Statistics on some classes of knot shadows

Franck Ramaharo
Département de Mathématiques et Informatique
Université d'Antananarivo
101 Antananarivo, Madagascar
franck.ramaharo@gmail.com

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Abstract

The present paper is concerned with the enumeration of the state diagrams for some classes of knot shadows endowed with the usual connected sum operation. We focus on shadows that are recursively generated by knot shadows with up to 3 crossings, and for which the enumeration problem is solved with the help of generating polynomials.

Keywords: knot shadow, state diagram, generating polynomial.

1 Introduction

Let a mathematical knot be identified with its regular projection onto the sphere S^2. The corresponding representation, called *shadow*, is a planar quadrivalent diagram without the usual under/over information at every crossing [3, 6, 8, 14]. We can split each crossing of the diagram in one of two ways as shown in Figure 1.

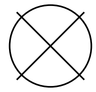
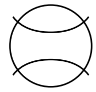
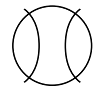
(a) Type 0 split (b) Type 1 split

Figure 1: Two types of splits.

By *state* is meant one of the obtained diagram with each crossing being split either by a *type* 0 or a *type* 1 *split*. A state can be seen as a collection of disjoint non-intersecting closed curves called *circles*. For a state S, we let $|S|$ denote the number of its circles. Then, for a knot diagram K with m crossings, we define the following statistics by summing over all states S:

$$K(x) = \sum_{S} x^{|S|} = \sum_{k\geq 0} \sigma(m,k)\, x^k, \tag{1}$$

where $\sigma(m,k)$ count the occurrence of the states with p circles, with $\sigma(m,0) = 0$ for all m. For the sake of simplicity, we call the state-sum formula (1) the *generating polynomial*. In fact, it is a simplified approach to the so-called Kauffman bracket polynomial [4, 12]. We intentionally omit the split variables indicating the over- and under-crossing structure since the summation is calculated with respect to the shadow diagram. The generating polynomial is only intended as a tool at enumerating the state diagrams, and no attempt is made here to investigate its topological property. Moreover, we have the following simplified rule which is then iteratively applied to all crossings in the diagram:

 $(x) =$  $(x) +$  $(x),$ (2)

In this paper, we mainly focus on the distribution of the number $\sigma(m,k)$ defined in (1) in terms of generating polynomial for some particular classes of knot shadows.

We organize the paper as follows. In section 2, we construct a recursive definition of some classes of knot shadows and define the associated closure operation. In section 2, we established the generating polynomial for the knots introduced in section 3. Then in section 4, we establish the generating polynomial for the closure of the same knots.

2 Background

Throughout this paper, unless explicitly stated otherwise, the generic term "knot (diagram)" refers to a shadow drawn on the sphere S^2. The simplest mathematical knot is the unknot which is a closed loop with no crossings in it. We say that two knots are the same, if one can be continuously deformed to the other so long as no new crossings are introduced and no crossings are removed. Such deformation is called a *planar isotopy*. A practical illustration would be to consider the corresponding diagram as a "highly deformable rubber" as suggested by Collins [1, p. 12]. To set up our framework, we introduce the following deformation which preserve as well the crossings configuration.

Definition 1 (Denton and Doyle [3])**.** When we have a loop on the outside edge of the diagram, we can redraw this loop around the other side of the diagram by pulling the entire loop around across the far side of the sphere without affecting the constraints on any of the already existing crossings (see Figure 2). The move is called a *type 0 move on the sphere*, denoted $0S^2$.

We can then define an equivalence relation on the set of knot shadows such that two knots lie in the same equivalence class if they have the same number of crossings, and if one can be transformed to the other by a finite sequence of OS^2 moves (modulo planar isotopy). Restricting ourself to the shadows of up to 3 crossings, we give in Table 1 all the possible combination of knot under the $0S^2$ move for each given number of crossings. We shall refer to this set of knots as *elementary knots*. For our arguments, we next associate these shadow diagrams with the following operations.

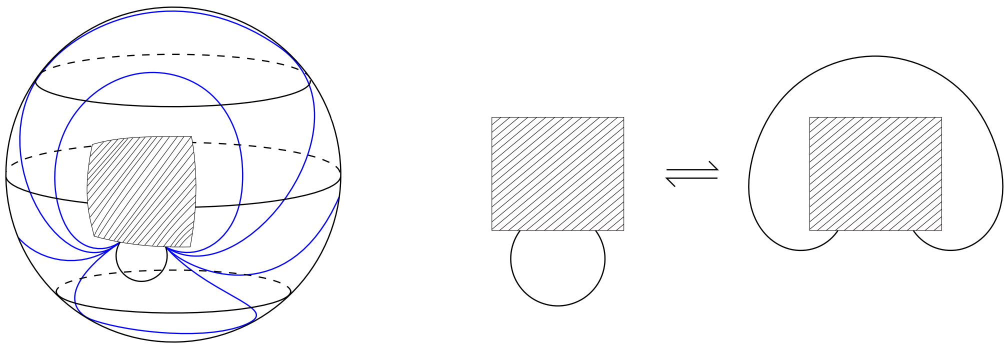


Figure 2: Denton and Doyle type 0 move on the sphere.

Definition 2. The *connected sum* of two knots K and K', denoted by $K\#K'$, is the knot obtained by removing a small arc from each knot and then connecting the four endpoints by two new arcs in such a way that no new crossings are introduced [10].

Analogously, the *disconnected sum* or *disjoint union* of two knots K and K', denoted by $K \sqcup K'$, is the diagram obtained by placing the two diagrams inside two non-intersecting domains on the sphere [13, p. 15].

Example 3. Consider the following connected sum:

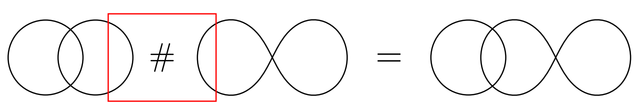


Figure 3: The connected sum of a 1-*link* and a 1-*twist loop* gives a 1-*twist link*.

