ON CERTAIN COMBINATORIAL EXPANSIONS OF DESCENT POLYNOMIALS AND THE CHANGE OF GRAMMARS

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Abstract. In this paper, we study certain combinatorial expansions of descent polynomials by using the change of context-free grammars method. We provide a unified approach to study the γ-positivity and the partial γ-positivity of the descent polynomials of several combinatorial structures, including the descent polynomials of permutations, derangements, Stirling permutations, Legendre-Stirling permutations and Jacobi-Stirling permutations. Moreover, we study a group action on Stirling permutations and Jacobi-Stirling permutations due to Foata and Strehl.

Keywords: Eulerian polynomials; Jacobi-Stirling permutations; Context-free grammars

1. Introduction

In the past decades, symbolic methods have been developed extensively in combinatorics, including umbral calculus [32], species [26, 38]), generating trees [13, 37] and context-free grammars [8, 14]. In this paper, we shall study descent polynomials by using context-free grammars.

For an alphabet A, let $\mathbb{Q}[[A]]$ be the rational commutative ring of formal power series in monomials formed from letters in A. Following Chen [8], a *context-free grammar* over A is a function $G : A \to \mathbb{Q}[[A]]$ that replaces a letter in A by a formal function over A. The formal derivative D is a linear operator defined with respect to a context-free grammar G. More precisely, the derivative $D = D_G$: $\mathbb{Q}[[A]] \to \mathbb{Q}[[A]]$ is defined as follows: for $x \in A$, we have $D(x) = G(x)$; for a monomial u in $\mathbb{Q}[[A]]$, $D(u)$ is defined so that D is a derivation, and for a general element $q \in \mathbb{Q}[[A]]$, $D(q)$ is defined by linearity. For example, if $A = \{x, y\}$ and $G = \{x \to xy, y \to y\}$, then $D(x) = xy, D^2(x) = D(xy) = xy^2 + xy$. The Chen's grammar has been found extremely useful in studying various combinatorial structures, including set partitions, permutations, increasing trees, perfect matchings and so on. The reader is referred to [10, 20, 30] for recent progress on this subject.

Let $f(x) = \sum_{i=0}^{n} f_i x^i$ be a symmetric polynomial, i.e., $h_i = h_{n-i}$. Then $f(x)$ can be expanded uniquely as $f(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_k x^k (1 + x)^{n-2k}$, and it is said to be γ-positive if $\gamma_k \geq 0$ (see [21]). The γ-positivity provides a natural approach to study symmetric and unimodal polynomials in combinatorics (see [4, 6, 27, 35] for instance).

The purpose of this paper is to present a systematic method for studying certain expansions of symmetric polynomials and asymmetric polynomials. The following definition is fundamental.

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Definition 1. *Let $p_n(x,y,z)$ be a three-variable polynomial. Suppose $p_n(x,y,z)$ has the following expansion:*

$$p_n(x,y,z)=\sum_i\sum_j \gamma_{n,i,j}q_{n,i,j}(x,y,z)r_{n,i,j}(x+y),$$

where $q_{n,i,j}(x,y,z)$ are three-variable polynomials and $r_{n,i,j}(x)$ are one-variable polynomials. If the coefficient of z^i in the expansion is a γ-positive polynomial for any i, then we say that $p_n(x,y,z)$ is a partial γ-positive polynomial.

The partial γ-positive polynomials occur very often in the study of the expansions of multivariable generalization of symmetric polynomials and asymmetric polynomials, see [27, 33, 35] for instance.

Let $[n]=\{1,2,\ldots,n\}$. Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]$ and let $\pi=\pi(1)\pi(2)\cdots\pi(n)\in\mathfrak{S}_n$. Let $\mathrm{Des}\,(\pi)=\{i\in[n]:\pi(i)>\pi(i+1)\}$. For a subset $S\subseteq[n]$, we define the *characteristic monomial* u_S in the noncommutative monomial variables $u_S=u_1u_2\cdots u_{n-1}$, where

$$u_i=\begin{cases} a, & \text{if } i\notin S;\\ b, & \text{if } i\in S.\end{cases}$$

The *ab*-index of $\mathfrak{S}_n$ is defined by

$$\Psi_n(a,b)=\sum_{\pi\in\mathfrak{S}_n}u_{\mathrm{Des}\,(\pi)}.$$

A classical result says that there exists a polynomial $\Phi_n(c,d)$ in the noncommuting variable c and d such that

$$\Psi_n(a,b)=\Phi_n(a+b,ab+ba). \tag{1}$$

The polynomial $\Phi_n(c,d)$ is called the *cd*-index of $\mathfrak{S}_n$ (see [25] for instance). Motivated by (1), the type of change of grammars considered in this paper is given as follows:

$$\begin{cases} u=a+b,\\ v=ab,\end{cases}$$

where a,b,u,v are commuting variables. We show that descent polynomials can be systematically studied by using the change of context-free grammars method.

This paper is organized as follows. In Section 2, we collect some definitions, notation and results that will be needed throughout this paper. In Section 3, we study derangement polynomials of type B. In Section 4, we study the descent polynomials of Stirling permutations. In Section 5, we study the descent polynomials of Legendre-Stirling permutations. In Section 6, we study the descent polynomials of Jacobi-Stirling permutations.

2. Preliminary

A *descent* (resp. *an ascent*) of a permutation $\pi\in\mathfrak{S}_n$ is a position i such that $\pi(i)>\pi(i+1)$ (resp. $\pi(i)<\pi(i+1)$), where $1\le i\le n-1$. Denote by $\mathrm{des}\,(\pi)$ and $\mathrm{asc}\,(\pi)$ the numbers of descents and ascents of π, respectively. Then the equations

$$A_n(x)=\sum_{\pi\in\mathfrak{S}_n}x^{\mathrm{des}\,(\pi)+1}=\sum_{\pi\in\mathfrak{S}_n}x^{\mathrm{asc}\,(\pi)+1}=\sum_{k=1}^n\left\langle {n\atop k}\right\rangle x^k,$$

define the *Eulerian polynomial* $A_n(x)$ and the *Eulerian number* $\left\langle {n \atop k} \right\rangle$ (see [36, A008292]). The following identity was attributed to Euler:

$$\sum_{n=0}^{\infty} n^k x^n = \frac{A_k(x)}{(1-x)^{k+1}}.$$

We say that a permutation $\pi \in \mathfrak{S}_n$ is a *derangement* if $\pi(i) \neq i$ for any $i \in [n]$. Let $\mathcal{D}_n$ be the set of derangements in $\mathfrak{S}_n$. An *excedance* of a permutation $\pi \in \mathfrak{S}_n$ is a position i such that $\pi(i) > i$, where $1 \leq i \leq n-1$. Denote by $\operatorname{exc}(\pi)$ the number of excedances of π. For any $n \geq 1$, the *derangement polynomial* is defined by

$$d_n(x) = \sum_{\pi \in \mathcal{D}_n} x^{\operatorname{exc}(\pi)}.$$

The *hyperoctahedral group* B_n is the group of signed permutations of $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all i, where $\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. Set $\pi(0) = 0$. For each $\pi \in B_n$, we define $\operatorname{des}_B(\pi) = \#\{i \in \{0, 1, 2, \ldots, n-1\} \mid \pi(i) > \pi(i+1)\}$. Let

$$B_n(x) = \sum_{\pi \in B_n} x^{\operatorname{des}_B(\pi)} = \sum_{k=0}^{n} B(n,k) x^k.$$

The polynomial $B_n(x)$ is called an *Eulerian polynomial of type* B, while $B(n,k)$ is called an *Eulerian number of type* B (see [36, A060187]).

The γ-positivity of $A_n(x)$ was first studied by Foata and Schützenberger [18]. Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$ with $\pi(0) = \pi(n+1) = 0$. An index $i \in [n]$ is a *peak* (resp. *double descent*) of π if $\pi(i-1) < \pi(i) > \pi(i+1)$ (resp. $\pi(i-1) > \pi(i) > \pi(i+1)$). Let $a(n,k)$ be the number of permutations in $\mathfrak{S}_n$ with k peaks and without double descent. Foata and Schützenberger [18] discovered that

$$A_n(x) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} a(n,k) x^k (1+x)^{n+1-2k}. \tag{2}$$

Moreover, the numbers $a(n,k)$ satisfy the recurrence

$$a(n,k) = ka(n-1,k) + (2n-4k+4)a(n-1,k-1),$$

with the initial conditions $a(1,1) = 1$ and $a(1,k) = 0$ for $k \geq 2$ (see [36, A101280]). Subsequently, Foata and Strehl [19] introduced the well known *Foata-Strehl group action*, by which they partition $\mathfrak{S}_n$ into equivalence classes, so that for each class C,

$$\sum_{\pi \in \mathrm{C}} x^{\operatorname{des}(\pi)} = x^i (1+x)^{n-1-2i}.$$

The γ-positivity of $B_n(x)$ was extensively studied by Petersen [31] and Chow [11]. The γ-positivity of $d_n(x)$ was studied by Shin and Zeng [35]. In recent years, γ-positivity attracted much attention, see [4, 6, 27, 33, 34] and references therein.

The idea of change of grammars is illustrated in the proof of the following well known result.

Proposition 2 ([18, 31, 35])**.** *The polynomials $A_n(x), B_n(x)$ and $d_n(x)$ are all γ-positive.*

Proof. We divide the proof into three parts.

(i) Following Dumont [14, Section 2.1], if $A = \{x, y\}$ and

$$G = \{x \to xy, y \to xy\},$$

then $D^n(x) = \sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle x^k y^{n+1-k}$ for $n \geq 1$. Note that $D(xy) = xy(x+y)$ and $D(x+y) = 2xy$. Set $u = xy$ and $v = x + y$. Then $D(u) = uv$ and $D(v) = 2u$. It is easy to verify that if $A = \{x, u, v\}$ and

$$G = \{x \to u, u \to uv, v \to 2u\},$$

then there exist nonnegative integers $\widehat{a}(n, k)$ such that

$$D^n(x) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \widehat{a}(n, k) u^k v^{n+1-2k}. \tag{3}$$

From $D^{n+1}(x) = D(D^n(x))$, we see that numbers $\widehat{a}(n, k)$ and $a(n, k)$ satisfy the same recurrence relation and initial conditions. Thus $\widehat{a}(n, k) = a(n, k)$. When $y = 1$, then (3) reduces to (2).

(ii) According to [28, Theorem 10], if $A = \{x, y\}$ and $G = \{x \to xy^2, y \to x^2 y\}$, then $D^n(xy) = \sum_{k=0}^{n} B(n, k) x^{2k+1} y^{2n-2k+1}$. Note that

$$D(xy) = xy(x^2 + y^2), D(x^2 + y^2) = 4x^2y^2.$$

Set $u = xy$ and $v = x^2 + y^2$. Then $D(u) = uv, D(v) = 4u^2$. It is routine to check that if $A = \{u, v\}$ and

$$G = \{u \to uv, v \to 4u^2\}, \tag{4}$$

then there exist nonnegative integers $b(n, k)$ such that $D^n(u) = u \sum_{k=0}^{\lfloor n/2 \rfloor} b(n, k) u^{2k} v^{n-2k}$. When $y = 1$, we get $u = x$ and $v = 1 + x^2$. Hence $B_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} b(n, k) x^k (1 + x)^{n-2k}$.

(iii) For $\pi \in \mathfrak{S}_n$, we define $\operatorname{fix}(\pi) = \#\{i \in [n] : \pi(i) = i\}$ and $\operatorname{dc}(\pi) = \#\{i \in [n] : \pi(i) < i\}$. Following Dumont [14, Section 2.2], if $A = \{x, y, z, e\}$ and $G = \{x \to xy, y \to xy, z \to xy, e \to ez\}$, then

$$D^n(e) = e \sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{exc}(\pi)} y^{\operatorname{dc}(\pi)} z^{\operatorname{fix}(\pi)}.$$

We set $u = xy, v = x + y$. Then $D(u) = uv, D(v) = 2u$. If $A = \{e, z, u, v\}$ and

$$G = \{e \to ez, z \to u, u \to uv, v \to 2u\},$$

then there exist nonnegative integers $d(n, k)$ such that $D^n(e)|_{z=0} = e \sum_{k=1}^{\lfloor n/2 \rfloor} d(n, k) u^k v^{n-2k}$. When $y = 1$, we get $u = x$ and $v = 1 + x$. Then for $n \geq 2$, we get

$$d_n(x) = \sum_{k=1}^{\lfloor n/2 \rfloor} d(n, k) x^k (1 + x)^{n-2k}.$$

This completes the proof. □

In the following sections, we will focus our attention on certain multivariate extension of asymmetric polynomials. In particular, using modified Foata and Strehl's group action, we consider the group action on derangements of B_n, Stirling permutations and Jacobi-Stirling permutations.

