

FOLLOWING SCHUBERT VARIETIES UNDER FEIGIN'S DEGENERATION OF THE FLAG VARIETY

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ABSTRACT. We describe the effect of Feigin's flat degeneration of the type A flag variety on its Schubert varieties. In particular, we study when they stay irreducible and in several cases we are able to encode reducibility of the degenerations in terms of symmetric group combinatorics. As a side result, we obtain an identification of some Schubert varieties with Richardson varieties in higher rank partial flag varieties.

1. INTRODUCTION

Let G be a simple Lie group and let $P \subset G$ be a parabolic subgroup. In [Fei12], Feigin introduced a flat degeneration of the flag variety G/P , which is equipped with an action of the M -fold product of the additive group of the field (M being the dimension of a maximal unipotent subgroup of G). These degenerations of flag varieties (and some generalizations in type A) have been in the past years intensively studied from many different perspectives (see, for example, [Fei11], [CIFR12], [CIL15], [Fou16], [CIFF⁺17], [LS]).

In this paper, we deal with the effect of Feigin's degeneration on the Schubert varieties inside $\mathcal{F}\ell_n := SL_n/B$, for B the Borel subgroup of upper triangular matrices. In [Fei12] it is shown that in type A the degeneration $\mathcal{F}\ell_n^a$ of $\mathcal{F}\ell_n$ can be embedded in the product of projective spaces, exactly as $\mathcal{F}\ell_n$, and that the defining ideal is generated by degenerate Plücker relations. More precisely, the defining ideal $\mathcal{I}_{\mathcal{F}\ell_n}$ of $\mathcal{F}\ell_n$ is generated by Plücker relations and the defining ideal $\mathcal{I}_{\mathcal{F}\ell_n^a}$ is obtained as the initial ideal $\text{in}_{\mathbf{w}}(\mathcal{I}_{\mathcal{F}\ell_n})$ with respect to a weight vector \mathbf{w} (whose components are indexed by Plücker coordinates), as described in Section 2.2.1. On the other hand, if $v \in S_n$ is a permutation, it is well-known that the ideal \mathcal{I}_v of the Schubert variety $X_v = \overline{BvB}/B \subseteq \mathcal{F}\ell_n$ is generated by the Plücker relations together with certain Plücker coordinates (see §2.3 for a more precise formulation). Thus it is natural to ask what happens to \mathcal{I}_v under Feigin's degeneration, that is to investigate $\text{in}_{\mathbf{w}}(\mathcal{I}_v)$.

From the first non-trivial example, it is already clear that not all Schubert varieties under Feigin's degeneration will stay irreducible: for $n = 3$, indeed, one of the six Schubert varieties degenerates to a reducible variety. Therefore, a consistent part of this paper is directed towards understanding this reducibility phenomenon.

We should mention here that what we refer to as *Feigin's degeneration* is in fact a modified version of his original construction, which was coming from Lie theory. The version we deal with in this paper is the one which has been studied in [CIL15]. The variety one obtains in this way is isomorphic to Feigin's original degeneration, but in some sense it behaves better with respect to Schubert varieties. In fact, Caldero noticed in [Cal02] that it does not exist a (flat) toric degeneration of the flag variety under which all Schubert varieties degenerate to toric varieties. For $n = 3$ (which is the only case, apart from $n = 2$, in which $\mathcal{F}\ell_n^a$ is toric) our version of the degeneration

preserves irreducibility of all but one Schubert varieties, while two Schubert varieties would become reducible under Feigin’s original definition. This is why we feel that in this setting the definition we use is sort of optimal.

Before focusing on Schubert varieties which become reducible after degenerating, we first describe some cases in which it is easy to show that they stay irreducible (see Section 3). In particular, we prove that there is a class of Schubert varieties (indexed by permutations which are less or equal than a distinguished Coxeter element) whose defining ideals are not affected by the degeneration (see Proposition 1).

Section 4 is devoted to sufficient conditions on the permutation v such that the initial ideal $\text{in}_{\mathbf{w}}(\mathcal{I}_v)$ is not prime. The strategy is to look for Plücker relations whose initial term is a monomial when considered modulo the Plücker coordinates which vanish on $X_v^a := V(\text{in}_{\mathbf{w}}(\mathcal{I}_v))$, which coincide with the ones vanishing on X_v . The efficiency of some of the conditions we give is then tested by looking at the $n = 4$ and $n = 5$ examples, for which we can detect all initial ideals containing monomials (see Tables 1 and 2).

In previous joint work with Cerulli Irelli [CIL15], the second author proved that the degenerate flag variety $\mathcal{F}\ell_n^a$ can be embedded in the flag variety SL_{2n-2}/P of partial flags in \mathbb{C}^{2n-2} consisting of odd dimensional spaces (that is, $P = P_{\omega_1+\omega_3+\dots+\omega_{2n-3}}$). Under this embedding, it was shown in [CIL15] that $\mathcal{F}\ell_n^a$ is isomorphic to a Schubert variety. From this fact (together with classical results) one could obtain a new proof of projective normality, Frobenius splitting, and rationality of the singularities of $\mathcal{F}\ell_n^a$. In Section 5 we further exploit such an isomorphism and study the effect of Feigin’s degeneration on Schubert varieties inside SL_{2n-2}/P . The idea is to show irreducibility of the degeneration of some Schubert variety by proving that the above-mentioned embedding sends it to a Richardson variety. Although our main focus is the analysis of Plücker relations (cf. Sections 4 and 3), for which there is no need to move to a higher rank (partial) flag variety, we decided to have a section on Richardson varieties. By comparing Proposition 1 with Lemma 5 we obtain a realization of some Richardson varieties inside SL_{2n-2}/P as Schubert varieties in a lower rank (complete) flag variety.

The last section of the paper deals with Schubert divisors, that is Schubert varieties of codimension one in $\mathcal{F}\ell_n$. By applying our reducibility criteria from Section 4, we are able to prove that if n is even all Schubert divisors become reducible, while for n odd this happens for all but one. In this case, the remaining divisor is shown to be isomorphic to a Richardson variety inside SL_{2n-2}/P , and hence irreducible.

We want to point out that our paper is very different in spirit from [Fou16], where irreducible flat degenerations of Schubert varieties corresponding to some special Weyl group elements (*triangular elements*) are produced by considering PBW-degenerations of Demazure modules $V_w(\lambda)$ and then realizing the desired degeneration as the closure of an appropriate \mathbb{G}_a^M -orbit inside $\mathbb{P}(V_w(\lambda))$. So for any Schubert variety which is indexed by a triangular element (see [Fou16, Definition 1]) one can construct a flat irreducible degeneration via Fourier’s procedure, while in this article we fix the degeneration (Feigin’s) of the whole flag variety and study its effect on Schubert varieties (which are hence simultaneously degenerated).

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2. PRELIMINARIES AND NOTATION

2.1. Symmetric group combinatorics. The combinatorics of the symmetric group controls many geometric properties of $\mathcal{F}\ell_n$ and its Schubert varieties, therefore we spend a little time here introducing the notation we will need later on.

For any two positive integers $i, j \in \mathbb{Z}_{\geq 1}$, with $i \leq j$ we denote by $[i, j] := \{a \in \mathbb{Z} \mid i \leq a \leq j\}$. Moreover, we use the short hand notation $[j] := [1, j]$. We write $\binom{[n]}{k}$ for the set of subsets of cardinality k inside $[n]$.

Let $n \geq 2$ and denote by S_n the symmetric group. Recall that the symmetric group S_n admits a presentation as a Coxeter group, with set of simple reflections $\{s_i \mid i = 1, \dots, n-1\}$, for s_i the transposition $(i, i+1)$. We will use the standard terminology and say that a product $s_{i_1} \dots s_{i_r}$ is a reduced expression for $v \in S_n$ if $v = s_{i_1} \dots s_{i_r}$ and all other expressions of v as a product of simple reflections $v = s_{j_1} \dots s_{j_t}$ are such that $t \geq r$. In this case $r = \ell(w)$ is called the *length* of w . We denote by \leq the Bruhat order on S_n and recall the following equivalent characterization (see, for example, [BB05, Theorem 2 2.1.5]): For $v \in S_n$ and $i, j \in [n]$ set

$$(2.1) \quad w^{i,j} = \#\{a \in [i] \mid w(a) \geq j\}.$$

Then

$$(2.2) \quad v \leq u \Leftrightarrow v^{i,j} \leq u^{i,j} \text{ for all } i, j.$$

In the following we will also need that if $v \in S_n$ and $i \in [n-1]$, then

$$vs_i < v \Leftrightarrow v(i) > v(i+1),$$

or, equivalently,

$$s_i v < v \Leftrightarrow v^{-1}(i) > v^{-1}(i+1).$$

The symmetric group S_n acts on $\binom{[n]}{k}$ for any k : if $I = \{i_1, \dots, i_k\} \subset \binom{[n]}{k}$ then

$$v(I) := \{v(i_1), \dots, v(i_k)\}.$$

This action is transitive and the Bruhat order induces a partial order on $\binom{[n]}{k}$, which has the following description (see, for instance, [BB05, Proposition 2.4.8])

$$(2.3) \quad u(I) \leq v(I) \Leftrightarrow u(i) < v(i) \text{ for all } i \in [k].$$

We will sometimes write elements $v \in S_n$ as $[v(1), v(2), \dots, v(n)]$. This is referred to as *one-line* notation.

2.1.1. Sequences. In the following sections, we will also need to deal with sequences (i_1, \dots, i_k) rather than sets $\{i_1, \dots, i_k\}$. We denote by $\mathcal{S}(n, k)$ the set of sequences of k pairwise distinct numbers between 1 and n .

Given two sequences $I_1 = (i_1^{(1)}, \dots, i_k^{(1)}) \in \mathcal{S}(n, k)$, $I_2 = (i_1^{(2)}, \dots, i_l^{(2)}) \in \mathcal{S}(n, l)$ such that $I_1 \cap I_2 = \emptyset$, we denote by $(I_1, I_2) := (i_1^{(1)}, \dots, i_k^{(1)}, i_1^{(2)}, \dots, i_l^{(2)}) \in \mathcal{S}(n, k+l)$ the sequence obtained by concatenation.

If $L \in \mathcal{S}(n, d)$ and $J = (j_1, \dots, j_k) \in \mathcal{S}(n, e)$, then the sequence $L' = (L \setminus (l_{r_1}, \dots, l_{r_k})) \cup (j_1, \dots, j_k) \in \mathcal{S}(n, d)$ is obtained from L by substituting the subsequence $(l_{r_1}, \dots, l_{r_k})$ with (j_1, \dots, j_k) , that is $l'_a = l_a$ if $a \notin \{r_1, \dots, r_k\}$ and $l'_a = j_b$ if $a = r_b$.

There is a forgetful map

$$F : \mathcal{S}(n, k) \rightarrow \binom{[n]}{k}, \quad (i_1, \dots, i_k) \mapsto \{i_1, \dots, i_k\}.$$

By abuse of notation, if $I \in \mathcal{S}(n, k)$ and $v \in S_n$, we will write $I \leq v([\#I])$ instead of $F(I) \leq v([\#I])$ (and $I \geq v([\#I])$, $I \not\leq v([\#I])$, etc., will have an analogous meaning).

2.1.2. A special Coxeter element. The Coxeter element $c = s_{n-1}s_{n-2} \cdots s_2s_1 \in S_n$ will play an important role later on. Observe, that in the one-line notation

$$c = [n, 1, 2, 3, \dots, n-1]$$

so that, by (2.3), for $I \in \binom{[n]}{d}$

$$(2.4) \quad I \leq c([d]) \Leftrightarrow I = [d-1] \cup \{b\} \text{ for } d \leq b \leq n.$$

2.2. Basics on the flag variety. Let $n \geq 2$. In this paper we deal with the variety \mathcal{F}_n of complete flags in \mathbb{C}^n . Let $(e_i)_{1 \leq i \leq n}$ denote the standard basis of \mathbb{C}^n . Let $B \subset SL_n$ be the Borel subgroup of upper triangular matrices. The group SL_n acts on \mathcal{F}_n and we can identify the flag variety with the quotient SL_n/B by looking at the SL_n -orbit of the standard flag E_\bullet with

$$E_i := \text{span}_{\mathbb{C}}\{e_1, \dots, e_i\} \quad (i = 1, \dots, n-1).$$

Recall that under the left action of B , the flag variety decomposes as a union of cells indexed by the elements of the symmetric group S_n :

$$SL_n/B = \bigsqcup_{v \in S_n} BvB/B$$

where, by abuse of notation, v in BvB/B denotes the corresponding permutation matrix in SL_n . We denote by X_v the Schubert variety $\overline{BvB/B}$.