We point out that the connected sum # and the disconnected sum $\sqcup$ are both associative and commutative [19, p. 61]. Moreover, for any knot K, we have $K\#U = K$, where U denote the unknot.

Our framework will make extensive use of the following special notation:

Notation 4. Let K be a knot, and let n be a nonnegative integer.

1. $K_n := \underbrace{K\#K\#\cdots\#K}_{n \text{ copies}}$ with $K_0 = U$. We say that the knot K_n is *generated* by K, and the knot K is the *generator* of K_n.

2. $K^n := \underbrace{K \sqcup K \sqcup \cdots \sqcup K}_{n \text{ copies}}$ with $K^0 = \varnothing$ (the empty knot).

In this paper, we establish the generating polynomials of the knots that are generated by the elementary knots. To begin with, we pay a special attention to the following series of knots.

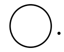
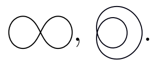
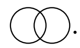
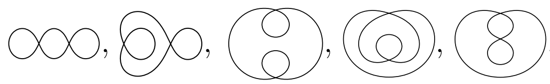
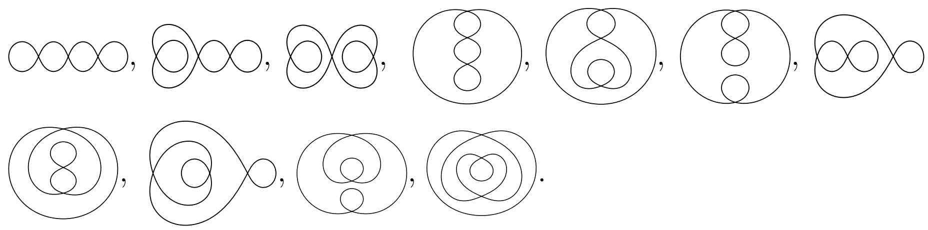
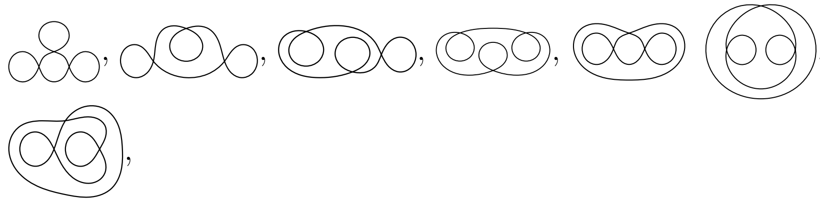
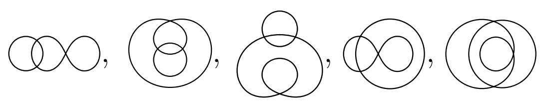
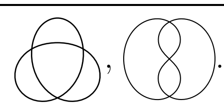
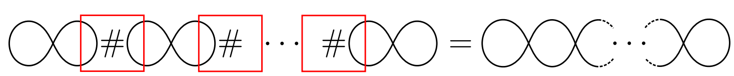
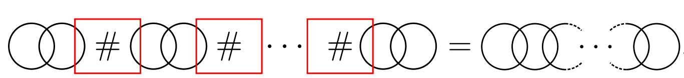
$\mathcal{S}_0$ (*unknots*)	
$\mathcal{S}_1$ (1-*twist loops*)	
$\mathcal{S}_{2,1}$ (1-*links*)	
$\mathcal{S}_{2,2}$ (2-*twist loops*)	
$\mathcal{S}_{3,1}$ (3-*twist loops*)	
$\mathcal{S}_{3,2}$ (3-*ears*)	
$\mathcal{S}_{3,3}$ (1-*twist links*)	
$\mathcal{S}_{3,4}$ (*trefoils*)	

Table 1: Knot shadow with at most 3 crossings

Definition 5 (Twist loop)**.** A *twist loop* is a knot obtained by twisting the unknot. We refer to a twist loop of n half twists as n-*twist loop* [16, 17]. We let T_n denote an n-twist loop, with

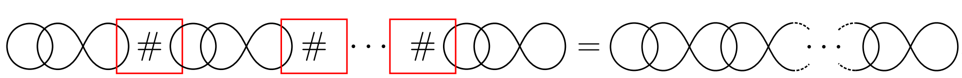
$T_n :=$  (3)

Definition 6 (Link)**.** An n-*link* is a knot which consists of $n+1$ linear interlocking circles. We let L_n denote an n-link, with

$L_n :=$  (4)

The knot L_1 is also referred to as *Hopf link*.

Definition 7 (Twist link)**.** We construct an n-*twist link* is by interlocking n series of 1-twist loops, starting from the unknot. We let W_n denote an n-twist link, with

$W_n :=$  (5)

where the generator is obtained by twisting the Hopf link.

Definition 8 (Hitch knot)**.** Ashley [2, #50, p. 14] describes the *half hitch* as "tied with one end of a rope being passed around an object and secured to its own standing part with a single hitch", see Figure 4.

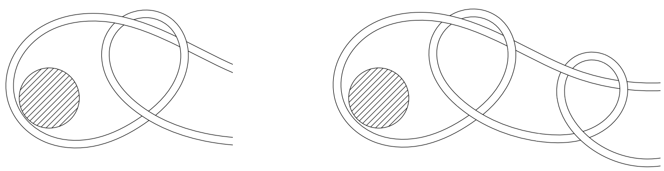
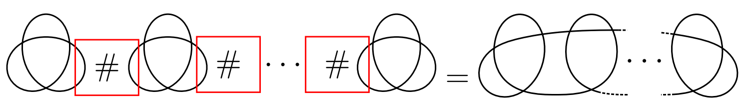


Figure 4: Single half hitch and double half hitches

We define a *n-hitch knot*, denoted by H_n, as the shadow obtained by joining together the two loose ends of a thread of n half hitches. The corresponding connected sum is given by

$H_n :=$  (6)

The knot H_1 is known as *trefoil*.

