Parabolic orbits of 2-nilpotent elements for classical groups

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Abstract

We consider the conjugation-action of an arbitrary standard parabolic subgroup of the symplectic or the orthogonal group on the variety of nilpotent complex elements of nilpotency degree 2 in its Lie algebra. By translating the setup to a representation-theoretic context in the language of a symmetric quiver, we show that these actions admit only a finite number of orbits. We specify systems of representatives for the orbits for each parabolic in a combinatorial way by so-called (enhanced) symplectic/orthogonal oriented link patterns and deduce information about numerology and dimensions. Our results are restricted to the nilradical, then.

1 Introduction

Let G be a classical complex group of rank n. Then G is either the general linear group $\mathrm{GL}_n(K)$ or the symplectic group $\mathrm{SP}_{2l}(K)$, where $n = 2l$ or the orthogonal group $\mathrm{O}_n(K)$, where $K = \mathbf{C}$. Let $\mathfrak{g}$ be the corresponding Lie algebra.

The study of the adjoint action of (subgroups of) G on $\mathfrak{g}$ and numerous variants thereof is a well-established and much considered task in algebraic Lie theory. Employing methods of geometric invariant theory, a classical topic is the study of orbits and their closures, which is also known as the *vertical* problem [10].

One famous example of a classification problem alike is the study of GL_n-conjugation (or SL_n-conjugation, this doesn't make a difference) on the variety of complex matrices of square–size n. A complete system of representatives up to conjugation is given by the Jordan canonical form [9] which dates back to the 19^{th} century. This system of representatives is given by continuous parameters, the eigenvalues of the matrix, and discrete parameters. In order to determine the latter, it suggests itself to restrict the action to the nilpotent cone, namely to GL_n-conjugation on the set of nilpotent matrices.

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The number of conjugacy classes of nilpotent matrices is finite and can be described combinatorially by partitions of n.

One generalization of this setup is obtained by restricting the acting group from G to parabolic subgroups $P \subseteq G$. In particular, the Borel subgroup B is considered, then, and the question about a variety admitting only finitely many orbits is closely related to the concept of so-called spherical varieties [5]. One example of a parabolic action can be found in [8], where Hille and Röhrle prove a finiteness criterion for the number of orbits of parabolic conjugation on the unipotent radical of $\mathfrak{g}$.

Another adaption of the above setup is given by restricting the nilpotent cone $\mathcal{N}$ of nilpotent matrices to certain subvarieties. For example, Melnikov parametrizes the Borel-orbits in the variety of 2-nilpotent elements in the nilradical $\mathfrak{n}$ of $\mathfrak{g} = \mathrm{Lie}(\mathrm{GL}_n(K))$ in [12] which is inspired by the study of orbital varieties. A parametrization in the symplectic setup ias published by Barnea and Melnikov in [2].

In this article, we consider the algebraic subvariety $\mathcal{N}(2)$ of 2–nilpotent elements of the nilpotent cone of $\mathfrak{g}$, namely

$$\mathcal{N}(2) = \mathcal{N}(2, G) = \{x \in \mathfrak{g} | \, x^2 = 0\}.$$

Every parabolic subgroup P of G acts on $\mathcal{N}(2)$. It is known that the number of orbits is always finite, since Panyushev shows finiteness for the Borel-action in [15]. In case $G = \mathrm{GL}_n(K)$, a parametrization of the P-orbits and a description of their degenerations is given in [4] and [3] for each parabolic subgroup $P \subset G$.

Our first goal in this article is to prove in a different manner that there are only finitely many P-orbits in $\mathcal{N}(2)$ for the remaining classical groups, that is, for types B, C and D. We approach the problem in a way closely related to [4] from a quiver-theoretic point of view - but instead of translating to the representation variety of a quiver with relations of a special dimension vector, we translate the orbits to certain (sets of) representations of a symmetric quiver with relations of a fixed dimension vector. In this setup we show that there are only finitely many of the latter.

Our second goal is to parametrize all orbits explicitly. The approach via a symmetric quiver makes it possible to classify the orbits by representations, and thus, by combinatorial data. We are able to calculate the dimensions of the orbits by means of representation theoretic methods. In the last section, we restrict our results to the action of P on the nilradical $\mathfrak{n}(2)$ of 2-nilpotent upper-triangular matrices in $\mathfrak{g}$ and obtain complete parametrizations, here.

Afterwards, we restrict our results to the nilradical $\mathfrak{n}$ and obtain parametrizations of parabolic orbits in $\mathcal{N}(2) \cap \mathfrak{n}$. The parametrization coincides with the parametrization by so-called symplectic link patterns of [2] in the symplectic case, even though the methods used to prove it are different.

Acknowledgments: The authors would like to thank Giovanna Carnovale for her input concerning the results and methods of this work. Furthermore, the first author thanks Martin Bender for discussions about Lie-theoretical background.

2 Classical groups and Lie algebras

Let K be the field of complex numbers $K := \mathbf{C}$ and let n be an integer. We consider the complex classical groups, that is, the general linear group $\mathrm{GL}_n := \mathrm{GL}_n(K)$, the symplectic group $\mathrm{SP}_n := \mathrm{SP}_n(K)$, whenever $n = 2l$ for some integer l, and the orthogonal group $\mathrm{O}_n := \mathrm{O}_n(K)$. The corresponding Lie algebras are denoted by $\mathfrak{gl}_n := \mathfrak{gl}_n(K)$, $\mathfrak{sp}_n := \mathfrak{sp}_n(K)$ and $\mathfrak{o}_n := \mathfrak{o}_n(K)$.

In general, given a vector space V endowed with a non-degenerate bilinear form $\langle -, - \rangle$, let us denote by $\mathrm{Sym}(V)$ the group of symmetries of the vector space V which preserve $\langle -, - \rangle|_{V \times V}$. Then $\mathrm{Sym}(V)$ equals either the symplectic group $\mathrm{SP}(V)$ or the orthogonal group $\mathrm{O}(V)$, depending on whether $(V, \langle -, - \rangle)$ is symplectic or orthogonal). We define $\mathfrak{sym}(V) := \mathrm{Lie}(\mathrm{Sym}(V))$.

Let l be an integer, then we denote by $J = J_l$ the $l \times l$ anti-diagonal matrix with every entry on the anti-diagonal being 1:

$$J_l = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}$$

It is easy to see (and well–known) that $J^{-1} = J$ and that the conjugate $J\,{}^T\!AJ$ by J of the *transpose* ${}^T\!A$ of a matrix $A \in K^{l \times l}$ is given by "the transpose of A with respect to the anti-diagonal". For example, for $l = 2$, given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$J\,{}^T\!AJ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} d & b \\ c & a \end{bmatrix}.$$

We set

$${}^{\mathfrak{T}}\!A := J\,{}^T\!AJ.$$

In this notation, it is easy to write down the elements of the symplectic and orthogonal Lie algebras.

2.1 Symplectic group

Let V be an $n = 2l$–dimensional complex vector space. Let us fix a basis of V and a bilinear form $F = F_V : V \times V \to K$, $F(v, w) = \langle v, w \rangle$, associated with the matrix (still denoted by F)

$$F = \begin{bmatrix} 0 & J_l \\ -J_l & 0 \end{bmatrix}. \tag{2.1}$$

The symplectic group SP_n consists of those matrices $A \in \mathrm{GL}_n$ which preserve this bilinear form (i.e. $\langle Av, Aw \rangle = \langle v, w \rangle$); in other words A satisfies the relation

$${}^T\!AFA = F.$$

The Lie algebra $\mathfrak{sp}_n$ of SP_n consists of those matrices $a \in \mathfrak{gl}_n$ which fulfill

$$ {}^{T}aF + Fa = 0. \tag{2.2} $$

We write the matrix a into four $l \times l$ blocks $a = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, so that condition (2.2) translates into the following equations

$$ a = \begin{bmatrix} A & B = {}^{\mathfrak{T}}B \\ C = {}^{\mathfrak{T}}C & D = -\,{}^{\mathfrak{T}}A \end{bmatrix}. \tag{2.3} $$

In particular, $\mathfrak{sp}_n$ has dimension $l^2 + l(l+1) = l(n+1)$. The intersection of $\mathfrak{sp}_n$ with the Borel subalgebra $\mathfrak{b}_n := \mathfrak{b}_n(K)$ of upper-triangular matrices is a solvable subalgebra of $\mathfrak{sp}_n$ of dimension $l(l+1) = l^2 + l$. Since $\mathfrak{sp}_n$ is a Lie algebra of type C_l, the number of positive roots is l^2 and the number of simple roots is l; we hence see that $\mathfrak{b}(\mathfrak{sp}_n) := \mathfrak{sp}_n \cap \mathfrak{b}_n$ is a solvable subalgebra of maximal dimension and hence a Borel subalgebra. Again, this is the advantage of working with the form F given by (2.1).

We will see in Subsection 4.1 that the same holds for the standard Borel subgroup B of SP_n which equals the intersection of the standard Borel subgroup of GL_n with SP_n. In the same manner, the standard parabolic subgroups SP_n are exactly given by the intersections $P \cap \mathrm{SP}_n$, where P are upper-block standard parabolic subgroups of GL_n.

2.2 Orthogonal group

Let V be an n–dimensional complex vector space (where n can be even or odd). Let us fix a basis of V and let us choose the non–degenerate bilinear form on V associated with the matrix $F = J_n$. The orthogonal group O_n consists of those matrices $A \in \mathrm{GL}_n$ for which ${}^{T}AFA = F$ holds true. The Lie algebra $\mathfrak{o}_n$ consists of those matrices $a \in \mathfrak{gl}_n$ satisfying (2.2) which translates into the relation

$$ a = -\,{}^{\mathfrak{T}}a. \tag{2.4} $$

In particular, $\mathfrak{o}_n$ has dimension $\frac{n(n-1)}{2}$. The intersection of $\mathfrak{o}_n$ with the Borel subalgebra $\mathfrak{b}_n$ of upper-triangular matrices, is a solvable subalgebra of $\mathfrak{o}_n$.