Analogously, also B_- , the Borel subgroup of lower triangular matrices acts by left multiplication on SL_n/B , providing the decomposition:

$$SL_n/B = \bigsqcup_{u \in S_n} B_-uB/B.$$

We denote by X^u the opposite Schubert variety $\overline{B_-uB/B}$. In §5, we will also consider Richardson varieties $X_v^u := X_v \cap X^u$.

2.2.1. Plücker relations. Our main reference for Plücker coordinates and relations is [Ful97], while we refer to [Fei12] for the degenerate Plücker relations.

We start by recalling the Plücker embedding of a Grassmannian. Recall that $(e_i)_{1 \leq i \leq n}$ is the standard basis of \mathbb{C}^n , so that

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < i_2 < \dots < i_k \leq n\}$$

is a basis of $\wedge^k \mathbb{C}^n$. Let $(\wedge^k \mathbb{C}^n)^*$ be the dual vector space, then the Plücker coordinate $p_{i_1, \dots, i_k} \in (\wedge^k \mathbb{C}^n)^*$ for $1 \leq i_1 < i_2 < \dots < i_k \leq n$ is defined to be the basis element dual to $e_{i_1} \wedge \dots \wedge e_{i_k}$. For $i_1, \dots, i_k \in [n]$ pairwise distinct, but not necessarily increasing, the Plücker coordinate p_{i_1, \dots, i_k} has the following property

$$p_{\sigma(i_1), \dots, \sigma(i_k)} = (-1)^{\ell(\sigma)} p_{i_1, \dots, i_k} \quad \text{for all } \sigma \in S_n.$$

Denote by p_I the Plücker coordinate corresponding to a sequence $I = (i_1, \dots, i_k) \in \mathcal{S}(n, k)$. In the following sections it will be sometimes convenient to simplify notation and index some Plücker coordinates by a set instead of a sequence. This has to be

interpreted as being indexed by the sequence obtained by arranging the elements of the set in an increasing order.

We have obtained in this way the Plücker embedding

$$(2.5) \quad \mathrm{Gr}(k, \mathbb{C}^n) \hookrightarrow \mathbb{P}(\wedge^k \mathbb{C}^n).$$

The flag variety is embedded in the product of Grassmannians

$$\mathcal{F}\ell_n \hookrightarrow \mathrm{Gr}(1, \mathbb{C}^n) \times \mathrm{Gr}(2, \mathbb{C}^n) \times \cdots \times \mathrm{Gr}(n-1, \mathbb{C}^n).$$

By composing with the embedding (2.5) for each Grassmannian in the product, we get

$$\mathcal{F}\ell_n \hookrightarrow \mathbb{P}\mathbb{C}^n \times \mathbb{P}(\wedge^2 \mathbb{C}^n) \times \cdots \times \mathbb{P}(\wedge^{n-1} \mathbb{C}^n).$$

Denote by $\mathcal{I}_{\mathcal{F}\ell_n}$ the ideal of $\mathcal{F}\ell_n$ in $\mathbb{C}[p_{i_1, \dots, i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n, k \in [n-1]]$ with respect to this embedding. Then $\mathcal{I}_{\mathcal{F}\ell_n}$ is generated by elements in

$$\{R_{(j_1, \dots, j_e), (l_1, \dots, l_d)}^k \mid e \leq d, k \in [e]\}$$

given by

$$R_{J,L}^k = p_J p_L - \sum_{1 \leq r_1 < \cdots < r_k \leq d} p_{J'} p_{L'},$$

where $L = (l_1, \dots, l_d) \in \mathcal{S}(n, d)$, $J = (j_1, \dots, j_e) \in \mathcal{S}(n, e)$, $L' = (L \setminus (l_{r_1}, \dots, l_{r_k})) \cup (j_1, \dots, j_k)$ and $J' = (J \setminus (j_1, \dots, j_k)) \cup (l_{r_1}, \dots, l_{r_k})$. The elements $R_{J,L}^k$ will be referred to as Plücker relations. To simplify notation we set

$$(2.6) \quad \mathcal{L}_{J,L}^k = \left\{ (J', L') \mid \begin{array}{l} \exists 1 \leq r_1 < \cdots < r_k \leq \#L, \\ J' = (J \setminus (j_1, \dots, j_k)) \cup (l_{r_1}, \dots, l_{r_k}), \\ L' = (L \setminus (l_{r_1}, \dots, l_{r_k})) \cup (j_1, \dots, j_k) \end{array} \right\}.$$

The weight vector $\mathbf{w} \in \mathbb{R}^{\binom{n}{1} + \cdots + \binom{n}{n-1}}$ is defined componentwise by setting for $I = \{i_1, \dots, i_k\} \in \binom{[n]}{k}$

$$\mathbf{w}_I = \#\{r \mid k \leq i_r \leq n-1\}.$$

Then the initial ideal $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_{\mathcal{F}\ell_n})$ is generated by the initial forms $\mathrm{in}_{\mathbf{w}}(R_{J,L}^k)$ by [Fei12, Theorem 3.13]. They are of form

$$\mathrm{in}_{\mathbf{w}}(R_{J,L}^k) = p_J p_L - \sum_{\substack{(J', L') \in \mathcal{L}_{J,L}^k \\ \{l_{r_1}, \dots, l_{r_k}\} \cap [e, d-1] = \emptyset}} p_{J'} p_{L'},$$

where the leading term is non-zero, only if

$$(2.7) \quad \{j_1, \dots, j_k\} \cap [e, d-1] = \emptyset.$$

We can choose J, L in such a way that (2.7) holds. Observe that for $q = d$, we always have $\mathrm{in}_{\mathbf{w}}(R_{J,L}^k) = R_{J,L}^k$ since the condition (2.7) is empty.

Definition 1 ([Fei12]). The *degenerate flag variety* is the vanishing of the ideal $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_{\mathcal{F}\ell_n})$, that is

$$\mathcal{F}\ell_n^a := V(\mathrm{in}_{\mathbf{w}}(\mathcal{I}_{\mathcal{F}\ell_n})) \subset \mathbb{P}\mathbb{C}^n \times \mathbb{P}(\wedge^2 \mathbb{C}^n) \times \cdots \times \mathbb{P}(\wedge^{n-1} \mathbb{C}^n).$$

Remark 1. Feigin's original definition, valid for any simple Lie group, was different from the one we have just given, which is a characterization of the type A degenerate flag variety by [Fei12, Theorem 3.13]. As already mentioned in the introduction, we modify Feigin definition to match the one considered in [CIL15]. Explicitly, to obtain our degeneration from Feigin's original one, a global shift by -1 (modulo n) to all indices is needed.

2.3. Ideals for Schubert varieties and their degeneration. Recall the following property of initial ideals.

Lemma 1. Consider two ideals $\mathcal{I}, \mathcal{J} \subset \mathbf{k}[x_1, \dots, x_n]$ and $\mathbf{u} \in \mathbb{R}^n$. Let $\text{in}_{\mathbf{u}}(\mathcal{I}) = (\text{in}_{\mathbf{u}}(f_1), \dots, \text{in}_{\mathbf{u}}(f_r))$ and $\text{in}_{\mathbf{u}}(\mathcal{J}) = (\text{in}_{\mathbf{u}}(g_1), \dots, \text{in}_{\mathbf{u}}(g_s))$. Then

$$\text{in}_{\mathbf{u}}(\mathcal{I} + \mathcal{J}) = (\text{in}_{\mathbf{u}}(f_i), \text{in}_{\mathbf{u}}(g_j) \mid 1 \leq i \leq r, 1 \leq j \leq s) = \text{in}_{\mathbf{u}}(\mathcal{I}) + \text{in}_{\mathbf{u}}(\mathcal{J}).$$

For $v \in S_n$ the defining ideal of the Schubert variety $X_v \subset \mathcal{F}\ell_n$ is given by the vanishing of $(p_I)_{I \not\leq v(\#\#I)}$. It is shown in [LLM98, §10.12] (see also [KR87, Theorem 3]) that by embedding $X_v \hookrightarrow \mathbb{P}\mathbb{C}^n \times \mathbb{P}(\wedge^2 \mathbb{C}^n) \times \dots \times \mathbb{P}(\wedge^{n-1} \mathbb{C}^n)$, we obtain the ideal

$$(2.8) \quad \mathcal{I}_v := \mathcal{I}_{\mathcal{F}\ell_n} + (p_I)_{I \not\leq v(\#\#I)}$$

of $\mathbb{C}[p_{i_1, \dots, i_d} \mid 1 \leq i_1 < i_2 < \dots < i_d \leq n, d \in [n-1]]$. Note that $\text{in}_{\mathbf{w}}(p_I) = p_I$ for all $I \subset \mathcal{S}(n, d)$, for all $d \in [n-1]$. As by [Fei12, Theorem 3.13] we know that $\text{in}_{\mathbf{w}}(\mathcal{I}_{\mathcal{F}\ell_n}) = (\text{in}_{\mathbf{w}}(R_{J,L}^k))_{k,J,L}$, we deduce the following from Lemma 1

$$(2.9) \quad \text{in}_{\mathbf{w}}(\mathcal{I}_v) = (\text{in}_{\mathbf{w}}(R_{J,L}^k))_{k,J,L} + (p_I)_{I \not\leq v(\#\#I)}.$$

Hence, Plücker coordinates indexed by increasing sequences (or, in our convention, sets in $\bigcup_{r=1}^{n-1} \binom{[n]}{r}$) form a Gröbner basis with respect to \mathbf{w} for the ideals of Schubert varieties. Feigin's degeneration of the flag variety induces therefore a degeneration $X_v^a \subset \mathcal{F}\ell_n^a$ of any Schubert variety $X_v \subset \mathcal{F}\ell_n$:

$$(2.10) \quad X_v^a := V(\text{in}_{\mathbf{w}}(\mathcal{I}_v)) \subset \mathbb{P}\mathbb{C}^n \times \mathbb{P}(\wedge^2 \mathbb{C}^n) \times \dots \times \mathbb{P}(\wedge^{n-1} \mathbb{C}^n).$$

3. EXAMPLES OF IRREDUCIBLE X_v^a

The following lemma provides a first class of examples where the degeneration X_v^a of the Schubert variety X_v stays irreducible.

Lemma 2. Let $v \in S_n$ be the minimal representative of the longest word in $S_n / \langle s_1, \dots, s_i, s_{i+r}, \dots, s_{n-1} \rangle$ for some $i \geq 1$ and $r \geq 0$ such that $i + r < n - 1$. Then

$$X_v^a \cong \mathcal{F}\ell_r^a.$$

Proof. First note that written in one-line notation v is of form

$$v = [1, 2, \dots, i, i+r, i+r-1, \dots, i+1, i+r+1, \dots, n].$$

So $v(j) = j$ for $j \in [i] \cup [i+r+1, n]$ and $v(i+k) = i+r-k+1$ for $k \in [r]$. For the Schubert variety we have $X_v \cong \mathcal{F}\ell_r$, i.e. the only non-vanishing Plücker coordinates besides $p_{[s]}$ for $s \leq n-1$ are associated with the index sets in

$$\mathcal{J}_v = \{I \mid I = \{[i] \cup \{l_1, \dots, l_s\}, s \in [r-1], l_j \in [i+1, i+r] \forall j\}.$$

We want to show that such an isomorphism survives the degeneration.