3. Derangement polynomials of type B

3.1. Basic definitions and notation.

Let $\pi \in B_n$. We say that $i \in [n]$ is a *weak excedance* of π if $\pi(i) = i$ or $\pi(|\pi(i)|) > \pi(i)$ (see [7, p. 431]). Let $\mathrm{wexc}\,(\pi)$ be the number of weak excedances of π. From [7, Theorem 3.15],

$$B_n(x) = \sum_{\pi \in B_n} x^{\mathrm{wexc}\,(\pi)}.$$

A *fixed point* of $\pi \in B_n$ is an index $i \in [n]$ such that $\pi(i) = i$. A *derangement of type B* is a signed permutation $\pi \in B_n$ with no fixed points. Let $\mathcal{D}_n^B$ be the set of derangements of B_n. Following [12], the *derangement polynomials of type B* are defined by

$$d_0^B(x) = 1,\ d_n^B(x) = \sum_{\pi \in \mathcal{D}_n^B} x^{\mathrm{wexc}\,(\pi)}.$$

The first few terms of $d_n^B(x)$ are given as follows:

$$d_1^B(x) = 1, d_2^B(x) = 1 + 4x, d_3^B(x) = 1 + 20x + 8x^2, d_4^B(x) = 1 + 72x + 144x^2 + 16x^3.$$

Given $\pi \in \mathcal{D}_n^B$. Then $\mathrm{wexc}\,(\pi) = \#\{i \in [n] \mid \pi(|\pi(i)|) > \pi(i)\}$. We say that i is an *anti-excedance* of π if $\pi(|\pi(i)|) < \pi(i)$. Let $\mathrm{aexc}\,(\pi)$ be the number of anti-excedances of π. We say that i is a *singleton* if $(\bar{i})$ is a cycle of π. Let $\mathrm{single}\,(\pi)$ be the number of singletons of π. Hence

$$\mathrm{wexc}\,(\pi) + \mathrm{aexc}\,(\pi) + \mathrm{single}\,(\pi) = n.$$

In the following discussion, we always write π by using its *standard cycle decomposition*, in which each cycle is written with its largest entry last and the cycles are written in ascending order of their last entry. For example, $\overline{3}51\overline{7}2\overline{6}\ \overline{4} \in \mathcal{D}_7^B$ can be written as $(\overline{6})(\overline{7},\overline{4})(\overline{3},1)(2,5)$. Let $(c_1, c_2, \ldots, c_i)$ be a cycle of π. We say that c_j is called

- a *cycle ascent* in the cycle if $c_j < c_{j+1}$, where $1 \le j < i$;
- a *cycle descent* in the cycle if $c_j > c_{j+1}$, where $1 \le j \le i$ and we set $c_{i+1} = c_1$;
- a *cycle double ascent* in the cycle if $c_{j-1} < c_j < c_{j+1}$, where $1 < j < i$;
- a *cycle double descent* in the cycle if $c_{j-1} > c_j > c_{j+1}$, where $1 < j < i$;

Denote by $\mathrm{cda}\,(\pi)$ (resp. $\mathrm{cdd}\,(\pi)$) the number of cycle double ascents (resp. cycle double descents) of π. As pointed out by Chow [12, p. 819], if $\pi \in \mathcal{D}_n^B$ with no singletons, then $\mathrm{wexc}\,(\pi)$ equals the number cycle ascents and $\mathrm{aexc}\,(\pi)$ equals the number of cycle descents.

3.2. Main results.

Let

$$E_n(x,y,z) = \sum_{\pi \in \mathcal{D}_n^B} x^{\mathrm{wexc}\,(\pi)} y^{\mathrm{aexc}\,(\pi)} z^{\mathrm{single}\,(\pi)}.$$

Very recently, we discovered the following lemma.

Lemma 3 ([30]). *If $A = \{x, y, z, e\}$ and*

$$G = \{x \to xy^2, y \to x^2y, z \to x^2y^2z^{-3}, e \to ez^4\}, \tag{5}$$

then

$$D^n(e) = eE_n(x^2, y^2, z^4). \tag{6}$$

Now we present the first main result of this paper.

Theorem 4. *The polynomial $E_n(x,y,z)$ is a partial γ-positive polynomial. More precisely, for $n\geq 0$, we have*

$$E_n(x,y,z)=\sum_{i=0}^{n} z^i \sum_{j=0}^{\lfloor (n-i)/2\rfloor} g_n(i,j)(xy)^j(x+y)^{n-i-2j}, \tag{7}$$

where the numbers $g_n(i,j)$ satisfy the recurrence relation

$$g_{n+1}(i,j)=g_n(i-1,j)+4(1+i)g_n(i+1,j-1)+2jg_n(i,j)+4(n+2-i-2j)g_n(i,j-1), \tag{8}$$

with the initial conditions $g_1(1,0)=1$ and $g_1(1,j)=0$ for $j\neq 0$.

Proof. Consider the grammar (5). If we set $s=e, t=z^4, u=xy$ and $v=x^2+y^2$, then

$$D(s)=st, D(t)=4u^2,\ D(u)=uv,\ D(v)=4u^2.$$

Thus, if $A=\{s,t,v,u\}$ and

$$G=\{s\to st, t\to 4u^2, u\to uv, v\to 4u^2\}. \tag{9}$$

then

$$D^n(s)=s\sum_{i,j,k} f_n(i,j,k)4^{j+k}t^{i-j}u^{2(j+k)}v^{n-i-j-2k},$$

where i,j,k range nonnegative integers satisfying $i\geq j$ and $i+j+k=n$. Setting $i-j=\alpha$ and $j+k=\beta$, we have $\alpha+2\beta=i+j+2k=n+k$. Hence $k=\alpha+2\beta-n$. Therefore,

$$D^n(s)=s\sum_{\alpha=0}^{n} t^\alpha \sum_{\beta=0}^{\lfloor (n-\alpha)/2\rfloor} \sum_{\substack{k\leq\beta \\ k=\alpha+2\beta-n}} f_n(\alpha+\beta-k,\beta-k,k)4^\beta u^{2\beta}v^{n-\alpha-2\beta}.$$

Setting

$$g_n(\alpha,\beta)=\sum_{\substack{k\leq\beta \\ k=\alpha+2\beta-n}} f_n(\alpha+\beta-k,\beta-k,k)4^\beta,$$

we obtain

$$D^n(s)=s\sum_{\alpha=0}^{n}\sum_{\beta=0}^{\lfloor (n-\alpha)/2\rfloor} g_n(\alpha,\beta)t^\alpha u^{2\beta}v^{n-\alpha-2\beta}. \tag{10}$$

Comparing (6) with (10), we get (7). Notice that

$$\begin{aligned} D^{n+1}(s) &= D(D^n(s)) \\ &= D\left(s\sum_{\alpha=0}^{n}\sum_{\beta=0}^{\lfloor (n-\alpha)/2\rfloor} g_n(\alpha,\beta)t^\alpha u^{2\beta}v^{n-\alpha-2\beta}\right) \\ &= s\sum_{\alpha,\beta} g_n(\alpha,\beta)(t^{\alpha+1}u^{2\beta}v^{n-\alpha-2\beta}+4\alpha t^{\alpha-1}u^{2\beta+2}v^{n-\alpha-2\beta} \\ &\quad +2\beta t^\alpha u^{2\beta}v^{n+1-\alpha-2\beta}+4(n-\alpha-2\beta)t^\alpha u^{2\beta+2}v^{n-1-\alpha-2\beta}). \end{aligned}$$

By comparing coefficients on the both sides of $D^{n+1}(s)=D(D^n(s))$, we get (8). □

Let $g_n(x,y)=\sum_{i,j} g_n(i,j)x^iy^j$. Multiplying both sides of the recurrence relation (8) by x^iy^j and summing over all i,j, we get that the polynomials $g_n(x,y)$ satisfy the recurrence relation

$$g_{n+1}(x,y)=(x+4ny)g_n(x,y)+4y(1-x)\frac{\partial}{\partial x}g_n(x,y)+2y(1-4y)\frac{\partial}{\partial y}g_n(x,y),$$

with the initial condition $g_0(x,y)=1$. The first few of terms of $g_n(x,y)$ are given as follows:

$$g_1(x,y)=x,\ g_2(x,y)=x^2+4y,\ g_3(x,y)=x^3+12xy+8y,\ g_4(x,y)=x^4+32xy+24x^2y+16y+80y^2.$$

Let $a_n(x)=\sum_{k\geq 1} a(n,k)x^k$, where the numbers $a(n,k)$ are defined by (2).

Corollary 5. *For ≥ 0, we have*

$$g_{n+1}(x,y)=xg_n(x,y)+\sum_{k=0}^{n-1}\binom{n}{k}2^{n+1-k}g_k(x,y)a_{n-k}(y).$$

Proof. Let G be the grammar (9). Note that $D(t)=4u^2, D^2(t)=8u^2v$. Assume that

$$D^n(t)=2^{n+1}\sum_{k\geq 1}\widehat{a}(n,k)u^{2k}v^{n+1-2k}.$$

for $n\geq 1$. Hence

$$\begin{aligned}D^{n+1}(t)&=D(D^n(t))\\&=D\left(2^{n+1}\sum_{k\geq 1}\widehat{a}(n,k)u^{2k}v^{n+1-2k}\right)\\&=2^{n+2}\sum_{k\geq 1}\widehat{a}(n,k)\left(ku^{2k}v^{n+2-2k}+2(n+1-2k)u^{2k+2}v^{n-2k}\right).\end{aligned}$$

Therefore, $\widehat{a}(n+1,k)=k\widehat{a}(n,k)+2(n+3-2k)\widehat{a}(n,k-1)$. It is clear that $\widehat{a}(1,1)=1$ and $\widehat{a}(1,k)=0$ for $k\neq 1$. Since $a(n,k)$ and $\widehat{a}(n,k)$ satisfy the same recurrence relation and initial conditions, so they agree. Using the *Leibniz's formula*, we get

$$D^{n+1}(s)=D^n(st)=tD^n(s)+\sum_{k=0}^{n-1}\binom{n}{k}D^k(s)D^{n-k}(t),$$

which yields the desired recurrence relation. □

Let

$$g_n=\sum_{i=0}^{n}\sum_{j=0}^{\lfloor (n-i)/2\rfloor} g_n(i,j).$$

The first few terms of g_n are $g_0=1, g_1=1, g_2=5, g_3=21, g_4=153, g_5=1209$. It should be noted that the numbers g_n appear as A182825 in [36].

Theorem 6. *For $n\geq 1$, we have*

$$g_n(i,j)=\#\{\pi\in\mathcal{D}_n^B \mid \mathrm{single}\,(\pi)=i, \mathrm{wexc}\,(\pi)=j, \mathrm{cda}\,(\pi)=0\}. \tag{11}$$

Proof. By using Foata-Strehl's group action (see [19, 27] for instance), we define the action φ_i on $\mathcal{D}_n^B$ as follows. Let $c=(c_1,c_2,\ldots,c_i)$ be a cycle of $\pi\in\mathcal{D}_n^B$. Since $c_i=\max\{c_1,c_2,\ldots,c_i\}$, we set $c_0=+\infty$ and $\widetilde{c}=(c_0,c_1,c_2\ldots,c_i)$.

- If c_k is a cycle double ascent in c, then $\varphi_i(\widetilde{c})$ is obtained by deleting c_k and then inserting c_k between c_j and c_{j+1}, where j is the largest index satisfying $0 \le j < k$ and $c_j > c_k > c_{j+1}$;
- If c_k is a cycle double descent in c, then $\varphi_i(\widetilde{c})$ is obtained by deleting c_k and then inserting c_k between c_j and c_{j+1}, where j is the smallest index satisfying $k < j < i$ and $c_j < c_k < c_{j+1}$;

Given $\pi \in \mathcal{D}_n^B$. We define the Foata-Strehl's group action on $\mathcal{D}_n^B$ by

$$\varphi_i'(\pi) = \begin{cases} \varphi_i(\pi), & \text{if } i \text{ is a cycle double ascent or cycle double descent;} \\ \pi, & \text{otherwise.} \end{cases}$$

It is clear that the φ_i''s are involutions and that they commute. For any subset $S \subseteq [n]$, we may define the function $\varphi_S' : \mathcal{D}_n^B \mapsto \mathcal{D}_n^B$ by $\varphi_S'(\pi) = \prod_{i \in S} \varphi_i'(\pi)$. Hence the group $\mathbb{Z}_2^{2n}$ acts on $\mathcal{D}_n^B$ via the function φ_S', where $S \subseteq [n]$. Let $\mathrm{Dasc}\,(\pi)$ and $\mathrm{Ddes}\,(\pi)$ denote the sets of cycle double ascents and cycle double descents of π, respectively. Let $S = S(\pi) = \mathrm{Dasc}\,(\pi) \cup \mathrm{Ddes}\,(\pi)$. Note that $\mathrm{Dasc}\,(\varphi_S'(\pi)) = \mathrm{Ddes}\,(\pi)$, $\mathrm{Ddes}\,(\varphi_S'(\pi)) = \mathrm{Dasc}\,(\pi)$. We call two permutations in $\mathcal{D}_n^B$ are equivalent if one can be obtained from the other by using φ_S'. It is clear that each equivalence class contains exactly one element with no cycle double ascent. This completes the proof. □

4. Stirling permutations

4.1. **Basic definitions and notation.**

The *Stirling numbers of the second kind* $\left\{ {n \atop k} \right\}$ count the number of ways to partition $[n]$ into k non-empty subsets. Counting all functions from $[n]$ to $\{1, 2, \ldots, x\}$ yields

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \prod_{i=0}^{k-1} (x - i).$$

In [23], Gessel and Stanley considered the polynomial $C_k(x)$ defined by

$$\sum_{n=0}^{\infty} \left\{ {n+k \atop n} \right\} x^n = \frac{C_k(x)}{(1-x)^{2k+1}},$$

and they found that $C_k(x)$ is the descent polynomial of Stirling permutations of order k. The first few terms of $C_k(x)$ are given as follows:

$$C_1(x) = x, C_2(x) = x + 2x^2, C_3(x) = x + 8x^2 + 6x^3, C_4(x) = x + 22x^2 + 58x^3 + 24x^4.$$

A *Stirling permutation* of order n is a permutation of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ such that for each i, $1 \le i \le n$, all entries between the two occurrences of i are larger than i. Denote by $\mathcal{Q}_n$ the set of *Stirling permutations* of order n. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{Q}_n$ and set $\sigma_0 = \sigma_{2n+1} = 0$. For $0 \le i \le 2n$, we say that an index i is a *descent* (resp. *ascent*, *plateau*) of σ if $\sigma_i > \sigma_{i+1}$ (resp. $\sigma_i < \sigma_{i+1}$, $\sigma_i = \sigma_{i+1}$). Let $\mathrm{des}\,(\sigma), \mathrm{asc}\,(\sigma)$ and $\mathrm{plat}\,(\sigma)$ be the numbers of descents, ascents and plateaus of σ, respectively. A classical result of Bóna [5] says that descents, ascents and plateaus have the same distribution over $\mathcal{Q}_n$, i.e.,

$$C_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{des}\,\sigma} = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{asc}\,\sigma} = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{plat}\,\sigma}. \tag{12}$$

Let $C_n(x) = \sum_{k=1}^{n} C(n,k)x^k$. Chen and Fu [10, Theorem 2.3] discovered that if $A = \{x, y\}$ and $G = \{x \to xy^2, y \to xy^2\}$, then $D^n(x) = \sum_{k=1}^{n} C(n,k)x^k y^{2n+1-k}$.