- If $n = 2l$, the dimension of such a solvable subalgebra is easily seen to be $\frac{n(n-1)}{2} - l(l-1) = l(2l-1) - l(l-1) = l^2$. Since $\mathfrak{o}_n$ is a Lie algebra of type D_l, the number of positive roots is $l(l-1)$ and the number of simple roots is l; we hence see that $\mathfrak{b}(\mathfrak{o}_n) := \mathfrak{o}_n \cap \mathfrak{b}_n$ is a solvable subalgebra of maximal dimension and hence a Borel subalgebra.
- Similarly, if $n = 2l + 1$, the dimension of $\mathfrak{b}(\mathfrak{o}_n) := \mathfrak{o}_n \cap \mathfrak{b}_n$ is easily seen to be $\frac{n(n-1)}{2} - (l(l-1) + l) = (2l+1)l - l^2 = l^2 + l$. Since $\mathfrak{o}_n$ is a Lie algebra of type B_l, the number of positive roots is l^2 and the number of simple roots is l; we hence see that $\mathfrak{b}(\mathfrak{o}_n)$ is a solvable subalgebra of maximal dimension and hence a Borel subalgebra.

This is the advantage of working with the form F given by J_n.

In the same manner as in the symplectic case, we will see in Subsection 4.1 that the standard Borel subgroup B of O_n equals the intersection of the standard Borel subgroup of GL_n with O_n. Furthermore, the standard parabolic subgroups of O_n are exactly given by the intersections $P \cap \mathrm{O}_n$, where P is an arbitrary upper-block standard parabolic subgroup of GL_n.

3 Representation theory of (symmetric) quivers

We include basic knowledge about the representation theory of finite-dimensional algebras via finite quivers [1] before we introduce the notion of a symmetric quiver and discuss its representations. This theoretical background will be necessary later on to prove our main results.

A *finite quiver* Q is a directed graph $Q = (Q_0, Q_1, s, t)$, such that Q_0 is a finite set of *vertices* and Q_1 is a finite set of *arrows*, whose elements are written as $\alpha\colon s(\alpha) \to t(\alpha)$. The *path algebra* KQ is defined as the K-vector space with a basis consisting of all paths in Q, that is, sequences of arrows $\omega = \alpha_s \dots \alpha_1$, such that $t(\alpha_k) = s(\alpha_{k+1})$ for all $k \in \{1, \dots, s-1\}$; formally included is a path ε_i of length zero for each $i \in Q_0$ starting and ending in i. The multiplication is defined as the concatenation of paths $\omega = \alpha_s ... \alpha_1$ and $\omega' = \beta_t ... \beta_1$, that is,

$$\omega \cdot \omega' = \begin{cases} \alpha_s ... \alpha_1 \beta_t ... \beta_1, & \text{if } t(\beta_t) = s(\alpha_1); \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathrm{rad}(KQ)$ be the *path ideal* of KQ, that is, the (two-sided) ideal generated by all paths of positive lengths. An ideal $I \subseteq KQ$ is called *admissible* if there exists an integer s with $\mathrm{rad}(KQ)^s \subset I \subset \mathrm{rad}(KQ)^2$. If this is the case for an ideal I, then the algebra KQ/I is finite-dimensional.

We denote by $\mathrm{rep}(KQ)$ the abelian K-linear category of all representations of Q (which is equivalent to the category of KQ-modules). In more detail, the objects are given as finite-dimensional *(K-)representations* of Q which are given by tuples

$$((M_i)_{i \in Q_0}, (M_\alpha\colon M_i \to M_j)_{(\alpha\colon i \to j) \in Q_1}),$$

where the M_i are K-vector spaces, and the M_α are K-linear maps. A *morphism of representations* $M = ((M_i)_{i \in Q_0}, (M_\alpha)_{\alpha \in Q_1})$ and $M' = ((M'_i)_{i \in Q_0}, (M'_\alpha)_{\alpha \in Q_1})$ consists of a tuple of K-linear maps $(f_i\colon M_i \to M'_i)_{i \in Q_0}$, such that $f_j M_\alpha = M'_\alpha f_i$ for every arrow $\alpha\colon i \to j$ in Q_1.

Let us denote by $\mathrm{rep}(KQ/I)$ the category of representations of Q bound by I: For a representation M and a path ω in Q as above, we denote $M_\omega = M_{\alpha_s} \cdot \ldots \cdot M_{\alpha_1}$. A representation M is called *bound by I*, if $\sum_\omega \lambda_\omega M_\omega = 0$ whenever $\sum_\omega \lambda_\omega \omega \in I$. The category $\mathrm{rep}(KQ/I)$ is equivalent to the category of finite-dimensional KQ/I-representations.

Let M be a KQ/I-representation, let $B_i \subseteq \epsilon_i M$ be a K-basis of $\epsilon_i M$ and let B be the disjoint union of these sets B_i. We define the *coefficient quiver* $\Gamma(M) = \Gamma(M, B)$ of M

with respect to the basis B to be the quiver with exactly one vertex for each element of B, such that for each arrow $\alpha \in Q_1$ and every element $b \in B_{s(\alpha)}$ we have

$$M_\alpha(b) = \sum_{c \in B_{t(\alpha)}} \lambda^\alpha_{b,c} c$$

with $\lambda^\alpha_{b,c} \in K$. For each $\lambda^\alpha_{b,c} \neq 0$ we draw an arrow $b \to c$ with label α. Thus, the quiver reflects the coefficients corresponding to the representation M with respect to the chosen basis B.

Given a representation $M \in \mathrm{rep}(KQ)$, its *dimension vector* $\underline{\dim} M \in \mathbf{N}Q_0$ is defined by $(\underline{\dim} M)_i = \dim_k M_i$ for $i \in Q_0$. Given a fixed dimension vector $\underline{d} \in \mathbf{N}Q_0$, we denote by $\mathrm{rep}(KQ/I, \underline{d})$ the full subcategory of $\mathrm{rep}(KQ/I)$ which consists of representations of dimension vector $\underline{d}$.

For certain finite-dimensional algebras $\mathcal{A} := KQ/I$, a convenient tool for the classification of the indecomposable representations (up to isomorphism) is the *Auslander–Reiten quiver* $\Gamma(\mathcal{A})$ of $\mathrm{rep}(\mathcal{A})$. Its vertices $[M]$ are given by the isomorphism classes of indecomposable representations of $\mathrm{rep}(\mathcal{A})$; the arrows between two such vertices $[M]$ and $[M']$ are parametrized by a basis of the space of so-called irreducible maps $f \colon M \to M'$.

By defining the affine space $\mathrm{R}_{\underline{d}}(KQ) := \bigoplus_{\alpha\colon i \to j} \mathrm{Hom}_K(K^{d_i}, K^{d_j})$, one realizes that its points m naturally correspond to representations $M \in \mathrm{rep}(KQ, \underline{d})$ with $M_i = K^{d_i}$ for $i \in Q_0$. Via this correspondence, the set of such representations bound by I corresponds to a closed subvariety $\mathrm{R}_{\underline{d}}(KQ/I) \subset \mathrm{R}_{\underline{d}}(KQ)$.

The algebraic group $\mathrm{GL}_{\underline{d}} = \prod_{i \in Q_0} \mathrm{GL}_{d_i}$ acts on $\mathrm{R}_{\underline{d}}(KQ)$ and on $\mathrm{R}_{\underline{d}}(KQ/I)$ via base change, furthermore the $\mathrm{GL}_{\underline{d}}$-orbits $\mathcal{O}_M$ of this action are in bijection to the isomorphism classes of representations M in $\mathrm{rep}(KQ/I, \underline{d})$.

3.1 Symmetric quivers

We introduce the notion of symmetry for a finite quiver and obtain an additional datum as follows: A *symmetric quiver* is a pair (Q, σ) where Q is a finite quiver and $\sigma : Q_0 \cup Q_1 \to Q_0 \cup Q_1$ is an involution, such that $\sigma(Q_0) = Q_0$, $\sigma(Q_1) = Q_1$ and every arrow $i \xrightarrow{\alpha} j$ is sent to the arrow $\sigma(j) \xrightarrow{\sigma(\alpha)} \sigma(i)$.

In this article, we represent the action of σ by adding the symbol $*$. For example,

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b^*} 2^* \xrightarrow{a^*} 1^*$$

is the symmetric quiver (Q, σ) with underlying quiver Q being equioriented of type A_5, such that σ acts on Q by sending an elment $x \in Q_0 \cup Q_1$ to x^*; the vertex 3 is fixed by σ.

A *symmetric (K-)representation* of a symmetric quiver (Q, σ) is a representation $M = (\{M_p\}_{p \in Q_0}, \{M_\alpha\}_{\alpha \in Q_1})$ in $\mathrm{rep}(KQ)$ endowed with a non–degenerate bilinear form $\langle -, - \rangle : \bigoplus_{p \in Q_0} M_p \times \bigoplus_{q \in Q_0} M_q \to K$, such that:

(i) The equation

$$\langle -,-\rangle|_{M_p \times M_q} = 0, \tag{3.1}$$

holds true, unless $q = \sigma(p)$;

(ii) The equation

$$\langle M_\alpha(v), w\rangle + \langle v, M_{\sigma(\alpha)}(w)\rangle = 0 \tag{3.2}$$

holds true for every $v \in M_p$, $w \in M_{\sigma(q)}$ and for every arrow $p \xrightarrow{\alpha} q \in Q_1$.

