# Relationships between solid spherical and toroidal harmonics

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We derive new relationships expressing solid spherical harmonics as series of toroidal harmonics and vice versa. The expansions include regular and irregular spherical harmonics, ring and axial toroidal harmonics of even and odd parity about the plane of the torus. The expansion coefficients are given in terms of a recurrence relation. As an example application we apply one of the expansions to express the potential of a charged conducting torus on a basis of spherical harmonics.

arXiv:1802.03484v2 [math-ph] 25 Feb 2018

## I. Introduction

Laplace's equation in spherical coordinates has been well documented, while in toroidal coordinates it has only been investigated significanlty recently. The Laplacian is only partially separable in toroidal coordinates, so their corresponding solutions, solid[1] toroidal harmonics difficult to apply to problems even involving the torus. Toroidal harmonics have been applied to the express the magnetic field around a superconducting torus [1], the field of a magnetized torus [2], and low frequency electromagnetic or acoustic scattering of a point charge or dipole near a torus [3] [4]. Spherical harmonics, but not toroidal harmonics, have been used to express the gravitational potential of a solid torus [5]. The orientation of ring washer with a spherical colloid in the center in a uniform external field has been investigated [6]; a combination of spherical and toroidal harmonics would be ideal to study this arrangement. Expansions of toroidal harmonics in terms of spherical harmonics are well known for degree 0, as are the expansions of some low degree (0 and 1) spherical harmonics in terms of toroidal harmonics. In this document we derive new expansions which cover all degrees and orders, including the inverse expansions.

Toroidal coordinates and harmonics are closely related to bispherical coordinates; the coordinates essentially differ by a real/imaginary interchange of the focal distance, and the basis functions differ by a shift of the separation constant by $\pm 1/2$. So the formulae relating spherical/bispherical harmonics (which also appear to be unknown) could possibly be derived by making substitutions to the spherical/toroidal expansions, but here we just focus on toroidal harmonics.

## II. Preliminaries

### A. Toroidal coordinates

First define spherical and cylindrical coordinates[2]:

$$r=\sqrt{x^2+y^2+z^2},\qquad \rho=\sqrt{x^2+y^2},\qquad \theta=\mathrm{acos}\frac{z}{r},\qquad \phi=\mathrm{atan2}(y,x) \tag{1}$$

Then toroidal coordinates $(\eta,\sigma,\phi)$ with a focal ring radius $a$ are defined as

$$\eta=\frac{1}{2}\log\frac{(\rho+a)^2+z^2}{(\rho-a)^2+z^2},\qquad \sigma=\mathrm{sign}(z)\mathrm{acos}\frac{r^2-a^2}{\sqrt{(r^2+a^2)^2-4\rho^2a^2}}, \tag{2}$$

with ranges $\eta\in[0,\infty)$, $\sigma\in[-\pi,\pi]$. $d_-$ is the distance to the closest side of the focal ring and $d_+$ is the distance to the farthest side of the ring. $\eta$ corresponds to the torus size and $\sigma$ to the angle around the minor axis. For convenience we also define:

$$\beta=\cosh\eta=\frac{r^2+a^2}{\sqrt{(r^2+a^2)^2-4\rho^2a^2}}=\frac{\chi}{\sqrt{\chi^2-1}} \tag{3}$$

$$\chi=\coth\eta=\frac{r^2+a^2}{2\rho a}=\frac{\beta}{\sqrt{\beta^2-1}} \tag{4}$$

With ranges $\beta\in[1,\infty)$, $\chi\in[1,\infty)$.

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[1] "Solid" means the full solution to Laplace's equation. For example, solid spherical harmonics include the radial part. We will generally omit this term and it should be assumed that all harmonic functions mentioned here are solid.

[2] atan2 is a similar to the arctangent but provides correct results in all four quadrants of $x$ and $y$.