4.2. **Main results.**

Define

$$C_n(x,y,z) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{asc}\,\sigma} y^{\mathrm{des}\,(\sigma)} z^{\mathrm{plat}\,\sigma}.$$

The following lemma will be used in our discussion, which implies (12)

Lemma 7 ([9])**.** *If $A = \{x, y, z\}$ and*

$$G = \{x \to xyz, y \to xyz, z \to xyz\}, \tag{13}$$

then $D^n(x) = C_n(x, y, z)$.

Proof. We first introduce a grammatical labeling of $\sigma \in \mathcal{Q}_n$ as follows:

(L_1) If i is an ascent, then put a superscript label x right after σ_i;
(L_2) If i is a descent, then put a superscript label y right after σ_i;
(L_3) If i is a plateau, then put a superscript label z right after σ_i.

Note that the weight of σ is given by $w(\sigma) = x^{\mathrm{asc}\,(\sigma)} y^{\mathrm{des}\,(\sigma)} z^{\mathrm{plat}\,(\sigma)}$. We proceed by induction on n. For $n = 1$, we have $\mathcal{Q}_1 = \{{}^x1^z1^y\}$ and $\mathcal{Q}_2 = \{{}^x1^z1^x2^z2^y, {}^x1^x2^z2^y1^y, {}^x2^z2^y1^z1^y\}$. Note that $D(x) = xyz, D^2(x) = D(xyz) = xy^2z^2 + x^2yz^2 + x^2y^2z$. Then the weight of ${}^x1^z1^y$ is given by $D(x)$, and the sum of weights of the elements in $\mathcal{Q}_2$ is given by $D^2(x)$. Hence the result holds for $n = 1, 2$. Suppose we get all labeled permutations in $\mathcal{Q}_{n-1}$, where $n \geq 3$. Let σ' be obtained from $\sigma \in \mathcal{Q}_{n-1}$ by inserting the pair nn. Then the changes of labeling are illustrated as follows:

$$\cdots \sigma_i^x \sigma_{i+1} \cdots \mapsto \cdots \sigma_i^x n^z n^y \sigma_{i+1} \cdots ;$$

$$\cdots \sigma_i^y \sigma_{i+1} \cdots \mapsto \cdots \sigma_i^x n^z n^y \sigma_{i+1} \cdots ;$$

$$\cdots \sigma_i^z \sigma_{i+1} \cdots \mapsto \cdots \sigma_i^x n^z n^y \sigma_{i+1} \cdots .$$

In each case, the insertion of nn corresponds to the operator D. It is easy to check that the action of D on elements of $\mathcal{Q}_{n-1}$ generates all elements of $\mathcal{Q}_n$. This completes the proof. □

We can now present the second main result of this paper.

Theorem 8. *For $n \geq 1$, we have*

$$C_n(x,y,z) = \sum_{i=1}^{n} x^i \sum_{j=0}^{\lfloor (2n+1-i)/2 \rfloor} \gamma_{n,i,j} (yz)^j (y+z)^{2n+1-i-2j}. \tag{14}$$

Let $\gamma_n(x,y) = \sum_{i=1}^{n} \sum_{j \geq 0} \gamma_{n,i,j} x^i y^j$. Then the polynomials $\gamma_n(x,y)$ satisfy the recurrence relation

$$\gamma_{n+1}(x,y) = (4n+2)xy\gamma_n(x,y) + xy(1-2x)\frac{\partial}{\partial x}\gamma_n(x,y) + xy(1-4y)\frac{\partial}{\partial y}\gamma_n(x,y), \tag{15}$$

with the initial condition $\gamma_1(x,y) = xy$.

Proof. We first consider a change of the grammar (13). Setting $w = x, u = yz$ and $v = y + z$, we get $D(w) = wu, D(u) = wuv, D(v) = 2wu$. If $A = \{w, u, v\}$ and

$$G = \{w \to wu, u \to wuv, v \to 2wu\},$$

then it is routine to verify that

$$D^n(w) = \sum_{i=1}^{n} \sum_{j=1}^{\lfloor (2n+1-i)/2 \rfloor} \gamma_{n,i,j} w^i u^j v^{2n+1-i-2j}.$$

Then upon taking $w = x, u = yz$ and $v = y + z$, we get (14). Note that

$$D^{n+1}(w) = D\left(\sum_{i,j} \gamma_{n,i,j} w^i u^j v^{2n+1-i-2j}\right)$$
$$= \sum_{i,j} \gamma_{n,i,j} \left(iw^i u^{j+1} v^{2n+1-i-2j} + jw^{i+1} u^j v^{2n+2-i-2j} + 2(2n+1-i-2j) w^{i+1} u^{j+1} v^{2n-i-2j}\right).$$

Then the numbers $\gamma_{n,i,j}$ satisfy the recurrence relation

$$\gamma_{n+1,i,j} = i\gamma_{n,i,j-1} + j\gamma_{n,i-1,j} + 2(2n+4-i-2j)\gamma_{n,i-1,j-1}, \tag{16}$$

with the initial conditions $\gamma_{1,1,1} = 1$ and $\gamma_{1,i,j} = 0$ for $(i, j) \neq (1, 1)$. Multiplying both sides of this recurrence relation by $x^i y^j$ and summing over all i, j, we get (15). □

The first few terms of $\gamma_n(x, y)$ are given as follows:

$$\gamma_1(x, y) = xy,$$
$$\gamma_2(x, y) = xy^2 + x^2 y,$$
$$\gamma_3(x, y) = x^3 y + 4x^2 y^2 + xy^3 + 2x^3 y^2,$$
$$\gamma_4(x, y) = x^4 y + 11x^3 y^2 + 11x^2 y^3 + xy^4 + 8x^4 y^2 + 14x^3 y^3.$$

Recall that the Eulerian numbers satisfy the recurrence relation

$$\left\langle {n+1 \atop i} \right\rangle = i \left\langle {n \atop i} \right\rangle + (n+2-i) \left\langle {n \atop i-1} \right\rangle,$$

with the initial conditions $\left\langle {1 \atop 1} \right\rangle = 1$ and $\left\langle {1 \atop i} \right\rangle = 0$ for $i \neq 1$. We have the following result.

Proposition 9. *For $n \geq 1$, we have $\gamma_{n,i,n+1-i} = \left\langle {n \atop i} \right\rangle$.*

Proof. Set $F(n, i) = \gamma_{n,i,n+1-i}$. Then $F(n, i-1) = \gamma_{n,i-1,n+2-i}$. Using (16), it is easy to verify that $\gamma_{n,i,j} = 0$ for $i + j \leq n$. Thus the numbers $F(n, i)$ satisfy the recurrence relation

$$F(n+1, i) = iF(n, i) + (n+2-i)F(n, i-1),$$

which yields the desired result. □

4.3. **Partial γ-coefficients.**

Let $\sigma=\sigma_0\sigma_1\sigma_2\cdots\sigma_{2n}\sigma_{2n+1}\in\mathcal{Q}_n$, where $\sigma_0=\sigma_{2n+1}=0$. A *double ascent* (resp. *double descent*, *left ascent-plateau*, *descent-plateau*) of σ is an index i such that $\sigma_{i-1}<\sigma_i<\sigma_{i+1}$ (resp. $\sigma_{i-1}>\sigma_i>\sigma_{i+1}$, $\sigma_{i-1}<\sigma_i=\sigma_{i+1}$, $\sigma_{i-1}>\sigma_i=\sigma_{i+1}$), where $i\in[2n-1]$. Denote by $\operatorname{dasc}(\sigma)$ (resp. $\operatorname{ddes}(\sigma)$, $\operatorname{laplat}(\sigma)$, $\operatorname{desp}(\sigma)$) the number of double ascents (resp. double descents, left ascent-plateaus, descent-plateaus) of σ.

Theorem 10. *For $n\geq 1$, we have*

$$\gamma_{n,i,j}=\#\{\sigma\in\mathcal{Q}_n\mid \operatorname{des}(\sigma)=i,\operatorname{laplat}(\sigma)=j,\operatorname{desp}(\sigma)=0\}.$$

Proof. Let $\mathcal{Q}_{n;i,j}=\{\sigma\in\mathcal{Q}_n\mid \operatorname{des}(\sigma)=i,\operatorname{laplat}(\sigma)=j,\operatorname{desp}(\sigma)=0\}$. Let $\sigma\in\mathcal{Q}_n$. We first define two operations on σ. For any $0\leq k\leq 2n$, let $\theta_{n+1,k}(\sigma)$ denote the element of $\mathcal{Q}_{n+1}$ obtained from σ by inserting the pair $(n+1)(n+1)$ between σ_k and σ_{k+1}, and let $\psi_n(\sigma)$ denote the element of $\mathcal{Q}_{n-1}$ obtained from σ by deleting the pair nn. We define

$$\begin{aligned}\operatorname{Des}(\sigma)&=\{k\in[2n]\mid \sigma_k>\sigma_{k+1}\},\\ \operatorname{Laplat}(\sigma)&=\{k\in[2n]\mid \sigma_{k-1}<\sigma_k=\sigma_{k+1}\},\\ \operatorname{Dasc}(\sigma)&=\{k\in[2n]\mid \sigma_{k-1}<\sigma_k<\sigma_{k+1}\}.\end{aligned}$$

For any $\sigma\in\mathcal{Q}_{n;i,j}$, we have $|\operatorname{Des}(\sigma)|+2|\operatorname{Laplat}(\sigma)|+|\operatorname{Dasc}(\sigma)|=2n+1$, since $\operatorname{desp}(\sigma)=0$. Thus, $|\operatorname{Dasc}(\sigma)|=2n+1-i-2j$.

For any $\sigma\in\mathcal{Q}_{n+1;i,j}$, denote by $r=r(\sigma)$ the index of the first occurrence of $n+1$ in σ. In other words, $\sigma_r=\sigma_{r+1}=n+1$. Then we partition the set $\mathcal{Q}_{n+1;i,j}$ into four subsets:

$$\begin{aligned}\mathcal{Q}^1_{n+1;i,j}&=\{\sigma\in\mathcal{Q}_{n+1;i,j}\mid \sigma_{r-1}>\sigma_{r+2}\}\\ \mathcal{Q}^2_{n+1;i,j}&=\{\sigma\in\mathcal{Q}_{n+1;i,j}\mid \sigma_{r-2}<\sigma_{r-1}=\sigma_{r+2}\}\\ \mathcal{Q}^3_{n+1;i,j}&=\{\sigma\in\mathcal{Q}_{n+1;i,j}\mid \sigma_{r-1}<\sigma_{r+2}<\sigma_{r+3}\}\\ \mathcal{Q}^4_{n+1;i,j}&=\{\sigma\in\mathcal{Q}_{n+1;i,j}\mid \sigma_{r-2}>\sigma_{r-1}=\sigma_{r+2}\}.\end{aligned}$$

Claim 1. There is a bijection $\phi_1:\mathcal{Q}^1_{n+1;i,j}\mapsto\{(\sigma,k)\mid\sigma\in\mathcal{Q}_{n;i,j-1}\text{ and }k\in\operatorname{Des}(\sigma)\}$.

For any $\sigma\in\mathcal{Q}^1_{n+1;i,j}$, note that $\psi_{n+1}(\sigma)\in\mathcal{Q}_{n;i,j-1}$ and $r(\sigma)-1\in\operatorname{Des}(\psi_{n+1}(\sigma))$. We define the map $\phi_1:\mathcal{Q}^1_{n+1;i,j}\mapsto\{(\sigma,k)\mid\sigma\in\mathcal{Q}_{n;i,j-1}\text{ and }k\in\operatorname{Des}(\sigma)\}$ by letting

$$\phi_1(\sigma)=(\psi_{n+1}(\sigma),r(\sigma)-1).$$

The inverse of ϕ_1^{-1} is given by $\phi_1^{-1}(\sigma,k)=\theta_{n+1,k}(\sigma)$.

Claim 2. There is a bijection $\phi_2:\mathcal{Q}^2_{n+1;i,j}\mapsto\{(\sigma,k)\mid\sigma\in\mathcal{Q}_{n;i-1,j}\text{ and }k\in\operatorname{Laplat}(\sigma)\}$.