A representation $(M, \langle -,-\rangle)$ of a symmetric quiver (Q, σ) is called *symplectic*, if the bilinear form is skew–symmetric and it is called *orthogonal*, if the bilinear form is symmetric.

Let (Q, σ) be a symmetric quiver and let I be an ideal of KQ, such that $\sigma \cdot I \subset I$. The involution σ induces an involution on the algebra $\mathcal{A} := KQ/I$ and we can consider symplectic and orthogonal representations of the algebra $\mathcal{A}$: these are symplectic or orthogonal representations of $\mathcal{A}$ which are annihilated by the ideal I. We denote the categories of symmetric, symplectic and orthogonal representations by $\mathrm{srep}(\mathcal{A})$ and make sure that it will always be clear from the context, which one is meant. The restriction to the full subcategory of representations of a fixed dimension vector $\underline{d}$ is denoted by $\mathrm{srep}(\mathcal{A}, \underline{d})$. Analogously to the non-symmetric case, we associate a variety $\mathrm{SR}_{\underline{d}}(\mathcal{A})$ to this category.

4 Parabolic actions and symmetric quivers

Let $G \in \{\mathrm{SP}_n, \mathrm{O}_n\}$ where $n = 2l$ in the symplectic case and $n \in \{2l, 2l+1\}$ in the orthogonal case for some integer $l \in \mathbf{N}$ and let $\mathfrak{g}$ be the corresponding symplectic or orthogonal Lie algebra. Let P be a standard parabolic subgroup of G, that is, a subgroup of G which contains the standard Borel subgroup; its actual structure will be examined in Subsection 4.1.

We consider the algebraic variety $\mathcal{N}(2)$ of 2–nilpotent elements of $\mathfrak{g}$, that is,

$$\mathcal{N}(2) = \mathcal{N}(2, G) = \{x \in \mathfrak{g} |\, x^2 = 0\}.$$

Each parabolic subgroup P acts on $\mathcal{N}(2)$ via conjugation and our aim in this article is to prove by means of symmetric quiver representations that the action admits only a finite number of orbits. We thereby specify an explicit parametrization of the orbits.

4.1 Parabolic subgroups

Given a vector space V we denote its linear dual by $V^* = \{f : V \to K \text{ linear }\}$. If V is endowed with a given non–degenerate form $\langle -,-\rangle_V$, then we identify V and its dual V^* via the canonical map $v \mapsto \langle v, -\rangle_V$. Given another vector space W with a bilinear form

$\langle -,-\rangle_W$ and a linear map $f : V \to W$, there exists a unique linear map $f^* : W^* \to V^*$ such that $\langle f(v), w\rangle_W = \langle v, f^*(w)\rangle_V$, for every $v \in V$ and $w \in W$.

We often work in coordinates. To do so, we fix a basis of V and a basis of W. Let A be the matrix representing the linear map $f : V_p \to V_q$ in these bases. We usually denote by F_V the matrix associated with a bilinear form on a vector space V with respect to some basis. The matrix A^* representing f^* is given by

$$A^* = F_W^{-1} A F_V \tag{4.1}$$

Let $(V, \langle -,-\rangle)$ be a symmetric representation of a symmetric quiver (Q, σ). Let us fix a basis of V and let us denote by F the matrix representing the form $\langle -,-\rangle$ on V. Our default choice of F has been mentioned above: if V is symplectic, then F is represented by the matrix $\begin{bmatrix} 0 & J_l \\ -J_l & 0 \end{bmatrix}$, if V is orthogonal, then $F = J_n$.

The non–degenerate form $\langle -,-\rangle$ on V induces a non–degenerate form $\langle -,-\rangle|_{V_p \times V_{\sigma(p)}}$ for every $p \in Q_0$; we denote by F_p the corresponding matrix. An automorphism $\psi = (\psi_p)_{p \in Q_0} \in \prod_{p \in Q_0} \mathrm{GL}(V_p)$ of $(V, \langle -,-\rangle)$ is an automorphism of the quiver representation V, such that for every $p \in Q_0$ the following diagram

$$\begin{tikzcd} V_p \arrow[r, "\psi_p"] \arrow[rrr, bend right, "F_p"'] & V_p \arrow[r, "F_p"] & V_{\sigma(p)} \arrow[r, "\psi_{\sigma(p)}"] & V_{\sigma(p)} \end{tikzcd}$$

commutes. In other words, for every $v \in V_p$ and $w \in V_{\sigma(p)}$

$$\langle \psi_p(v), \psi_{\sigma(p)}(w)\rangle = \langle v, w\rangle.$$

If $p \neq \sigma(p)$, then $\psi_{\sigma(p)} = {}^T\psi^{-1}$ and if $p = \sigma(p)$, then ψ_p belongs to $\mathrm{Sym}(V_p)$.

One of the advantages of endowing V_p with the bilinear form represented by the matrix J_n is the following lemma:

Lemma 4.1. *The intersection of* $\mathrm{Sym}(V_p)$ *with the Borel subgroup B of* $\mathrm{GL}(V_p)$ *of upper-triangular matrices is a Borel subgroup B_{Sym} of* $\mathrm{Sym}(V_p)$*. This fact generalizes to parabolics: the standard parabolic subgroups of* $\mathrm{Sym}(V_p)$ *are given by the standard parabolics of* $\mathrm{GL}(V_p)$*, intersected with* $\mathrm{Sym}(V_p)$*.*

Thus, every standard parabolic subgroup P is determined by its block sizes, that is, $(b_1, ..., b_k, b_k, ..., b_1)$ for $n = 2l$ and $(b_1, ..., b_k, 1, b_k, ..., b_1)$ for $n = 2l + 1$. We set $\underline{b}_P := (b_1, ..., b_k)$ and call this vector the *block size vector* of P.

We notice that by having fixed the non–degenerate bilinear form on V either to be symmetric or to be skew–symmetric, forces all the groups $\mathrm{Sym}(V_p)$ to be all orthogonal or all symplectic. There is a more general construction due to Shmelkin [17] that allows to have both types of groups, but we do not need to work in this generality in this paper.

4.2 Stabilizers of standard flags

In order to approach our main goal, we translate the setup to the representation theory of a symmetric quiver. To do so, we begin by discussing stabilizers of (incomplete) standard flags. Let us look at an example first.

Example 4.2. *Let us consider the quiver Q_2*

$$\begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[r, "b"] & 3 \arrow[r, "b^*"] \arrow[loop above, "\alpha"] & 2^* \arrow[r, "a^*"] & 1^* \end{tikzcd}$$

and the algebra $\mathcal{A}(2) = KQ/(\alpha^2, b^*b)$. Let us consider the $\mathcal{A}$-representation M_0 given by

$$\begin{array}{ccccc} 1 \xrightarrow{a} & 2 \xrightarrow{b} & 3 & & \\ & 2 \xrightarrow{b} & 3 & & \\ & & 3 \xrightarrow{b^*} & 2^* & \\ & & 3 \xrightarrow{b^*} & 2^* \xrightarrow{a^*} & 1^* \end{array} \tag{4.2}$$

and the $\mathcal{A}(2)$-representation M_0' given by

$$\begin{array}{ccccc} 1 \xrightarrow{a} & 2 \xrightarrow{b} & 3 & & \\ & 2 \xrightarrow{b} & 3 & & \\ & & 3 & & \\ & & 3 \xrightarrow{b^*} & 2^* & \\ & & 3 \xrightarrow{b^*} & 2^* \xrightarrow{a^*} & 1^* \end{array} \tag{4.3}$$

In view of (3.2)*, in order for M to be symmetric, the arrows a, b of Q_2 must act by* 1 *and the arrows a^*, b^* must act as* -1.

The symmetric structure of M_0 (that is, the choice of a non-degenerate bilinear form) is induced by the symmetric structure on the vector space at vertex 3*. In this* 4*–dimensional vector space we consider the bilinear form given by the matrix*

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \pm 1 & 0 & 0 \\ \pm 1 & 0 & 0 & 0 \end{bmatrix}$$

(the sign $\pm$ depends if we work with a symplectic (-) or an orthogonal (+) representation).

The symplectic space $\mathrm{End}^{\mathrm{sym}}(M_0)$ *has dimension* 6 *and can be represented by the matrix*

$$\left[\begin{array}{cc|cc} a & c & f & g \\ 0 & b & e & f \\ \hline 0 & 0 & b & c \\ 0 & 0 & 0 & a \end{array}\right].$$

In orthogonal type, it is 4–dimensional and represented by

$$\left[\begin{array}{cc|cc} a & c & f & 0 \\ 0 & b & 0 & -f \\ \hline 0 & 0 & -b & -c \\ 0 & 0 & 0 & -a \end{array}\right].$$

In a similar way, we proceed for the orthogonal group O_5 *and look at the representation* M_0'. *Then, as above, the stabilizer is given by*

$$\left[\begin{array}{ccc|cc} a & c & e & f & 0 \\ 0 & b & d & 0 & -f \\ \hline 0 & 0 & 0 & -d & -e \\ 0 & 0 & 0 & -b & -c \\ 0 & 0 & 0 & 0 & -a \end{array}\right]$$

We have hence found the well known fact that the stabilizer of the complete standard flag is the Borel subgroup in both the symplectic and orthogonal setup.

Clearly, Example 4.2 generalizes as follows, the proof is straight forward.