From what we have observed, we know that the only non-trivial Plücker relations on X_v are $R_{J,L}^k$, where $F(J), F(L) \in \mathcal{J}_v$. We have a bijection

$$\mathcal{J}_v \rightarrow \bigcup_{s=1}^{r-1} \binom{[r]}{s}, \quad I \mapsto \tilde{I},$$

where if $I = [i] \cup \{l_1, l_2, \dots, l_s\}$, we set $\tilde{I} = \{l_1 - i, l_2 - i, \dots, l_s - i\}$. This induces a bijection between the set of Plücker coordinates $\neq p_{[s]}$, $s \in [n-1] \setminus [i+1, i+r]$, which are non-vanishing on X_v (that is, the ones involved in the relevant Plücker relations) and Plücker coordinates $(\tilde{p}_{\tilde{K}})$ which generate the coordinate ring of $\mathcal{F}\ell_r$.

Notice that for J, L with $F(J), F(L) \in \mathcal{J}_v$, the Plücker relation $R_{J,L}^k$ is not identically 0 if and only if $R_{\tilde{J},\tilde{L}}^k$ is not identically 0 (since this happens for $k \in [\#(L \setminus (L \cap J))] = [\#(\tilde{L} \setminus (\tilde{L} \cap \tilde{J}))]$).

We will show that such a bijection sends $\text{in}_{\mathbf{w}}(R_{J,L}^k)$ to $\text{in}_{\mathbf{w}}(R_{\tilde{J},\tilde{L}}^k)$ for any pair J, L with $F(J), F(L) \in \mathcal{J}_v$, and hence induces the desired isomorphism.

Let $L = ((1, \dots, i), (l_1, \dots, l_d)) > J = ((j_1, \dots, j_e), (1, \dots, i))$. Consider the relation $R_{J,L}^k$. Without loss of generality we can assume that J and L are chosen in such a way that $\text{in}_{\mathbf{w}}(R_{J,L}^k)$ contains the monomial $p_J p_L$. All other monomials $p_{J'} p_{L'}$ in $\text{in}_{\mathbf{w}}(R_{J,L}^k)$ are obtained from $p_J p_L$ by choosing $1 \leq r_1 < \dots < r_k \leq i + d$, such that $\{l_{r_1}, \dots, l_{r_k}\} \cap [i + e, i + d - 1] = \emptyset$, but this is of course the case if and only if $\{\tilde{l}_{r_1}, \dots, \tilde{l}_{r_k}\} \cap [e, d - 1] = \emptyset$.

Now the claim follows by Lemma 1. \square

Corollary 1. With assumptions being as in Lemma 2, X_v^a is irreducible.

Proof. By [Fei12, §5.1] the degenerate flag variety is the closure of a homogeneous space and therefore irreducible. As $X_v^a \cong \mathcal{F}\ell_r^a$ by Lemma 2 the claim follows. \square

Let $\underline{i} = \{i_1, \dots, i_r\} \subsetneq [n - 1]$. We set $m := \min\{\underline{i}\}$, $M := \max\{\underline{i}\}$, and $r := M - m + 1$. Let $v \in \langle s_{i_1}, \dots, s_{i_r} \rangle \subset S_n$ denote by \tilde{v} the element $\tilde{s}_{i_1 - m + 1} \cdots \tilde{s}_{i_r - m + 1} \in S_r$. In this notation, from the proof of Lemma 2 we can deduce the following result, which in this case allows one to reduce to smaller rank flag varieties.

Corollary 2. Let $\underline{i} = \{i_1, \dots, i_r\} \subsetneq [n]$ and $v \in \langle s_{i_1}, \dots, s_{i_r} \rangle \subset S_n$. Then for $X_v^a \subset \mathcal{F}\ell_n^a$ we have

$$X_v^a \cong X_{\tilde{v}}^a \subset \mathcal{F}\ell_r^a.$$

3.1. Degenerated vs. original Schubert varieties. In the following we present another instance in which a Schubert variety stays irreducible under Feigin's degeneration of $\mathcal{F}\ell_n$. In fact, for the class of varieties we deal with in this section a stronger property holds: the degeneration process does not touch them, that is X_v^a is isomorphic to the original Schubert variety X_v .

Recall that we denote by $c \in S_n$ the special Coxeter element $c = s_{n-1} s_{n-2} \cdots s_2 s_1$.

Proposition 1. Let $v \leq c$. Then $\mathcal{I}_v = \text{in}_{\mathbf{w}}(\mathcal{I}_v)$

Proof. Recall that $\mathcal{I}_v = (\{p_I\}_{I \not\leq v(\#[I])} \cup \{R_{J,L}^k\}_{k,J,L})$ with initial ideal given by (2.9). We will show that $R_{J,L}^k - \text{in}_{\mathbf{w}}(R_{J,L}^k) \in (p_I)_{I \not\leq v(\#[I])}$ for all k, J, L . If $R_{J,L}^k = \text{in}_{\mathbf{w}}(R_{J,L}^k)$ we are done. Otherwise we have

$$R_{J,L}^k - \text{in}_{\mathbf{w}}(R_{J,L}^k) = \sum_{\substack{(J',L') \in \mathcal{L}_{J,L}^k \\ \{l_{r_1}, \dots, l_{r_k}\} \cap [q, d-1] \neq \emptyset}} p_{J'} p_{L'} \neq 0.$$

We claim that in this case $L' \not\leq v([d])$ holds. Note that $\{l_{r_1}, \dots, l_{r_k}\} \cap [q, d - 1] \neq \emptyset$ implies in particular that there exists $x \in [q, d - 1]$ with $x \notin L' = (L \setminus (l_{r_1}, \dots, l_{r_k})) \cup (j_1, \dots, j_k)$. By (2.4),

$$v \leq c \Leftrightarrow v([d]) = [d - 1] \cup \{v(d)\} \text{ with } d \leq v(d) \leq n$$

it follows that $p_{L'} \in (p_I)_{I \not\leq v(\#[I])}$. And further, $R_{J,L}^k - \text{in}_{\mathbf{w}}(R_{J,L}^k) \in (p_I)_{I \not\leq v(\#[I])}$. \square

4. CRITERIA FOR REDUCIBILITY

In this section we examine when Schubert varieties become reducible after degenerating. We give a number of sufficient conditions for certain monomials of degree two to be contained in the initial ideal $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ for $w \in S_n$.

4.1. Relations between $\text{Gr}(1, \mathbb{C}^n)$ and $\text{Gr}(2, \mathbb{C}^n)$. We start the discussion by focusing on very special Plücker relations, namely those between Plücker coordinates on $\text{Gr}(1, \mathbb{C}^n)$ and on $\text{Gr}(2, \mathbb{C}^n)$. In this case, we can classify the $w \in S_n$ for which $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ contains a monomial of this form.

For $v \in S_n$ denote by \bar{v} the minimal length representative of the coset of v in $S_n/\langle s_2, s_3, \dots, s_{n-1} \rangle$ and $\bar{\bar{v}}$ the minimal length representative of the coset of v in $S_n/\langle s_1, s_3, s_4, \dots, s_{n-1} \rangle$.

Theorem 1. Let $v \in S_n$ and $1 < j < k \leq n$. Then $\text{in}_{\mathbf{w}}(\mathcal{I}_v)$ contains the monomial $p_{\{j\}}p_{\{1,k\}}$ if and only if v satisfies

$$s_{j-1}s_{j-2} \cdots s_2s_1 \leq \bar{v} \leq s_{k-2}s_{k-3} \cdots s_2s_1 \quad \text{and} \quad s_{k-1}s_{k-2} \cdots s_3s_2 \leq \bar{\bar{v}}.$$

The conditions on \bar{v} and $\bar{\bar{v}}$ in Theorem 1 are depicted for S_4 with $j = 2, k = 4$ in Figure 1.

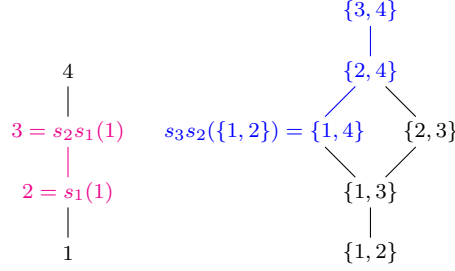


FIGURE 1. The Bruhat posets of $\text{Gr}(1, \mathbb{C}^4)$ and $\text{Gr}(2, \mathbb{C}^4)$ with intervals given by $s_1 \leq \bar{v} \leq s_2s_1$ and $s_3s_2 \leq \bar{\bar{v}}$ as in Theorem 1 for $j = 2, k = 4$.

Proof. To simplify notation, $a \in [n]$, we denote $p_a := p_{(a)}$, and for $a, b \in [n]$ we write $p_{a,b}$ instead of $p_{(a,b)}$. We will only consider Plücker coordinates corresponding to increasing sequences in this proof and hence adapt the signs.

Consider for $1 \leq i < j < k \leq n$ the Plücker relation $R_{(i),(j,k)}^1 = p_i p_{j,k} - p_j p_{i,k} + p_k p_{i,j}$. Note that if $\text{in}_{\mathbf{w}}(R_{(i),(j,k)}^1) = R_{(i),(j,k)}^1$ the relation will not produce a monomial in $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ for any $w \in S_n$ as \mathcal{I}_w does not contain monomials. Note that $R_{(i),(j,k)}^1 \neq \text{in}_{\mathbf{w}}(R_{(i),(j,k)}^1)$ only if $i = 1$. In this case

$$\text{in}_{\mathbf{w}}(p_1 p_{j,k} - p_j p_{1,k} + p_k p_{1,j}) = -p_j p_{1,k} + p_k p_{1,j}.$$

As $j < k$, if p_j vanishes on the Schubert variety X_v , then so does p_k . Hence, both monomials are zero on X_v . Similarly, if $p_{1,j}$ vanishes on X_v , then so does $p_{1,k}$. Our aim is to determine $v \in S_n$ such that one of the two terms of $\text{in}_{\mathbf{w}}(R_{(i),(j,k)}^1)$ lies in $(p_I)_{I \not\leq v(\#I)}$ but the other does not. In fact, if this case, the ideal $\text{in}_{\mathbf{w}}(\mathcal{I}_v)$ contains a monomial and we deduce that X_v^a is reducible. A priori, there are two cases for the restriction of p_k and $p_{1,k}$ to X_v :

- (1) $p_{1,k} \neq 0$ and $p_k = 0$,

(2) $p_{1,k} = 0$ and $p_k \neq 0$.

We will show that in fact the second case can never happen. Both cases yield conditions on \bar{v} and $\overline{\bar{v}}$ (keeping also in mind that we do not want p_j and $p_{1,j}$ to vanish). In the first case we have the following conditions

$$(4.1) \quad s_{j-1}s_{j-2} \cdots s_2s_1 \leq \bar{v} \leq s_{k-2}s_{k-3} \cdots s_2s_1 \quad \text{and} \quad s_{k-1}s_{k-2} \cdots s_3s_2 \leq \overline{\bar{v}},$$

respectively, in the second case we have

$$(4.2) \quad s_{k-1}s_{k-2} \cdots s_2s_1 \leq \bar{v} \quad \text{and} \quad s_{j-1}s_{j-2} \cdots s_3s_2 \leq \overline{\bar{v}} \leq s_{k-2}s_{k-3} \cdots s_3s_2.$$

Assume $v \in S_n$ is chosen such that the minimal length representatives of the cosets fulfill the inequalities in (4.2). Then

$$s_{k-1}s_{k-2} \cdots s_2s_1 \leq v \leq s_{k-2} \cdots s_2x$$

for some $x \in \langle s_1, s_3, \dots, s_{n-1} \rangle$. Observe that $s_{k-1} \cdots s_1(1) = k$ and

$$s_{k-2} \cdots s_2x(1) = \begin{cases} 1 & \text{if } s_1x > x \\ k-1 & \text{if } s_1x < x. \end{cases}$$

With the notation as in (2.1) this implies $(s_{k-1} \cdots s_1)^{1,k} = 1 > (s_{k-2} \cdots s_2x)^{1,k} = 0$. But $s_{k-1} \cdots s_1 \leq s_{k-2} \cdots s_2x$, contradicting (2.2). Hence, case (4.2) never applies. \square

Remark 2. Theorem 1 is enough to detect all Schubert varieties in $\mathcal{F}\ell_3 \hookrightarrow \text{Gr}(1, \mathbb{C}^3) \times \text{Gr}(2, \mathbb{C}^3)$ which become reducible under Feigin's degeneration. In fact, the only Schubert variety having this property is the one indexed by s_1s_2 . All the other permutations but the longest element (which indexes the Schubert variety corresponding to the irreducible variety $\mathcal{F}\ell_n^a$) are $\leq c = s_2s_1$ and hence, by Proposition 1, are irreducible.