For any $\sigma\in\mathcal{Q}^2_{n+1;i,j}$, note that $\psi_{n+1}(\sigma)\in\mathcal{Q}_{n;i-1,j}$ and $r(\sigma)-1\in\operatorname{Laplat}(\psi_{n+1}(\sigma))$. We define the map $\phi_2:\mathcal{Q}^2_{n+1;i,j}\mapsto\{(\sigma,k)\mid\sigma\in\mathcal{Q}_{n;i-1,j}\text{ and }k\in\operatorname{Laplat}(\sigma)\}$ by letting

$$\phi_2(\sigma)=(\psi_{n+1}(\sigma),r(\sigma)-1).$$

The inverse of ϕ_2^{-1} is given by $\phi_2^{-1}(\sigma,k)=\theta_{n+1,k}(\sigma)$.

Claim 3. There is a bijection $\phi_3:\mathcal{Q}^3_{n+1;i,j}\mapsto\{(\sigma,k)\mid\sigma\in\mathcal{Q}_{n;i-1,j-1}\text{ and }k\in\operatorname{Dasc}(\sigma)\}$.

For any $\sigma \in \mathcal{Q}^3_{n+1;i,j}$, note that $\psi_{n+1}(\sigma) \in \mathcal{Q}_{n;i-1,j-1}$ and $r(\sigma)-1 \in \mathrm{Dasc}\,(\psi_{n+1}(\sigma))$. We define the map $\phi_3 : \mathcal{Q}^3_{n+1;i,j} \mapsto \{(\sigma,k) \mid \sigma \in \mathcal{Q}_{n;i-1,j-1} \text{ and } k \in \mathrm{Dasc}\,(\sigma)\}$ by letting $\phi_3(\sigma) = (\psi_{n+1}(\sigma), r(\sigma)-1)$. The inverse of ϕ_3^{-1} is given by $\phi_3^{-1}(\sigma,k) = \theta_{n+1,k}(\sigma)$.

Claim 4. There is a bijection $\phi_4 : \mathcal{Q}^4_{n+1;i,j} \mapsto \{(\sigma,k) \mid \sigma \in \mathcal{Q}_{n;i-1,j-1} \text{ and } k \in \mathrm{Dasc}\,(\sigma)\}$.

Let $k \in [2n]$ and let $\sigma \in \mathcal{Q}_n$. We define a modified Foata-Strehl's group action φ_k as follows:

- If k is a double ascent then $\varphi_k(\sigma)$ is obtained by moving σ_k to the left of the second σ_k, which forms a new pleateau $\sigma_k\sigma_k$;
- If k is a descent-plateau then $\varphi_k(\sigma)$ is obtained by moving σ_k to the right of σ_j, where $j = \max\{s \in \{0,1,2,\ldots,k-1\} : \sigma_s < \sigma_k\}$.

For instance, if $\sigma = 2447887332115665$, then

$$\varphi_1(\sigma) = 4478873322115665,\ \varphi_4(\sigma) = 2448877332115665,$$

$\varphi_9 \circ \varphi_1(\sigma) = \sigma$ and $\varphi_6 \circ \varphi_4(\sigma) = \sigma$, where $\circ$ denote the composition operation.

For any $\sigma \in \mathcal{Q}^4_{n+1;i,j}$, note that the index $r(\sigma)-1$ is the unique descent-plateau in $\psi_{n+1}(\sigma)$ and $\varphi_{r(\sigma)-1} \circ \psi_{n+1}(\sigma) \in \mathcal{Q}_{n;i-1,j-1}$. Let $\sigma' = \varphi_{r(\sigma)-1} \circ \psi_{n+1}(\sigma)$. Read σ' from left to right and let p be the index of the first occurrence of the integer $\sigma_{r(\sigma)-1}$. Then $p \in \mathrm{Dasc}\,(\varphi_{r(\sigma)-1} \circ \psi_{n+1}(\sigma))$. Therefore, we define the map $\phi_4 : \mathcal{Q}^4_{n+1;i,j} \mapsto \{(\sigma,k) \mid \sigma \in \mathcal{Q}_{n;i-1,j-1} \text{ and } k \in \mathrm{Dasc}\,(\sigma)\}$ by letting $\phi_4(\sigma) = (\varphi_{r(\sigma)-1} \circ \psi_{n+1}(\sigma), p)$. For any $\sigma \in \mathcal{Q}_{n;i-1,j-1}$ and $k \in \mathrm{Dasc}\,(\sigma)$, the inverse of ϕ_4^{-1} is given by $\phi_4^{-1}(\sigma,k) = \theta_{n+1,r}(\varphi_k(\sigma))$, where r is the unique descent-plateau in $\varphi_k(\sigma)$. Thus ϕ_4 is the desired bijection.

In conclusion, we have

$$\begin{aligned}|\mathcal{Q}_{n+1;i,j}| &= |\mathcal{Q}^1_{n+1;i,j}| + |\mathcal{Q}^2_{n+1;i,j}| + |\mathcal{Q}^3_{n+1;i,j}| + |\mathcal{Q}^4_{n+1;i,j}| \\ &= i|\mathcal{Q}_{n;i,j-1}| + j|\mathcal{Q}_{n;i-1,j}| + 2(2n+4-i-2j)|\mathcal{Q}_{n;i-1,j-1}|.\end{aligned}$$

It is clear that $\mathcal{Q}_{1;1,1} = \{11\}$ and $\mathcal{Q}_{1,i,j} = \emptyset$ for $(i,j) \neq (1,1)$. So, $\gamma_{1;1,1} = 1 = |\mathcal{Q}_{1;1,1}|$ and $\gamma_{1;i,j} = 0 = |\mathcal{Q}_{1,i,j}|$ for $(i,j) \neq (1,1)$. By induction, we get $|\mathcal{Q}_{n+1;i,j}| = \gamma_{n+1,i,j}$ and this completes the proof. □

5. Legendre-Stirling permutations

5.1. **Basic definitions and notation.**

The Legendre-Stirling numbers of the second kind $\mathrm{LS}\,(n,k)$ first arose in the study of a certain differential operator related to Legendre polynomials (see [16]). The numbers $\mathrm{LS}\,(n,k)$ can be defined as follows:

$$x^n = \sum_{j=0}^{n} \mathrm{LS}\,(n,k) \prod_{i=0}^{k-1}(x - i(i+1)),$$

and satisfy the recurrence relation $\mathrm{LS}\,(n,k) = \mathrm{LS}\,(n-1,k-1) + k(k+1)\mathrm{LS}\,(n-1,k)$, with the initial conditions $\mathrm{LS}\,(0,0) = 1$ and $\mathrm{LS}\,(0,k) = 0$ for $k \geq 1$. Andrews and Littlejohn [3] discovered that $\mathrm{LS}\,(n,k)$ is the number of Legendre-Stirling set partitions of the set $\{1,1,2,2,\ldots,n,n\}$ into k blocks. Subsequently, Egge [15] considered the polynomial $L_k(x)$ defined by

$$\sum_{n=0}^{\infty} \mathrm{LS}\,(n+k,n)x^n = \frac{L_k(x)}{(1-x)^{3k+1}},$$

and found that $L_k(x)$ is the descent polynomial of Legendre-Stirling permutations of order k. The reader is referred to [2] for further properties of the Legendre-Stirling numbers.

For $n \geq 1$, let N_n denote the multiset $\{1, 1, \overline{1}, 2, 2, \overline{2}, \ldots, n, n, \overline{n}\}$, in which we have two unbarred copies and one barred copy of each integer i, where $1 \leq i \leq n$. In this section, we always assume that the elements of N_n are ordered by $\overline{1} = 1 < \overline{2} = 2 < \cdots < \overline{n} = n$. Here the $\overline{k} = k$ means that $\overline{k}k$ count as a plateau.

A *Legendre-Stirling permutation* of order n is a permutation of N_n such that if $i < j < k$, π_i and π_k are both unbarred and $\pi_i = \pi_k$, then $\pi_j > \pi_i$. Let LS_n denote the set of Legendre-Stirling permutations of order n. Let $\pi = \pi_1\pi_2 \cdots \pi_{3n} \in \mathrm{LS}_n$, and we always set $\pi_0 = \pi_{3n+1} = 0$. An index i is a *descent* (resp. *ascent*, *plateau*) of π if $\pi_i > \pi_{i+1}$ (resp. $\pi_i < \pi_{i+1}$, $\pi_i = \pi_{i+1}$). Hence the index $i = 1$ is always an ascent and $i = 3k$ is always a descent. Denote by $\mathrm{des}\,(\pi)$ (resp. $\mathrm{asc}\,(\pi)$, $\mathrm{plat}\,(\pi)$) the number of descents (resp. ascents, plateaus) of π. Let $L_n(x) = \sum_{\pi \in \mathrm{LS}_n} x^{\mathrm{des}\,(\pi)}$. The first few terms of $L_n(x)$ are given as follows:

$$L_1(x) = 2x, L_2(x) = 4x + 24x^2 + 12x^3, L_3(x) = 8x + 240x^2 + 984x^3 + 864x^4 + 144x^5.$$

Let LSD_n be the set of Legendre-Stirling permutations of the multiset $ND_n = N_n \setminus \{n, n\}$, i.e., $ND_n = \{1, 1, \overline{1}, 2, 2, \overline{2}, \ldots, n-1, n-1, \overline{n-1}, \overline{n}\}$.

5.2. Main results.

For $n \geq 1$, we define

$$H_n(x, y, z) = \sum_{\pi \in \mathrm{LSD}_n} x^{\mathrm{asc}\,(\pi)-1} y^{\mathrm{des}\,(\pi)-1} z^{\mathrm{plat}\,(\pi)},$$
$$L_n(x, y, z) = \sum_{\pi \in \mathrm{LS}_n} x^{\mathrm{asc}\,(\pi)} y^{\mathrm{des}\,(\pi)} z^{\mathrm{plat}\,(\pi)}.$$

Lemma 11. *Let $A = \{u, v, x, y, z\}$ and*

$$G_1 = \{x \to uv, y \to uv, z \to uv\}, \tag{17}$$

$$G_2 = \{x \to \frac{x^2y^2z}{uv}, y \to \frac{x^2y^2z}{uv}, z \to \frac{x^2y^2z}{uv}, u \to \frac{xyz^2}{v}, v \to \frac{xyz^2}{u}\}. \tag{18}$$

Then for $n \geq 1$, we have

$$D_1(D_2D_1)^{n-1}(x) = uvH_n(x, y, z),\ (D_2D_1)^n(x) = L_n(x, y, z),$$

where D_2D_1 is a composition operation, i.e., $(D_2D_1)^n(x) = D_2\left(D_1\left((D_2D_1)^{n-1}(x)\right)\right)$.

Proof. Note that every permutations in LS_n can be obtained from a permutation in LS_{n-1} by first inserting $\overline{n}$ between two entries, and then inserting the pair nn between two entries of this new permutation. We first introduce a grammatical labeling of $\pi \in \mathrm{LSD}_n$ as follows:

- (L_1) Put a superscript label u immediately before the entry $\overline{n}$ and a superscript label v right after $\overline{n}$;
- (L_2) If i is an ascent and $\pi_{i+1} \neq \overline{n}$, then put a superscript label x right after π_i;
- (L_3) If i is a descent and $\pi_i \neq \overline{n}$, then put a superscript label y right after π_i;
- (L_4) If i is a plateau, then put a superscript label z right after π_i.

Thus, the weight of $\pi \in \mathrm{LSD}_n$ is given by $w(\pi) = uvx^{\mathrm{asc}(\pi)-1}y^{\mathrm{des}(\pi)-1}z^{\mathrm{plat}(\pi)}$. We then introduce a grammatical labeling of $\pi \in \mathrm{LS}_n$ as follows:

(L_1) If i is an ascent, then put a superscript label x right after π_i;
(L_2) If i is a descent, then put a superscript label y right after π_i;
(L_3) If i is a plateau, then put a superscript label z right after π_i.

Thus, the weight of $\pi \in \mathrm{LS}_n$ is given by $w(\pi) = x^{\mathrm{asc}(\pi)}y^{\mathrm{des}(\pi)}z^{\mathrm{plat}(\pi)}$.

We proceed by induction on n. When $n = 1$, we have $\mathrm{LSD}_1 = \{{}^u\overline{1}^v\}$, $\mathrm{LS}_1 = \{{}^x\overline{1}^z1^z1^y, {}^x1^z1^z\overline{1}^y\}$. Note that $D_1(x) = uv, (D_2D_1)(x) = D_2(D_1(x)) = D_2(uv) = 2xyz^2$. Then the weight of $\overline{1}$ is given by $D_1(x)$, and the sum of weights of the elements in LS_1 is given by $(D_2D_1)(x)$. Hence the results hold for $n = 1$. Suppose we get all labeled permutations in LS_{n-1}, where $n \geq 2$. Let π' be obtained from $\pi \in \mathrm{LS}_{n-1}$ by inserting the entry $\overline{n}$ to a position with a label x, y or z. The changes of labeling are illustrated as follows:

$$\cdots \pi_i^x\pi_{i+1}\cdots \mapsto \cdots \pi_i^u\overline{n}^v\pi_{i+1}\cdots,$$

$$\cdots \pi_i^y\pi_{i+1}\cdots \mapsto \cdots \pi_i^u\overline{n}^v\pi_{i+1}\cdots,$$

$$\cdots \pi_i^z\pi_{i+1}\cdots \mapsto \cdots \pi_i^u\overline{n}^v\pi_{i+1}\cdots.$$

In each case, the insertion of $\overline{n}$ corresponds to the operator D_1 defined by (17). Let $\widehat{\pi}$ be obtained from $\pi \in \mathrm{LSD}_n$ by inserting the pair nn. We distinguish the following cases:

(c_1) If nn is inserted at a position with the label x, y or z, then the changes of labeling can be illustrated as follows:

$$\cdots{}^u\,\overline{n}^v\cdots\pi_i^x\pi_{i+1}\cdots \mapsto \cdots{}^x\,\overline{n}^y\cdots\pi_i^xn^zn^y\pi_{i+1}\cdots;$$

$$\cdots{}^u\,\overline{n}^v\cdots\pi_i^y\pi_{i+1}\cdots \mapsto \cdots{}^x\,\overline{n}^y\cdots\pi_i^xn^zn^y\pi_{i+1}\cdots;$$

$$\cdots{}^u\,\overline{n}^v\cdots\pi_i^z\pi_{i+1}\cdots \mapsto \cdots{}^x\,\overline{n}^y\cdots\pi_i^xn^zn^y\pi_{i+1}\cdots;$$

(c_2) If nn is inserted at a position with the label u, then the change of labeling is illustrated as follows: $\cdots{}^u\,\overline{n}^v\cdots \mapsto \cdots{}^x\,n^zn^z\overline{n}^y\cdots$.