Lemma 4.3. *Let* P *be a parabolic subgroup of* G *and let* M_P *be the (incomplete) standard flag corresponding to it. Then* $\mathrm{stab}_G(M_P) \cong P$. *In particular, if* M_0 *is the complete standard flag, then* $\mathrm{stab}_G(M_0) \cong B$ *as in Example 4.2.*

4.3 Translation

The translation from P–orbits in $\mathcal{N}(2)$ to the representation theory of a symmetric quiver is done precisely as in [4] and is based on a theorem on associated fibre bundles which we recall for the convenience of the reader. Its origin can be found in [16].

Theorem 4.4. *Let* G *be an algebraic group, let* X *and* Y *be* G*–varieties, and let* $\pi : X \to Y$ *be a* G*–equivariant morphism. Assume that* Y *is a single* G*–orbit,* $Y = Gy_0$. *Define* $H := Stab_G(y_0) = \{g \in G | \, g \cdot y_0 = y_0\}$ *and* $F := \pi^{-1}(y_0)$. *Then* X *is isomorphic to the associated fibre bundle* $G \times^H F$, *and the embedding* $\iota : F \to X$ *induces a bijection between* H*–orbits in* F *and* G*–orbits in* X *preserving orbit closures.*

Corollary 4.5. *With the notation of Theorem 4.4, given a point* $p \in F$, *we have* $\mathrm{stab}_H(p) = \mathrm{stab}_G(p)$

Proof. Since H is a subgroup of G, $\mathrm{Stab}_H(p) \subseteq \mathrm{Stab}_G(p)$; viceversa, since $H \cdot p = G \cdot p \cap F$, the reversed inclusion also holds. □

Let P be a standard parabolic subgroup of G of block size vector $\underline{b}_P = (b_1, ..., b_k)$ as in Subsection 4.1.

We define $\mathcal{A}(k)$ to be the algebra given by the quiver

$$\begin{tikzcd} Q_k: \ 1 \arrow[r, "a_1"] & 2 \arrow[r, "a_2"] & \cdots \arrow[r, "a_{k-1}"] & k \arrow[r, "a_k"] & \omega \arrow[loop above, "\alpha"] \arrow[r, "a_k^*"] & k^* \arrow[r, "a_{k-1}^*"] & \cdots \arrow[r, "a_2^*"] & 2^* \arrow[r, "a_1^*"] & 1^* \end{tikzcd}$$

with relations $\alpha^2 = a_k^* a_k = 0$. Notice that the $2k$ vertices of Q_k are colored; the choice of the color will be clear in a few lines. We consider the dimension vector

$$\mathbf{d}_P = ((d_P)_1, ..., (d_P)_k, (d_P)_\omega, (d_P)_{k^*}, ..., (d_P)_{1^*}) = (b_1, b_1 + b_2, \cdots, l, n, l, \cdots, b_2 + b_1, b_1)$$

and the variety $\mathrm{SR}_{\mathbf{d}_P}(\mathcal{A}(k))$. If the parabolic subgroup is clear from the context, we may abbreviate $(d_P)_i = d_i$ for $1 \leq i \leq k$ and $(d_P)_\omega = d_\omega$.

This variety is acted upon by the group

$$\mathrm{GL}_{\mathrm{sym}}(P) := \mathrm{GL}(d_1) \times \mathrm{GL}(d_2) \times \cdots \times \mathrm{GL}(l) \times \mathrm{Sym}(n)$$

where $\mathrm{Sym}(n)$ denotes either the symplectic or the orthogonal group on a vector space of dimension n. Inside the variety $\mathrm{SR}_{\mathbf{d}_P}(\mathcal{A}(k))$ we consider the open subset $\mathrm{SR}_{\mathbf{d}_P}(\mathcal{A}(k))^0$ corresponding to the full subcategory $\mathrm{srep}(\mathcal{A}(k), \mathbf{d}_P)^0$ of $\mathrm{srep}(\mathcal{A}(k), \mathbf{d}_P)$ of those representations whose linear maps associated with the arrows a_i and a_i^* have maximal rank.

In view of Theorem 4.4, we can now prove the following key lemma, analogous to [3, Lemma 3.1].

Lemma 4.6. *There is a bijection between isoclasses of symplectic/orthogonal $\mathcal{A}(k)$–modules in* $\mathrm{srep}(\mathcal{A}(k), \mathbf{d}_P)^0$ *and symplectic/orthogonal P–orbits in $\mathcal{N}(2)$. This bijection respects orbit closure relations and dimensions of stabilizers.*

Proof. Let $\widetilde{Q_k}$ be the quiver obtained from Q_k by removing the loop α and let $\widetilde{\mathcal{A}(k)}$ be the corresponding symmetric algebra without relations. By defining $\mathrm{SR}_{\mathbf{d}_P}(\widetilde{\mathcal{A}(k)})^0$ analougously to $\mathrm{SR}_{\mathbf{d}_P}(\mathcal{A}(k))^0$, we see that this variety is acted upon transitively by $\mathrm{GL}_{\mathrm{sym}}(P)$ and we denote the generating point by M_0, which is a (non-)complete standard flag. The embedding $\widetilde{\mathcal{A}(k)} \subset \mathcal{A}(k)$ induces a $\mathrm{GL}_{\mathrm{sym}}(P)$-equivariant projection

$$\pi : \mathrm{SR}_{\mathbf{d}_P}(\mathcal{A}(k))^0 \twoheadrightarrow \mathrm{SR}_{\mathbf{d}_P}\, p(\widetilde{\mathcal{A}(k)})^0$$

which is just given by forgetting the linear map associated with the loop α. The fiber of π equals the variety $\mathcal{N}(2)$.

The stabilizer of the symplectic/orthogonal representation M_0 is the parabolic subgroup P of the symplectic/orthogonal group, see Subsection 4.2. Thus, Theorem 4.4 proves the claim. $\square$

We are hence left to classify the isoclasses of symplectic/orthogonal representations of $\mathcal{A}(k)$ of dimension vector $\mathbf{d}_P$ with maximal rank maps, which in view of Remak–Krull–Schmidt theorem is analogous to classifying the unique decompositions of elements of srep($\mathcal{A}(k), \mathbf{d}_P)^0$ into indecomposable symplectic/orthogonal representations (up to isomorphism). Let M and M' be two points of srep($\mathcal{A}(k), \mathbf{d}_P)^0$ lying in different orbits. Since $\pi(M) = \pi(M')$ under the morphism of the proof of Lemma 4.6, the only difference beetween them is given by the action of the loop α. This means that the only part of the coefficient quivers of M and M' which differs is the subquiver which represents the loop α.

5 Representation theory of $\mathcal{A}(k)$

In this section, we look at the (symmetric) representation theory of the algebra $\mathcal{A}(k)$ corresponding to the symmetric quiver Q_k. With these considerations, we are able to prove explicit parametrizations of the parabolic orbits in $\mathcal{N}(2)$ in Section 6.

5.1 Indecomposable symmetric $\mathcal{A}(k)$–modules

The following proposition follows from [6, Section 3] by noticing that there are no band modules.

Proposition 5.1. *The algebra $\mathcal{A}(k)$ is a string algebra of finite representation type. In particular, the indecomposable $\mathcal{A}(k)$–modules are string modules and their isoclasses are parametrized by words with letters in the arrows of Q_k and their inverses, avoiding relations.*

Let us give names to the indecomposable $\mathcal{A}(k)$–modules (where $k + 1 := \omega$).

M_{ij}: For $1 \leq i \leq j \leq k + 1$, we denote by M_{ij} the string module associated with the word $a_i \cdots a_{j-1}$, i.e. it is the indecomposable module supported on vertices $i, i + 1, \cdots, j$; its coefficient quiver is given by

$$i \longrightarrow i+1 \longrightarrow \cdots \longrightarrow j-1 \longrightarrow j$$

M^*_{ij}: For $1 \leq i \leq j \leq k + 1$, we denote by M^*_{ij} the string module associated with the word $a^*_{j-1} \cdots a^*_i$, i.e. it is the indecomposable module supported on vertices $j^*, (j-1)^*, \cdots, i^*$; its coefficient quiver is given by

$$j^* \longrightarrow (j-1)^* \longrightarrow \cdots \longrightarrow (i+1)^* \longrightarrow i^*$$

D^+_{ij}: For $1 \leq i \leq j \leq k + 1$ we denote by D^+_{ij} the indecomposable associated with the word $a_i a_{i+1} \cdots a_k \alpha a^-_k a^-_{k-1} \cdots a^-_j$; its coefficient quiver is given by

$$\begin{tikzcd} i \arrow[r] & i+1 \arrow[r] & \cdots \arrow[r] & j \arrow[r] & j+1 \arrow[r] & \cdots \arrow[r] & k \arrow[r] & \omega \arrow[d] \\ & & & j \arrow[r] & j+1 \arrow[r] & \cdots \arrow[r] & k \arrow[r] & \omega \end{tikzcd}$$

D^-_{ij}: For $1 \leq i < j \leq k+1$, we denote by D^-_{ij} the indecomposable associated with the word $a_i a_{i+1} \cdots a_k \alpha^- a^-_k a^-_{k-1} \cdots a^-_j$; its coefficient quiver has the following form

$$\begin{tikzcd}
i \arrow[r] & i+1 \arrow[r] & \cdots \arrow[r] & j \arrow[r] & j+1 \arrow[r] & \cdots \arrow[r] & k \arrow[r] & \omega \\
 & & & j \arrow[r] & j+1 \arrow[r] & \cdots \arrow[r] & k \arrow[r] & \omega \arrow[u]
\end{tikzcd}$$

