian distribution on $W^{(n)} = \mathfrak{S}_n$ satisfies the CLT [8]. The W-double Eulerian analogue of Theorem 6.1 is currently open.

Problem 6.11. Find necessary and sufficient conditions on general sequences of finite Coxeter groups of increasing rank under which the double-Eulerian distribution satisfies a CLT.

Appendix A. Additional computational data

In this section, we present the experimental investigations of the asymptotics of permutation statistics. Assume one has computed explicit values of a permutation statistics $\mathrm{st} : \mathfrak{S}_n \longrightarrow \mathbb{N}$ for $2 \leq n \leq N$ for some N (in our case typically $6, 7$, or 8). One can then

(1) compute the generating functions $\mathcal{G}_{\mathrm{st}}(z)$, mean value and variance of the random variable X_{st} for $2 \leq n \leq N$, and
(2) use Lagrange interpolation on the $N-1$ data points to guess (Laurent) polynomial formulas for the mean and variance of X_{st} as a function of n.

As of today, the database `www.FindStat.org` [17] contains 1113 combinatorial statistics, including 285 permutation statistics. We have applied the above procedure to all these permutation statistics and searched for statistics $\mathrm{st} : \mathfrak{S}_n \longrightarrow \mathbb{N}$ such that the variance of the random variable $X^{(n)}_{\mathrm{st}}$ is the form $\mathbb{V}(X^{(n)}_{\mathrm{st}}) = f(n)/(an+b)^c$ with $a, b \in \{0, \pm 1, \pm 2\}$ and $c \in \{0, 1, 2, 3, 4, 5\}$ and polynomial $f \in \mathbb{Q}[n]$ such that the Lagrange interpolation had at least three more data points than the degree of f.

Among the 285 permutation statistics, there are 14 Mahonian statistics and 13 Eulerian statistics. On top of these we found additional statistics for which the Lagrange interpolation suggest variances of the above form and we list them below. Every table contains in its headline all statistics that yield one fixed random variable $X^{(n)}_{\mathrm{st}}$ followed by the interpolated mean value and variance for that random variable. Below we list numerical values for higher cumulants $\tilde{\kappa}^{(n)}_k = \tilde{\kappa}_k(X^{(n)}_{\mathrm{st}}) = \kappa_k(X^{(n)}_{\mathrm{st}}/s_n)$ normalized by the standard deviation $s_n = \kappa_2(X^{(n)}_{\mathrm{st}})^{1/2}$. To read this numerical information, recall that X_{st} satisfies the CLT if and only if for all $k \geq 3$, one has $\tilde{\kappa}^{(n)}_k \longrightarrow 0$ as $n \to \infty$.

Some of these distributions are well-known (e.g the number of fixed points `St000022`) and some are not hard to compute (such as the sum of the descent tops `St000111` or the sum of the descent bottoms `St000154`). Others seem unexpected at first glance (such as eigenvalues, indexed by permutations, of the random-to-random operator acting on the regular representation `St000500`). Finally, the computational data suggests central limit theorems for multiple of the below statistics.

`St000022`, `St000215`, `St000241`, `St000338`, `St000461`, `St000873`

$\mathbb{E}(X^{(n)}_{\mathrm{st}}) = 1$

$\mathbb{V}(X^{(n)}_{\mathrm{st}}) = 1$

n	$\tilde{\kappa}^{(n)}_3$	$\tilde{\kappa}^{(n)}_4$	$\tilde{\kappa}^{(n)}_5$	$\tilde{\kappa}^{(n)}_6$	$\tilde{\kappa}^{(n)}_7$	$\tilde{\kappa}^{(n)}_8$
5	1.00	1.00	1.00	0.000	-14.0	-118.
6	1.00	1.00	1.00	1.00	0.000	-20.0