(c_2) If nn is inserted at a position with the label v, then the change of labeling is illustrated as follows: $\cdots{}^u\,\overline{n}^v\cdots \mapsto \cdots{}^x\,\overline{n}^zn^zn^y\cdots$.

In each case, the insertion of the pair nn corresponds to the operator D_2 defined by (18). It is routine to check that the action of D_2D_1 on Legendre-Stirling permutations of LS_{n-1} generates all the Legendre-Stirling permutations of LS_n. This completes the proof. □

We can now present the third main result of this paper.

Theorem 12. *For $n \geq 1$, we have*

$$H_n(x,y,z) = \sum_{i=1}^{2n-2} z^i \sum_{j=0}^{\lfloor(3n-3-i)/2\rfloor} h_n(i,j)(xy)^j(x+y)^{3n-3-i-2j},$$

$$L_n(x,y,z) = \sum_{i=1}^{2n} z^i \sum_{j=1}^{\lfloor(3n+1-i)/2\rfloor} \ell_n(i,j)(xy)^j(x+y)^{3n+1-i-2j},$$

where the numbers $h_n(i,j)$ and $\ell_n(i,j)$ satisfy the recurrence relations

$$\ell_n(i,j) = 2h_n(i-2,j-1) + ih_n(i,j-2) + (j-1)h_n(i-1,j-1)+$$
$$2(3n+2-i-2j)h_n(i-1,j-2),$$
$$h_{n+1}(i.j) = (i+1)\ell_n(i+1,j) + (j+1)\ell_n(i,j+1) + 2(3n+1-i-2j)\ell_n(i,j),$$

with the initial conditions $h_1(0,0) = 1$ and $h_1(i,j) = 0$ for $(i,j) \neq (0,0)$, $\ell_1(2,1) = 2$ and $\ell_1(i,j) = 0$ for $(i,j) \neq (2,1)$.

Proof. Consider the grammars (17) and (18). Setting $a = x+y, b = xy$, we get

$$D_1(a) = 2uv, D_1(b) = auv, D_2(x) = \frac{zb^2}{uv}, D_2(y) = \frac{zb^2}{uv}, D_2(z) = \frac{zb^2}{uv},$$
$$D_2(u) = \frac{z^2b}{v}, D_2(v) = \frac{z^2b}{u}, D_2(a) = \frac{2zb^2}{uv}, D_2(b) = \frac{zab^2}{uv}.$$

Then the change of grammars are given as follows: $A = \{a,b,x,y,z,u,v\}$ and

$$G_3 = \{x \to uv, z \to uv, a \to 2uv, b \to auv\}, \tag{19}$$
$$G_4 = \{x \to \frac{zb^2}{uv}, y \to \frac{zb^2}{uv}, z \to \frac{zb^2}{uv}, u \to \frac{z^2b}{v}, v \to \frac{z^2b}{u}, a \to \frac{2zb^2}{uv}, b \to \frac{zab^2}{uv}\}. \tag{20}$$

It is routine to verify that there exist nonnegative integers $h_n(i,j)$ and $\ell_n(i,j)$ such that

$$D_3(D_4D_3)^{n-1}(x) = uv\sum_{i=1}^{2n-2} z^i \sum_{j=0}^{\lfloor (3n-3-i)/2\rfloor} h_n(i,j)b^ja^{3n-3-i-2j},$$
$$(D_4D_3)^n(x) = \sum_{i=1}^{2n} z^i \sum_{j=1}^{\lfloor (3n+1-i)/2\rfloor} \ell_n(i,j)b^ja^{3n+1-i-2j}.$$

Then upon taking $a = x+y$ and $b = xy$, we get the expansions of $H_n(x,y,z)$ and $L_n(x,y,z)$. From

$$D_4(D_3(D_4D_3)^{n-1}(x)) = D_4\left(\sum_{i,j} h_n(i,j)uvz^ib^ja^{3n-3-i-2j}\right)$$
$$= \sum_{i,j} h_n(i,j)(2z^{i+2}b^{j+1}a^{3n-3-i-2j} + iz^ib^{j+2}a^{3n-3-i-2j})+$$
$$\sum_{i,j} h_n(i,j)(jz^{i+1}b^{j+1}a^{3n-2-i-2j} + 2(3n-3-i-2j)z^{i+1}b^{j+2}a^{3n-4-i-2j}),$$

and

$$D_3((D_4D_3)^n(x)) = D_3\left(\sum_{i,j} \ell_n(i,j)z^ib^ja^{3n+1-i-2j}\right)$$
$$= uv\sum_{i,j} \ell_n(i,j)iz^{i-1}b^ja^{3n+1-i-2j} + uv\sum_{i,j} \ell_n(i,j)jz^ib^{j-1}a^{3n+2-i-2j}+$$
$$uv\sum_{i,j} \ell_n(i,j)2(3n+1-i-2j)z^ib^ja^{3n-i-2j},$$

we get the desired recurrence relations. In particular, $D_3(x) = uv, D_4D_3(x) = 2z^2b, D_3(2z^2b) = 4zbuv + 2z^2auv$. Thus, $h_1(0,0) = 1, h_2(1,1) = 4, h_2(2,0) = 2$ and $\ell_1(2,1) = 2$. This completes the proof. □

Using (19) and (20), it is not hard to verify that $\ell_n(i,j)$ satisfy the recurrence relation

$$\begin{aligned}\ell_{n+1}(i,j) = {} & i(i+1)\ell_n(i+1,j-2) + 2i(j-1)\ell_n(i,j-1) + j(j-1)\ell_n(i-1,j) + \\ & 2j\ell_n(i-2,j) + 4i(3n+5-i-2j)\ell_n(i,j-2) + 4(3n+5-i-2j)\ell(i-2,j-1) + \\ & 4(3n+6-i-2j)(3n+5-i-2j)\ell_n(i-1,j-2) + \\ & 2((2j-2)(3n+4-i-2j)+i+j-2)\ell_n(i-1,j-1),\end{aligned}$$

with the initial conditions $\ell_1(2,1) = 2$ and $\ell_1(i,j) = 0$ for $(i,j) \neq (2,1)$.

We define

$$h_n(x,y) = \sum_{i=1}^{2n-2} \sum_{j=0}^{\lfloor (3n-3-i)/2 \rfloor} h_n(i,j)x^iy^j, \ \ell_n(x,y) = \sum_{i=1}^{2n} \sum_{j=1}^{\lfloor (3n+1-i)/2 \rfloor} \ell_n(i,j)x^iy^j.$$

Using Theorem 12, multiplying both sides of the recurrence relations of $h_n(i,j)$ and $\ell_n(i,j)$ by x^iy^j and summing over all i, j, we get that

$$\ell_n(x,y) = xy(6ny-6y+2x)h_n(x,y) + xy^2(1-2x)\frac{\partial}{\partial x}h_n(x,y) + xy^2(1-4y)\frac{\partial}{\partial y}h_n(x,y),$$

$$h_{n+1}(x,y) = (6n+2)\ell_n(x,y) + (1-2x)\frac{\partial}{\partial x}\ell_n(x,y) + (1-4y)\frac{\partial}{\partial y}\ell_n(x,y).$$

The first few terms of the polynomials $h_n(x,y)$ and $\ell_n(x,y)$ are given as follows:

$$\begin{aligned}h_1(x,y) &= 1, \\ \ell_1(x,y) &= 2x^2y, \\ h_2(x,y) &= 4xy + 2x^2, \\ \ell_2(x,y) &= 4xy^3 + 8x^2y^2 + 12x^3y^2 + 4x^4y, \\ h_3(x,y) &= 4y^3 + 28xy^2 + 16x^2y + 52x^2y^2 + 40x^3y + 8x^4y + 4x^4.\end{aligned}$$

6. Jacobi-Stirling permutations

6.1. Definitions and notation.

The Jacobi-Stirling numbers $\mathrm{JS}\,(n,k;z)$ were discovered as a result of a problem involving the spectral theory of powers of the classical second-order Jacobi differential expression (see [3, 17]), and they can be defined as follows:

$$x^n = \sum_{k=0}^{n} \mathrm{JS}\,(n,k;z) \prod_{i=0}^{k-1}(x - i(z+i)).$$

In particular, $\mathrm{JS}\,(n,k;1) = \mathrm{LS}\,(n,k)$. The reader is referred to Andrews et al. [1] for further properties of the Jacobi-Stirling numbers. The Jacobi-Stirling polynomial of the second kind is defined by $f_k(n;z) = \mathrm{JS}\,(n+k, n; z)$. The coefficient $p_{k,i}(n)$ of z^i in $f_k(n;z)$ is called the Jacobi-Stirling coefficient of the second for $0 \le i \le k$. Gessel, Lin and Zeng [22] found a combinatorial interpretation of the polynomial $A_{k,i}(x)$ defined by

$$\sum_{n\ge 0} p_{k,i}(n)x^n = \frac{A_{k,i}(x)}{(1-x)^{3k-i+1}}.$$

Define the multiset $M_k = \{\overline{1}, 1, 1, \overline{2}, 2, 2, \ldots, \overline{k}, k, k\}$, in which we have two unbarred copies and one barred copy of each integer i, where $1 \le i \le k$. In this section, we always assume that the elements of M_k are ordered by

$$\overline{1} < 1 < \overline{2} < 2 < \cdots < \overline{k} < k.$$

A permutation of M_k is a *Jacobi-Stirling permutation* if for each i, $1 \le i \le k$, all entries between the two occurrences of the unbarred i are larger than i. Let JSP_k denote the set of Jacobi-Stirling permutations of M_k. For example, $\mathrm{JSP}_1 = \{\overline{1}11, 11\overline{1}\}$. Let $\pi = \pi_1\pi_2\cdots\pi_{3k} \in \mathrm{JSP}_k$. As usual, we always set $\pi_0 = \pi_{3k+1} = 0$. In the same way as in Legendre-Stirling permutation, we define

$$\begin{aligned}
\mathrm{des}\,(\pi) &= \#\{i \in [3n] \mid \pi_i > \pi_{i+1}\},\\
\mathrm{asc}\,(\pi) &= \#\{i \in \{0,1,2,\ldots,3n-1\} \mid \pi_i < \pi_{i+1}\},\\
\mathrm{plat}\,(\pi) &= \#\{i \in [3n-1] \mid \pi_i = \pi_{i+1}\}.
\end{aligned}$$

It follow from [22, Theorem 2] that

$$(1-x)^{3k+1}\sum_{n\ge 0} p_{k,0}(n)x^n = \sum_{\pi\in \mathrm{JSP}_k} x^{\mathrm{des}\,(\pi)}.$$

6.2. **Main results.**

Define

$$S_n(x,y,z) = \sum_{\pi\in\mathrm{JSP}_n} x^{\mathrm{asc}\,(\pi)} y^{\mathrm{des}\,(\pi)} z^{\mathrm{plat}\,(\pi)}.$$

The first few terms of $S_n(x,y,z)$ are given as follows:

$$\begin{aligned}
S_1(x,y,z) &= xy(x+y)z,\\
S_2(x,y,z) &= (xy)^2(3x^2+10xy+3y^2)z + xy(x^3+11x^2y+11xy^2+y^3)z^2,\\
S_3(x,y,z) &= (xy)^3(17x^3+119x^2y+119xy^2+17y^3)z+\\
&\quad (xy)^2(18x^4+284x^3y+644x^2y^2+284xy^2+18y^4)z^2+\\
&\quad (xy)(x^5+57x^4y+302x^3y^2+302x^2y^3+57xy^4+y^5)z^3.
\end{aligned}$$

Lemma 13. *Let* $A = \{x, y, z\}$ *and*

$$G_1 = \{x \to xy, y \to xy, z \to xy\},\ G_2 = \{x \to xyz, y \to xyz, z \to xyz\}. \tag{21}$$

Then for $n \ge 1$*, we have* $(D_2D_1)^n(x) = (D_2D_1)^n(y) = (D_2D_1)^n(z) = S_n(x,y,z)$.

Proof. We first introduce a grammatical labeling of $\pi \in \mathrm{JSP}_n$ as follows:

(L_1) If i is an ascent, then put a superscript label x right after π_i;

(L_2) If i is a descent, then put a superscript label y right after π_i;

(L_3) If i is a plateau, then put a superscript label z right after π_i.

Note that the weight of π is given by $w(\pi) = x^{\operatorname{asc}(\pi)} y^{\operatorname{des}(\pi)} z^{\operatorname{plat}(\pi)}$.