C^+_{ij}: For $1 \leq i \leq j \leq k+1$ we denote by C^+_{ij} the indecomposable module associated with the word $(a^*_j)^-(a^*_{j-1})^- \cdots (a^*_k)^- \alpha\, a^*_k a^*_{k-1} \cdots a^*_i$; its coefficient quiver is given by

$$\begin{tikzcd}
\omega \arrow[r] & k^* \arrow[r] & \cdots \arrow[r] & j^* \arrow[r] & (j-1)^* \arrow[r] & \cdots \arrow[r] & (i+1)^* \arrow[r] & i^* \\
\omega \arrow[u] \arrow[r] & k^* \arrow[r] & \cdots \arrow[r] & j^*
\end{tikzcd}$$

C^-_{ij}: For $1 \leq i < j \leq k+1$ we denote by C^-_{ij} is the indecomposable associated with the word $(a^*_j)^-(a^*_{j-1})^- \cdots (a^*_k)^- \alpha^- a^*_k a^*_{k-1} \cdots a^*_i$; its coefficient quiver is given by

$$\begin{tikzcd}
\omega \arrow[d] \arrow[r] & k^* \arrow[r] & \cdots \arrow[r] & j^* \arrow[r] & (j-1)^* \arrow[r] & \cdots \arrow[r] & (i+1)^* \arrow[r] & i^* \\
\omega \arrow[r] & k^* \arrow[r] & \cdots \arrow[r] & j^*
\end{tikzcd}$$

Z^+_{ij}: For $1 \leq i, j \leq k$ we denote by Z^+_{ij} the indecomposable associated with the word $a_i a_{i+1} \cdots a_k \alpha a^*_k \cdots a^*_j$; its coefficient quiver is given by

$$\begin{tikzcd}
i \arrow[r] & i+1 \arrow[r] & \cdots \arrow[r] & k \arrow[r] & \omega \arrow[d] \\
 & & & & \omega \arrow[r] & k^* \arrow[r] & \cdots \arrow[r] & j^*
\end{tikzcd}$$

Z^-_{ij}: For $1 \leq i, j \leq k$ we denote by Z^-_{ij} the indecomposable associated with the word $a_i a_{i+1} \cdots \alpha_k \alpha^- a^*_k \cdots a^*_j$; its coefficient quiver is given by

$$\begin{tikzcd}
i \arrow[r] & i+1 \arrow[r] & \cdots \arrow[r] & k \arrow[r] & \omega \\
 & & & & \omega \arrow[u] \arrow[r] & k^* \arrow[r] & \cdots \arrow[r] & j^*
\end{tikzcd}$$

Remark 5.2. All the modules above are non–isomorphic to each other, apart from $D^+_{k+1,k+1} \simeq C^+_{k+1,k+1}$ and $M_{k+1,k+1} \simeq M^*_{k+1,k+1}$.

We consider the involution σ of Q_k, which sends every vertex i to i^* and every arrow a to a^* and fixes ω and α (here we use the convention that $(-)^{**} = (-)$). Then (Q_k, σ) is a symmetric quiver and we can consider symmetric representations of $\mathcal{A}(k)$. The involution σ induces a duality on the category of representations of $\mathcal{A}(k)$ that we denote by ∇ (as in [7]).

Convention 5.3. *Given an indecomposable $\mathcal{A}(k)$–module M, we need to choose carefully the linear maps. Since we often work with its coefficient quiver, we fix one and for all a convention about these:*
The arrows of the coefficient quiver of M colored with $a_1, \cdots, a_k$ act as 1, while the arrows colored with $a_k^*, \cdots, a_1^*$ act as -1.

The two arrows $\omega_i \xrightarrow{\alpha_1} \omega_j$ and $\omega_{j^*} \xrightarrow{\alpha_2} \omega_{i^*}$ colored with ω (if they exist) have to satisfy the following conditions.

- *For V to be orthogonal, α_1 acts as 1 and α_2 as -1.*
- *For V to be symplectic, if $1 \leq i, j \leq n$ or $1 \leq i^*, j^* \leq n$, then α_1 and α_2 both act as 1, otherwise α_1 acts as 1 and α_2 as -1.*

Proposition 5.4. *With the above notation, we have: $\nabla M_{i,j} \simeq M^*_{i,j}$, $\nabla D^+_{i,j} \simeq C^+_{i,j}$, $\nabla D^-_{i,j} \simeq C^-_{i,j}$, $\nabla Z^+_{i,j} \simeq Z^+_{j,i}$, $\nabla Z^-_{i,j} \simeq Z^-_{j,i}$. In particular, $\nabla M_{k+1,k+1} \simeq M_{k+1,k+1}$ and $\nabla D^+_{k+1,k+1} \simeq D^+_{k+1,k+1}$*

Proof. Let M be an indecomposable module as listed above. The coefficient quiver of the dual ∇M of M is obtained from the coefficient quiver of M by reversing all the arrows, changing their sign and then making a reflection through the middle vertex $\omega = k + 1$. □

Thus, we obtain the following classification lemma.

Lemma 5.5. *The symplectic indecomposable representations of $\mathcal{A}(k)$ are $Z^\pm_{ii}$, $M_{ij} \oplus M^*_{ij}$, $D^\pm_{ij} \oplus C^\pm_{ij}$ (for $(i, j) \neq (k + 1, k + 1)$), $D_{k+1,k+1}$ and $Z^\pm_{ij} \oplus Z^\pm_{ji}$ (for $i \neq j$).*
The orthogonal indecomposable representations of $\mathcal{A}(k)$ are $M_{ij} \oplus M^*_{ij}$, $D^\pm_{ij} \oplus C^\pm_{ij}$, $Z^\pm_{ij} \oplus Z^\pm_{ji}$ and $M_{k+1,k+1}$.

In particular, there is only one indecomposable $\mathcal{A}(k)$–modules which can be endowed with an orthogonal structure.

Remark 5.6. The reason why an indecomposable $\mathcal{A}(k)$–module with symmetric dimension vector cannot be orthogonal, except for the case that it is one–dimensional, is the following: let M be such a (at least two–dimensional) module and let M_α be the linear map associated with the loop α. Such a map is a 2–nilpotent endomorphism of an orthogonal two–dimensional vector space. In order for M to be orthogonal, M_α must lie in the Lie algebra $\mathfrak{o}_2$ of O_2 and hence it must be zero, contradicting the fact that M is indecomposable.

For example, the following representation:

$$
\begin{tikzcd}
1 \arrow[r, "1"] & 2 \arrow[r, "1"] & 3 \arrow[r, "1"] & 4 \arrow[r, "1"] & \omega \arrow[dddd, bend left=60, "1"] & & & & \\
& 2 \arrow[r, "1"] & 3 \arrow[r, "1"] & 4 \arrow[r, "1"] & \omega \arrow[d, bend left, "1"] & & & & \\
& & 3 \arrow[r, "1"] & 4 \arrow[r, "1"] & \omega & & & & \\
& & & 4 \arrow[r, "1"] & \omega \arrow[dddd, bend right=60, "b"'] & & & & \\
& & & & \omega \arrow[r, "-1"] & 4^* & & & \\
& & & & \omega \arrow[r, "-1"] \arrow[d, bend right, "-1"'] & 4^* \arrow[r, "-1"] & 3^* & & \\
& & & & \omega \arrow[r, "-1"] & 4^* \arrow[r, "-1"] & 3^* \arrow[r, "-1"] & 2^* & \\
& & & & \omega \arrow[r, "-1"] & 4^* \arrow[r, "-1"] & 3^* \arrow[r, "-1"] & 2^* \arrow[r, "-1"] & 1^*
\end{tikzcd}
$$

is symplectic if $b = 1$ and orthogonal if $b = -1$.

5.2 Auslander–Reiten quiver of $\mathcal{A}(k)$

The algebra $\mathcal{A}(k)$ is a string algebra of finite representation–type, that is, it does only admit a finite number of isomorphism classes of indecomposable representations. Its Auslander–Reiten quiver can be obtained in several ways. We prefer to follow the treatment of Butler–Ringel [4] and get the following result.

Proposition 5.7. *The following are the Auslander–Reiten sequences of $\mathcal{A}(k)$:*

(i) Auslander–Reiten sequences starting with M_{ij}:

$$0 \longrightarrow M_{1,\omega} \longrightarrow Z^-_{1,1} \longrightarrow M^*_{1,\omega} \longrightarrow 0\,,$$

$$0 \longrightarrow M_{i,\omega} \longrightarrow M_{i-1,\omega} \oplus Z^-_{i,1} \longrightarrow Z^-_{i-1,1} \longrightarrow 0\,, \textit{ if } i > 1,$$

$$0 \longrightarrow M_{i,j} \longrightarrow M_{i,j-1} \oplus M_{i-1,j} \longrightarrow M_{i-1,j-1} \longrightarrow 0\,, \textit{ if } i > 1 \textit{ and } j \leq k,$$

$$0 \longrightarrow M_{i,i} = S_i \longrightarrow M_{i-1,i} \longrightarrow M_{i-1,i-1} \longrightarrow 0\,, \textit{ if } i = j > 1.$$