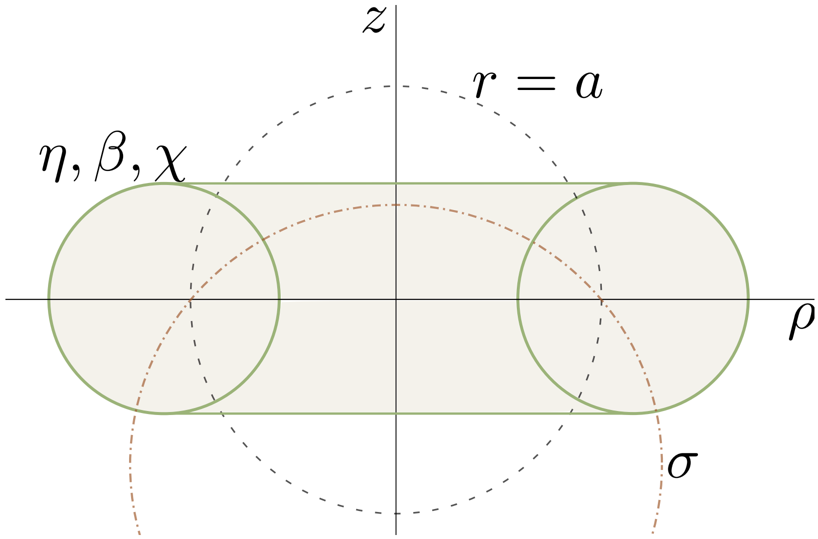


FIG. 1. Schematic of the coordinates used. $\eta, \beta, \chi$ all define the torus size, and $\sigma$ relates to the angle around the minor axis. The sphere at $r = a$ defines the boundary of convergence of the expansions of toroidal harmonics on a basis of spherical harmonics.

### B. Toroidal harmonics

Laplace's equation is partially separable in toroidal coordinates, meaning that solutions can be written as a product of functions of each coordinate but must also be multiplied by a coordinate-dependent prefactor. There are two variations of toroidal harmonics that essentially differ by normalization. We will present relationships between spherical harmonics and both types of toroidal harmonics for completeness. What we will call the 'standard' toroidal harmonics are more commonly used, while the 'alternative' toroidal harmonics were first used explicitly in 2006 [7]. These are:

$$\text{'standard' harmonics:}\quad \Delta \begin{Bmatrix} P^m_{n-1/2}(\beta) \\ Q^m_{n-1/2}(\beta) \end{Bmatrix} \begin{Bmatrix} \cos n\sigma \\ \sin n\sigma \end{Bmatrix} e^{\pm im\phi}, \qquad \Delta = \sqrt{2(\beta - \cos\sigma)} \tag{5}$$

$$\text{'alternate' harmonics:}\quad \sqrt{\frac{a}{\rho}} \begin{Bmatrix} P^n_{m-1/2}(\chi) \\ Q^n_{m-1/2}(\chi) \end{Bmatrix} \begin{Bmatrix} \cos n\sigma \\ \sin n\sigma \end{Bmatrix} e^{\pm im\phi}, \tag{6}$$

where the curly braces indicate any linear combination of their interior functions. $P^m_{n-1/2}(\beta)$ and $Q^m_{n-1/2}(\beta)$ are Legendre functions of half-integer degree, also called toroidal functions; see appendix for computation. Note the interchange of the indicies of the Legendre functions. We will use the term 'ring harmonics' for the ones containing $P^m_{n-1/2}(\beta)$ or $Q^n_{m-1/2}(\chi)$ as they are singular on the focal ring, and 'axial harmonics' for the ones containing $Q^m_{n-1/2}(\beta)$ or $P^n_{m-1/2}(\chi)$ as they are singular on the entire $z$ axis.

The standard and alternate harmonics are in fact identical up to a prefactor of $n$ and $m$. This can be seen from the Whipple formulae, which relate the Legendre functions $P$ and $Q$ of half integer degree. Expressed in toroidal coordinates the Whipple formulae are

$$\Delta P^m_{n-1/2}(\beta) = \frac{(-)^n 2/\sqrt{\pi}}{\Gamma(n-m+\frac{1}{2})} \sqrt{\frac{a}{\rho}} Q^n_{m-1/2}(\chi) \tag{7}$$

$$\Delta Q^m_{n-1/2}(\beta) = \frac{(-)^n \pi\sqrt{\pi}}{\Gamma(n-m+\frac{1}{2})} \sqrt{\frac{a}{\rho}} P^n_{m-1/2}(\chi). \tag{8}$$

Note for half integer arguments

$$\Gamma(n+\frac{1}{2}) = \sqrt{\pi}\frac{(2n-1)!!}{2^n}, \qquad \Gamma(-n+\frac{1}{2}) = \sqrt{\pi}\frac{(-2)^n}{(2n-1)!!} \qquad (n \geq 0). \tag{9}$$

### C. Green's function expansions

We will utilize three different expansions of the inverse distance in deriving the relationships between spherical and toroidal harmonics.