4.2. Monomials from other relations. Theorem 2 (1) to (5) provide sufficient conditions on $w \in S_n$ for the initial ideal $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ to contain a degree two monomial originating from a Plücker relation between Plücker coordinates on adjacent Grassmannians, that is $\text{Gr}(k, \mathbb{C}^n)$ and on $\text{Gr}(k+1, \mathbb{C}^n)$ for suitable k . Notice that here we are only producing sufficient conditions, so that for $k=1$ we clearly obtain a weaker result than Theorem 1. Theorem 2 (6) and (7) deal with Plücker relations between Plücker coordinates lying not necessary on adjacent Grassmannians.

Table 1 (resp. Table 2 in the appendix) show to which permutations $w \in S_4$ (resp. S_5) each one of the points of Theorem 2 applies. The computations for these were performed by Sage [Dev16] and Macaulay2 [GS].

Let $w \in S_n$. In the following, it will be convenient to set $w([0]) := \emptyset$. Moreover, since $\text{in}_{\mathbf{w}}(\mathcal{I}_e) = \mathcal{I}_e$, we can exclude the case $w = e$ right away in the following theorem.

Theorem 2. Let $w \in S_n \setminus \{e\}$. If one of the following conditions holds for w , then $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ contains a monomial of degree 2:

(1) there exist $i \in [n-1]$ with $ws_i > w$ and $j \in [n]$ such that

$$i, j \leq w(i), i \neq j \text{ and } i, j \notin w([i-1]) \cup \{w(i+1)\};$$

(2) there exist $i \in [3, n-1]$ with $ws_i > w$ and $l, x \in [n]$ with $x \neq i-1, l \leq w(i)$ and $w(i+1) \leq x, i-1$, such that

$$i-1, x \in w([i-1]) \cup \{w(i+1)\} \text{ and } l \notin w([i-1]) \cup \{w(i+1)\};$$

- (3) there exist $j \in [2, n-1]$ with $s_j w > w$ and $i \in [n-1], i < j$ such that

$$j \in w([i]), i \notin w([i]), \text{ and } j+1 \leq w(i+1);$$
- (4) there exists $i \in [n-2]$ with $s_i w < w$ and $j \in [n]$ such that

$$i, j \notin w([i+1]), j \leq w(i+2), i+1 \in w([i+1]) \text{ and } i+1 < j;$$
- (5) there exist $i \in [2, n-1]$ and $l \in [2, n], l > i$ with

$$i \notin w([i+1]), l \in w([i]), l > w(i+1) \text{ and } i > w(i+1);$$
- (6) for $i \in [n]$, minimal with $w(i) \neq i$, it holds $w(i) < n$ and, for the minimal $j \in [i+1, n-1]$ such that $w(j) > w(i)$, it holds $w(i) \notin [j-1]$;
- (7) for $i \in [n]$, minimal with $w(i) \neq i$, it holds $w(i) = n$ and, for the minimal $j \in [i+2, n-1]$, such that $w(j) > w(i+1)$, it holds $i \notin w([i+1, j-1])$.

Proof.

- (1) Assume there exist i, j fulfilling the conditions above. Let J be any sequence such that $F(J) = w([i-1]) \cup \{j\}$ and $j_1 = j$, and let L be any sequence such that $F(L) = w([i-1]) \cup \{i, w(i+1)\}$. Then the Plücker relation $R_{J,L}^1$ equals

$$p_J p_L - p_{(J \setminus (j)) \cup (i)} p_{(L \setminus (i)) \cup (j)} - p_{(J \setminus (j)) \cup (w(i+1))} p_{(L \setminus (w(i+1))) \cup (j)}.$$

Taking the initial form with respect to \mathbf{w} we obtain

$$\text{in}_{\mathbf{w}}(R_{J,L}^1) = p_J p_L - p_{(J \setminus (j)) \cup (w(i+1))} p_{(L \setminus (w(i+1))) \cup (j)}.$$

Restricting to X_w , we have $p_{(J \setminus (j)) \cup (w(i+1))} = p_{(w([i-1]), w(i+1))} = 0$ as $ws_i > w$ and so $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ contains the monomial $p_J p_L$.

- (2) Assume such i, l, x exist. Let J be any sequence such that $F(J) = (w([i-1]) \cup \{w(i+1)\}) \setminus \{i-1\}$ and $j_1 = x$, and let L be any sequence such that $F(L) = (w([i-1]) \cup \{w(i+1), l\}) \setminus \{x\}$ the Plücker relation $R_{J,L}^1$, i.e.

$$p_J p_L - p_{(J \setminus (x)) \cup (i-1)} p_{(L \setminus (i-1)) \cup (x)} - p_{(J \setminus (x)) \cup (l)} p_{(L \setminus (l)) \cup (x)}.$$

Taking the initial form with respect to \mathbf{w} we obtain

$$\text{in}_{\mathbf{w}}(R_{J,L}^1) = p_J p_L - p_{(J \setminus (x)) \cup (l)} p_{(L \setminus (l)) \cup (x)}.$$

Note that $(F(L) \setminus \{l\}) \cup \{x\} = w([i-1]) \cup \{w(i+1)\}$ and so restricting to X_w we have $p_{(L \setminus (l)) \cup (x)} = 0$ as $ws_i > w$. So $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ contains the monomial $p_J p_L$.

- (3) Assume such i and j exist and take J any sequence such that $F(J) = w([i])$ and $j_1 = j$, and L any sequence such that $F(L) = (w([i]) \cup \{i, j+1\}) \setminus \{j\}$. Note that $j \in w([i])$ and $s_j w > w$ imply $j+1 \notin w([i+1])$. Then

$$R_{J,L}^1 = p_J p_L - p_{(J \setminus (j)) \cup (i)} p_{(L \setminus (i)) \cup (j)} - p_{(J \setminus (j)) \cup (j+1)} p_{(L \setminus (j+1)) \cup (j)}.$$

Taking the initial form with respect to \mathbf{w} we obtain

$$\text{in}_{\mathbf{w}}(R_{J,L}^1) = p_J p_L - p_{(J \setminus (j)) \cup (j+1)} p_{(L \setminus (j+1)) \cup (j)}.$$

As $(J \setminus (j)) \cup (j+1) \not\leq w([\#J])$ restricting to X_w we have $p_{(w([i]) \setminus (j)) \cup (j+1)} = 0$. Hence, $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ contains the monomial $p_J p_L$.

- (4) Assume such i and j exist and consider L any sequence such that $F(L) = w([i+1]) \cup \{j\}$, and J any sequence such that $F(J) = s_i w([i+1]) = (w([i+1]) \setminus \{i+1\}) \cup \{i\}$ and $j_1 = i$. Then

$$R_{J,L}^1 = p_J p_L - p_{(J \setminus (i)) \cup (i+1)} p_{(L \setminus (i+1)) \cup (i)} - p_{(J \setminus (i)) \cup (j)} p_{(L \setminus (j)) \cup (i)}$$

Taking the initial form with respect to \mathbf{w} yields

$$\text{in}_{\mathbf{w}}(R_{J,L}^1) = p_J p_L - p_{(J \setminus (i)) \cup (j)} p_{(L \setminus (j)) \cup (i)}$$

Now $(J \setminus (i)) \cup (j) = (w([i+1]) \setminus (i+1)) \cup (j)$, but restricting to X_w we have $p_{(J \setminus (i)) \cup (j)} = 0$ as $j > i+1$. Hence, $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ contains the monomial $p_J p_L$.

- (5) Assume such i, l exist, take $J = w([i])$ and $L = (w([i+1]) \setminus \{l\}) \cup \{i\}$. Consider the relation $R_{J,L}^1$:

$$p_J p_L - p_{(J \setminus (l)) \cup (i)} p_{(L \setminus (i)) \cup (l)} - p_{(J \setminus (l)) \cup (w(i+1))} p_{(L \setminus (w(i+1))) \cup (l)}.$$

Taking the initial form with respect to \mathbf{w} yields

$$\text{in}_{\mathbf{w}}(R_{J,L}^1) = p_J p_L - p_{(J \setminus (l)) \cup (w(i+1))} p_{(L \setminus (w(i+1))) \cup (l)}.$$

Restricting to X_w we have $(F(L) \setminus \{w(i+1)\}) \cup \{l\} = (F(w([i+1]) \setminus \{w(i+1)\}) \cup \{l\}) \cup \{i\}$ and $p_{(w([i+1]) \setminus (w(i+1))) \cup (i)} = 0$ as $i > w(i+1)$. So $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ contains the monomial $p_J p_L$.

- (6) First note that $w(i) \neq i$ in particular implies $i < n$. Consider J any sequence such that $F(J) = w([i]) = [i-1] \cup \{w(i)\}$ with $j_1 = w(i)$. Let L be any sequence such that $F(L) = [j-1] \cup \{w(j)\}$. As $w(i) \notin [j-1]$ implies $w(i) > j-1$ and so $w(j) > w(i) > j-1$, then the set $[j-1] \cup \{w(j)\}$ has cardinality j . Then

$$R_{J,L}^1 = p_J p_L - p_{(w(j), [i-1])} p_{(L \setminus (w(j))) \cup (w(i))} - \sum_{r \in [i, j-1]} p_{(r, [i-1])} p_{(L \setminus (r)) \cup (w(i))}$$

Taking the initial form with respect to \mathbf{w} yields

$$\text{in}_{\mathbf{w}}(R_{J,L}^1) = p_J p_L - p_{(w(j), [i-1])} p_{(L \setminus (w(j))) \cup (w(i))}.$$

Since $w(j) > w(i)$, the coordinate $p_{(w(j), [i-1])}$ vanishes in the coordinate ring of X_w , so that $\text{in}_{\mathbf{w}}(R_{J,L}^1) \in \text{in}_{\mathbf{w}}(\mathcal{I}_w)$ is a monomial.

- (7) Consider J any sequence such that $F(J) = [i] \cup \{n\} = w([i]) \cup \{i\}$ such that $j_1 = i$, and let L be any sequence such that $F(L) = [i-1] \cup [i+1, j-1] \cup \{w(j), n\}$. Note that $L \leq w([j])$ as $i \notin w([i+1, j-1])$, and hence we get

$$R_{J,L}^1 = p_J p_L - p_{(w(j), w([i]))} p_{(L \setminus (w(j))) \cup (i)} - \sum_{r \in [i+1, j-1]} p_{(r, w([i]))} p_{(L \setminus (r)) \cup (i)}$$

with initial term $\text{in}_{\mathbf{w}}(R_{J,L}^1) = p_J p_L - p_{(w(j), w([i]))} p_{(L \setminus (w(j))) \cup (i)}$. Further observe that $w(j) > w(i+1) \geq i$, which implies that $p_{(w(j), w([i]))}$ vanishes in the coordinate ring of X_w . Then $R_{J,L}^1$ produces a monomial. \square

Remark 3. In principle, we could have assumed $i \in \{2, 3, \dots, n-1\}$ in Theorem 2 (2). Instead, we exclude the case $i = 2$, since it is never happens under the other assumptions, for which we would have $w(3) \leq 1$ and $ws_2(2) = w(3) > w(2)$ contradicting each other.

Remark 4. In the points (6) and (7) of Theorem 2, the j does not need to exist, in which case the criterion would just not apply.