(ii) Auslander–Reiten sequences starting with M^*_{ij}:

$$0 \longrightarrow M^*_{1,j} \longrightarrow M^*_{2,j} \oplus M^*_{1,j+1} \longrightarrow M^*_{2,j+1} \longrightarrow 0\,, \textit{ if } j \leq k-1,$$

$$0 \longrightarrow M^*_{1,k} \longrightarrow M^*_{2,k} \oplus P_\omega \longrightarrow C^+_{1,k} \longrightarrow 0\,,$$

$$0 \longrightarrow M^*_{1,\omega} \longrightarrow M^*_{2,\omega} \oplus C_{1,1} \longrightarrow C^-_{1,2} \longrightarrow 0\,,$$

$$0 \longrightarrow M^*_{i,i} = S_{i^*} \longrightarrow M^*_{i,i+1} \longrightarrow M^*_{i+1,i+1} \longrightarrow 0\,, \textit{ if } i < k,$$

$$0 \longrightarrow M^*_{k,k} = S_{k^*} \longrightarrow C^+_{1,k} \longrightarrow C^+_{1,\omega} \longrightarrow 0\,,$$

$$0 \longrightarrow S_\omega \longrightarrow M_{k,\omega} \oplus C^-_{1,\omega} \longrightarrow Z^-_{k,1} \longrightarrow 0\ ,$$

$$0 \longrightarrow M^*_{i,\omega} \longrightarrow M^*_{i+1,\omega} \oplus C^-_{1,i} \longrightarrow C^-_{1,i+1} \longrightarrow 0\ , \textit{ if } 1 < i < k,$$

$$0 \longrightarrow M^*_{k,\omega} \longrightarrow S_\omega \oplus C^-_{1,k} \longrightarrow C^-_{1,\omega} \longrightarrow 0\ ,$$

$$0 \longrightarrow M^*_{i,j} \longrightarrow M^*_{i+1,j} \oplus M^*_{i,j+1} \longrightarrow M^*_{i+1,j+1} \longrightarrow 0\ , \textit{ if } i > 1 \textit{ and } j < k.$$

(iii) Auslander–Reiten sequences starting with D^+_{ij}:

$$0 \longrightarrow D^+_{1,\omega} \longrightarrow D^+_{1,k} \longrightarrow M_{k,k} = S_k \longrightarrow 0\ ,$$

$$0 \longrightarrow D^+_{i,\omega} \longrightarrow D^+_{i-1,\omega} \oplus D^-_{i,k} \longrightarrow D^+_{i-1,k} \longrightarrow 0\ , \textit{ if } 1 < i \le k,$$

$$0 \longrightarrow D_{i,i} \longrightarrow D^+_{i-1,i} \oplus D^-_{i-1,i} \longrightarrow D_{i-1,i-1} \longrightarrow 0\ , \textit{ if } 1 < i \le \omega$$

$$0 \longrightarrow D^+_{1,j} \longrightarrow D^+_{1,j-1} \oplus M_{j,k} \longrightarrow M_{j-1,k} \longrightarrow 0\ , \textit{ if } 1 < j \le k,$$

$$0 \longrightarrow D^+_{i,j} \longrightarrow D^+_{i-1,j} \oplus D^+_{i,j-1} \longrightarrow D^+_{i-1,j-1} \longrightarrow 0\ , \textit{ if } 1 < i < j \le k.$$

(iv) Auslander–Reiten sequences starting with D^-_{ij}:

$$0 \longrightarrow D^-_{1,\omega} \longrightarrow D^-_{1,k} \oplus S_\omega \longrightarrow M_{k,\omega} \longrightarrow 0\ ,$$

$$0 \longrightarrow D^-_{i,\omega} \longrightarrow D^-_{i-1,\omega} \oplus D^-_{i,k} \longrightarrow D^-_{i-1,k} \longrightarrow 0\ , \textit{ if } 1 < i \le k,$$

$$0 \longrightarrow D^-_{1,j} \longrightarrow D^-_{1,j-1} \oplus M_{j,\omega} \longrightarrow M_{j-1,\omega} \longrightarrow 0\ , \textit{ if } 1 < j \le k,$$

$$0 \longrightarrow D^-_{i,j} \longrightarrow D^-_{i-1,j} \oplus D^-_{i,j-1} \longrightarrow D^-_{i-1,j-1} \longrightarrow 0\ , \textit{ if } 1 < i \le j \le k.$$

(v) Auslander–Reiten sequences starting with C^+_{ij}:

$$0 \longrightarrow C^+_{1,\omega} \longrightarrow Z^+_{k,1} = P_k \oplus C^+_{2,\omega} \longrightarrow Z^+_{k,2} \longrightarrow 0\ ,$$

$$0 \longrightarrow C^+_{i,\omega} \longrightarrow Z^+_{k,i} \oplus C^+_{i+1,\omega} \longrightarrow Z^+_{k,i+1} \longrightarrow 0\ , \textit{ if } 1 \le i \le k,$$

$$0 \longrightarrow C^+_{1,1} = P_\omega \longrightarrow C^+_{1,2} \oplus C^-_{1,2} \longrightarrow C_{2,2} \longrightarrow 0\ ,$$

$$0 \longrightarrow C^+_{i,j} \longrightarrow C^+_{i+1,j} \oplus C^+_{i,j+1} \longrightarrow C^+_{i+1,j+1} \longrightarrow 0\ , \textit{ if } 1 < i \le j \le n.$$

(vi) Auslander–Reiten sequences starting with C^-_{ij}:

$$0 \longrightarrow C^-_{i,\omega} \longrightarrow C^-_{i+1,\omega} \oplus Z^-_{k,i} \longrightarrow Z^-_{k,i+1} \longrightarrow 0\ , \textit{ if } 1 \le i < k,$$

$$0 \longrightarrow C^-_{k,\omega} \longrightarrow C_{\omega,\omega} \oplus Z^-_{k,k} \longrightarrow D^-_{k,\omega} \longrightarrow 0\ ,$$

$$0 \longrightarrow C^-_{i,j} \longrightarrow C^-_{i+1,j} \oplus C^-_{i,j+1} \longrightarrow C^-_{i+1,j+1} \longrightarrow 0\ , \textit{ if } 1 < i \le j \le k.$$

(vii) Auslander–Reiten sequences starting with Z^+_{ij} (note that $Z^+_{1j} = I_{j^*}$):

$$0 \longrightarrow Z^+_{i,1} = P_i \longrightarrow Z^+_{i-1,1} \oplus Z^+_{i,2} \longrightarrow Z^+_{i-1,2} \longrightarrow 0 \text{ , } \textit{if } i > 1,$$

$$0 \longrightarrow Z^+_{i,j} \longrightarrow Z^+_{i-1,j} \oplus Z^+_{i,j+1} \longrightarrow Z^+_{i-1,j+1} \longrightarrow 0 \text{ , } \textit{if } 1 < i,\, j \leq k,$$

(viii) Auslander–Reiten sequences starting with Z^-_{ij} (note that $Z_{i,\omega} = D_{i,\omega}$):

$$0 \longrightarrow Z^-_{1,j} \longrightarrow Z^-_{1,j+1} \oplus M^*_{j,\omega} \longrightarrow M^*_{j+1,\omega} \longrightarrow 0 \text{ ,}$$

$$0 \longrightarrow Z^-_{i,j} \longrightarrow Z^-_{i,j+1} \oplus Z^-_{i-1,j} \longrightarrow Z^-_{i-1,j+1} \longrightarrow 0 \text{ , } \textit{if } i > 1.$$

The resulting Auslander–Reiten quiver of $\mathcal{A}(k)$ has the shape of a "christmas tree"; its bottom part consists of pre–projective modules and its top consists of $k + 1$ periodic τ–orbits. The duality ∇ acts as a reflection through the vertical line formed by the self–dual $\mathcal{A}(k)$–modules $Z^{\pm}_{ii}$ and $D^+_{k+1,k+1}$. Figure 1 shows the Auslander–Reiten quiver of $\mathcal{A}(3)$.

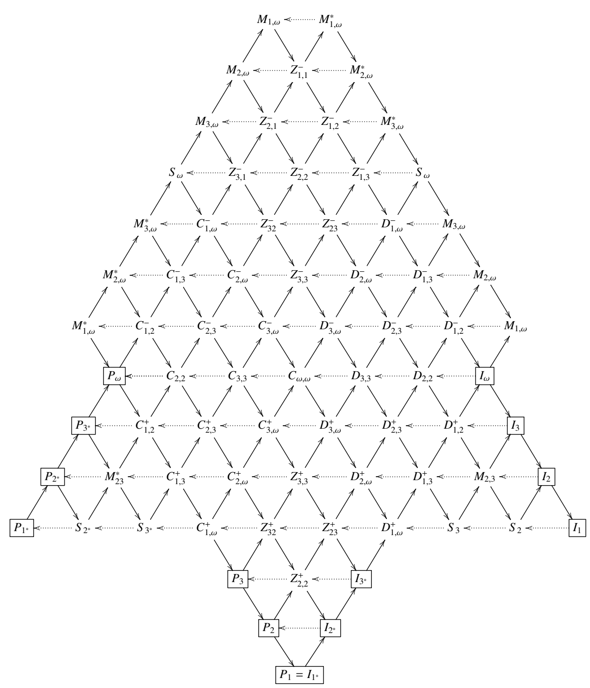


Figure 1: Auslander–Reiten quiver of $\mathcal{A}(3)$

6 Parametrization of orbits

It is known that every parabolic acts finitely on the variety $\mathcal{N}(2)$: Panyushev shows that the Borel subgroup acts finitely on $\mathcal{N}(2)$ in [15]. We aim to prove explicit parametrizations of the parabolic orbits by means of symmetric representations and, thus, in a very combinatorial way. This way, we hope to be able to calculate e.g. degenerations in a follow-up article by means of the used representation-theoretic methods. We begin by discussing symplectic orbits in Subsection 6.1 and deduce orthogonal orbits in Subsection 6.2. In each type, we start by discussing the Borel orbits and generalize the results to parabolic orbits afterwards.

6.1 Orbits in type C

Let $G = \mathrm{SP}_n$, where $n = 2l$ for some integer l.

Parametrization of Borel-orbits

We denote by B the standard Borel subgroup of G and consider the algebra $\mathcal{A}(l)$ and its symmetric representations as discussed in 5. Due to Lemma 4.6, we are interested in symplectic representations of dimension vector $\mathbf{d}_B = (1, 2, ..., l, 2l, l, ..., 2, 1)$.