Definition 9 (Overhand knot)**.** The *overhand knot* is a knot obtained by making a loop in a piece of cord and pulling the end through it. For instance, we see in Figure 5 a single and a two series of overhand knot.

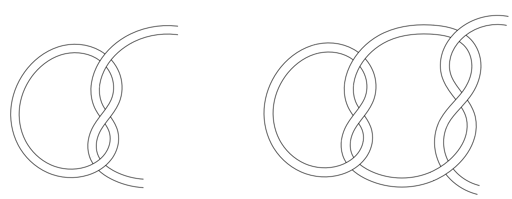
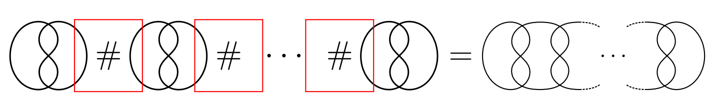


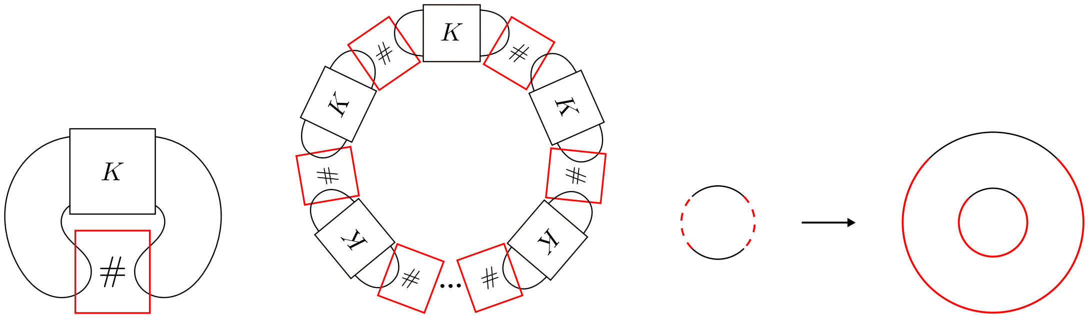
Figure 5: Single overhand knot, double overhand knot (*square knot*).

If as previously we join together the two loose ends of a n series of overhand knot, then we call the projected shadow an *n-overhand knot*, and we shall refer to such knot as O_n. We have

$O_n :=$  (7)

In the present representation, we also refer to the knot O_1 as *trefoil*.

Besides, we define as well the *closure* or the *closed connected sum* of a knot as the connected sum with itself as shown in Figure 6. The closure of the unknot, which is a disjoint union of two closed loops, is therefore the simplest of all the closure of knots. We let $\overline{K}$ denote a closure of the knot K.



(a) $\overline{K}$ (b) $\overline{K_n}$ (c) The unknot U and its closure $\overline{U}$

Figure 6: The closed connected sum.

Let us then present the *n-foil knot*, the *n-chain link*, the *n-twisted bracelet*, the *n-ringbolt hitching*, and the *n-sinnet of square knotting* which are respectively the closure of the n-twist loop, the n-link, the n-twist link, the n-hitch knot and the n-overhand knot. In what follows, we give the formal definition of these knots, and give the corresponding shadow diagrams.

Definition 10 (Foil knot)**.** An *n-foil* [17] is a knot obtained by winding n times around a circle in the interior of the torus, and 2 times around its axis of rotational symmetry [1, p. 107].

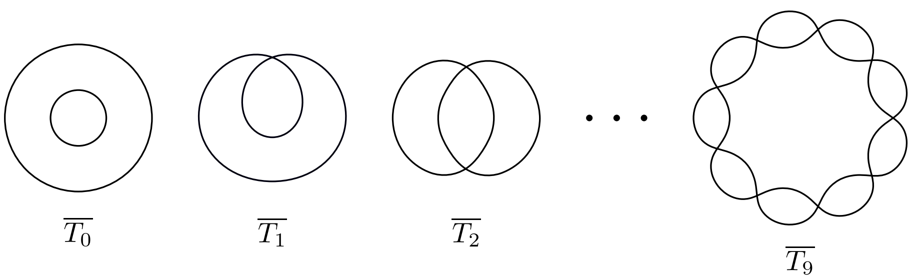


Figure 7: n-foil knots, $n = 0, 1, 2, 9$

Definition 11 (Chain link)**.** An *n-chain link* consists of n unknotted circles embedded in S^3, linked together in a closed chain [11].

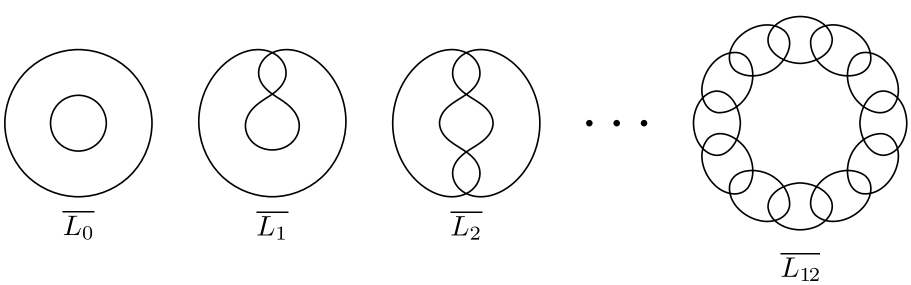


Figure 8: n-chain links, $n = 0, 1, 2, 12$.

Definition 12 (Twist bracelet)**.** An *n-twist bracelet* (or a *twisted n-chain link* [11]) consists of n twisted link intertwined together in a closed chain [15].