For points $\mathbf{r}_1$ and $\mathbf{r}_2$ with $r_1 < r_2$ , the spherical harmonic expansion of Green's function is

$$\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|} = \sum_{m=0}^{\infty} \epsilon_m (-)^m \sum_{n=m}^{\infty} \frac{r_1^n}{r_2^{n+1}} P_n^m(u_1) P_n^{-m}(u_2) \cos m(\phi_1-\phi_2), \qquad \epsilon_m = \begin{cases} 1 & m=0 \\ 2 & m>0 \end{cases} \tag{10}$$

with $u = \cos\theta$. Note that $P_n^{-m} = (-)^m \frac{(n-m)!}{(n+m)!} P_n^m$.
In terms of toroidal harmonics for $\beta_1 < \beta_2$ (point 2 closer to the focal ring)[8] :

$$\begin{aligned} \frac{1}{|\mathbf{r}_1-\mathbf{r}_2|} &= \frac{\Delta_1\Delta_2}{2\pi a} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} P_{n-1/2}^m(\beta_1) Q_{n-1/2}^{-m}(\beta_2) \exp[in(\sigma_1-\sigma_2) + im(\phi_1-\phi_2)] \\ &= \frac{\Delta_1\Delta_2}{2\pi a} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \epsilon_n \epsilon_m P_{n-1/2}^m(\beta_1) Q_{n-1/2}^{-m}(\beta_2) \cos n(\sigma_1-\sigma_2) \cos m(\phi_1-\phi_2). \end{aligned} \tag{11}$$

Also we have the 'cylindrical' expansion which converges in all space[9]:

$$\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|} = \frac{1}{\pi\sqrt{\rho_1\rho_2}} \sum_{m=0}^{\infty} \epsilon_m Q_{m-1/2}(\bar{\chi}) \cos m(\phi_1-\phi_2), \qquad \bar{\chi} = \frac{\rho_1^2+\rho_2^2+(z_1-z_2)^2}{2\rho_1\rho_2} \tag{12}$$

Which converges for all $\mathbf{r}_1 \neq \mathbf{r}_2$.

## III. Relationships between spherical and toroidal harmonics

### A. Expansions of toroidal harmonics

In our derivation it is more straightforward to first relate the spherical harmonics to the alternate toroidal harmonics, then use the Whipple formulae to transform the identities to relate to the standard toroidal harmonics. We omit the $\cos/\sin m\phi$ from all formulae as they can be tagged on at the end.
We can find the expansion of the ring harmonics (singular on the ring) with $n = 0$, in terms of spherical harmonics by equating the different expansions of Green's function.
Evaluating the spherical and cylindrical expansions of Green's function, (10) and (12), both at $\rho_2 = a, z_2 = 0$ ($\Rightarrow u_2 = 0$, $r_2 = a$, $\bar{\chi} = \chi$), and equating each $m^{th}$ term in the sum we have

$$\sqrt{\frac{a}{\rho}} Q_{m-1/2}(\chi) = \pi \sum_{k=m}^{\infty} (-)^m P_k^{-m}(0) \left(\frac{r}{a}\right)^k P_k^m(u) \qquad r<a \tag{13}$$

$$\text{with } P_k^{-m}(0) = \begin{cases} (-)^{(k+m)/2} \dfrac{(k-m-1)!!}{(k+m)!!} & k+m \text{ even} \\ 0 & k+m \text{ odd} \end{cases} \tag{14}$$

In this document $P_k^m(u)$ are defined without the phase $(-)^m$.
Similarly the expansion for $r > a$ can be found similarly by setting $\rho_1 = a, z_1 = 0$ in (10) and (12):

$$\sqrt{\frac{a}{\rho}} Q_{m-1/2}(\chi) = \pi \sum_{k=m}^{\infty} (-)^m P_k^{-m}(0) \left(\frac{a}{r}\right)^{k+1} P_k^m(u) \qquad r>a \tag{15}$$

These series only contain terms with $k + m$ and are therefore symmetric about $z$. Note that $\sqrt{\frac{a}{\rho}} Q_{m-1/2}(\chi)\cos(m\phi)$ is the potential of a thin ring with sinusoidal charge distribution. [3]
The result (14) has been generalized to the Helmholtz equation (harmonic time dependence), as a spherical wave function expansion of a circular ring with current distribution expressed as a Fourier series [10].
The $n = 1$ toroidal harmonics can be obtained from these $n = 0$ expansions by applying the differential operators $\partial_z$ and $r\partial_r$, which both preserve harmonicity. The $n = 0$ functions are the potential of a thin ring of charge with

[3] Note that there are two solutions for the $\sigma$ dependence as it is a second order differential equation, but the toroidal functions containing $\sin\sigma$) for $n = 0$ are zero because $\sin 0\sigma = 0$. Actually the second independent solution for $n = 0$ has $\sigma$ dependence of just $\sigma$, but this cannot be used as it has a discontinuity in space at $\sigma = 0, \pi$.