Let us begin with an example.

Example 6.1. *Figure 2 shows the complete list of isoclasses of symplectic representations in* $\mathrm{srep}(\mathcal{A}(2), \mathbf{d}_B)^0$*, where* $n = 4 = 2l$ *and* B *is the Borel subgroup.*

This observation leads us to the following definition.

Definition 6.2. *A* symplectic oriented link pattern *(solp for short) of size* l *consists of a set of* $2l$ *colored vertices* $1, 2, \cdots, l, 1^*, 2^*, \cdots, l^*$ *together with a collection of oriented arrows between these vertices, such that*

- *SO1 if there is an arrow from vertex* i *to vertex* j*, then there is an arrow from vertex* j^* *to vertex* i^* *(with the convention that* $s^{**} = s$*);*
- *SO2 every vertex is touched by at most one arrow;*
- *SO3 there are no loops (that is, no arrows from a vertex to itself).*

We denote by Solp_l *the set of solps of size* l*.*

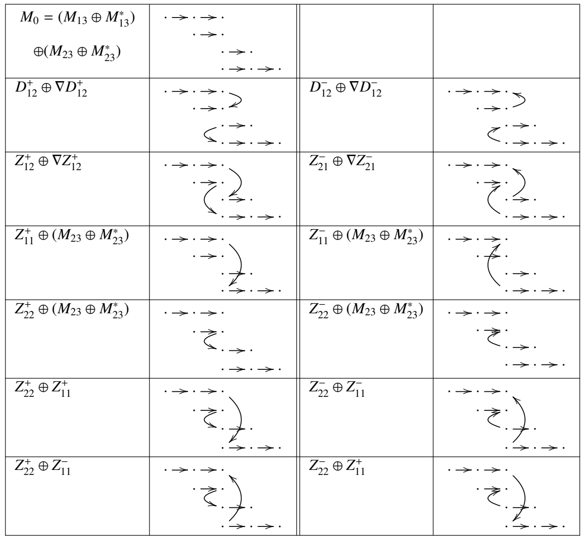
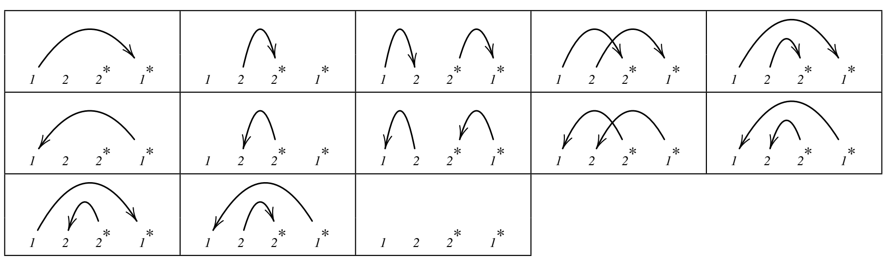


Figure 2: Isoclasses of symplectic representations in $\mathrm{srep}(\mathcal{A}(2), \mathbf{d}_B)^0$

Example 6.3. *The collection of solps of size* 2 *is given by*



We obtain a parametrization of symplectic Borel-orbits in $\mathcal{N}(2)$.

Theorem 6.4. *The B–orbits in the variety $\mathcal{N}(2) \subseteq \mathfrak{sp}_n$ are in bijection with the set* Solp_l *of solps of size l.*

Proof. By Krull–Remak–Schmidt, there is an obvious bijection between the set of solps of size l and the set of symplectic representations in $\mathrm{srep}(\mathcal{A}(l), \mathbf{d}_B)^0$ up to isomorphism which maps an isomorphism class $[M]$ of a symplectic representation M to the subquiver of the coefficient quiver of M induced by M_α. By Lemma 4.6, the claim follows. □

We can, thus, count the number of orbits.

Proposition 6.5. *Let s_l be the cardinality of* Solp_l*. Then the sequence $\{s_l\}$ is determined by*

- $s_0 = 1$,
- $s_1 = 3$,
- $s_l = 3s_{l-1} + 4(l-1)s_{l-2}$.

Proof. We divide the set Solp_l into the subset of symmetric link patterns where vertex 1 is not touched by any arrow and its complement. □

The sequence $1, 3, 13, 63, 345, 2043, ...$ of numbers of slps is classified in OEIS as A202837 [14].

Remark 6.6. For GL_n, the oriented link patterns considered in [4] only have to satisfy conditions $SO2$ and $SO3$, that is, the 2–nilpotency conditions. We hence see that solps are special oriented link patterns as defined in [4]. This is not surprising; indeed the following fact is known by [11]: if two symplectic elements are conjugate under the Borel of GL_n, then they are conjugate under the Borel of SP_n, as well.

Generalization to parabolic orbits

Let us consider the action of a parabolic subgroup $P \subseteq G$ of block sizes $(b_1, ..., b_k)$ on $\mathcal{N}(2)$. As before, we define combinatorial data in order to parametrize the orbits.

Definition 6.7. *An* enhanced symplectic oriented link pattern *(esolp. for short) of size k of type $(b_1, ..., b_k)$ consists of a set of $2k$ colored vertices $1, 2, \cdots, k, 1^*, 2^*, \cdots, k^*$ together with a collection of oriented arrows between these vertices. Denote by x_i the number of sources at vertex i and by y_i the number of targets of arrows at vertex i. Then the following conditions define an esolp:*

ESO1 for two different vertices i, j with $i \neq j^*$, the number of arrows from i to j equals the number of arrows from vertex j^* to vertex i^* (with the convention that $s^{**} = s$);

ESO2 for each vertex i, i^*: $0 \leq b_i - x_i - y_i = b_i - x_{i^*} - y_{i^*}$.

Note that loops are allowed here.

Theorem 6.8. *There is a natural bijection between the set of P-orbits in $\mathcal{N}(2) \subseteq \mathfrak{sp}_n$ and the set of esolps of size k and of type $(b_1, ..., b_k)$.*

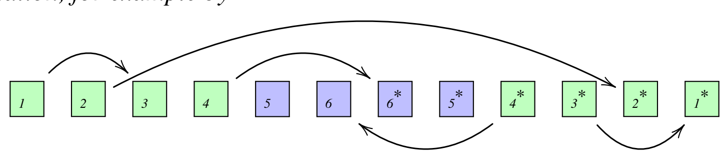
Proof. In an analogue manner to Theorem 6.4, by Krull–Remak–Schmidt, there is a bijection between the set of esolps of size k of type $(b_1, ..., b_k)$ and the set of isoclasses of symplectic representations of $\mathcal{A}(k)$ in $\mathrm{srep}(\mathcal{A}(k), \mathbf{d}_P)^0$ which maps an isomorphism class $[M]$ of a symplectic representation M to the subquiver of the coefficient quiver of M induced by M_α and then restricts the latter as follows: all vertices $i \in \{1, 2, ..., l\}$ which correspond to direct indecomposable summands M with $M_{i-1} = 0$ but $M_i \neq 0$ are glued together to vertex i. All vertices $i^* \in \{1^*, ..., l^*\}$ which correspond to direct indecomposable summands M with $M_{(i-1)^*} = 0$ but $M_{i^*} \neq 0$ are glued together to vertex i^*. This way, we obtain $2k$ vertices and vizualize all given arrows at the smaller pattern. In terms of representations, this means:

Indecomposable direct summand	Multiplicity given by
$M_{i,k+1} \oplus M^*_{i,k+1}$	$b_i - x_i - y_i$
$D^+_{i,j} \oplus C^+_{i,j}$, where $i \leq j$	number of arrows $i \to j$ = number of arrows $j^* \to i^*$
$D^-_{i,j} \oplus C^-_{i,j}$, where $i < j \neq k+1$	number of arrows $j \to i$ = number of arrows $i^* \to j^*$
$Z^+_{i,j} \oplus Z^+_{j,i}$, where $i \neq j$	number of arrows $i \to j^*$ = number of arrows $j \to i^*$
$Z^-_{ij} \oplus Z^-_{j,i}$, where $i \neq j$	number of arrows $j^* \to i$ = number of arrows $i^* \to j$
$Z^+_{i,i}$	number of arrows $i \to i^*$
$Z^-_{i,i}$	number of arrows $i^* \to i$

By Lemma 4.6, the claim follows. $\square$

Clearly, solps are special esolps: they are of size l and of type $(1, ..., 1)$, such that we obtain the classification of Borel-orbits.

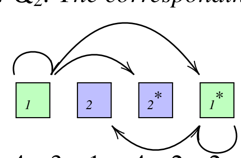
Example 6.9. *Let P be the symplectic parabolic subgroup of block sizes $(4, 2)$, thus, $b_1 = 4$ and $b_2 = 2$. Then a symplectic representation of dimension vector $(4, 6, 12, 6, 4)$ is represented by a pattern of 12 coloured vertices which represents the map M_α of the representation, for example by*



This pattern corresponds to the indecomposable direct sum decomposition

$$(D^+_{1,1} \oplus C^+_{1,1}) \oplus Z^+_{1,1} \oplus (Z^+_{1,2} \oplus Z^-_{1,2}) \oplus (M_{2,3} \oplus M^*_{2,3})$$

of a representation of the quiver Q_2. The corresponding esolp is given by



We have seen that $b_1 - x_1 - y_1 = 4 - 3 - 1 = 4 - 2 - 2 = b_1 - x_{1^*} - y_{1^*}$ and $b_2 - x_2 - y_2 = 2 - 0 - 1 = 1 = 2 - 0 - 1 = b_2 - x_{2^*} - y_{2^*}$ which give us the multiplicity of the indecomposables $M_{1,3} \oplus M^*_{1,3}$ and $M_{2,3} \oplus M^*_{2,3}$.