4.2.1. *Efficiency of the various criteria from Theorem 2.* We want to comment here on how efficient the various points of Theorem 2 are, based on the data we have collected for S_4 (see Table 1) and S_5 (see Table 2). The data can be found at the homepage: <http://www.mi.uni-koeln.de/~lbossing/schubert/>.

For $n = 4$, there are 11 permutations w such that at least one Plücker relation degenerates to a monomial. In the S_5 -case, this happens for 85 permutations.

Among the criteria collected in Theorem 2, point (6) seems to be the most powerful: it detects 9 out of 11 permutations for S_4 , and 65 out of 85 for S_5 . To cover the missing two permutations for S_4 it is enough to combine Theorem 2 (6) with one of the points (1),(4),(7) and one between (2) and (5). So that it is enough to apply three of our criteria to find all $w \in S_4$ such that $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ contains a Plücker relation which degenerates to a monomial.

Theorem 2(1) picks 9 out of 11 permutations in S_4 , and 64 out of 85 for S_5 .

Theorem 2 (3) covers 8 out of 11 permutations yielding monomial initial ideals for S_4 and 57 out of 85 for S_5 .

Theorem 2 (4) detects 4 permutations for S_4 and 36 permutations for S_5 .

Theorem 2 (2) and (5) both finds 2 permutations for $n = 4$ and 22 for $n = 5$, but the elements they see are different.

Finally, Theorem 2 (7) applies to only one permutation, resp. 8 permutations, in the $n = 4$, resp. $n = 5$, case, but it is necessary to cover all the permutations in S_5 containing monomial degenerate Plücker relations. For example, it is the only one among our criteria which can be applied to $s_1s_2s_3s_4s_3s_1s_2s_1$.

4.3. **Plücker relations not degenerating to monomials.** In this section we study some cases in which none of the Plücker relations produces a monomial in the defining ideal $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$. Clearly, this does not have to be equivalent to the irreducibility of the degeneration, but it turns out to be the case for $n = 3$ (by Remark 2) and $n = 4$ (by *Macaulay2* [GS] computations). We do not know whether such an equivalence holds in general.

We have seen in §3.1, that if $v \leq c = s_{n-1}s_{n-2} \cdots s_2s_1$, then the initial ideal $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ coincides with \mathcal{I}_w . In the following proposition we will show that if we multiply c on the right by simple reflections s_{k_1}, \dots, s_{k_r} which commute pairwise and each appear at most once, then none of the Plücker relations degenerates to a monomial in $\text{in}_{\mathbf{w}}(\mathcal{I}_{cs_{k_1} \cdots s_{k_r}})$.

Table 1 (resp. Table 2 in the appendix) show which statements apply to which elements of S_4 (resp. S_5).

Proposition 2. For any $h \in [n - 1]$, none of the Plücker relations degenerates to a monomial in $\text{in}_{\mathbf{w}}(\mathcal{I}_{cs_h})$.

Proof. First of all notice that if $h = 1$, then $cs_1 < c$ and the claim follows from Proposition 1, which says that $\text{in}_{\mathbf{w}}(\mathcal{I}_c) = \mathcal{I}_c$.

If $h \in [2, n - 1]$, then $cs_h > c$. In this case, if $J \leq c([\#J])$ and $L \leq c([\#L])$, then $\text{in}_{\mathbf{w}}(R_{J,L}^a)$ being a monomial on $X_{cs_h}^a$ implies that it is a monomial on X_c^a too. But this is not possible, again by Proposition 1. Therefore we can assume that $L \not\leq c([\#L])$ or $J \not\leq c([\#J])$. We set $k := h - 1 \in [n - 2]$ for convenience.

Recall that for any $i \in [k] \cup [k + 2, n - 1]$

$$\begin{aligned} cs_{k+1}/\langle s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_{n-1} \rangle &= s_r \cdots s_i / \langle s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_{n-1} \rangle \\ &= c / \langle s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_{n-1} \rangle. \end{aligned}$$

w one-line	w red. word	mono	(1)	(2)	(3)	(4)	(5)	(6)	(7)
[1, 2, 3, 4]	1	—	—	—	—	—	—	—	—
[1, 2, 4, 3]	s_3	—	—	—	—	—	—	—	—
[1, 3, 2, 4]	s_2	—	—	—	—	—	—	—	—
[1, 3, 4, 2]	s_2s_3	×	×	—	×	—	—	×	—
[1, 4, 2, 3]	s_3s_2	—	—	—	—	—	—	—	—
[1, 4, 3, 2]	$s_2s_3s_2$	—	—	—	—	—	—	—	—
[2, 1, 3, 4]	s_1	—	—	—	—	—	—	—	—
[2, 1, 4, 3]	s_3s_1	—	—	—	—	—	—	—	—
[2, 3, 1, 4]	s_1s_2	×	×	—	×	—	—	×	—
[2, 3, 4, 1]	$s_1s_2s_3$	×	×	—	×	×	—	×	—
[2, 4, 1, 3]	$s_3s_1s_2$	×	×	—	×	—	—	×	—
[2, 4, 3, 1]	$s_1s_2s_3s_2$	×	×	—	×	×	—	×	—
[3, 1, 2, 4]	s_2s_1	—	—	—	—	—	—	—	—
[3, 1, 4, 2]	$s_2s_3s_1$	×	—	—	×	—	—	×	—
[3, 2, 1, 4]	$s_1s_2s_1$	—	—	—	—	—	—	—	—
[3, 2, 4, 1]	$s_1s_2s_3s_1$	×	×	—	—	×	—	×	—
[3, 4, 1, 2]	$s_2s_3s_1s_2$	×	×	×	×	—	×	×	—
[3, 4, 2, 1]	$s_1s_2s_3s_1s_2$	×	×	—	×	—	—	×	—
[4, 1, 2, 3]	$s_3s_2s_1$	—	—	—	—	—	—	—	—
[4, 1, 3, 2]	$s_2s_3s_2s_1$	—	—	—	—	—	—	—	—
[4, 2, 1, 3]	$s_3s_1s_2s_1$	—	—	—	—	—	—	—	—
[4, 2, 3, 1]	$s_1s_2s_3s_2s_1$	×	×	—	—	×	—	—	×
[4, 3, 1, 2]	$s_2s_3s_1s_2s_1$	×	—	×	—	—	×	—	—
[4, 3, 2, 1]	$s_1s_2s_3s_1s_2s_1$	—	—	—	—	—	—	—	—
24		11	9	2	8	4	2	9	1

TABLE 1. Applying Theorem 2 to S_4

In one-line notation $cs_{k+1} = [n, 1, \dots, k-1, k+1, k, k+2, \dots, n-1]$. Hence, if $I \leq cs_{k+1}([\#I])$, but $I \not\leq c([\#I])$, then $\#I = k+1$ and it must hold

$$(4.3) \quad F(I) = [k-1] \cup \{k+1, i\} \text{ with } i \in [k+2, n].$$

Therefore a Plücker $R_{J,L}^m$ can produce a monomial in $\text{in}_{\mathbf{w}}(\mathcal{I}_{cs_h})$ only if J is a sequence such that $F(J) = [k-1] \cup \{k+1, j\}$ with $j_1 = j$ or $F(L) = [k-1] \cup \{k+1, l\}$ for $j, l \in [k+2, n]$. If $\#J = \#L$, then $\text{in}_{\mathbf{w}}(R_{J,L}^m) = R_{J,L}^m$, hence we only have to consider the case $\#J < \#L$.

Let $\#L = p > k+1$, then by (4.3) we have $F(J) = [k-1] \cup \{k+1, j\}$ and $F(L) = [p-1] \cup \{l\}$ for $j_1 = j \in [k+2, n]$ and $l \in [p, n]$. Note that $j \in J$ is the only possible element to swap for elements in L non-trivially, so that we impose $j \notin L$ (otherwise $R_{J,L}^m = 0$ for any m). Remember that we may assume $j \in [p, n]$. Then

$$(4.4) \quad \text{in}_{\mathbf{w}}(R_{J,L}^1) = pJpL - \mathcal{P}_{(J \setminus \{j\}) \cup \{l\}} \mathcal{P}_{(L \setminus \{l\}) \cup \{j\}} - \mathcal{P}_{(J \setminus \{j\}) \cup \{k\}} \mathcal{P}_{(L \setminus \{k\}) \cup \{j\}}.$$

As $[k-1] \cup \{k+1, l\} \leq cs_{k+1}([k+1])$ and $[k-1, p-1] \cup \{j\} \leq cs_{k+1}([p])$ at least two terms are non-zero on $X_{cs_{k+1}}$.

Now, assume $\#L = k+1$ and $\#J = q < k+1$. Then we have

$$F(L) = [k-1] \cup \{k+1, l\} \text{ and } F(J) = [q-1] \cup \{j\},$$

for $j = j_1, l \in [k+2, n]$ and $j \notin L$ in order for the relation to be non-trivial. We obtain

$$(4.5) \quad \text{in}_{\mathbf{w}}(R_{J,L}^1) = pJpL - \mathcal{P}_{(J \setminus \{j\}) \cup \{k+1\}} \mathcal{P}_{(L \setminus \{k+1\}) \cup \{j\}} - \mathcal{P}_{(J \setminus \{j\}) \cup \{l\}} \mathcal{P}_{(L \setminus \{l\}) \cup \{j\}}.$$

As $[q-1] \cup \{l\} \leq cs_{k+1}([q])$ and $[k-1] \cup \{k+1, j\} \leq cs_{k+1}([k+1])$, the relation $R_{J,L}^1$ does not degenerate to a monomial. \square

Corollary 3. Let $h \in [n-1]$. Then $\text{in}_{\mathbf{w}}(\mathcal{I}_{cs_h})$ is a pure difference ideal in the quotient $\mathbb{C}[p_I]/(p_I \mid I \not\leq cs_i(\#I))$.

Proof. First note that if $h = 1$, then by Proposition 1 $\text{in}_{\mathbf{w}}(\mathcal{I}_{cs_1}) = \mathcal{I}_{cs_1}$. The Plücker relations involving non-vanishing Plücker coordinates on X_{cs_1} are for $q < p \leq j < l \leq n$ the following pure differences

$$p_{[q-1] \cup \{j\}} p_{[p-1] \cup \{l\}} - p_{[q-1] \cup \{l\}} p_{[p-1] \cup \{j\}}.$$

Notice that the index sets of the Plücker coordinates in the above equation (as well as in the rest of this proof) are sets, and hence by convention, as sequences they are arranged in an increasing order, while in the proof of the previous result we always had $j = j_1$. This only affect the relation by a global sign.

If $h \in [2, n-1]$, we can set again $k := h-1$. In the proof of Proposition 2 we have seen in equations (4.4) and (4.5) the form of the additional relations for cs_{k+1} . Note that in (4.4) we have $[k-1] \cup [k+1, p-1] \cup \{j, l\} \not\leq cs_{k+1}([p])$ and hence, the middle term vanishes on $X_{cs_{k+1}}$. Similarly observe for (4.5) that $[k-1] \cup \{j, l\} \not\leq cs_{k+1}([k+1])$ as $j, l \geq k+2$. So all generators of $\text{in}_{\mathbf{w}}(\mathcal{I}_{cs_{k+1}})$ are pure differences in $\mathbb{C}[p_I]/(p_I \mid I \not\leq cs_{k+1}(\#I))$. \square

Remark 5. Note that while $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ and \mathcal{I}_w have the same generators for $w \leq c$, this is not true for cs_{k+1} with $k \geq 1$. Here taking the initial ideal with respect to \mathbf{w} modifies the generators.

The following proposition generalizes Proposition 2 to a product of pairwise distinct commuting simple reflections.

Proposition 3. Take $k_1, \dots, k_r \in [n-1]$ with $|k_i - k_j| > 1$ for all $i \neq j$, then none of the Plücker relations degenerates to a monomial in $\text{in}_{\mathbf{w}}(\mathcal{I}_{cs_{k_1} \dots cs_{k_r}})$.