`St000029`, `St000030`

$\mathbb{E}(X^{(n)}_{\mathrm{st}}) = \frac{1}{6}(n-1)(n+1)$

$\mathbb{V}(X^{(n)}_{\mathrm{st}}) = \frac{1}{90}(n+1)(n^2+\frac{7}{2})$

n	$\tilde{\kappa}^{(n)}_3$	$\tilde{\kappa}^{(n)}_4$	$\tilde{\kappa}^{(n)}_5$	$\tilde{\kappa}^{(n)}_6$	$\tilde{\kappa}^{(n)}_7$	$\tilde{\kappa}^{(n)}_8$
6	-0.283	-0.362	0.425	0.858	-1.70	-6.60
7	-0.244	-0.339	0.344	0.685	-1.61	-3.33
8	-0.216	-0.313	0.282	0.560	-1.15	-2.06

St000039, St000223, St000356, St000358

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = \tfrac{1}{12}(n-2)(n-1)$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \tfrac{1}{180}(n-2)(n^2+\tfrac{11}{2}n-\tfrac{1}{2})$$

n	$\tilde\kappa_3^{(n)}$	$\tilde\kappa_4^{(n)}$	$\tilde\kappa_5^{(n)}$	$\tilde\kappa_6^{(n)}$	$\tilde\kappa_7^{(n)}$	$\tilde\kappa_8^{(n)}$
6	0.564	-0.0574	-0.887	-1.46	0.411	10.7
7	0.494	-0.0267	-0.614	-1.01	0.133	6.31
8	0.448	-0.00899	-0.458	-0.746	-0.0523	3.56

St000054, St000740

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = \tfrac{1}{2}(n+1)$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \tfrac{1}{12}(n-1)(n+1)$$

n	$\tilde\kappa_3^{(n)}$	$\tilde\kappa_4^{(n)}$	$\tilde\kappa_5^{(n)}$	$\tilde\kappa_6^{(n)}$	$\tilde\kappa_7^{(n)}$	$\tilde\kappa_8^{(n)}$
5	0.000	-1.30	0.000	7.75	0.000	-102.
6	0.000	-1.27	0.000	7.46	0.000	-96.7
7	0.000	-1.25	0.000	7.29	0.000	-93.8

St000060

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = \tfrac{2}{3}(n-\tfrac{1}{2})$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \tfrac{1}{18}(n-2)(n+1)$$

n	$\tilde\kappa_3^{(n)}$	$\tilde\kappa_4^{(n)}$	$\tilde\kappa_5^{(n)}$	$\tilde\kappa_6^{(n)}$	$\tilde\kappa_7^{(n)}$	$\tilde\kappa_8^{(n)}$
5	-0.600	-0.800	3.00	0.400	-29.4	55.6
6	-0.588	-0.729	2.79	0.102	-25.9	52.7
7	-0.581	-0.690	2.68	-0.0439	-24.2	51.0

St000111, St000471

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = \tfrac{1}{3}(n-1)(n+1)$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \tfrac{1}{36}(n+2)(n+1)^2$$

n	$\tilde\kappa_3^{(n)}$	$\tilde\kappa_4^{(n)}$	$\tilde\kappa_5^{(n)}$	$\tilde\kappa_6^{(n)}$	$\tilde\kappa_7^{(n)}$	$\tilde\kappa_8^{(n)}$
6	-0.251	-0.216	0.309	0.206	-0.935	0.430
7	-0.235	-0.193	0.258	0.165	-0.718	-0.180
8	-0.222	-0.174	0.219	0.136	-0.549	-0.104

St000154, St000472

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = \tfrac{1}{6}(n-1)(n+1)$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \tfrac{1}{36}(n-1)(n+1)^2$$

n	$\tilde\kappa_3^{(n)}$	$\tilde\kappa_4^{(n)}$	$\tilde\kappa_5^{(n)}$	$\tilde\kappa_6^{(n)}$	$\tilde\kappa_7^{(n)}$	$\tilde\kappa_8^{(n)}$
6	0.323	-0.228	-0.461	0.165	1.70	1.35
7	0.294	-0.202	-0.364	0.142	1.14	0.316
8	0.270	-0.182	-0.297	0.122	0.825	0.151