6.2 Orbits in types B and D

Let $G = \mathrm{O}_n$, where $n \in \{2l, 2l + 1\}$. We denote by B the standard Borel subgroup of G and consider the algebra $\mathcal{A}(l)$ as discussed in Section 5. We are interested in orthogonal representations of dimension vector $\mathbf{d}_B = (1, 2, ..., l, n, l, ..., 2, 1)$ by Lemma 4.6.

Parametrization of Borel-orbits

As before, we begin with an example.

Example 6.10. *Figure 3 shows the complete list of isoclasses of orthogonal representations in* $\mathrm{srep}(\mathcal{A}(2), \mathbf{d}_B)^0$*, where* $n = 4 = 2l$*. The isoclasses of orthogonal representations in* $\mathrm{srep}(\mathcal{A}(2), \mathbf{d}_B)^0$ *corresponding to* O_5 *are depicted in Figure 4. Note that the dimension vector* $\mathbf{d}_B$ *equals* $(1, 2, 4, 2, 1)$ *for* O_4 *and* $(1, 2, 5, 2, 1)$ *for* O_5.

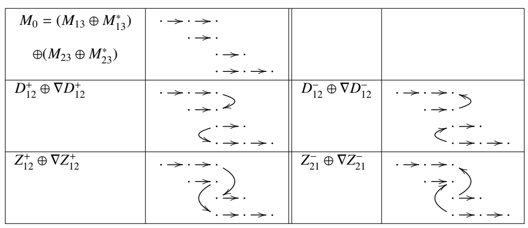


Figure 3: Isoclasses of orthogonal representations in $\mathrm{srep}(\mathcal{A}(2), \mathbf{d}_B)^0$

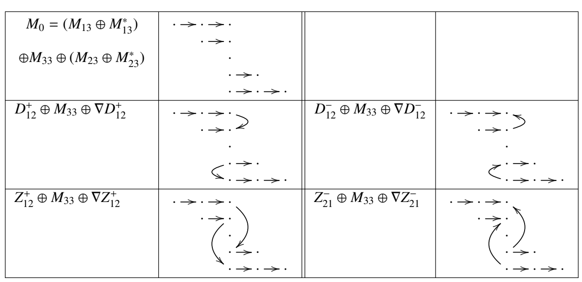


Figure 4: Isoclasses of orthogonal representations in $\mathrm{srep}(\mathcal{A}(2), \mathbf{d}_B)^0$

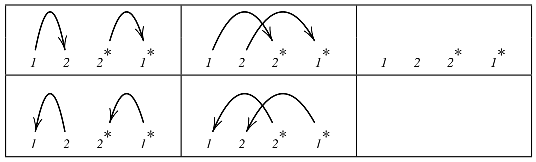
This observation leads us to the following definitions.

Definition 6.11. *An* orthogonal oriented link pattern *(oolp for short) of size l is a solp of size l, such that:*

O4 There are no arrows from i to i^* (with the convention that $i^{**} = i$).

We denote by Oolp_l *the set of oolps of size l.*

Example 6.12. *The collection of oolps of size* 2 *is given by*



As in the symplectic case, the parametrization of the Borel-orbits in $\mathcal{N}(2)$ follows straight away.

Theorem 6.13. *The B–orbits in the variety* $\mathcal{N}(2) \subseteq \mathfrak{o}_n$*, where* $n \in \{2l, 2l+1\}$*, are in bijection with the set* Oolp_l *of oolps of size l.*

Proof. In a similar manner to Theorem 6.4, there is an obvious bijection between the set Oolp_l of oolps of size l and the set of isoclasses of orthogonal representations of $\mathcal{A}(l)$ in $\mathrm{srep}(\mathcal{A}(l), \mathbf{d}_B)^0$ which maps an isomorphism class $[M]$ of an orthogonal representation to the subquiver of the coefficient quiver of M induced by M_α. As before, the claim follows from Lemma 4.6. The fact that the Borel orbits for O_{2l} and O_{2l+1} are classified by the same parametrizing set is due to the fact that each of the orthogonal representations in $\mathrm{srep}(\mathcal{A}(l), \mathbf{d}_B)^0$ for odd type has $M_{l+1,l+1}$ as a direct summand. This representation determines a fixed point as visualized in Figure 4 and the diagram representing M_α can thus be restricted to $2l$ vertices. □

Proposition 6.14. *Let* o_l *be the cardinality of* Oolp_l*. Then the sequence* $\{o_l\}$ *is determined by*

- $o_0 = 1$,
- $o_1 = 1$,
- $o_l = o_{l-1} + 4(l-1)o_{l-2}$.

Proof. We divide the set Oolp_l into the subset of oolps where vertex 1 is not touched by any arrow and its complement. □

The sequence $1, 1, 5, 13, 73, 281, 1741, \ldots$ which gives $\{o_l\}$ is classified in OEIS as A115329 [13].

Remark 6.15. As before, we see that oolps are special oriented link patterns. As in the symplectic case, this fact also follows from [11]: if two orthogonal elements are conjugate under the Borel of GL_n, then they are conjugate under the Borel of O_n, as well.

Generalization to parabolic orbits

Let us consider the action of a parabolic subgroup P of O_n on the variety $\mathcal{N}(2)$ of 2-nilpotent elements in $\mathfrak{o}_n$.

Definition 6.16. *An* enhanced orthogonal oriented link pattern *(eoolp for short) of size k of type* $(b_1, ..., b_k)$ *is an esolp of the* $2k$ *colored vertices* $1, 2, \cdots, k, 1^*, 2^*, \cdots, k^*$*, such that*

EO3 There are no arrows from i to i^* *(with the convention that* $i^{**} = i$*).*

The classification of parabolic orbits follows similarly to the considerations in 6.8.

Theorem 6.17. *There is a natural bijection between the set of P-orbits in* $\mathcal{N}(2) \subseteq \mathfrak{g}$ *and the set of eoolps of size k and of type* $(b_1, ..., b_k)$*.*

Clearly, oolps are special eoolps, they are of size l and of type $(1, ..., 1)$, such that we obtain the classification of Borel-orbits.

7 Restriction to the nilradical

If we restrict a parabolic action on $\mathcal{N}(2)$ to the nilradical $\mathfrak{n}(2)$ of 2-nilpotent upper-triangular matrices in the given Lie algebra, then we still have a parabolic action. The parametrization of the orbits can be obtained from our parametrizations of Section 6 straight away. Note that the action of the Borel subgroup in the symplectic case is parametrized in [2], where the authors also derive a description of the orbit closures and look at applications to orbital varieties in detail.

Definition 7.1. *A* symplectic link pattern *(slp for short) of size k is a symplectic oriented link pattern, such that every arrow goes from right to left, i.e. is of the form* $i \to j$ *where* $i > j$ *or* $i^* \to j^*$ *where* $i < j$ *or* $i^* \to j$*. In the same way, the natural notion of* orthogonal link pattern *(orlp for short)* enhanced symplectic *and* enhanced orthogonal link pattern *is obtained.*

Note that the sets of (enhanced) symplectic and (enhanced) orthogonal link patterns are obtained by taking all such oriented patterns and deleting the orientation. This is due to the fact that all representations named $-^-$ produce arrows which are oriented from right to left.

Corollary 7.2. *There is a bijection between the parabolic orbits in the symplectic nilradical* $\mathfrak{n}(2)$ *(or orthogonal* $\mathfrak{n}(2)$*, resp.) and the set of enhanced symplectic (or orthogonal, resp.) link patterns.*

Proof. The fact that all arrows are oriented from right to left corresponds equivalently to the nilpotent map at the loop being upper-triangular. Thus, these are directly the nilpotent elements contained in the nilradical of the particular Lie algebra. □

We end the section with giving an example.

Example 7.3. *For $l = 2$, the symplectic Borel-orbits in the nilradical of $\mathfrak{sp}_4$ are parametrized by slps which are emphasized with blue background colour in Figure 5, where their occurences as representations can be seen in detail.*

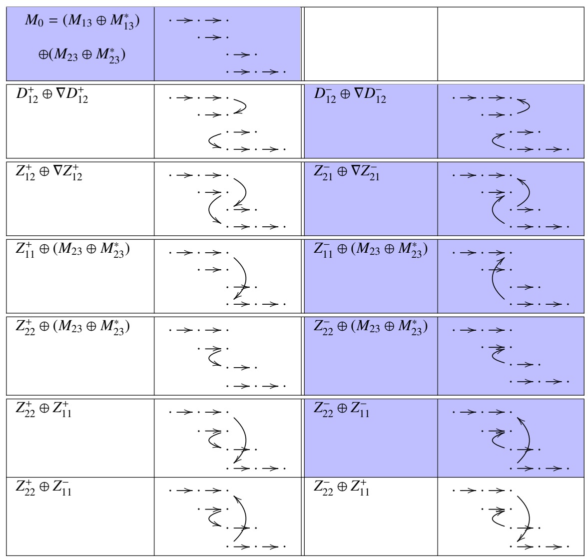
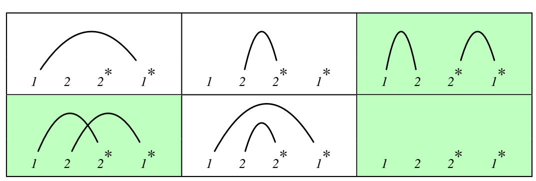


Figure 5: Representations corresponding to slps

Thus, the following is a list of all solps and all oolps (the latter are marked with green colour).



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