Proof. We may assume $k_1 < k_2 < \dots < k_r$ without loss of generality. Moreover, since we are multiplying by pairwise distinct commuting reflections, and as Plücker relations only involve pairs of Grassmannians, it is enough to consider the cases $r = 1, 2$. The case $r = 1$ was dealt with in Proposition 2, so we are left with $r = 2$.

We consider two cases: firstly, we deal with the case $k_1 = 1$, and then we suppose $k_1 \neq 1$.

If $k_1 = 1$, $cs_1 < c$ can be identified with the Coxeter element $\tilde{c} = \tilde{s}_{n-2} \dots \tilde{s}_1$ in S_{n-1} (via $s_i \mapsto \tilde{s}_{i-1}$ for $i \in [2, n-1]$). In this case, $cs_1s_{k_2} \in \langle s_2, \dots, s_{n-1} \rangle$ and, by Corollary 2, we have $\text{in}_{\mathbf{w}}(\mathcal{I}_{cs_1s_{k_2}}) = \text{in}_{\mathbf{w}}(\mathcal{I}_{\tilde{c}\tilde{s}_{k_2}})$. We then apply Proposition 2 to obtain the claim.

Now denote $k_1 := k + 1$ and $k_2 := g + 1$ and recall, that by assumption $k < g + 1$. As in the proof of Proposition 2, we only have to deal with Plücker relations $R_{J,L}^m$ with $\#J \neq \#L$, where $J \not\leq cs_{k+1}s_{g+1}([\#J])$ or $L \not\leq cs_{k+1}s_{g+1}([\#L])$. We can further reduce to the case $\#J = k + 1$, $j_1 = j$, and $\#L = g + 1$, otherwise the Plücker relations are the same as the ones considered in Proposition 2, and the result has been proven above.

Consider relations $R_{J,L}^m$ with $\#J = k + 1$, $\#L = g + 1$ and $J \leq cs_{k+1}s_{g+1}([k + 1])$, $J \not\leq c([k + 1])$ and $L \leq cs_{k+1}s_{g+1}([g + 1])$, $L \not\leq c([g + 1])$. We have shown in Proposition 2 that in this case it must hold

$$F(J) = [k - 1] \cup \{k + 1, j\}, \quad F(L) = [g - 1] \cup \{g + 1, l\}$$

with $j \in [k + 2, n]$ and $l \in [g + 2, n]$. In order for the relation to be non-trivial we may assume $j \notin L$. Since $k + 1 \in [g - 1]$, the only relation to be considered is

$$\begin{aligned} R_{J,L}^1 &= p_J p_L - p_{(J \setminus (j)) \cup (l)} p_{(L \setminus (l)) \cup (j)} - p_{(J \setminus (j)) \cup (g+1)} p_{(L \setminus (g+1)) \cup (j)} \\ &\quad - \sum_{r \in [k+1, g-1]} p_{(J \setminus (j)) \cup (r)} p_{(L \setminus (r)) \cup (j)}. \end{aligned}$$

It degenerates to

$$\text{in}_{\mathbf{w}}(R_{J,L}^1) = p_J p_L - p_{(J \setminus (j)) \cup (l)} p_{(L \setminus (l)) \cup (j)} - p_{(J \setminus (j)) \cup (g+1)} p_{(L \setminus (g+1)) \cup (j)}.$$

The monomial $p_{(J \setminus (j)) \cup (l)} p_{(L \setminus (l)) \cup (j)}$ does not vanish on the coordinate ring of $X_{cs_{k+1}s_{l+1}}$ (and thus of $X_{cs_{k_1} \dots s_{k_r}}$). Hence, $\text{in}_{\mathbf{w}}(R_{J,L}^1)$ is not monomial and this finishes the proof. \square

Lemma 3 below shows that the Coxeter word $c = s_{n-1} \dots s_2 s_1$ is in fact special among all Coxeter words regarding the degeneration.

Lemma 3. Let $w \in S_n$ have a reduced expression $\underline{w} = s_{i_r} \dots s_{i_1}$ with $i_k \neq i_l$ for all $k \neq l$. Then none of the Plücker relations degenerates to a monomial in $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ if and only if $w \leq c$.

Proof. " \Leftarrow " by Proposition 1.

" \Rightarrow " Assume $w = s_{i_r} \dots s_{i_1}$ is a product of pairwise distinct simple reflections. First note that $w \not\leq c$ implies there exists an $i_k \in \{i_1, \dots, i_r\}$ such that $i_k + 1 = i_l$ for $l < k$. We choose $i = i_k$, such that k is minimal with this property. In particular, if there exists t with $i_t + 1 = i$ then $t < k$. Since s_i commutes with all reflections s_{i_m} with $m > k$, as in this case $i_m \neq i \pm 1$ by minimality of k , we observe

$$w = s_i s_{i_r} \dots s_{i_{k+1}} s_{i_{k-1}} \dots s_{i_1} \in s_i \langle s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_{n-1} \rangle.$$

We deduce that $w([i]) = [i - 1] \cup \{i + 1\}$. Moreover, notice $w(i + 1) \geq i + 2$, since $i + 1$ is moved only by s_i and s_{i+1} , but we apply s_{i+1} first and by hypothesis there

are no other occurrences of s_{i+1} . We can now produce the degree two monomial in $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ by choosing as J any sequence such that $F(J) = w([i])$ and $j_1 = i + 1$, and as L any sequence with $F(L) = [i] \cup \{i + 2\}$, so that

$$R_{J,L}^1 = pJP_L - p_{(J \setminus (i+1)) \cup (i+2)} p_{(L \setminus (i+2)) \cup (i+1)} - p_{(J \setminus (i+1)) \cup (i)} p_{(L \setminus (i)) \cup (i+1)},$$

$$\text{in}_{\mathbf{w}}(R_{J,L}^1) = pJP_L - p_{(J \setminus (i+1)) \cup (i+2)} p_{(L \setminus (i+2)) \cup (i+1)}.$$

As $[i - 1] \cup \{i + 2\} \not\leq w([i])$ the second term vanishes on X_w . \square

4.4. More and more monomials. If we can write a permutation $u \in S_n$ as a product of two permutations v, w belonging to two distinct parabolic subgroups which centralize each other, then we can check how a Plücker relation degenerates on \mathcal{I}_u by looking at the ideals \mathcal{I}_v and \mathcal{I}_w . Lemma 4 concerns defining ideals for Schubert varieties and allows us to deduce Corollary 4, which suggests an inductive procedure on n to find Schubert varieties that become reducible under Feigin's degeneration.

Lemma 4. Let $v, w \in S_n$ assume there exist two sets of simple reflections $\mathcal{S}_v = \{s_{i_1}, \dots, s_{i_r}\}$ and $\mathcal{S}_w = \{s_{j_1}, \dots, s_{j_s}\}$ such that $|i_h - j_l| > 1$ for all $h \in [r], l \in [s]$ with $v \in \langle \mathcal{S}_v \rangle$ and $w \in \langle \mathcal{S}_w \rangle$. Then for all sequences J, L with $k \leq \#J$ we have

$$R_{J,L}^k|_{X_{vw}} = R_{J,L}^k|_{X_v} \text{ or } R_{J,L}^k|_{X_{vw}} = R_{J,L}^k|_{X_w}.$$

Corollary 4. Let $v, w \in S_n$ assume there exist two sets of simple reflections $\mathcal{S}_v = \{s_{i_1}, \dots, s_{i_r}\}$ and $\mathcal{S}_w = \{s_{j_1}, \dots, s_{j_s}\}$ such that $|i_h - j_l| > 1$ for all $h \in [r], l \in [s]$ with $v \in \langle \mathcal{S}_v \rangle$ and $w \in \langle \mathcal{S}_w \rangle$. Then

- (1) None of the $R_{J,L}^k$ degenerates to a monomial nor in $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ neither in $\text{in}_{\mathbf{w}}(\mathcal{I}_v)$, if and only if none of the $R_{J,L}^k$ degenerates to a monomial in $\text{in}_{\mathbf{w}}(\mathcal{I}_{vw})$.
- (2) If $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ or $\text{in}_{\mathbf{w}}(\mathcal{I}_v)$ contains a monomial degenerate Plücker relation, then so does $\text{in}_{\mathbf{w}}(\mathcal{I}_{vw})$.

Remark 6. From the previous corollary we see that the bigger n is, the more Schubert varieties become reducible after degenerating them à la Feigin, since there are several ways of embedding S_m into S_n for $m < n$ as a parabolic subgroup. Indeed, the number of permutations $v \in S_n$ such that at least one Plücker relation degenerates to a monomial in $\text{in}_{\mathbf{w}}(\mathcal{I}_v)$ is 0,1,11,85 for $n = 2, 3, 4, 5$, respectively. As a curiosity, we mention here that there is exactly one sequence in the On-Line Encyclopedia of Integer Sequences [Slo, Sequence A129180] whose first four terms are 0, 1, 11, 85, namely the *Total area below all Schroeder paths of semilength n* .

5. DEGENERATE SCHUBERT AND RICHARDSON VARIETIES

In this section we explore how degenerate Schubert varieties behave under the embedding of the degenerate flag variety $\mathcal{F}\ell_n^a$ into a larger partial flag variety given by Cerulli Irelli and the second author in [CIL15].

5.1. Degenerate flag varieties and flag varieties of higher rank. We start by introducing some notation and recalling the main result of [CIL15].

Let ω_i denote the i -th fundamental weight for SL_{2n-2} and consider the parabolic subgroup $P := P_{\omega_1 + \omega_3 + \dots + \omega_{2n-3}}$ of SL_{2n-2} . Then, SL_{2n-2}/P is the variety of (partial) flags in \mathbb{C}^{2n-2} whose points are flags of vector spaces of odd dimensions. Its Schubert varieties \tilde{X}_w are indexed by minimal length coset representatives $w \in S_{2n-2}/W_P$, where W_P is the Weyl group of the Levi of P . More precisely, if $\tilde{s}_i \in S_{2n-2}$ denotes

the simple transposition $(i, i + 1)$, then $W_P = \langle \tilde{s}_2, \tilde{s}_4, \dots, \tilde{s}_{2n-4} \rangle$. Let $w_n \in S_{2n-2}$ be defined by

$$w_n(i) = \begin{cases} r & \text{if } i = 2r, r \geq 1, \\ n + r - 1 & \text{if } i = 2r - 1, r \in [n - 1]. \end{cases}$$

The following Theorem can be found in [CIL15].

Theorem 3 ([CIL15]). The degenerate flag variety $\mathcal{F}\ell_n^a$ is isomorphic to the Schubert variety $\tilde{X}_{w_n} \subset SL_{2n-2}/P$.

5.1.1. *Translation into Plücker coordinates.* We describe here the isomorphism of Theorem 3 in terms of Plücker coordinates. Recall that whenever we index Plücker coordinates by a set, we really mean the associated sequence obtained by increasingly ordering the elements of the given set.