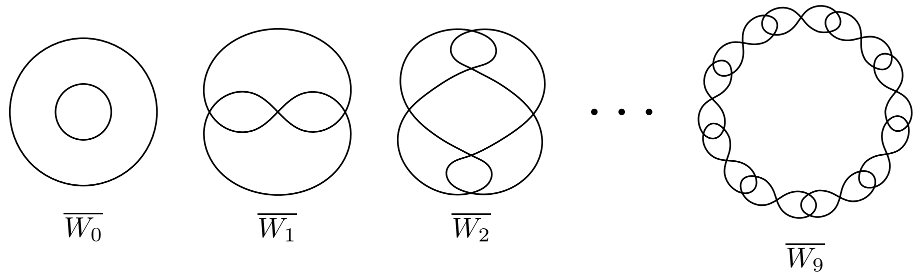


Figure 9: n-twist bracelets, $n = 0, 1, 2, 12$.

Definition 13 (Ringbolt hitching)**.** By *n-ringbolt hitching*, we mean a series of n half hitches that form a ridge around a ring or loop.

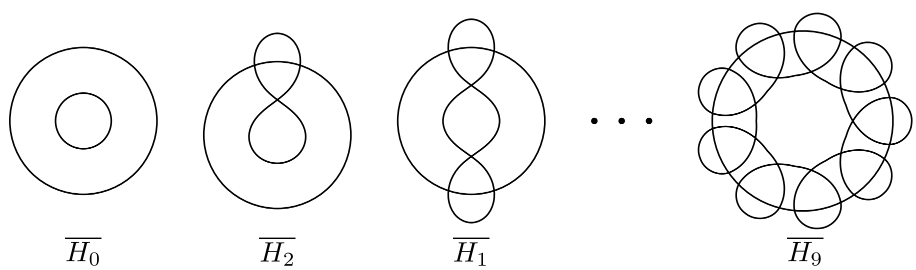


Figure 10: n-twisted bracelet, $n = 0, 1, 2, 9$.

Definition 14 (Sinnet of square knotting)**.** Ashley [2, #2906, p. 471] defines a *chain sinnet* as a knot which are made of one or more strands that are formed into successive loops, which are tucked though each other. Here, we borrow the term *n-sinnet of square knotting* to describe a closed chain of n overhand knot, see Figure 11.

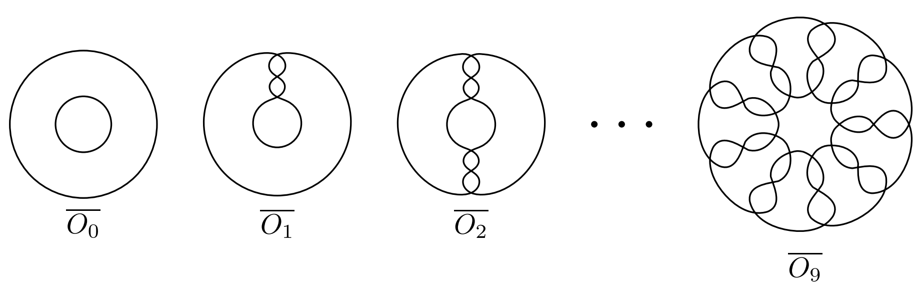
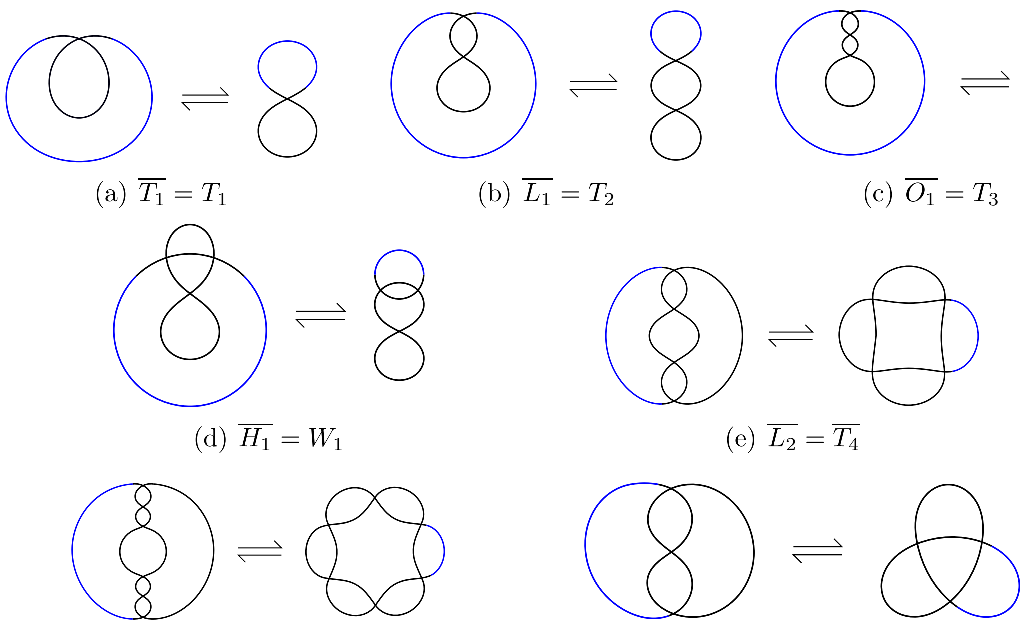


Figure 11: n-sinnet of square knotting, $n = 0, 1, 2, 9$.

Remark 15*.* The pairs of knots shown in Figure 12 are equivalent under the OS^2 move. The $0S^2$ move does not remove nor create a crossing, therefore a complete split leads to the same state diagram. We then expect that knots belonging to same equivalence class have equals generating polynomials.



(a) $\overline{T_1} = T_1$ (b) $\overline{L_1} = T_2$ (c) $\overline{O_1} = T_3$

(d) $\overline{H_1} = W_1$ (e) $\overline{L_2} = \overline{T_4}$

(f) $\overline{O_2} = \overline{T_6}$ (g) $\overline{W_1} = O_1 = H_1$

Figure 12: Equivalent knots under the $0S^2$ move.