St000213, St000325, St000470, St000702

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = \tfrac{1}{2}(n+1)$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \tfrac{1}{12}(n+1)$$

n	$\tilde\kappa_3^{(n)}$	$\tilde\kappa_4^{(n)}$	$\tilde\kappa_5^{(n)}$	$\tilde\kappa_6^{(n)}$	$\tilde\kappa_7^{(n)}$	$\tilde\kappa_8^{(n)}$
5	0.000	-0.200	0.000	0.000	0.000	10.8
6	0.000	-0.171	0.000	0.140	0.000	-2.27

St000235, St000673

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = (n-1)$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = 1$$

n	$\tilde\kappa_3^{(n)}$	$\tilde\kappa_4^{(n)}$	$\tilde\kappa_5^{(n)}$	$\tilde\kappa_6^{(n)}$	$\tilde\kappa_7^{(n)}$	$\tilde\kappa_8^{(n)}$
5	-1.00	1.00	-1.00	0.000	14.0	-118.
6	-1.00	1.00	-1.00	1.00	0.000	-20.0

St000236

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = 2$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = 2(n-1)^{-1}(n-2)$$

n	$\tilde\kappa_3^{(n)}$	$\tilde\kappa_4^{(n)}$	$\tilde\kappa_5^{(n)}$	$\tilde\kappa_6^{(n)}$	$\tilde\kappa_7^{(n)}$	$\tilde\kappa_8^{(n)}$
5	0.272	-0.556	-1.09	0.741	8.89	9.46
6	0.395	-0.266	-0.865	-0.713	2.49	12.5

St000242

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = (n-2)$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = 2(n-1)^{-1}(n-2)$$

n	$\tilde\kappa_3^{(n)}$	$\tilde\kappa_4^{(n)}$	$\tilde\kappa_5^{(n)}$	$\tilde\kappa_6^{(n)}$	$\tilde\kappa_7^{(n)}$	$\tilde\kappa_8^{(n)}$
5	-0.272	-0.556	1.09	0.741	-8.89	9.46
6	-0.395	-0.266	0.865	-0.713	-2.49	12.5

St000246, St000304, St000692, St000868

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = \tfrac{1}{4}(n-1)n$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \tfrac{1}{36}(n-1)n(n+\tfrac{5}{2})$$

n	$\tilde\kappa_3^{(n)}$	$\tilde\kappa_4^{(n)}$	$\tilde\kappa_5^{(n)}$	$\tilde\kappa_6^{(n)}$	$\tilde\kappa_7^{(n)}$	$\tilde\kappa_8^{(n)}$
5	0.000	-0.468	0.000	1.13	0.000	-6.40
6	0.000	-0.377	0.000	0.750	0.000	-3.55
7	0.000	-0.317	0.000	0.539	0.000	-2.18

St000279

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = 1$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \tfrac{1}{6}(n-1)(n+4)$$

n	$\tilde\kappa_3^{(n)}$	$\tilde\kappa_4^{(n)}$	$\tilde\kappa_5^{(n)}$	$\tilde\kappa_6^{(n)}$	$\tilde\kappa_7^{(n)}$	$\tilde\kappa_8^{(n)}$
5	3.20	12.3	49.6	165.	18.0	-7640.
6	4.41	27.4	211.	1790.	15200.	113000.
7	5.79	53.5	679.	10300.	171000.	2.91e6

St000355, St000359

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = \tfrac{1}{12}(n-2)(n-1)$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \tfrac{1}{60}(n-2)(n^2-\tfrac{1}{3}n+\tfrac{1}{3})$$

n	$\tilde\kappa_3^{(n)}$	$\tilde\kappa_4^{(n)}$	$\tilde\kappa_5^{(n)}$	$\tilde\kappa_6^{(n)}$	$\tilde\kappa_7^{(n)}$	$\tilde\kappa_8^{(n)}$
6	0.761	-0.0432	-1.96	-3.81	4.92	54.8
7	0.704	0.0300	-1.41	-2.96	1.52	31.3
8	0.663	0.0757	-1.07	-2.38	0.0499	19.1