an azimuthal charge density of $\cos(m\phi)$, but with no variation in the $z$ or $\rho$ directions. We would naturally expect the $n=1$ functions to be the potential of a ring of dipoles pointing in the $z$ or $\rho$ directions. Differentiating a point charge in the $z$ direction produces a dipole along $z$, so applying $\partial_z$ to a ring will produce a ring of dipoles. We use the following derivatives:

$$a\frac{\partial\chi}{\partial z} = \sqrt{\chi^2-1}\sin\sigma; \qquad \frac{\mathrm{d}Q^n_{m-1/2}(\chi)}{\mathrm{d}\chi} = \frac{Q^{n+1}_{m-1/2}(\chi)}{\sqrt{\chi^2-1}} + \frac{n\chi}{\chi^2-1}Q^n_{m-1/2}(\chi), \tag{16}$$

and that $\partial_z$ is also a ladder operator for the spherical harmonics:

$$a\frac{\partial}{\partial z}\left[\left(\frac{r}{a}\right)^n P^m_n(u)\right] = (n+m)\left(\frac{r}{a}\right)^{n-1} P^m_{n-1}(u) \tag{17}$$

$$a\frac{\partial}{\partial z}\left[\left(\frac{a}{r}\right)^{n+1} P^m_n(u)\right] = -(n-m+1)\left(\frac{r}{a}\right)^{n+2} P^m_{n+1}(u) \tag{18}$$

Then applying $a\partial_z$ to the toroidal harmonic expansion for $r<a$ (14):

$$a\frac{\partial}{\partial z}\sqrt{\frac{a}{\rho}}Q_{m-1/2}(\chi) = \sqrt{\frac{a}{\rho}}Q^1_{m-1/2}(\chi)\sin\sigma = \pi\sum_{k=m}^{\infty}(-)^m P^{-m}_k(0)(k+m)\left(\frac{r}{a}\right)^{k-1} P^m_{k-1}(u) \tag{19}$$

$$= \pi\sum_{k=m}^{\infty}(-)^m P^{-m}_{k+1}(0)(k+m+1)\left(\frac{r}{a}\right)^{n} P^m_k(u) \tag{20}$$

And for $r>a$:

$$\sqrt{\frac{a}{\rho}}Q^1_{m-1/2}(\chi)\sin\sigma = -\pi\sum_{k=m}^{\infty}(-)^m P^{-m}_k(0)(k-m+1)\left(\frac{a}{r}\right)^{k+2} P^m_{k+1}(u) \tag{21}$$

$$= \pi\sum_{k=m}^{\infty}(-)^m P^{-m}_{k+1}(0)(k+m+1)\left(\frac{a}{r}\right)^{k+1} P^m_k(u) \tag{22}$$

These series only contain terms with $n+m$ odd, and are therefore antisymmetric about $z$.
To produce toroidal harmonics with dipoles oriented outwards from the origin, we can apply $r\partial_r$ ($\partial_\rho$ cannot be used as it doesn't preserve harmonicity). This operator should turn a ring of charge on the $xy$ plane into a ring of dipoles pointing inward. We use the following derivatives:

$$r\frac{\partial\chi}{\partial r} = \sqrt{\chi^2-1}\cos\sigma; \qquad r\frac{\partial}{\partial r}\sqrt{\frac{a}{\rho}} = \frac{-1}{2}\sqrt{\frac{a}{\rho}} \tag{23}$$

Applying $r\partial_r$ to the $n=0$ toroidal harmonic expansion for $r<a$ (14) and rearranging:

$$r\frac{\partial}{\partial r}\sqrt{\frac{a}{\rho}}Q_{m-1/2}(\chi) = \sqrt{\frac{a}{\rho}}\left[Q^1_{m-1/2}(\chi)\cos\sigma - \frac{1}{2}Q_{m-1/2}(\chi)\right] = \sum_{n=m}^{\infty}(-)^m P^{-m}_n(0)n\left(\frac{r}{a}\right)^n P^m_n(u) \tag{24}$$

$$\Rightarrow\sqrt{\frac{a}{\rho}}Q^1_{m-1/2}(\chi)\cos\sigma = \sum_{n=m}^{\infty}(-)^m P^{-m}_n(0)\left(n+\frac{1}{2}\right)\left(\frac{r}{a}\right)^n P^m_n(u). \tag{25}$$

And for $r>a$ we find:

$$\sqrt{\frac{a}{\rho}}Q^1_{m-1/2}(\chi)\cos\sigma = -\sum_{n=m}^{\infty}(-)^m P^{-m}_n(0)\left(n+\frac{1}{2}\right)\left(\frac{a}{r}\right)^{n+1} P^m_n(u). \tag{26}$$

It is interesting that for $m=0$, this series starts from $n=0$ which means that the toroidal harmonic has a monopole moment - its corresponding charge distribution has a net charge.