We proceed by induction on n. For $n = 1$, we have $\mathrm{JSP}_1 = \{{}^x\overline{1}^x 1^z 1^y,\ {}^x 1^z 1^y \overline{1}^y\}$. Note that

$$(D_2D_1)(x) = (D_2D_1)(y) = (D_2D_1)(z) = xy^2z + x^2yz.$$

Hence the result holds for $n = 1$. Note that any permutation of JSP_n is obtained from a permutation of JSP_{n-1} by first inserting the element $\overline{n}$ and then inserting the pair nn.

We first insert $\overline{n}$ and the changes of labeling are illustrated as follows:

$$\cdots \pi_i^x \pi_{i+1} \cdots \mapsto \cdots \pi_i^x \overline{n}^y \pi_{i+1} \cdots;$$
$$\cdots \pi_i^y \pi_{i+1} \cdots \mapsto \cdots \pi_i^x \overline{n}^y \pi_{i+1} \cdots;$$
$$\cdots \pi_i^z \pi_{i+1} \cdots \mapsto \cdots \pi_i^x \overline{n}^y \pi_{i+1} \cdots.$$

In each case, the insertion of $\overline{n}$ corresponds to the operator D_1. We then insert the pair nn and the changes of labeling are illustrated as follows:

$$\cdots \pi_i^x \pi_{i+1} \cdots \mapsto \cdots \pi_i^x n^z n^y \pi_{i+1} \cdots;$$
$$\cdots \pi_i^y \pi_{i+1} \cdots \mapsto \cdots \pi_i^x n^z n^y \pi_{i+1} \cdots;$$
$$\cdots \pi_i^z \pi_{i+1} \cdots \mapsto \cdots \pi_i^x n^z n^y \pi_{i+1} \cdots.$$

In each case, the insertion of the pair nn corresponds to the operator D_2. It is easy to check that the action of D_2D_1 on Jacobi-Stirling permutations of JSP_{n-1} generates all te Jacobi-Stirling permutations of JSP_n. This completes the proof. □

We can now present the fouth main result of this paper.

Theorem 14. *For $n \geq 1$, we have*

$$S_n(x, y, z) = \sum_{i=1}^{n} z^i \sum_{j=1}^{\lfloor (3n+1-i)/2 \rfloor} s_n(i, j)(xy)^j (x + y)^{3n+1-i-2j}, \tag{22}$$

where the numbers $s_n(i, j)$ satisfy the recurrence relation

$$\begin{aligned} s_{n+1}(i, j) = {} & i(i + 1)s_n(i + 1, j - 2) + i(2j - 1)s_n(i, j - 1) + 4i(3n + 5 - i - 2j)s_n(i, j - 2)+ \\ & j^2 s_n(i - 1, j) + (4(j - 1)(3n + 4 - i - 2j) + 6n + 6 - 2i - 2j)s_n(i - 1, j - 1)+ \\ & 4(3n + 6 - i - 2j)(3n + 5 - i - 2j)s_n(i - 1, j - 2), \end{aligned}$$

with the initial conditions $s_0(1, 0) = 1$ and $s_0(i, j) = 0$ for $(i, j) \neq (1, 0)$.

Proof. Consider the grammars (21). Setting $a = z, b = x + y, c = xy$, we get $D_1(a) = c, D_1(b) = 2c, D_1(c) = bc$ and $D_2(a) = ac, D_2(b) = 2ac, D_2(c) = abc$. Then the change of grammars are given as follows: $A = \{a, b, c\}$ and

$$G_3 = \{a \to c, b \to 2c, c \to bc\},\ G_4 = \{a \to ac, b \to 2ac, c \to abc\}. \tag{23}$$

Thus, we have $(D_4D_3)^n(a) = \sum_{i,j,k} T_n(i, j, k)a^i b^j c^k$. This expansion can be written in the form

$$(D_4D_3)^n(z) = \sum_{i,j,k} T_n(i, j, k)z^i (x + y)^j (xy)^k.$$

By Lemma 13, we see that the degree of each term of $\sum_{i,j,k} T_n(i,j,k) z^i (x+y)^j (xy)^k$ is $3n+1$ and $1 \leq \deg(z) \leq n$ is from 1 to n. Thus, we can set $s_n(i,j) = T_n(i,j,k)$ and write $(D_4D_3)^n(a)$ as follows:

$$(D_4D_3)^n(a) = \sum_{i=1}^{n} \sum_{j=0}^{\lfloor (3n+1-i)/2 \rfloor} s_n(i,j) a^i c^j b^{3n+1-i-2j}. \tag{24}$$

Then upon taking $a = z, b = x+y$ and $c = xy$, we get (22). It follows from (23) that $s_n(i,j) \geq 0$. In particular, since $D_4D_3(a) = D_4(c) = abc$ and $(D_4D_3)^2(a) = a(3b^2c^2 + 4c^3) + a^2(b^3c + 8bc^2)$, we have $s_1(1,1) = 1, s_2(1,2) = 3, s_2(1,3) = 4, s_2(2,1) = 1, s_2(2,2) = 8$. For convenience, set $k = 3n+1-i-2j$. Note that $D_3(D_4D_3)^n(a) = \sum_{i,j} s_n(i,j)(ia^{i-1}c^{j+1}b^k + ja^ic^jb^{k+1} + 2ka^ic^{j+1}b^{k-1})$. It follows that

$$\begin{aligned}
&D_4\left(D_3(D_4D_3)^n(a)\right) \\
&= D_4\left(\sum_{i,j} s_n(i,j)(ia^{i-1}c^{j+1}b^k + ja^ic^jb^{k+1} + 2ka^ic^{j+1}b^{k-1})\right) \\
&= \sum_{i,j} s_n(i,j)\left(i(i-1)a^{i-1}c^{j+2}b^k + i(j+1)a^ic^{j+1}b^{k+1} + 2ika^ic^{j+2}b^{k-1}\right) + \\
&\sum_{i,j} s_n(i,j)\left(ija^ic^{j+1}b^{k+1} + j^2a^{i+1}c^jb^{k+2} + 2j(k+1)a^{i+1}c^{j+1}b^k\right) + \\
&\sum_{i,j} s_n(i,j)\left(2ika^ic^{j+2}b^{k-1} + 2(j+1)ka^{i+1}c^{j+1}b^k + 4k(k-1)a^{i+1}c^{j+2}b^{k-2}\right).
\end{aligned}$$

On the other hand, $(D_4D_3)^{n+1}(a) = \sum_{i,j} s_{n+1}(i,j)a^ic^jb^{k+3}$. Comparing the coefficient of $a^ic^jb^{k+3}$ in both sides of $(D_4D_3)^{n+1}(a) = D_4\left(D_3(D_4D_3)^n(a)\right)$, we get the desired recurrence relation. □

Define $\mathrm{JSP}_{n,k} = \{\pi \in \mathrm{JSP}_n : \mathrm{plat}(\pi) = k\}$. Let $\vartheta(\pi)$ be the permutation obtained from $\pi \in \mathrm{JSP}_n$ by deleting all of the first unbarred i from left to right, where $i \in [n]$. For example, $\vartheta(1331\overline{1}\ \overline{2}2442\overline{4}\ \overline{3}) = 31\overline{1}\ \overline{2}42\overline{4}\ \overline{3}$. Let $\widehat{\mathrm{JSP}}_{n,n} = \{\vartheta(\pi) : \pi \in \mathrm{JSP}_{n,n}\}$. Note that $\#\widehat{\mathrm{JSP}}_{n,n} = (2n)!$. Then ϑ is a bijection from $\mathrm{JSP}_{n,n}$ to $\mathfrak{S}_{2n}$. Therefore,

$$\sum_{j=1}^{\lfloor (2n+1)/2 \rfloor} s_n(n,j)(xy)^j(x+y)^{2n+1-2j} = \sum_{\pi \in \mathfrak{S}_{2n}} x^{\mathrm{des}(\pi)+1} y^{\mathrm{asc}(\pi)+1}.$$

Let JSPD_n denote the set of Jacobi-Stirling permutations of the multiset $MD_n = M_n \setminus \{n,n\}$. In particular, $\mathrm{JSPD}_1 = \{\overline{1}\}$ and $\mathrm{JSPD}_2 = \{\overline{2}\ \overline{1}11, \overline{1}\ \overline{2}11, \overline{1}1\overline{2}1, \overline{1}11\overline{2}, \overline{2}11\overline{1}, 1\overline{2}1\overline{1}, 11\overline{2}\ \overline{1}, 11\overline{1}\ \overline{2}\}$. We define

$$T_n(x,y,z) = \sum_{\pi \in \mathrm{JSPD}_n} x^{\mathrm{asc}(\pi)} y^{\mathrm{des}(\pi)} z^{\mathrm{plat}(\pi)}.$$

Along the same lines of the proof of Theorem 14, one can derive that for $n \geq 1$,

$$D_1(D_2D_1)^{n-1}(x) = D_1(D_2D_1)^{n-1}(y) = D_1(D_2D_1)^{n-1}(z) = T_n(x,y,z),$$

where

$$T_n(x,y,z) = \sum_{i=0}^{n-1} z^i \sum_{j=1}^{\lfloor (3n-1-i)/2 \rfloor} t_n(i,j)(xy)^j(x+y)^{3n-1-i-2j}.$$

In particular,

$$\sum_{j=1}^{\lfloor 2n-1\rfloor/2} t_n(n-1,j)(xy)^j(x+y)^{2n-2j}=\sum_{\pi\in\mathfrak{S}_{2n-1}} x^{\operatorname{des}(\pi)+1}y^{\operatorname{asc}(\pi)+1}.$$

Define

$$s_n(x,y)=\sum_{i=1}^{n}\sum_{j=0}^{\lfloor(3n+1-i)/2\rfloor} s_n(i,j)x^iy^j,$$

$$t_n(x,y)=\sum_{i=0}^{n-1}\sum_{j=1}^{\lfloor(3n-1-i)/2\rfloor} t_n(i,j)x^iy^j.$$

It follows from (23) that

$$D_3(D_4D_3)^{n-1}(a)=\sum_{i=0}^{n-1}\sum_{j=1}^{\lfloor(3n-1-i)/2\rfloor} t_n(i,j)a^ic^jb^{3n-1-i-2j}. \tag{25}$$

Combining (24) and (25), we get the following result.

Proposition 15. *For $n\geq 1$, the numbers $s_n(i,j)$ and $t_n(i,j)$ satisfy the recurrence relation*

$$s_n(i,j)=it_n(i,j-1)+jt_n(i-1,j)+2(3n+2-i-2j)t_n(i-1,j-1),$$
$$t_{n+1}(i,j)=(i+1)s_n(i+1,j-1)+js_n(i,j)+2(3n+3-i-2j)s_n(i,j-1),$$

with the initial conditions $t_1(0,1)=1$ and $t_1(i,j)=0$ for $(i,j)\neq(0,1)$. Equivalently, the polynomials $s_n(x,y)$ and $t_n(x,y)$ satisfy the recurrence relatoion

$$s_n(x,y)=2(3n-1)xyt_n(x,y)+xy(1-2x)\frac{\partial}{\partial x}t_n(x,y)+xy(1-4y)\frac{\partial}{\partial y}t_n(x,y),$$
$$t_{n+1}(x,y)=2(3n+1)ys_n(x,y)+y(1-2x)\frac{\partial}{\partial x}s_n(x,y)+y(1-4y)\frac{\partial}{\partial y}s_n(x,y),$$

with the initial condition $t_1(x,y)=y$.

The first few terms of the polynomials $s_n(x,y)$ and $t_n(x,y)$ are given as follows:

$$s_1(x,y)=xy,$$
$$t_2(x,y)=xy+2xy^2+y^2,$$
$$s_2(x,y)=x^2y+8x^2y^2+3xy^2+4xy^3,$$
$$t_3(x,y)=x^2y+22x^2y^2+16x^2y^3+8xy^2+3y^3+40xy^3+4y^4.$$

6.3. **Partial γ-coefficients and a modified Foata-Strehl's group action.**

For the grammars (23), notice that the insertion of $\overline{n}$ corresponds to the operator D_3, and the insertion of the pair nn corresponds to the operator D_4. Figure 1 provides a diagram of the grammars (23). Using this diagram, we discover some statistics on Jacobi-Stirling permutations, and then we can present combinatorial interpretations of the numbers $s_n(i,j)$ and $t_n(i,j)$.

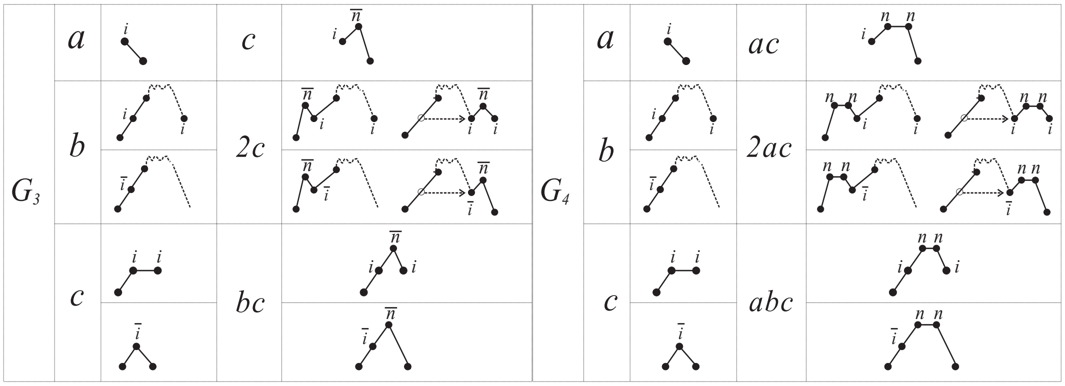

Figure 1.