3 The generating polynomial for the connected sum

The present section is devoted to computing the generating polynomials of the previously introduced knots.

3.1 Preliminaries

Regarding the knot operations $\#$ and $\sqcup$, we have the following immediate results:

$$U(x) = x \text{ and } U^n(x) = x^n.$$

A first extension is as follows.

Proposition 16. *For an arbitrary knot K and the unknot U, the following property holds*

$$(K \sqcup U)(x) = xK(x). \tag{8}$$

Proof. The initial diagram is accompanied with the unknot and so are each resulting diagram after a series of splits. So if $K(x) = \sum_S x^{|S|}$, then $(K \sqcup U)(x) = \sum_S x^{|S|+1} = x \sum_S x^{|S|}$. $\square$

Corollary 17. *Let n be a nonnegative integer, and let K be an arbitrary knot. Then*

$$\left(K \sqcup U^n\right)(x) = x^n K(x).$$

We can generalize Proposition 16 as follows

Proposition 18. *For two arbitrary knots K and K', the following equality holds*

$$(K \sqcup K')(x) = K(x).K'(x). \tag{9}$$

Proof. Let us first compute the states of K. Each of these states is accompanied with the diagram of K'. Hence we have

$$(K \sqcup K')(x) = \sum_{i\geq 1} \left(U^{k_i} \sqcup K'\right)(x),$$

where i runs over all the states of K', and k_i is the corresponding number of circles. Finally by Corollary 17 we obtain

$$(K \sqcup K')(x) = \sum_{i\geq 1} x^{k_i} K'(x),$$

and we conclude by noting that $K(x) = \sum_{i\geq 1} x^{k_i}$. □

Proposition 19. *For two arbitrary knots K and K', the following equality holds*

$$(K\#K')(x) = x^{-1}K(x).K'(x). \tag{10}$$

Proof. If we first compute the states of the knot K', then for each of these states, there exists exactly one circle which is connected to K'. In terms of polynomial, it means

$$(K \sqcup K')(x) = \sum_{i\geq 1} x^{-1}\left(U^{k_i} \sqcup K'\right)(x),$$

where, as previously, i runs over all the states of K, and k_i is the corresponding number of circles. The result immediately follows from formula (9). □

Corollary 20. *Let K, K' and K'' be three knots. Then*

$$(K\#K')(x) = (K'\#K)(x), \tag{11}$$
$$\big((K\#K')\#K''\big)(x) = \big(K\#(K'\#K'')\big)(x). \tag{12}$$

We will make extensive use of the following particular case

Corollary 21. *For an arbitrary knot K and a nonnegative integer n, we have*

$$K_n(x) = x\big(x^{-1}K(x)\big)^n. \tag{13}$$

Remark 22. Formula (13) suggests that in order to compute the generating polynomial of the knot K_n, we simply have to compute that of the generator K. This formula also means that the generating polynomials associated with knots generated by elementary knots that lie in the same equivalent class are exactly the same, regardless of the choice of the arcs at which the connected sum is performed.

Theorem 23. *The generating function for the sequence* $\{K_n(x)\}_{n\geq 0}$ *with respect to the occurrence of the generator* K *(marked by* y*) and the state diagrams (marked by* x*) is*

$$K(x;y) := \frac{x^2}{x - yK(x)}.$$

Proof. We write $K(x;y) := \sum_{n\geq 0} K_n(x)y^n$, and the result follows from formula (13). □

The results in subsection 3.2 – 3.6 are all then immediate application of Corollary 21 and Theorem 23. For each of the concerned knots, we give the generating polynomial and the associated generating function. If available, we also give the A-records and the short definition from the *On-Line Encyclopedia of Integer Sequences* (OEIS) [18].

3.2 Twist loop

Let $T_n(x) := \sum_{k\geq 0} t\,(n,k)\,x^k$ denote the generating polynomial of the n-twist loop.

Theorem 24. *The generating polynomial of the* n*-twist loop is given by the recurrence relation*

$$T_n(x) = (x+1)T_{n-1}(x), \tag{14}$$

and is expressed by the closed form formula

$$T_n(x) = x(x+1)^n. \tag{15}$$

Proof. See Figure 13, then apply Corollary 21. □

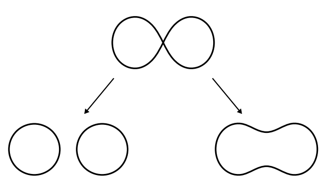


Figure 13: The states of the 1-twist loop, $T_1(x) = x^2 + x$.

Corollary 25. *The generating function for the sequence* $\{T_n(x)\}_{n\geq 0}$ *is given by*

$$T(x;y) := \frac{x}{1 - y(x+1)}. \tag{16}$$

Combining expressions (14) and (15), we obtain the following recurrence relation:

$$\begin{cases} t(n,0) = 0,\ t(n,1) = 1, & n \geq 0; \\ t(n,k) = t(n-1,k-1) + t(n-1,k), & k \geq 1,\ n \geq 0. \end{cases} \tag{17}$$

The values for the array $(t(n,k))_{n\geq 0,\ k\geq 0}$ are given in Table 2 for small value of n and k, with $n \geq k-1$. The result is a horizontal-shifted Pascal's triangle [18, A007318]

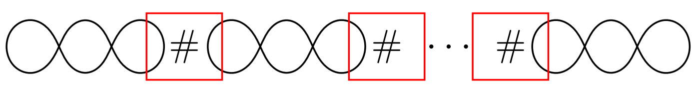
$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	1												
1	0	1	1											
2	0	1	2	1										
3	0	1	3	3	1									
4	0	1	4	6	4	1								
5	0	1	5	10	10	5	1							
6	0	1	6	15	20	15	6	1						
7	0	1	7	21	35	35	21	7	1					
8	0	1	8	28	56	70	56	28	8	1				
9	0	1	9	36	84	126	126	84	36	9	1			
10	0	1	10	45	120	210	252	210	120	45	10	1		
11	0	1	11	55	165	330	462	462	330	165	55	11	1	
12	0	1	12	66	220	495	792	924	792	495	220	66	12	1

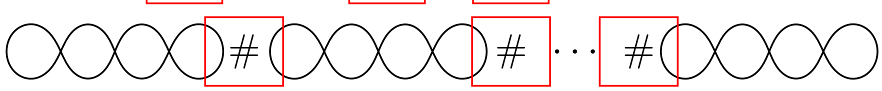
Table 2: Values of $t(n,k)$ for $0 \le n \le 12$ and $0 \le k \le 13$.