Now we derive the formulae for general $n$ by induction, applying $r\partial_r$ to the $n^{th}$ harmonic (with $\sin n\sigma$ or $\cos n\sigma$)to derive the expansion for $n+1$.
We use the following formulae:

$$r\frac{\partial}{\partial r}\cos n\sigma = \frac{n\chi}{\sqrt{\chi^2-1}}\sin\sigma\sin n\sigma, \qquad r\frac{\partial}{\partial r}\sin n\sigma = \frac{-n\chi}{\sqrt{\chi^2-1}}\sin\sigma\cos n\sigma, \tag{27}$$

$$2\cos\sigma\cos n\sigma = \cos(n-1)\sigma - \cos(n+1)\sigma, \qquad \cos\sigma\cos n\sigma + \sin\sigma\sin n\sigma = \cos(n-1)\sigma$$

$$2\cos\sigma\sin n\sigma = \sin(n-1)\sigma + \sin(n+1)\sigma, \qquad \cos\sigma\sin n\sigma - \sin\sigma\cos n\sigma = \sin(n-1)\sigma \tag{28}$$

$$Q^{n+1}_{m-1/2}(\chi) = \frac{-2n\chi}{\sqrt{\chi^2-1}}Q^n_{m-1/2}(\chi) + (m^2-(n-\frac{1}{2})^2)Q^{n-1}_{m-1/2}(\chi) \tag{29}$$

Applying $r\partial_r$ to the the symmetric and antisymmetric harmonics and rearranging to express the result as a sum of harmonic functions we obtain:

$$r\frac{\partial}{\partial r}\sqrt{\frac{a}{\rho}}Q^n_{m-1/2}(\chi)\frac{\cos}{\sin}(n\sigma)$$
$$=\frac{1}{2}\sqrt{\frac{a}{\rho}}\left[Q^{n+1}_{m-1/2}(\chi)\frac{\cos}{\sin}(n+1)\sigma-Q^n_{m-1/2}(\chi)\frac{\cos}{\sin}n\sigma+\left(m^2-\frac{(2n-1)^2}{4}\right)Q^{n-1}_{m-1/2}(\chi)\frac{\cos}{\sin}(n-1)\sigma\right] \tag{30}$$

Now assume the symmetric and antisymmetric toroidal harmonics can be expanded as regular spherical harmonics for $r < a$, as

$$\sqrt{\frac{a}{\rho}}Q^n_{m-1/2}(\chi)\cos n\sigma = \pi(-)^m\sum_{k=m}^{\infty} c^m_{nk}P_k^{-m}(0)\left(\frac{r}{a}\right)^k P_k^m(u) \qquad r<a \tag{31}$$
$$\sqrt{\frac{a}{\rho}}Q^n_{m-1/2}(\chi)\sin n\sigma = \pi(-)^m\sum_{k=m}^{\infty} s^m_{nk}P_{k+1}^{-m}(0)\left(\frac{r}{a}\right)^k P_k^m(u) \qquad r<a \tag{32}$$

Plugging these expansions in to (30) and rearranging gives a recurrence relation for $c^m_{nk}$ and $s^m_{nk}$:

$$c^m_{n+1,k}=(2k+1)c^m_{nk}+\left(\frac{(2n-1)^2}{4}-m^2\right)c^m_{n-1,k} \tag{33}$$
$$s^m_{n+1,k}=(2k+1)s^m_{nk}+\left(\frac{(2n-1)^2}{4}-m^2\right)s^m_{n-1,k} \tag{34}$$

with initial values that we have determined above for the $n = 0, 1$ expansions:

$$c^m_{0,k}=1,\ c^m_{1,k}=k+\frac{1}{2}, \qquad s^m_{0,k}=0,\ s^m_{1,k}=k+m+1. \tag{35}$$

We were unable to find a closed form solution for $c^m_{nk}$ or $s^m_{nk}$, or find any links with known sequences on the OEIS. The first few orders for $m = 0$ are:

$$c^0_{2,k}=2(k^2+k+\frac{3}{8})$$
$$c^0_{3,k}=2(2k+1)(k^2+k+\frac{15}{16})$$
$$c^0_{4,k}=2(4k^4+8k^3+15k^2+11k+\frac{105}{32})$$
$$c^0_{5,k}=2(2k+1)(4k^4+8k^3+\frac{109}{4}k^2+\frac{93}{4}k+\frac{945}{64})$$

$$s^0_{2,k}=(k+1)(2k+1)$$
$$s^0_{3,k}=4(k+1)(k^2+k+\frac{13}{16})$$
$$s^0_{4,k}=4(k+1)(2k+1)(k^2+k+\frac{76}{32})$$
$$s^0_{5,k}=4(k+1)(4k^4+8k^3+\frac{107}{4}k^2+\frac{91}{4}k+\frac{789}{64})$$