Let $\pi = \pi_1\pi_2\cdots\pi_{3n} \in \mathrm{JSP}_n$. As usual, set $\pi_0 = \pi_{3n+1} = 0$. An *unbarred descent* of π is an index $i \in [3n]$ such that $\pi_i > \pi_{i+1}$ and π_i is unbarred. A *double ascent* (resp. *peak*, *left ascent-plateau*) of π is an index i such that $\pi_{i-1} < \pi_i < \pi_{i+1}$ (resp. $\pi_{i-1} < \pi_i > \pi_{i+1}$, $\pi_{i-1} < \pi_i = \pi_{i+1}$), where $i \in [3n-1]$. It is clear that if i is a peak, then π_i is barred. A *barred double descent* of π is an index $i \in [3n]$ such that $\pi_{i-1} > \pi_i > \pi_{i+1}$ and π_i is barred. A *descent plateau* of π is an index i such that $\pi_{i-1} > \pi_i = \pi_{i+1}$. In the same way, we define the same statistics on JSPD_n.

Let

$$\begin{aligned} \mathrm{ubdes}\,(\pi) &= \#\{i \mid \pi_i > \pi_{i+1},\ \pi_i \text{ is unbarred}\},\\ \mathrm{dasc}\,(\pi) &= \#\{i \mid \pi_{i-1} < \pi_i < \pi_{i+1}\},\\ \mathrm{expk}\,(\pi) &= \#\{i \mid \pi_{i-1} < \pi_i > \pi_{i+1} \text{ or } \pi_{i-1} < \pi_i = \pi_{i+1}\},\\ \mathrm{bddes}\,(\pi) &= \#\{i \mid \pi_{i-1} > \pi_i > \pi_{i+1},\ \pi_i \text{ is barred}\},\\ \mathrm{desp}\,(\pi) &= \#\{i \mid \pi_{i-1} > \pi_i = \pi_{i+1}\}. \end{aligned}$$

We can now state the following result.

Theorem 16. *For $n \geq 1$, we have*

$$s_n(i,j) = \#\{\pi \in \mathrm{JSP}_n : \mathrm{ubdes}\,(\pi) = i, \mathrm{expk}\,(\pi) = j, \mathrm{bddes}\,(\pi) = 0, \mathrm{desp}\,(\pi) = 0\},$$

$$t_n(i,j) = \#\{\pi \in \mathrm{JSPD}_n : \mathrm{ubdes}\,(\pi) = i, \mathrm{expk}\,(\pi) = j, \mathrm{bddes}\,(\pi) = 0, \mathrm{desp}\,(\pi) = 0\}.$$

Given a permutation $\pi \in \mathrm{JSPD}_n$. For any $k \in \{0, 1, \ldots, 3n-2\}$, let $\theta_{n,k}(\pi)$ be the permutation in JSP_n obtained from π by inserting the pair nn between π_k and π_{k+1}, and let $\psi_{\overline{n}}(\pi)$ denote the permutation in JSP_{n-1} obtained from π by deleting the entry $\overline{n}$.

Given a permutation $\pi \in \mathrm{JSP}_n$. For any $k \in \{0, 1, \ldots, 3n\}$, let $\theta_{\overline{n+1},k}(\pi)$ be the permutation in JSPD_{n+1} obtained from π by inserting $\overline{n+1}$ between π_k and π_{k+1}, and let $\psi_n(\pi)$ denote the permutation in JSPD_n obtained from π by deleting the pair nn.

We define

$$\mathrm{JSP}_{n;i,j} = \{\pi \in \mathrm{JSP}_n : \mathrm{ubdes}\,(\pi) = i, \mathrm{expk}\,(\pi) = j, \mathrm{bddes}\,(\pi) = \mathrm{desp}\,(\pi) = 0\},$$
$$\mathrm{JSPD}_{n;i,j} = \{\pi \in \mathrm{JSPD}_n : \mathrm{ubdes}\,(\pi) = i, \mathrm{expk}\,(\pi) = j, \mathrm{bddes}\,(\pi) = \mathrm{desp}\,(\pi) = 0\}.$$

In order to prove Theorem 16, we need two lemmas.

Lemma 17. *For $n \geq 1$, we have*

$$|\mathrm{JSPD}_{n+1;i,j}| = (i+1)|\mathrm{JSP}_{n;i+1,j-1}| + j|\mathrm{JSP}_{n;i,j}| + 2(3n+3-i-2j)|\mathrm{JSP}_{n;i,j-1}|.$$

Proof. We define

$$\begin{aligned} \mathrm{Ubdes}(\pi) &= \{k \mid \pi_k > \pi_{k+1},\ \pi_k \text{ is unbarred}\},\\ \mathrm{Expk}(\pi) &= \{k \mid \pi_{k-1} < \pi_k > \pi_{k+1} \text{ or } \pi_{k-1} < \pi_k = \pi_{k+1}\},\\ \mathrm{Dasc}(\pi) &= \{k \mid \pi_{k-1} < \pi_k < \pi_{k+1}\}. \end{aligned}$$

For $\pi \in \mathrm{JSPD}_{n;i,j}$, we have $|\mathrm{Ubdes}(\pi)| + 2|\mathrm{Expk}(\pi)| + |\mathrm{Dasc}(\pi)| = 3n-1$, since $\mathrm{bddes}(\pi) = \mathrm{desp}(\pi) = 0$.

For any $\pi \in \mathrm{JSPD}_{n+1;i,j}$, let $r = r(\pi)$ be the index such that $\pi_r = \overline{n+1}$. We now partition the set $\mathrm{JSPD}_{n+1;i,j}$ into the following six subsets:

$$\begin{aligned} \mathrm{JSPD}^1_{n+1;i,j} &= \{\pi \in \mathrm{JSPD}_{n+1;i,j} \mid \pi_{r-1} > \pi_{r+1},\ \pi_{r-1} \text{ is unbarred}\},\\ \mathrm{JSPD}^2_{n+1;i,j} &= \{\pi \in \mathrm{JSPD}_{n+1;i,j} \mid \pi_{r-2} < \pi_{r-1} > \pi_{r+1},\ \pi_{r-1} \text{ is barred}\},\\ \mathrm{JSPD}^3_{n+1;i,j} &= \{\pi \in \mathrm{JSPD}_{n+1;i,j} \mid \pi_{r-2} < \pi_{r-1} = \pi_{r+1}\},\\ \mathrm{JSPD}^4_{n+1;i,j} &= \{\pi \in \mathrm{JSPD}_{n+1;i,j} \mid \pi_{r-1} < \pi_{r+1} < \pi_{r+2}\},\\ \mathrm{JSPD}^5_{n+1;i,j} &= \{\pi \in \mathrm{JSPD}_{n+1;i,j} \mid \pi_{r-2} > \pi_{r-1} = \pi_{r+1}\},\\ \mathrm{JSPD}^6_{n+1;i,j} &= \{\pi \in \mathrm{JSPD}_{n+1;i,j} \mid \pi_{r-2} > \pi_{r-1} > \pi_{r+1},\ \pi_{r-1} \text{ is barred}\}. \end{aligned}$$

Claim 1. There is a bijection

$$\phi_1 : \mathrm{JSPD}^1_{n+1;i,j} \mapsto \{(\pi, k) \mid \pi \in \mathrm{JSP}_{n;i+1,j-1} \text{ and } k \in \mathrm{Ubdes}(\sigma)\}.$$

For any $\pi \in \mathrm{JSPD}^1_{n+1;i,j}$, notice that $\psi_{\overline{n+1}}(\pi) \in \mathrm{JSP}_{n;i+1,j-1}$ and $r(\pi)-1 \in \mathrm{Ubdes}(\psi_{\overline{n+1}}(\pi))$. Thus, we define the map ϕ_1 by letting $\phi_1(\pi) = (\psi_{\overline{n+1}}(\pi), r(\pi)-1)$. Then the inverse of ϕ_1 is given by $\phi_1^{-1}(\pi, k) = \theta_{\overline{n+1},k}(\pi)$.

Claim 2. There is a bijection

$$\phi_2 : \mathrm{JSPD}^2_{n+1;i,j} \cup \mathrm{JSPD}^3_{n+1;i,j} \mapsto \{(\sigma, k) \mid \sigma \in \mathrm{JSP}_{n;i,j} \text{ and } k \in \mathrm{Expk}(\pi)\}.$$

For any $\pi \in \mathrm{JSPD}^2_{n+1;i,j} \cup \mathrm{JSPD}^3_{n+1;i,j}$, notice that $\psi_{\overline{n+1}}(\pi) \in \mathrm{JSP}_{n;i,j}$ and $r(\sigma)-1 \in \mathrm{Expk}(\psi_{\overline{n+1}}(\sigma))$. Thus, we define the map ϕ_2 by letting $\phi_2(\pi) = (\psi_{\overline{n+1}}(\pi), r(\pi)-1)$. Then the inverse of ϕ_2 is given by $\phi_2^{-1}(\pi, k) = \theta_{\overline{n+1},k}(\pi)$.

Claim 3. There is a bijection $\phi_3 : \mathrm{JSPD}^4_{n+1;i,j} \mapsto \{(\pi, k) \mid \pi \in \mathrm{JSP}_{n;i,j-1} \text{ and } k \in \mathrm{Dasc}(\pi)\}$.

For any $\pi \in \mathrm{JSPD}^4_{n+1;i,j}$, notice that $\psi_{\overline{n+1}}(\pi) \in \mathrm{JSP}_{n;i,j-1}$ and $r(\pi) \in \mathrm{Dasc}(\psi_{\overline{n+1}}(\pi))$. Thus, we define the map ϕ_3 by letting $\phi_3(\pi) = (\psi_{\overline{n+1}}(\pi), r(\pi))$. Then the inverse of ϕ_3 is given by $\phi_3^{-1}(\pi, k) = \theta_{\overline{n+1},k-1}(\pi)$.

Claim 4. There is a bijection

$$\phi_4 : \mathrm{JSPD}^5_{n+1;i,j} \cup \mathrm{JSPD}^6_{n+1;i,j} \mapsto \{(\pi, k) \mid \pi \in \mathrm{JSP}_{n;i,j-1} \text{ and } k \in \mathrm{Dasc}(\pi)\}.$$

Let $k \in \{0, 1, \ldots, 3n+1\}$ and let $\pi = \pi_1\pi_2\ldots\pi_{3n+1} \in \mathrm{JSPD}_{n+1}$. We define a modified Foata-Strehl's group action φ_k as follows:

- If k is a double ascent, then $\varphi_k(\pi)$ is obtained from π by deleting π_k and then inserting π_k immediately before the integer π_j, where $j=\min\{s\in\{k+1,k+2,\ldots,3n+2\}\mid \pi_s\leq\pi_k\}$;
- If k satisfies either (i) it is a descent-plateau or (ii) it is a double descent and π_k is barred, then $\varphi_k(\pi)$ is obtained from π by deleting π_k and then inserting π_k right after the integer π_j, where $j=\max\{s\in\{0,1,2,\ldots,k-1\}:\pi_s<\pi_k\}$.

For any $\pi\in\mathrm{JSPD}^{\,5}_{n+1;i,j}$, notice that the index $r(\pi)-1$ is the unique descent-plateau of $\psi_{\overline{n+1}}(\pi)$ and $\varphi_{r(\pi)-1}\circ\psi_{\overline{n+1}}(\pi)\in\mathrm{JSP}_{\,n;i,j-1}$. Read $\varphi_{r(\pi)-1}\circ\psi_{\overline{n+1}}(\pi)$ from left to right and let p be the index of the first occurrence of the integer $\pi_{r(\pi)-1}$. Then $p\in\mathrm{Dasc}\,(\varphi_{r(\pi)-1}\circ\psi_{\overline{n+1}}(\pi))$.

For any $\pi\in\mathrm{JSPD}^{\,6}_{n+1;i,j}$, notice that $\pi_{r(\pi)-1}$ has a bar and the index $r(\pi)-1$ is a double-descent in $\psi_{\overline{n+1}}(\pi)$, and $\varphi_{r(\pi)-1}\circ\psi_{\overline{n+1}}(\pi)\in\mathrm{JSP}_{\,n;i,j-1}$. Read $\varphi_{r(\pi)-1}\circ\psi_{\overline{n+1}}(\pi)$ from left to right and let p be the index of the occurrence of the integer $\pi_{r(\pi)-1}$. Then

$$p\in\mathrm{Dasc}\,(\varphi_{r(\pi)-1}\circ\psi_{\overline{n+1}}(\pi)).$$

Therefore, we define the map ϕ_4 by letting $\phi_4(\sigma)=(\varphi_{r(\pi)-1}\circ\psi_{\overline{n+1}}(\pi),p)$, and the inverse of ϕ_4 is given by $\phi_4^{-1}(\pi,k)=\theta_{\overline{n+1},r-1}\circ\varphi_k(\pi)$, where $r-1$ is the unique descent-plateau or barred double descent of $\varphi_k(\pi)$.