St000357, St000360

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = \frac{1}{12}(n-2)(n-1)$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \frac{1}{60}(n-2)(n^2+\frac{13}{6}n-\frac{43}{6})$$

n	$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$
5	1.46	2.27	2.30	-8.03	-80.0	-388.
6	1.28	1.84	2.19	-1.74	-33.2	-195.
7	1.14	1.49	1.74	-0.229	-15.6	-92.6
8	1.03	1.24	1.35	0.0677	-8.65	-48.7

St000462, St000463, St000866, St000961

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = \frac{1}{4}(n-2)(n-1)$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \frac{1}{36}(n-2)(n+\frac{1}{2})(n+3)$$

n	$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$
5	0.142	-0.674	-0.142	2.53	0.222	-22.3
6	0.0754	-0.482	-0.0309	1.11	-0.329	-5.89
7	0.0446	-0.376	-0.00857	0.690	-0.0656	-2.78
8	0.0284	-0.311	-0.00278	0.483	-0.0204	-1.74

St000500

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = n$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = (n-1)(n+2)$$

n	$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$
5	1.05	0.847	-0.436	-5.40	-20.6	-46.2
6	1.08	1.01	0.287	-2.76	-13.4	-45.6

St000619

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = \frac{1}{2}n$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \frac{1}{12}n$$

n	$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$
5	0.000	-0.240	0.000	1.92	0.000	-39.4
6	0.000	-0.200	0.000	0.000	0.000	10.8
7	0.000	-0.171	0.000	0.140	0.000	-2.27

St000724

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = \frac{2}{3}(n+1)$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \frac{1}{18}(n-2)(n+1)$$

n	$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$
5	-0.600	-0.800	3.00	0.400	-29.4	55.6
6	-0.588	-0.729	2.79	0.102	-25.9	52.7
7	-0.581	-0.690	2.68	-0.0439	-24.2	51.0

St000756

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = n$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \frac{1}{2}(n-1)n$$

n	$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$
5	0.632	-0.100	-1.35	-2.07	2.91	24.0
6	0.689	0.0889	-1.04	-2.25	-0.0675	15.8
7	0.727	0.222	-0.790	-2.18	-1.63	9.59

St000809

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = \frac{1}{12}(n-1)(n+4)$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \frac{1}{180}(n^3+\frac{7}{2}n^2+\frac{7}{2}n+16)$$

n	$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$
6	0.194	-0.153	-0.338	-0.203	0.840	2.65
7	0.232	-0.102	-0.308	-0.277	0.396	1.92
8	0.253	-0.0672	-0.273	-0.292	0.202	1.59

St000825

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = \frac{1}{2}(n-1)n$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \frac{1}{18}(n-1)n(n+7)$$

n	$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$
5	0.000	-0.143	0.000	-0.109	0.000	0.0396
6	0.000	-0.101	0.000	-0.0304	0.000	-0.160
7	0.000	-0.0800	0.000	0.00121	0.000	-0.125

St000830

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = \frac{1}{3}(n-1)(n+1)$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \frac{2}{45}(n+1)(n^2+\frac{7}{2})$$

n	$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$
6	-0.283	-0.362	0.425	0.858	-1.70	-6.60
7	-0.244	-0.339	0.344	0.685	-1.61	-3.33
8	-0.216	-0.313	0.282	0.560	-1.15	-2.06

St000962

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = \frac{1}{4}(n-4)(n-3)$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \frac{5}{8}(n-4)(n-3)$$

n	$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$
6	0.998	-0.174	-3.91	-6.08	28.4	174.
7	0.548	-0.589	-1.70	1.43	13.9	-0.347
8	0.311	-0.590	-0.679	1.87	3.98	-14.6

St001084

$$\mathbb{E}(X_{\mathrm{st}}^{(n)}) = \frac{1}{6}(n-2)$$
$$\mathbb{V}(X_{\mathrm{st}}^{(n)}) = \frac{1}{18}(n-2)(n-\frac{1}{2})$$

n	$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$
6	1.57	1.38	-4.43	-29.2	-59.7	307.
7	1.54	1.30	-4.36	-27.7	-53.6	300.

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