$c^m_{nk}$ and $s^m_{nk}$ can be computed accurately by the forward recurrence on $n$.
For $r > a$ the expansions look almost identical:

$$\sqrt{\frac{a}{\rho}}Q^n_{m-1/2}(\chi)\cos n\sigma = \pi(-)^{n+m}\sum_{k=m}^{\infty} c^m_{nk}P_k^{-m}(0)\left(\frac{a}{r}\right)^{k+1} P_k^m(u) \qquad r>a \tag{36}$$
$$\sqrt{\frac{a}{\rho}}Q^n_{m-1/2}(\chi)\sin n\sigma = \pi(-)^{n+m+1}\sum_{k=m}^{\infty} s^m_{nk}P_{k+1}^{-m}(0)\left(\frac{a}{r}\right)^{k+1} P_k^m(u) \qquad r>a \tag{37}$$

Finally we apply the Whipple formulae (8) to express these expansions in terms of standard toroidal harmonics:

$$\Delta P^m_{n-1/2}(\beta)\cos n\sigma = 2(-)^m \sum_{k=m}^{\infty} d^m_{nk} P^{-m}_k(0) \begin{cases} (-)^n \left(\frac{r}{a}\right)^k P^m_k(u) & r < a \\ \left(\frac{a}{r}\right)^{k+1} P^m_k(u) & r > a \end{cases} \tag{38}$$

$$\Delta P^m_{n-1/2}(\beta)\sin n\sigma = 2(-)^m \sum_{k=m}^{\infty} t^m_{nk} P^{-m}_{k+1}(0) \begin{cases} (-)^n \left(\frac{r}{a}\right)^k P^m_k(u) & r < a \\ -\left(\frac{a}{r}\right)^{k+1} P^m_k(u) & r > a \end{cases} \tag{39}$$

where

$$d^m_{nk} = \frac{\sqrt{\pi}}{\Gamma(n-m+\frac{1}{2})} c^m_{nk}, \qquad t^m_{nk} = \frac{\sqrt{\pi}}{\Gamma(n-m+\frac{1}{2})} s^m_{nk} \tag{40}$$

which both follow the same recurrence

$$(n-m+1/2) d^m_{n+1,k} = (2k+1) d^m_{nk} + (n+m-1/2) d^m_{n-1,k} \tag{41}$$

$$d^m_{0,k} = \frac{\sqrt{\pi}}{\Gamma(-m+\frac{1}{2})}, \quad d^m_{1,k} = \left(k+\frac{1}{2}\right)\frac{\sqrt{\pi}}{\Gamma(-m+\frac{3}{2})}, \qquad t^m_{0,k} = 0, \quad t^m_{1,k} = (k+m+1)\frac{\sqrt{\pi}}{\Gamma(-m+\frac{3}{2})}. \tag{42}$$

The numerical importance in defining $d^m_{nk}$ and $t^m_{nk}$ is seen in the next section.

### B. Expansions of spherical harmonics

The expansions of spherical harmonics in terms of axial toroidal harmonics can be found from inserting the spherical expansion of the ring harmonics in to the toroidal harmonic expansion of Green's function (11), and then equating this to the spherical expansion. We first use a trigonometric identity to express the toroidal Green's function expansion as

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{\Delta_1\Delta_2}{2\pi a} \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \epsilon_k \epsilon_m P^m_{k-1/2}(\beta_1) Q^{-m}_{k-1/2}(\beta_2) [\cos(k\sigma_1)\cos(k\sigma_2) + \sin(k\sigma_1)\sin(k\sigma_2)] \cos m(\phi_1 - \phi_2) \tag{43}$$

we can then insert the regular expansions for point 2 (39), and equate to the spherical expansion of Green's function (10):

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{\Delta_2}{a} \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \epsilon_k \epsilon_m Q^{-m}_{k-1/2}(\beta_2) \frac{(-)^k/\sqrt{\pi}}{\Gamma(k-m+\frac{1}{2})} \sum_{n=m}^{\infty} \left[\cos(k\sigma_2) c^m_{kn} P^{-m}_n(0) + \sin(k\sigma_2) s^m_{kn} P^{-m}_{n+1}(0)\right] \times (-)^m \left(\frac{r_1}{a}\right)^n P^m_n(u_1) \cos m(\phi_1 - \phi_2) \tag{44}$$

$$= \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \epsilon_m (-)^m \frac{r_1^n}{r_2^{n+1}} P^{-m}_n(u_1) P^m_n(u_2) \cos m(\phi_1 - \phi_2) \tag{45}$$

Equating the coefficients of the spherical harmonics of point 1 for each degree and order, and noting that $Q^{-m}_{n-1/2} = (-)^m \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n+m+\frac{1}{2})} Q^m_{n-1/2}$, we find the expansion of irregular spherical harmonics ($P^m_n(0)$ and $P^m_{n+1}(0)$ are alternately zero for $n+m$ even or odd):

$$\left(\frac{a}{r}\right)^{n+1} P^m_n(u) = \frac{\Delta}{\pi}(-)^m \begin{cases} P^m_n(0) \sum_{k=0}^{\infty} \epsilon_k (-)^k d^{-m}_{kn} Q^m_{k-1/2}(\beta) \cos k\sigma & n+m \text{ even} \\ 2P^m_{n+1}(0) \sum_{k=1}^{\infty} (-)^k t^{-m}_{kn} Q^m_{k-1/2}(\beta) \sin k\sigma & n+m \text{ odd} \end{cases} \tag{46}$$