Remark 26. Let $K \in \mathcal{S}_{2,2} \cup \mathcal{S}_{3,1} \cup \mathcal{S}_{3,2}$. Then by Remark 22, we have

$$K_n(x) = \begin{cases} (T_2)_n(x), & \text{if } K \in \mathcal{S}_{2,2}; \\ (T_3)_n(x), & \text{if } K \in \mathcal{S}_{3,1} \cup \mathcal{S}_{3,2}, \end{cases} \tag{18}$$

with

$(T_2)_n :=$  ;

$(T_3)_n :=$  .

Here, it is immediate that for all nonnegative integer i, $(T_i)_n = T_{in}$. Let us then introduce the following calculations to complete our results.

$2n$-twist loop: let $T_{2n}(x) := \sum_{k \ge 0} t_2(n,k) x^k$.

1. Generating polynomial:
$$T_{2n}(x) = x\,(x+1)^{2n}. \tag{19}$$

2. Generating function:
$$T_2(x;y) = \frac{x}{1 - y(x^2 + 2x + 1)}. \tag{20}$$

3. Distribution of $t_2(n,k)$: see Table 3.

$$\begin{cases} t_2(n,0) = 0,\ t_2(n,1) = 1,\ t_2(n,2) = 2n, & n \ge 0; \\ t_2(n,k) = t_2(n-1,k-2) + 2t_2(n-1,k-1) + t_2(n-1,k), & k \ge 2,\ n \ge 0. \end{cases} \tag{21}$$

When $k \geq 1$, the triangle $(t_2(n,k))_{n\geq 0}$ is a horizontal-shifted even-numbered rows of Pascal's triangle [18, A034870].

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1														
1	0	1	2	1												
2	0	1	4	6	4	1										
3	0	1	6	15	20	15	6	1								
4	0	1	8	28	56	70	56	28	8	1						
5	0	1	10	45	120	210	252	210	120	45	10	1				
6	0	1	12	66	220	495	792	924	792	495	220	66	12	1		
7	0	1	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14	1

Table 3: Values of $t_2(n,k)$ for $0 \leq n \leq 7$ and $0 \leq k \leq 15$.

$3n$-twist loop: let $T_{3n}(x) := \sum_{k\geq 0} t_3(n,k)x^k$.

1. Generating polynomial:
$$T_{3n}(x) = x\,(x+1)^{3n}\,. \tag{22}$$

2. Generating function:
$$T_3(x;y) := \frac{x}{1-y(x^3+3x^3+3x^2+1)}. \tag{23}$$

3. Distribution of $t_3(n,k)$: see Table 4 (also, refer back to Table 2).
$$\begin{cases} t_3(n,0)=0,\ t_3(n,1)=1,\ t_3(n,2)=3n,\ t_3(n,3)=\dfrac{3n(3n-1)}{2}, & n\geq 0; \\ t_3(n,k)=t_3(n-1,k-3)+3t_3(n-1,k-2) & \\ \qquad\qquad +3t_3(n-1,k-1)+t_3(n-1,k), & k\geq 3, n\geq 0. \end{cases} \tag{24}$$

The triangle $(t_3(n,k))_{k\geq 1,\ n\geq 0}$ is given by $\binom{3n}{k-1}$ [18, A007318$(3n,k-1)$].

3.3 Link

Let $L_n(x) := \sum_{k\geq 0} \ell(n,k)x^k$ denote the generating polynomial of the n-link.

Theorem 27. *The generating polynomial of the n-link is given by the recurrence relation*
$$L_n(x) = (2x+2)\,L_{n-1}(x), \tag{25}$$

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	0	1															
1	0	1	3	3	1												
2	0	1	6	15	20	15	6	1									
3	0	1	9	36	84	126	126	84	36	9	1						
4	0	1	12	66	220	495	792	924	792	495	220	66	12	1			
5	0	1	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	105	15	1

Table 4: Values of $t_3(n,k)$ for $0 \leq n \leq 5$ and $0 \leq k \leq 16$.

and is expressed by the closed form formula

$$L_n(x) = x(2x+2)^n. \tag{26}$$

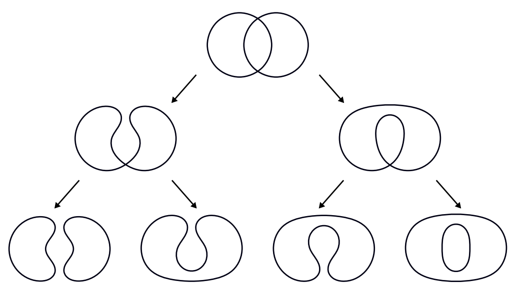


Figure 14: The states of the 1-link, $L_1(x) = 2x^2 + 2x$.