Let $J \in \binom{[2n-2]}{2k-1}$, with $k \in [n-1]$, then $J \leq w_n([2k-1]) = [k-1] \cup [n, n+k-1]$ if and only if

$$(5.1) \quad [k-1] \subset J \subset [k+n-1].$$

In order to give the translation of the isomorphism in terms of coordinate rings, we need to set some notation. Let $k \in [n-1]$, we denote by $\{\leq w_n\}^{(2k-1)}$ the set of $J \in \binom{[2n-2]}{2k-1}$, with $J \leq w_n([2k-1])$. There is hence a bijection

$$(5.2) \quad \{\leq w_n\}^{(2k-1)} \rightarrow \binom{[n]}{k}, \quad J \mapsto \tau_k(J \setminus [k-1])$$

where $\tau_k : [n+k-1] \rightarrow [n]$ is given by

$$\tau_k(j) \mapsto \begin{cases} j & \text{if } j \in [k, n], \\ j - n & \text{if } j \in [n+1, n+k-1]. \end{cases}$$

For a sequence $I = (i_1, \dots, i_k) \in \mathcal{S}(n, k)$ we set $\tau_k(I) := (\tau_k(i_1), \dots, \tau_k(i_k)) \in \mathcal{S}(n, k)$. If $\rho_k : [n] \rightarrow [k, n+k-1]$ is given by

$$\rho_k(j) \mapsto \begin{cases} j & \text{if } j \in [k, n], \\ j + n & \text{if } j \in [k-1], \end{cases}$$

then the inverse map to (5.2) is given by

$$\binom{[n]}{k} \rightarrow \{\leq w_n\}^{(2k-1)}, \quad I \mapsto [k-1] \cup \rho_k(I).$$

On the level of sequences, this lifts to a map

$$\begin{aligned} \mathcal{S}(n, k) & \xrightarrow{\tilde{\rho}_k} \{J \in \mathcal{S}(2n-2, 2k-1) \mid F(J) \in \{\leq w_n\}^{(2k-1)}\}, \\ (i_1, \dots, i_k) & \mapsto (1, 2, \dots, k-1, \rho_k(i_1), \dots, \rho_k(i_k)) \end{aligned}$$

Fix an ordered basis $(\tilde{e}_j)_{j \in [2n-2]}$ of \mathbb{C}^{2n-2} , then the linear algebraic description of \tilde{X}_{w_n} is

$$\tilde{X}_{w_n} = \left\{ \begin{array}{l} \{0\} \subset W_1 \subset W_3 \subset \dots \subset W_{2n-3} \\ \left. \begin{array}{l} W_{2k-1} \in \text{Gr}(2k-1, \mathbb{C}^{2n-2}) \\ \text{span}_{\mathbb{C}}\{\tilde{e}_j \mid j \in [k-1]\} \subset W_{2k-1}, \\ W_{2k-1} \subset \text{span}_{\mathbb{C}}\{\tilde{e}_j \mid j \in [n+k-1]\}. \end{array} \right\} \end{array} \right.$$

Denote by $(e_i)_{i \in [n]}$ an ordered basis for \mathbb{C}^n . For $k \in [n-1]$ define the projection operator (which we also denote by π_k as in [CIL15])

$$\begin{aligned} \pi_k : \text{span}_{\mathbb{C}}\{\tilde{e}_j \mid j \in [n+k-1]\} &\rightarrow \mathbb{C}^n = \text{span}_{\mathbb{C}}\{e_i \mid i \in [n]\}, \\ \tilde{e}_j &\mapsto \begin{cases} e_{\tau_k(j)} & \text{if } j \in [k, n+k-1], \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Then there is an isomorphism, that we denote by the same symbol, of algebraic varieties

$$\tilde{X}_{w_n}^{(2k-1)} := \left\{ \begin{array}{l} U \in \text{Gr}(2k-1, \mathbb{C}^{2n-2}) \\ \left. \begin{array}{l} \text{span}_{\mathbb{C}}\{\tilde{e}_j \mid j \in [2i-2]\} \subset U, \\ U \subset \text{span}_{\mathbb{C}}\{\tilde{e}_j \mid j \in [n+2k-2]\}. \end{array} \right\} \xrightarrow{\pi_k} \text{Gr}(k, \mathbb{C}^n), \\ U \mapsto \pi_k(U) \end{array} \right.$$

and the desired isomorphism (cf. [CIL15]) is given by

$$(5.3) \quad \xi : \tilde{X}_{w_n} \rightarrow \mathcal{F}\ell_n^a, \quad (W_{2k-1})_{k \in [n-1]} \mapsto (\pi_k(W_{2k-1}))_{k \in [n-1]}.$$

Remark 7. In [CIL15], an embedding of $\zeta : \mathcal{F}\ell_n \hookrightarrow SL_{2n-2}/P$ is given, and hence the isomorphism from Theorem 3 is rather the inverse of the isomorphism ξ we consider here. We prefer to work with ξ instead of ζ since in this way we obtain an induced map from the coordinate ring of $\mathcal{F}\ell_n^a$ to the coordinate ring of \tilde{X}_{w_n} , which we make explicit in the following.

For SL_{2n-2}/P we also have an embedding into the product of Grassmannians

$$SL_{2n-2}/P \hookrightarrow \text{Gr}(1, \mathbb{C}^{2n-2}) \times \text{Gr}(3, \mathbb{C}^{2n-2}) \times \dots \times \text{Gr}(2n-3, \mathbb{C}^{2n-2}),$$

and hence a Plücker embedding. Plücker coordinates for $\text{Gr}(2k-1, \mathbb{C}^{2n-2})$ with $k \in [n-1]$ are denoted by \tilde{p}_J , $J \in \mathcal{S}(2n-2, 2k-1)$. Let $I = (i_1, \dots, i_k)$ then

$$\pi_k^* : \mathbb{C}[\text{Gr}(k, n)] \rightarrow \mathbb{C}[\tilde{X}_w^{(2k-1)}], \quad p_I \mapsto \tilde{p}_{\tilde{\rho}_k(I)}.$$

As π_k^* is compatible with Plücker relations, we have an isomorphism

$$\xi^* : \mathbb{C}[\mathcal{F}\ell_n^a] \rightarrow \mathbb{C}[\tilde{X}_{w_n}], \quad p_I \mapsto \pi_{\#I}^*(p_I).$$

Notice that even if I is ordered increasingly, $\tilde{\rho}_k(I)$ needs not be ordered increasingly. To get an increasing sequence we have to multiply by some sign. While keeping track of the sign is fundamental to check that Plücker relations are satisfied, it is not relevant to us, as we only deal with vanishing of certain Plücker coordinates, which of course vanish independently of their sign.

5.2. Richardson varieties in SL_{2n-2}/P . Let $u, v \in S_{2n-2}$ be minimal length coset representatives of S_{2n-2}/W_P and assume that $u \leq v$. We denote by $\tilde{X}_v^u := \tilde{X}_v \cap \tilde{X}^u \subseteq SL_{2n-2}/P$ the corresponding Richardson variety. Recall that its defining ideal in $\mathbb{C}[p_I \mid \#I \equiv 1 \pmod{2}, I \subset [2n-2]]$ is

$$(5.4) \quad \mathcal{I}_v^u = (R_{J,L}^k) + (p_I)_{I \not\leq v([\#I])} + (p_I)_{I \not\leq u([\#I])}.$$

In the following we will show that for appropriate permutations $x \in S_n$, $u, v \in S_{2n-2}$ with $u \leq v \leq w_n$, the isomorphism ξ^* induces an isomorphism between the coordinate rings

$$\mathbb{C}[X_x^a] \rightarrow \mathbb{C}[\tilde{X}_v^u].$$

To stress out the fact that such an isomorphism really comes from the embedding ζ , we will express it as $\zeta(X_x^a) = \tilde{X}_v^u$.

Since $\mathbb{C}[X_x^a] = \mathbb{C}[\mathcal{F}\ell_n^a]/(p_I \mid I \not\leq x([\#I]))$ and $\mathbb{C}[\tilde{X}_v^u] = \mathbb{C}[SL_{2n-2}/P]/(p_K \mid K \not\leq v([\#K]), K \not\leq u([\#K]))$, the claim will be proven by verifying that

$$(5.5) \quad ((K \leq v([\#K]) \text{ and } K \geq u([\#K])) \Rightarrow \tau_k(K \setminus [k-1]) \leq x([k]),$$

where $k := \frac{\#K+1}{2}$, and the opposite direction

$$(5.6) \quad I \leq x([\#I]) \Rightarrow \left(\begin{array}{l} [k-1] \cup \rho_{\#I}(I) \leq v([n-1+\#I]) \\ [k-1] \cup \rho_{\#I}(I) \geq u([n-1+\#I]) \end{array} \right).$$

An important role will be played by the following permutation $y_n \in S_{2n-2}$:

$$y_n(i) = \begin{cases} 1 & \text{if } i = 1, \\ r+1 & \text{if } i = 2r, r \in [n-1], \\ n+r-1 & \text{if } i = 2r-1, r \in [n-1]. \end{cases}$$

Notice that for any $m \in [n-1]$

$$\tilde{s}_m \tilde{s}_{m-1} \dots \tilde{s}_1 y_n(i) = \begin{cases} m+1 & \text{if } i = 1, \\ r & \text{if } i = 2r, r \in [m], \\ r+1 & \text{if } i = 2r, r \in [m+1, n-1], \\ n+r-1 & \text{if } i = 2r-1, r \in [n-1], \end{cases}$$

and, by (2.2), $y_n < \tilde{s}_m \tilde{s}_{m-1} \dots \tilde{s}_1 y_n \leq w_n$.

Lemma 5. Let $m \in [n-1]$ and $x := s_m s_{m-1} \dots s_1 \in S_n$. Then,

$$\zeta(X_x^a) = \tilde{X}_{\tilde{s}_m \tilde{s}_{m-1} \dots \tilde{s}_1 y_n}^{y_n}.$$

Proof. Let $I \in \binom{[n]}{k}$. Then, by (2.4), $I \leq x([k])$ if and only if

$$I = \begin{cases} [k-1] \cup \{i\}, i \in [k, m+1] & \text{if } k \leq m, \\ [k] & \text{if } k > m. \end{cases}$$

On the other hand, let $K \in \binom{[2k-1]}{2k-1}$, then both $K \leq \tilde{s}_m \tilde{s}_{m-1} \dots \tilde{s}_1 y_n([2k-1])$ and $K \geq y_n([2k-1])$ hold if and only if

$$K = \begin{cases} [k-1] \cup [n+1, n+k-1] \cup \{i\}, i \in [k, m+1] & \text{if } k \leq m, \\ [k] \cup [n+1, n+k-1] & \text{if } k > m. \end{cases}$$

These two facts imply (5.5) and (5.6). \square

Combining Lemma 5 with Proposition 1 we obtain the following corollary.

Corollary 5. Let $x = s_m s_{m-1} \dots s_1 \leq c$ and consider the Schubert variety $X_x \subset \mathcal{F}l_n$. Then there is an isomorphism

$$X_v \cong \tilde{X}_{\tilde{s}_m \tilde{s}_{m-1} \dots \tilde{s}_1 y_n}^{y_n} \subset SL_{2n-2}/P.$$

6. SCHUBERT DIVISORS

In this section we focus on Schubert divisors and apply the results from previous sections to them. In this case we can completely answer the question whether or not they stay irreducible under the degeneration.

Let $w_0 \in S_n$ be the longest element, then all Schubert divisors are indexed by permutations of the form $w = w_0 s_i$ for $i \in [n-1]$. Note that

$$w(k) = \begin{cases} n - k + 1 & \text{if } k \neq i, i + 1, \\ n - i & \text{if } k = i, \\ n - i + 1 & \text{if } k = i + 1. \end{cases}$$

The following Theorem 4 is an application of Theorem 2 (1) and (2).

Theorem 4. Let $n > 2$ and $w \in S_n$ be such that $ws_i = w_0$. If n is odd assume $i \neq \frac{n+1}{2}$, for even n there is no additional assumption. Then X_w^a is reducible.

Proof. We consider four cases separately: $i < \frac{n}{2}$, $i = \frac{n}{2}$, $i \geq \frac{n+3}{2}$, and $i = \frac{n+2}{2}$. Notice that they cover all possibilities, since $i > \frac{n}{2}$ together with the assumption $i \neq \frac{n+1}{2}$ implies $i > \frac{n+1}{2}$, hence $i \geq \frac{n+2}{2}$. We will deal with the first two cases by applying Theorem 2 (1), while we will use Theorem 2 (2) for the remaining two.

First of all, notice that $w_0 = ws_i > w$.

Case 1: If $i < \frac{n}{2}$, then

$$(6.1) \quad w(k) = n - k + 1 \geq n - i + 2 > \frac{n}{2} + 2 > i, \quad \text{for any } k \leq i - 1,$$

$$w(i) = n - i > \frac{n}{2} > i,$$

and

$$(6.2) \quad w(i+1) = n - i + 1 > \frac{n}{2} + 1 > i.$$

We conclude that $i \notin w([i+1])$ and we can hence apply Theorem 2 (1) with $j = w(i)$.