For the regular spherical harmonics, a similar derivation gives

$$\left(\frac{r}{a}\right)^{n} P^m_n(u) = \frac{\Delta}{\pi}(-)^m \begin{cases} P^m_n(0) \sum_{k=0}^{\infty} \epsilon_k d^{-m}_{kn} Q^m_{k-1/2}(\beta) \cos k\sigma & n+m \text{ even} \\ -2P^m_{n+1}(0) \sum_{k=1}^{\infty} t^{-m}_{kn} Q^m_{k-1/2}(\beta) \sin k\sigma & n+m \text{ odd} \end{cases} \tag{47}$$

Note that $d_{kn}^{-m} = \frac{2^{k+m}}{(2k+2m-1)!!} c_{kn}^{-m}$, $t_{kn}^{-m} = \frac{2^{k+m}}{(2k+2m-1)!!} s_{kn}^{-m}$, $s_{kn}^{-m} = \frac{k-m+1}{k+m+1} s_{kn}^{m}$, $c_{kn}^{-m} = c_{kn}^{m}$.
Again these can be re-expressed in terms of alternative toroidal harmonics using the Whipple formulae. For $n = 0$, (47) corresponds to Heine's expansion and (46) to Green's function expansion. Also (47) has been derived for $n = 1$, $m = 0, 1$, and used in the context of low frequency plane wave scattering [3]. The rest are presumably unknown. Again the coefficients $d_{kn}^{m}, t_{kn}^{m}$ do not seem to have simple closed forms (for fixed $n$ and $m$), but some low orders are

$$d_{k0}^{0} = 1, \qquad t_{k1}^{0} = 4k, \qquad d_{k2}^{0} = 4k^2 + 1, \qquad t_{k3}^{0} = \frac{8}{9}(4k^3 + 5k). \tag{48}$$

$d_{kn}^{m}$ only appear in the series for $n + m$ even, while $t_{kn}^{m}$ only for $n + m$ odd.

**Region of convergence**
We can determine the boundary of convergence of expansions (47) and (46) from the behaviour of the $k^{th}$ term in the series as $k \to \infty$. The Legendre functions grow as [11] pg 191 ((49) is presented for later):

$$\lim_{k\to\infty} P_{k-1/2}(\cosh\eta) = \frac{e^{k\eta}}{\sqrt{(2k-1)\sinh\eta}} \tag{49}$$

$$\lim_{k\to\infty} Q_{k-1/2}(\cosh\eta) = \frac{\sqrt{\pi}e^{-k\eta}}{\sqrt{(2k-1)\sinh\eta}} \tag{50}$$

The series coefficients $d_{kn}^{m}$ and $t_{kn}^{m}$ are less than the sequence $e_{k+1} = (2n+1)/k e_k + e_{k-1}$ (with the same initial values), which itself grows slower than $k^{2n+1}$. $\Gamma(k + m - 1/2)/k!$ is also bounded by a polynomial in $k$ (degree $m$). And $\beta - \sqrt{\beta^2 - 1} \leq 1$. By the ratio test, the series converges everywhere except $\beta = 1$ (the $z$-axis and at $r = \infty$).
Numerically, the expansions (46) and (47) converge slowly away from the focal ring - near the $z$ axis and far from the origin. In these cases the series terms grow significantly in magnitude before converging, which sacrifices accuracy because the series can only be accurate to the last digit of the largest term in the series.

### C. Existence of expansions

For (solid) prolate and oblate spheroidal harmonics, the internal spheroidal harmonics (finite at the origin, diverge at $r = \infty$), can be written as a sum of internal spherical harmonics, and vice versa [12]. The same applies to the external spherical and spheroidal harmonics. The internal spheroidal harmonics cannot be expressed in terms of external spherical harmonics (and vice versa).
But the toroidal harmonics do not follow the same notion of internal and external. We have shown that the ring toroidal harmonics can be written as a series of either internal or external spherical harmonics. This is due to the fact that they are finite at both the origin and at $r = \infty$. However the axial toroidal functions are singular at the origin and at infinity, which means that they cannot be expanded as a series of spherical harmonics at all (They could however be expressed as a series of spherical harmonics of the second kind, which contain $Q_n^m(\cos\theta)$ and are singular on the $z$ axis).
Also, neither internal and external spherical harmonics can be expressed as a series of ring harmonics, because any series of ring harmonics will diverge inside some toroidal boundary, and this boundary must enclose any intrinsic singularity of the function being expanded. The external spherical harmonics are singular at the origin, so this toroidal boundary must cover the origin, however, this torus will then extend to all space. The internal spherical harmonics cannot be expanded for a similar reason - the torus must extend to all space to cover the singularity at $r = \infty$.
However, both internal and external spherical harmonics can be expanded with toroidal line harmonics, because any series of axial toroidal harmonics will diverge *outside* some torus. For the internal spherical harmonics, this toroidal boundary may extend up to infinity since that is where the singularity lies. For external spherical harmonics the toroidal boundary may extend to the origin. And as shown mathematically in the previous section, the toroidal boundary actually does extend to all space.
To summarise, the expansions relating toroidal and spherical harmonics are all non-invertible.