Proof. See Figure 14. □

Corollary 28. *The generating function for the sequence* $\{L_n(x)\}_{n\geq 0}$ *is given by*

$$L(x;y) := \frac{x}{1-y(2x+2)}. \tag{27}$$

The expression of the polynomial $L_n(x)$ suggests that

$$\ell(n,k) = 2^n t(n,k) = 2^n \binom{n}{k-1}, \ k \geq 1, \ n \geq 0 \ [18, \text{A038208}].$$

Therefore, the corresponding polynomial coefficients satisfy the following recurrence relation:

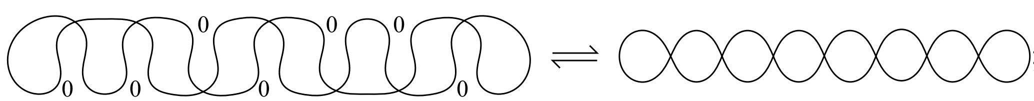
$$\begin{cases} \ell(n,0) = 0, \ \ell(n,1) = 2^n, & n \geq 0; \\ \ell(n,k) = 2^n\big(t(n-1,k) + t(n-1,k-1)\big) & \\ \qquad\quad = \ell(n-1,k) + \ell(n-1,k-1), & k \geq 1, \ n \geq 0. \end{cases} \tag{28}$$

Therefore we have Table 5 giving the numbers $\ell(n,k)$, for $0 \leq n \leq 9$ and $0 \leq k \leq 10$.

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10
0	0	1									
1	0	2	2								
2	0	4	8	4							
3	0	8	24	24	8						
4	0	16	64	96	64	16					
5	0	32	160	320	320	160	32				
6	0	64	384	960	1280	960	384	64			
7	0	128	896	2688	4480	4480	2688	896	128		
8	0	256	2048	7168	14336	17920	14336	7168	2048	256	
9	0	512	4608	18432	43008	64512	64512	43008	18432	4608	512

Table 5: Values of $\ell(n, k)$ for $0 \le n \le 9$ and $0 \le k \le 10$.

Remark 29. We have $L_n(x) = 2^n T_n(x)$. We interpret the factor 2^n as follows: at each pair of crossings making up a "link part", if we split one crossing in either type 0 or 1, then the resulting knot has equals generating with a twist loop. For example, we take $n = 7$. Letting the splits series be identified by a 0-1's string, we obtain some of the possible splits:

- 0000000: 

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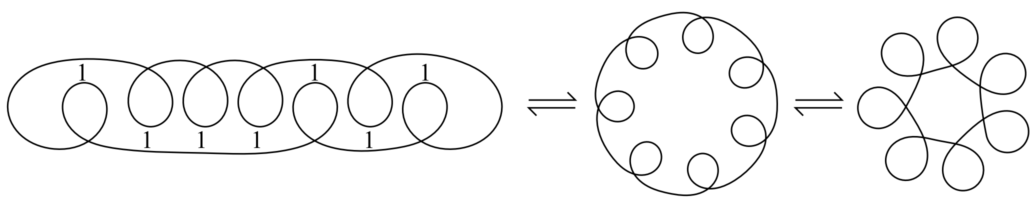
- 1111111: 

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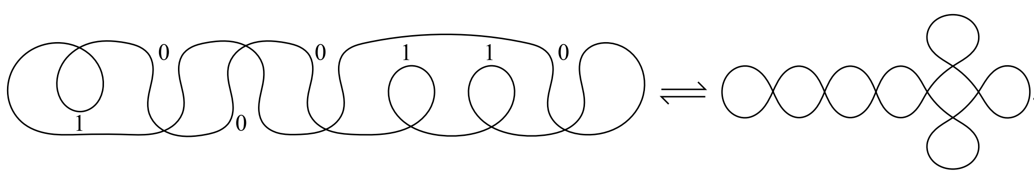
- 1000110: 

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3.4 Twist link

Let $W_n(x) := \sum_{k \ge 0} w(n, k) x^k$ denote the generating polynomial of the n-twist link.

Theorem 30. *The generating polynomial of the n-twist link is given by the recurrence relation*

$$W_n(x) = (2x^2 + 4x + 2) W_{n-1}(x), \tag{29}$$

and is expressed by the closed form formula

$$W_n(x) = x \left(2x^2 + 4x + 2\right)^n . \tag{30}$$

Proof. Note that $W_1 = L_1 \# T_1$. Then by Proposition 19 we have

$$\begin{aligned} W_1(x) &= x^{-1} L_1(x) T_1(x) \\ &= x^{-1}\left(2x^2+2x\right)\left(x^2+x\right) \\ &= 2x^3+4x^2+2x. \end{aligned}$$

We conclude by Corollary 21. □

Corollary 31. *The generating function for the sequence* $\{W_n(x)\}_{n\geq 0}$ *is given by*

$$W(x;y) := \frac{x}{1-y(2x^2+4x+2)}. \tag{31}$$

The recurrence relation (29) along with the closed formula (30) allow us to write:

$$\begin{cases} w(n,0)=0,\ w(n,1)=2^n,\ w(n,2)=n2^{n+1}, & n\geq 0; \\ w(n,k)=2w(n-1,k-2)+4w(n-1,k-1)+2w(n-1,k), & k\geq 2,\ n\geq 0. \end{cases} \tag{32}$$

For $k \geq 1$, we obtain $w(n,k) = 2^n\binom{2n}{k-1}$ [18, A139548]. Hence we have Table 6.

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	1												
1	0	2	4	2										
2	0	4	16	24	16	4								
3	0	8	48	120	160	120	48	8						
4	0	16	128	448	896	1120	896	448	128	16				
5	0	32	320	1440	3840	6720	8064	6720	3840	1440	320	32		
6	0	64	768	4224	14080	31680	50688	59136	50688	31680	14080	4224	768	64

Table 6: Values of $w(n,k)$ for $0 \leq n \leq 6$ and $0 \leq k \leq 13$.

3.5 Hitch knot

Let $H_n(x) := \sum_{k\geq 0} h(n,k)x^k$ denote the generating polynomial of the n-hitch knot.