Case 2: If $i = \frac{n}{2}$, then (6.1) and (6.2) still hold, but $w(i) = i$, so that $i \notin w([i-1]) \cup \{w(i+1)\}$, but we cannot choose $j = w(i)$. Nevertheless, (6.1) and (6.2) imply that any j with $j \leq i-1 < i = w(i)$ (which exists, since $n > 2$) fulfills the hypotheses of Theorem 2 (1).

Case 3: Let $i \geq \frac{n+3}{2}$, so that $n \leq 2i - 3$ and $n - i + 2 \leq 2i - 3 - i + 2 = i - 1$. Note further that $w(i+1) = n+i-1 \leq \frac{n+3}{2} - 1 \leq i-1$. Thus $w(n-i+2) = i-1 \in w([i-1])$ and we can apply Theorem 2 (2) with $l = w(i)$ and $x = w(i+1)$.

Case 4: Consider $i = \frac{n}{2} + 1$. In this case, $w(i+1) = n - i + 1 = \frac{n}{2} = i - 1 \in w([i-1]) \cup \{w(i+1)\}$ and we can apply Theorem 2 (2) with x any element in $w([i-1])$ and $l = w(i)$. \square

For flag varieties $\mathcal{F}\ell_n$ with n odd, the next proposition explains why the case of w_0s_i for $i = \frac{n+1}{2}$ is special. This is another instance, of a degenerate Schubert variety being isomorphic to a Richardson variety in SL_{2n-2}/P . However, unlike the degenerate Schubert varieties of form X_v^a , for $v \leq c$, this one is not isomorphic to the original Schubert variety.

Proposition 4. Let $i \geq 2$ and $n = 2i - 1$. Then $\zeta(X_{w_0s_i}^a) = \tilde{X}_{w_n}^{\tilde{s}_{2i-1}}$.

Proof. First note that $w_0s_i([i]) = \{n-i\} \cup [n-i+2, n] = \{i-1\} \cup [i+1, n]$ and $w_0([i]) = [n-i+1, n] = [i, n]$. Let $J \in \binom{[n]}{k}$, then $J \not\leq w_0s_i([k]) = [n-k+1, n]$ if and only if $k = i$ and $J = [i, n]$.

On the other hand, recall that $w_n([2k-1]) = [k-1] \cup [n+k-1, n]$ and

$$\tilde{s}_{2i-1}([2k-1]) = \begin{cases} [2k-1] & \text{if } k \neq i, \\ [2i-2] \cup \{2i\} & \text{if } k = i. \end{cases}$$

If $K \in \binom{[2n-2]}{2k-1}$ is such that $K \leq w_n([2k-1])$, then $K \not\leq \tilde{s}_{2i-1}([2k-1])$ if and only if $k = i$ and $K = [2i-1] = [n]$.

At this point the claim follows from $\pi_i^*(p_{[i,n]}) = \tilde{p}_{[i-1] \cup \rho_i([i,n])} = \tilde{p}_{[n]}$. \square

Corollary 6. (1) If n is even, then all Schubert divisors $X_{w_0s_i} \subset \mathcal{F}\ell_n$ become reducible under Feigin's degeneration.

(2) If n is odd, then the Schubert divisor $X_{w_0s_{\frac{n+1}{2}}} \subset \mathcal{F}\ell_n$ stays irreducible under Feigin's degeneration, while all the others become reducible.

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w one-line	w red. word	mono.	(1)	(2)	(3)	(4)	(5)	(6)	(7)
[2, 1, 3, 4, 5]	s_1	—	—	—	—	—	—	—	—
[2, 1, 3, 5, 4]	s_4s_1	—	—	—	—	—	—	—	—
[2, 1, 4, 3, 5]	s_3s_1	—	—	—	—	—	—	—	—
[2, 1, 4, 5, 3]	$s_3s_4s_1$	×	×	—	×	—	—	—	—
[2, 1, 5, 3, 4]	$s_4s_3s_1$	—	—	—	—	—	—	—	—
[2, 1, 5, 4, 3]	$s_3s_4s_3s_1$	—	—	—	—	—	—	—	—
[2, 3, 1, 4, 5]	s_1s_2	×	×	—	×	—	—	×	—
[2, 3, 1, 5, 4]	$s_4s_1s_2$	×	×	—	×	—	—	×	—
[2, 3, 4, 1, 5]	$s_1s_2s_3$	×	×	—	×	×	—	×	—
[2, 3, 4, 5, 1]	$s_1s_2s_3s_4$	×	×	—	×	×	—	×	—
[2, 3, 5, 1, 4]	$s_4s_1s_2s_3$	×	×	—	×	×	—	×	—
[2, 3, 5, 4, 1]	$s_1s_2s_3s_4s_3$	×	×	—	×	×	—	×	—
[2, 4, 1, 3, 5]	$s_3s_1s_2$	×	×	—	×	—	—	×	—
[2, 4, 1, 5, 3]	$s_3s_4s_1s_2$	×	×	—	×	—	—	×	—
[2, 4, 3, 1, 5]	$s_1s_2s_3s_2$	×	×	—	×	×	—	×	—
[2, 4, 3, 5, 1]	$s_1s_2s_3s_4s_2$	×	×	—	×	×	—	×	—
[2, 4, 5, 1, 3]	$s_3s_4s_1s_2s_3$	×	×	×	×	×	×	×	—
[2, 4, 5, 3, 1]	$s_1s_2s_3s_4s_2s_3$	×	×	—	×	×	—	×	—
[2, 5, 1, 3, 4]	$s_4s_3s_1s_2$	×	×	—	×	—	—	×	—
[2, 5, 1, 4, 3]	$s_3s_4s_3s_1s_2$	×	×	—	×	—	—	×	—
[2, 5, 3, 1, 4]	$s_4s_1s_2s_3s_2$	×	×	—	×	×	—	×	—
[2, 5, 3, 4, 1]	$s_1s_2s_3s_4s_3s_2$	×	×	—	×	×	—	×	—
[2, 5, 4, 1, 3]	$s_3s_4s_1s_2s_3s_2$	×	×	×	×	×	×	×	—
[2, 5, 4, 3, 1]	$s_1s_2s_3s_4s_2s_3s_2$	×	×	—	×	×	—	×	—
[3, 1, 2, 4, 5]	s_2s_1	—	—	—	—	—	—	—	—
[3, 1, 2, 5, 4]	$s_4s_2s_1$	—	—	—	—	—	—	—	—
[3, 1, 4, 2, 5]	$s_2s_3s_1$	×	—	—	×	—	—	×	—
[3, 1, 4, 5, 2]	$s_2s_3s_4s_1$	×	—	—	×	×	—	×	—
[3, 1, 5, 2, 4]	$s_4s_2s_3s_1$	×	—	—	×	—	—	×	—

w one-line	w red. word	mono.	(1)	(2)	(3)	(4)	(5)	(6)	(7)
[3, 1, 5, 4, 2]	$s_2s_3s_4s_3s_1$	×	—	—	×	×	—	×	—
[3, 2, 1, 4, 5]	$s_1s_2s_1$	—	—	—	—	—	—	—	—
[3, 2, 1, 5, 4]	$s_4s_1s_2s_1$	—	—	—	—	—	—	—	—
[3, 2, 4, 1, 5]	$s_1s_2s_3s_1$	×	×	—	—	×	—	×	—
[3, 2, 4, 5, 1]	$s_1s_2s_3s_4s_1$	×	×	—	—	×	—	×	—
[3, 2, 5, 1, 4]	$s_4s_1s_2s_3s_1$	×	×	—	—	×	—	×	—
[3, 2, 5, 4, 1]	$s_1s_2s_3s_4s_3s_1$	×	×	—	—	×	—	×	—
[3, 4, 1, 2, 5]	$s_2s_3s_1s_2$	×	×	×	×	—	×	×	—
[3, 4, 1, 5, 2]	$s_2s_3s_4s_1s_2$	×	×	—	×	×	×	×	—
[3, 4, 2, 1, 5]	$s_1s_2s_3s_1s_2$	×	×	—	×	—	—	×	—
[3, 4, 2, 5, 1]	$s_1s_2s_3s_4s_1s_2$	×	×	—	×	—	—	×	—
[3, 4, 5, 1, 2]	$s_2s_3s_4s_1s_2s_3$	×	×	×	×	—	—	×	—
[3, 4, 5, 2, 1]	$s_1s_2s_3s_4s_1s_2s_3$	×	×	—	×	—	—	×	—
[3, 5, 1, 2, 4]	$s_4s_2s_3s_1s_2$	×	×	×	×	—	×	×	—
[3, 5, 1, 4, 2]	$s_2s_3s_4s_3s_1s_2$	×	×	—	×	×	×	×	—
[3, 5, 2, 1, 4]	$s_4s_1s_2s_3s_1s_2$	×	×	—	×	—	—	×	—
[3, 5, 2, 4, 1]	$s_1s_2s_3s_4s_3s_1s_2$	×	×	—	×	—	—	×	—
[3, 5, 4, 1, 2]	$s_2s_3s_4s_1s_2s_3s_2$	×	×	×	×	—	—	×	—
[3, 5, 4, 2, 1]	$s_1s_2s_3s_4s_1s_2s_3s_2$	×	×	—	×	—	—	×	—
[4, 1, 2, 3, 5]	$s_3s_2s_1$	—	—	—	—	—	—	—	—
[4, 1, 2, 5, 3]	$s_3s_4s_2s_1$	×	—	—	×	—	—	×	—
[4, 1, 3, 2, 5]	$s_2s_3s_2s_1$	—	—	—	—	—	—	—	—
[4, 1, 3, 5, 2]	$s_2s_3s_4s_2s_1$	×	×	—	—	×	—	×	—
[4, 1, 5, 2, 3]	$s_3s_4s_2s_3s_1$	×	—	×	×	—	×	×	—
[4, 1, 5, 3, 2]	$s_2s_3s_4s_2s_3s_1$	×	—	—	×	—	—	×	—
[4, 2, 1, 3, 5]	$s_3s_1s_2s_1$	—	—	—	—	—	—	—	—
[4, 2, 1, 5, 3]	$s_3s_4s_1s_2s_1$	×	—	—	×	—	—	×	—
[4, 2, 3, 1, 5]	$s_1s_2s_3s_2s_1$	×	×	—	—	×	—	—	—
[4, 2, 3, 5, 1]	$s_1s_2s_3s_4s_2s_1$	×	×	—	—	×	—	×	—

w one-line	w red. word	mono.	(1)	(2)	(3)	(4)	(5)	(6)	(7)
[5, 3, 2, 4, 1]	$s_1 s_2 s_3 s_4 s_3 s_1 s_2 s_1$	×	—	—	—	—	—	—	×
[5, 3, 4, 1, 2]	$s_2 s_3 s_4 s_1 s_2 s_3 s_2 s_1$	×	×	×	×	—	—	—	×
[5, 3, 4, 2, 1]	$s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_2 s_1$	×	×	—	×	—	—	—	×
[5, 4, 1, 2, 3]	$s_3 s_4 s_2 s_3 s_1 s_2 s_1$	×	—	×	—	—	×	—	—
[5, 4, 1, 3, 2]	$s_2 s_3 s_4 s_2 s_3 s_1 s_2 s_1$	×	—	—	—	—	×	—	—
[5, 4, 2, 1, 3]	$s_3 s_4 s_1 s_2 s_3 s_1 s_2 s_1$	×	—	×	—	—	×	—	—
[5, 4, 2, 3, 1]	$s_1 s_2 s_3 s_4 s_2 s_3 s_1 s_2 s_1$	—	—	—	—	—	—	—	—
[5, 4, 3, 1, 2]	$s_2 s_3 s_4 s_1 s_2 s_3 s_1 s_2 s_1$	×	—	×	—	—	—	—	—
[5, 4, 3, 2, 1]	$s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_1 s_2 s_1$	—	—	—	—	—	—	—	—
120		85	64	22	57	36	22	65	8

Table 2: Initial ideals $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ (see §2.3) for $w \in S_5$ and which criteria for monomials apply.

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