## IV. Application: Charged conducting torus

We demonstrate a simple application of these expansions in expressing the potential of a torus on a spherical basis.
Consider a perfectly conducting torus held at potential $V_0$. The hole radius is $\rho_-$ and outer radius (maximum extent

of the torus on the $xy$ plane) is $\rho_+$. The focal ring radius and surface parameter $\beta = \beta_0$ can be obtained from

$$a = \sqrt{\rho_+\rho_-}, \quad \beta_0 = \frac{\rho_+ + \rho_-}{\rho_+ - \rho_-}. \tag{51}$$

To find the potential, we first expand $V_0$ on toroidal harmonics:

$$V_0 = V_0 \frac{\Delta}{\pi} \sum_{n=0}^{\infty} \epsilon_n Q_{n-1/2}(\beta) \cos n\sigma \tag{52}$$

The potential outside the torus can be found by matching $V_{out}(\beta = \beta_0) = V_0$ and is given by [13]:

$$V_{out} = V_0 \frac{\Delta}{\pi} \sum_{n=0}^{\infty} \epsilon_n \frac{Q_{n-1/2}(\beta_0)}{P_{n-1/2}(\beta_0)} P_{n-1/2}(\beta) \cos n\sigma \tag{53}$$

Substituting the expansion of toroidal in terms of spherical harmonics and rearranging the summation order:

$$V_{out} = \frac{2V_0}{\pi} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \epsilon_n \frac{Q_{n-1/2}(\beta_0)}{P_{n-1/2}(\beta_0)} d_{nk} P_k(0) \begin{cases} (-)^n \left(\frac{r}{a}\right)^k P_k(u) & \text{inside} \\ \left(\frac{a}{r}\right)^{k+1} P_k(u) & \text{outside} \end{cases} \tag{54}$$

We now briefly study the convergence of these series. Using the asymptotic formulae (49) and (50), we can show that the toroidal harmonic series (53) converges for $\eta < 2\eta_0$, so the singularity of the potential lies within this smaller torus. This means that there is an intermediate spherical annulus (that also contains this singularity) where neither spherical harmonic series (54) converges, because a power series in $r$ cannot converge at any radius whose shell contains a singularity. This problem of divergence is also encountered in the problem of the gravity of a solid torus, and a solution for the potential in the intermmediate region is given in terms of a series of both regular and irregular spherical harmonics [5]. Numerical tests show that the annulus doesn't contain the whole torus $\eta < 2\eta_0$ - the exact radii of the annulus could be found with a method similar to that in [14].

### A. Computation of toroidal functions

The Legendre functions of half-integer degree have the following series representations [15] [8]:

$$P^m_{n-1/2}(x) = \frac{\sqrt{2\pi}(x^2-1)^{m/2}(x+1)^{-n-m-1/2}}{\Gamma(n-m+\frac{1}{2})(2n-1)!!} \sum_{k=0}^{\infty} \frac{(2(n+m+k)-1)!!(2(n+k)-1)!!}{k!(m+k)!2^{m+k}} \left(\frac{x-1}{x+1}\right)^k \tag{A1}$$

$$Q^m_{n-1/2}(x) = \pi(-)^m \frac{(x^2-1)^{m/2}}{(2x)^{n+m+1/2}} \sum_{k=0}^{\infty} \frac{(4k+2n+2m-1)!!}{(2k)!!(2k+2n)!!} \frac{1}{(2x)^{2k}} \qquad x > 1, n \geq 0, m \geq 0 \tag{A2}$$

However, these series are not ideal for all values of $x$, and it is much faster to compute the functions by recurrence (using the series to compute the initial values). Advanced methods of calculating these functions are detailed in [16]. Matlab functions used in calculations in testing the results in this document are attached as supplementary material.

**Acknowledgments.** This research was funded by a Victoria University of Wellington doctoral scholarship.

**Author contributions.** MM carried out the derivations, and wrote the paper in consultation with ECLR. Both agreed on final submission.


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