In conclusion, we get that

$$\begin{aligned}|\mathrm{JSPD}_{\,n+1;i,j}|&=|\mathrm{JSPD}^{\,1}_{n+1;i,j}|+|\mathrm{JSPD}^{\,2}_{n+1;i,j}|+|\mathrm{JSPD}^{\,3}_{n+1;i,j}|+\\&\quad|\mathrm{JSPD}^{\,4}_{n+1;i,j}|+|\mathrm{JSPD}^{\,5}_{n+1;i,j}|+|\mathrm{JSPD}^{\,6}_{n+1;i,j}|\\&=(i+1)|\mathrm{JSP}_{\,n;i+1,j-1}|+j|\mathrm{JSP}_{\,n;i,j}|+2(3n+3-i-2j)|\mathrm{JSP}_{\,n;i,j-1}|.\end{aligned}$$

This completes the proof. □

Lemma 18. *For $n\geq 1$, we have*

$$|\mathrm{JSP}_{\,n;i,j}|=i|\mathrm{JSPD}_{\,n;i,j-1}|+j|\mathrm{JSPD}_{\,n;i-1,j}|+2(3n+2-i-2j)|\mathrm{JSPD}_{\,n;i-1,j-1}|.$$

Proof. For any $\pi\in\mathrm{JSP}_{\,n;i,j}$, we have $|\mathrm{Ubdes}\,(\pi)|+2|\mathrm{Expk}\,(\pi)|+|\mathrm{Dasc}\,(\pi)|=3n+1$, since $\mathrm{bddes}\,(\pi)=\mathrm{desp}\,(\pi)=0$. Let $r=r(\pi)$ be the index of the first occurrence of the entry n, i.e., $\pi_r=\pi_{r+1}=n$. We partition the set $\mathrm{JSP}_{\,n;i,j}$ into the following six subsets:

$$\begin{aligned}\mathrm{JSP}^{\,1}_{n;i,j}&=\{\pi\in\mathrm{JSP}_{\,n;i,j}\mid \pi_{r-1}>\pi_{r+2},\ \pi_{r-1}\text{ is unbarred}\}\\\mathrm{JSP}^{\,2}_{n;i,j}&=\{\pi\in\mathrm{JSP}_{\,n;i,j}\mid \pi_{r-2}<\pi_{r-1}>\pi_{r+2},\ \pi_{r-1}\text{ is barred}\}\\\mathrm{JSP}^{\,3}_{n;i,j}&=\{\pi\in\mathrm{JSP}_{\,n;i,j}\mid \pi_{r-2}<\pi_{r-1}=\pi_{r+2}\}\\\mathrm{JSP}^{\,4}_{n;i,j}&=\{\pi\in\mathrm{JSP}_{\,n;i,j}\mid \pi_{r-1}<\pi_{r+2}<\pi_{r+3}\}\\\mathrm{JSP}^{\,5}_{n;i,j}&=\{\pi\in\mathrm{JSP}_{\,n;i,j}\mid \pi_{r-2}>\pi_{r-1}=\pi_{r+2}\}\\\mathrm{JSP}^{\,6}_{n;i,j}&=\{\pi\in\mathrm{JSP}_{\,n;i,j}\mid \pi_{r-2}>\pi_{r-1}>\pi_{r+2},\ \pi_{r-1}\text{ is barred}\}.\end{aligned}$$

Claim 1. There is a bijection $\Phi_1:\mathrm{JSP}^{\,1}_{n;i,j}\mapsto\{(\pi,k)\mid \pi\in\mathrm{JSPD}_{\,n;i,j-1}\text{ and }k\in\mathrm{Ubdes}\,(\sigma)\}$.

For any $\pi\in\mathrm{JSP}^{\,1}_{n;i,j}$, notice that $\psi_n(\pi)\in\mathrm{JSPD}_{\,n;i,j-1}$ and $(r(\pi)-1)\in\mathrm{Ubdes}\,(\psi_n(\pi))$. Thus, we define the map Φ_1 by letting $\Phi_1(\pi)=(\psi_n(\pi),r(\pi)-1)$. Then the inverse of Φ_1 is given by $\Phi_1^{-1}(\pi,k)=\theta_{n,k}(\pi)$.

Claim 2. There is a bijection

$$\Phi_2 : \mathrm{JSP}^{\,2}_{n;i,j} \cup \mathrm{JSP}^{\,3}_{n;i,j} \mapsto \{(\pi,k) \mid \pi \in \mathrm{JSPD}_{\,n;i-1,j} \text{ and } k \in \mathrm{Expk}\,(\pi)\}.$$

For any $\pi \in \mathrm{JSP}^{\,2}_{n;i,j} \cup \mathrm{JSP}^{\,3}_{n;i,j}$, notice that $\psi_n(\pi) \in \mathrm{JSPD}_{\,n;i-1,j}$ and $r(\pi)-1 \in \mathrm{Expk}\,(\psi_n(\pi))$. Thus, we define the map Ψ_2 by letting $\Psi_2(\pi) = (\psi_n(\pi), r(\pi)-1)$. Then the inverse of Φ_2 is given by $\Phi_2^{-1}(\pi,k) = \theta_{n,k}(\pi)$.

Claim 3. There is a bijection $\Phi_3 : \mathrm{JSP}^{\,4}_{n;i,j} \mapsto \{(\pi,k) \mid \pi \in \mathrm{JSPD}_{\,n;i-1,j-1} \text{ and } k \in \mathrm{Dasc}\,(\pi)\}$.

For any $\pi \in \mathrm{JSP}^{\,4}_{n;i,j}$, notice that $\psi_n(\pi) \in \mathrm{JSPD}_{\,n;i-1,j-1}$ and $r(\pi) \in \mathrm{Dasc}\,(\psi_n(\pi))$. Thus, we define the map Φ_3 by letting $\Phi_3(\pi) = (\psi_n(\pi), r(\pi))$. Then the inverse of Φ_3 is given by

$$\Phi_3^{-1}(\pi,k) = \theta_{n,k-1}(\pi).$$

Claim 4. There is a bijection

$$\Phi_4 : \mathrm{JSP}^{\,5}_{n;i,j} \cup \mathrm{JSP}^{\,6}_{n;i,j} \mapsto \{(\pi,k) \mid \pi \in \mathrm{JSPD}_{\,n;i-1,j-1} \text{ and } k \in \mathrm{Dasc}\,(\pi)\}.$$

Let $k \in \{0,1,\ldots,3n\}$ and let $\pi = \pi_1\pi_2\ldots\pi_{3n} \in \mathrm{JSP}_{\,n}$. We define a modified Foata-Strehl's group action φ_k as follows:

- If k is a double ascent then $\varphi_k(\pi)$ is obtained from π by deleting π_k and then inserting π_k immediately before the integer π_j, where $j = \min\{s \in \{k+1, k+2, \ldots, 3n+1\} : \pi_s \le \pi_k\}$;
- If k satisfies either (i) it is a descent-plateau or (ii) it is a double descent and π_k is barred, then $\varphi_k(\pi)$ is obtained from π by deleting π_k and then inserting π_k right after the integer π_j, where $j = \max\{s \in \{0,1,2,\ldots,k-1\} : \pi_s < \pi_k\}$.

For any $\pi \in \mathrm{JSP}^{\,5}_{n;i,j}$, note that the index $r(\pi)-1$ is the unique descent-plateau in $\psi_n(\pi)$ and

$$\varphi_{r(\pi)-1} \circ \psi_n(\pi) \in \mathrm{JSPD}_{\,n;i-1,j-1}.$$

Read the permutation $\varphi_{r(\pi)-1} \circ \psi_n(\pi)$ from left to right and let p be the index of the first occurrence of the integer $\pi_{r(\pi)-1}$. Then $p \in \mathrm{Dasc}\,(\varphi_{r(\pi)-1} \circ \psi_n(\pi))$.

For any $\pi \in \mathrm{JSP}^{\,6}_{n;i,j}$, notice that $\pi_{r(\pi)-1}$ is barred and the index $r(\pi)-1$ is a double-descent of $\psi_n(\pi)$, and $\varphi_{r(\pi)-1} \circ \psi_n(\pi) \in \mathrm{JSPD}_{\,n;i-1,j-1}$. Read $\varphi_{r(\pi)-1} \circ \psi_n(\pi)$ from left to right and let p be the index of the first occurrence of the integer $\pi_{r(\pi)-1}$. Then $p \in \mathrm{Dasc}\,(\varphi_{r(\pi)-1} \circ \psi_n(\pi))$.

Therefore, we define the map Φ_4 by letting $\Phi_4(\pi) = (\varphi_{r(\pi)-1} \circ \psi_n(\pi), p)$. Then the inverse of Φ_4 is given by $\Phi_4^{-1}(\pi,k) = \theta_{n,r-1} \circ \varphi_k(\pi)$, where $r-1$ is the unique descent-plateau or barred double descent of $\varphi_k(\pi)$.

In conclusion, we get that

$$\begin{aligned}|\mathrm{JSP}_{\,n;i,j}| &= |\mathrm{JSP}^{\,1}_{n;i,j}| + |\mathrm{JSP}^{\,2}_{n;i,j}| + |\mathrm{JSP}^{\,3}_{n;i,j}| + |\mathrm{JSP}^{\,4}_{n;i,j}| + |\mathrm{JSP}^{\,5}_{n;i,j}| + |\mathrm{JSP}^{\,6}_{n;i,j}| \\ &= i|\mathrm{JSPD}_{\,n+1;i,j-1}| + j|\mathrm{JSPD}_{\,n;i-1,j}| + 2(3n+2-i-2j)|\mathrm{JSPD}_{\,n;i-1,j-1}|.\end{aligned}$$

This complete the proof. □

A proof Theorem 16:

Proof. Notice that $\mathrm{JSPD}_{\,1;0,1} = \{\bar{1}\}$ and $\mathrm{JSP}_{\,1;1,1} = \{\bar{1}11\}$. Moreover, $\mathrm{JSPD}_{\,1;i,j} = \emptyset$ for any $(i,j) \neq (0,1)$ and $\mathrm{JSP}_{\,1;i,j} = \emptyset$ for any $(i,j) \neq (1,1)$. So,

$$t_1(0,1) = 1 = |\mathrm{JSPD}_{\,1;0,1}| \text{ and } s_1(1,1) = 1 = |\mathrm{JSP}_{\,1;1,1}|.$$

Combining Proposition 15, Lemma 17 and Lemma 18, we obtain

$$\begin{aligned}|\mathrm{JSP}_{n;i,j}| &= i|\mathrm{JSPD}_{n;i,j-1}| + j|\mathrm{JSPD}_{n;i-1,j}| + 2(3n+2-i-2j)|\mathrm{JSPD}_{n;i-1,j-1}| \\ &= it_n(i,j-1) + jt_n(i-1,j) + 2(3n+2-i-2j)t_n(i-1,j-1) \\ &= s_n(i,j),\end{aligned}$$

$$\begin{aligned}|\mathrm{JSPD}_{n+1;i,j}| &= (i+1)|\mathrm{JSP}_{n;i+1,j-1}| + j|\mathrm{JSP}_{n;i,j}| + 2(3n+3-i-2j)|\mathrm{JSP}_{n;i,j-1}| \\ &= (i+1)s_n(i+1,j-1) + js_n(i,j) + 2(3n+3-i-2j)s_n(i,j-1) \\ &= t_{n+1}(i,j).\end{aligned}$$

This complete the proof. □

Let $[\overline{k}] = \{\overline{1}, \overline{2}, \ldots, \overline{k}\}$. For any subset $S \subseteq [\overline{k}]$, let $M_{k,S} = M_k \setminus S$. Denote by $\mathrm{JSP}_{k,S}$ the set of Jacobi-Stirling permutations of $M_{k,S}$. Let

$$\mathrm{JSP}_{k,i} = \bigcup_{\substack{S \subseteq [\overline{k}] \\ |S|=i}} \mathrm{JSP}_{k,S}.$$

We define

$$\mathrm{JSP}_{k,i}(x,y,z) = \sum_{\pi \in \mathrm{JSP}_{k,i}} x^{\mathrm{asc}\,(\pi)} y^{\mathrm{des}\,(\pi)} z^{\mathrm{plat}\,(\pi)}.$$

It is clear that

$$\begin{aligned}\mathrm{JSP}_{k,k}(x,y,z) &= \sum_{\pi \in \mathcal{Q}_k} x^{\mathrm{asc}\,(\pi)} y^{\mathrm{des}\,(\pi)} z^{\mathrm{plat}\,(\pi)}, \\ \mathrm{JSP}_{k,0}(x,y,z) &= \sum_{\pi \in \mathrm{JSP}_k} x^{\mathrm{asc}\,(\pi)} y^{\mathrm{des}\,(\pi)} z^{\mathrm{plat}\,(\pi)}.\end{aligned}$$

Based on empirical evidence, we propose the following conjecture.

Conjecture 19. *For any $k \geq 1$ and $1 \leq i \leq k-1$, the polynomial $\mathrm{JSP}_{k,i}(x,y,z)$ is a partial γ-positive polynomial.*

7. Concluding remarks

In this paper, we introduce the change of grammars method and we show that it is an effective method for studying γ-positivity and partial γ-positivity. Along the same lines, one may study multivariate extensions of asymmetric polynomials, such as multivariate orthogonal polynomials. Since there is a bijection between Stirling permutations and perfect matchings (see [29] for instance), it would be interesting to study the partial γ-positivity of multivariate polynomials of perfect matchings. Recall that a perfect matching of $[2n]$ can be seen as a fixed-point free involution in $\mathfrak{S}_{2n}$. Let I_n be the set of all involutions in $\mathfrak{S}_n$ and $I_n(x) = \sum_{\pi \in \mathrm{I}_n} x^{\mathrm{des}\,(\pi)}$. It is now well known that $I_n(x)$ is symmetric and unimodal (see [6, 24] for instance). Let

$$I_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a(n,k) x^k (1+x)^{n-1-2k}.$$

Guo and Zeng [24] conjectured that $a(n,k) \geq 0$. This conjecture still open. We end our paper by proposing the following.

Problem 20. *Is there a statistic* st *on* I_n *that makes the polynomial* $\sum_{\pi\in\mathrm{I}_n} x^{\mathrm{des}\,(\pi)}y^{\mathrm{asc}\,(\pi)}z^{\mathrm{st}\,(\pi)}$ *a partial* γ*-positive polynomial?*

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