Theorem 32. *The generating polynomial of the n-hitch knot is given by the recurrence relation*

$$H_n(x) = (x^2+4x+3)H_{n-1}(x), \tag{33}$$

and is expressed by the closed form formula

$$H_n(x) = x\left(x^2+4x+3\right)^n. \tag{34}$$

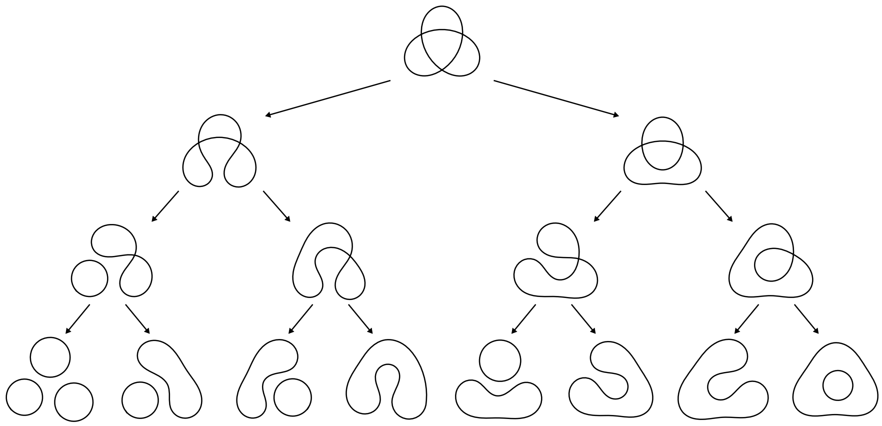

Figure 15: The states of the 1-hitch, $H_1(x) = x^3 + 4x^2 + 3x$.

Proof. See Figure 15. □

Corollary 33. *The generating function for the sequence* $\{H_n(x)\}_{n\geq 0}$ *is given by*

$$H(x;y) := \frac{x}{1 - y\left(x^2 + 4x + 3\right)}.$$

By (33) and (34), we deduce the following recurrence relation:

$$\begin{cases} h(n,0) = 0,\ h(n,1) = 3^n,\ h(n,2) = 4(n-1)3^{n-2}, & n \geq 0; \\ h(n,k) = h(n-1,k-2) + 4h(n-1,k-1) + 3h(n-1,k), & k \geq 2,\ n \geq 0. \end{cases} \tag{35}$$

The recurrence relation (35) is used to generate the entries in Table 7. Particularly, we recognize the following sequences:

- $h(n,1) = 3^n$, the powers of 3 [18, A000244];
- $h(n,2) = 4n3^{n-1}$, the sum of the lengths of the drops in all ternary words of length $n+1$ on $\{0,1,2\}$ [18, A120908];
- $h(n,n+1) = \sum_{k=0}^{n} \binom{n}{k}^2 3^k$ [18, A069835];
- $h(n,2n+1) = 1$, the all 1's sequence [18, A000012];
- $h(n,2n-1) = n(8n-5)$ [18, A139272];
- $h(n,2n) = 4n$, the multiples of 4 [18, A008586].

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	1												
1	0	3	4	1										
2	0	9	24	22	8	1								
3	0	27	108	171	136	57	12	1						
4	0	81	432	972	1200	886	400	108	16	1				
5	0	243	1620	4725	7920	8430	5944	2810	880	175	20	1		
6	0	729	5832	20898	44280	61695	59472	40636	19824	6855	1640	258	24	1

Table 7: Values of $h(n,k)$ for $0 \leq n \leq 5$ and $0 \leq k \leq 11$.

3.6 Overhand knot

Theorem 34. *The generating polynomial of the n-overhand knot is given by the recurrence relation*

$$O_n(x) = (x^2 + 4x + 3)O_{n-1}(x), \tag{36}$$

and is expressed by the closed form formula

$$O_n(x) = x\left(x^2 + 4x + 3\right)^n. \tag{37}$$

Proof. See Figure 16. ∎

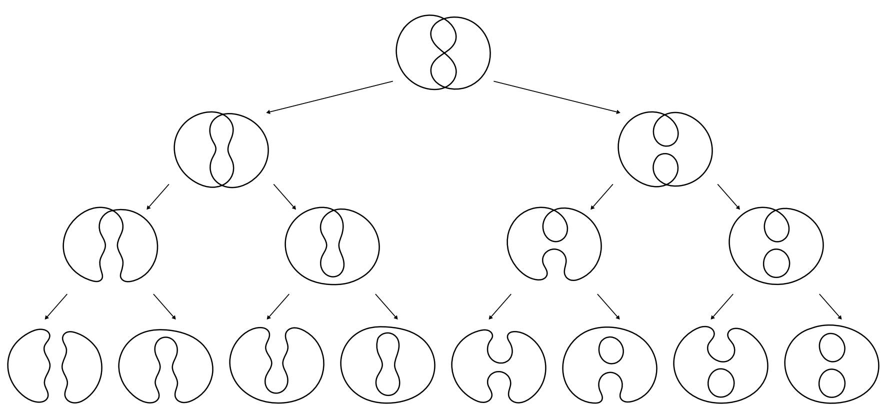

Figure 16: The states of the 1-overhand knot, $O_1(x) = x^3 + 4x^2 + 3x$.

Corollary 35. *The generating function for the sequence $\left\{O_n(x)\right\}_{n\geq 0}$ is given by*

$$O(x;y) := \frac{x}{1 - y\left(x^2 + 4x + 3\right)}. \tag{38}$$

The results are as expected since the knots H_1 and O_1 are equivalent under the OS^2 move. Thus we obtain the same statistics as the hitch knot (subsection 3.5).

4 The generating polynomial for the closure

4.1 Preliminaries

Let K be a knot diagram. The state diagrams of the closure of K can be illustrated as in Figure 17. Therefore, there exists two polynomials $\alpha_K, \beta_K \in \mathbb{Z}[x]$ such that we can write

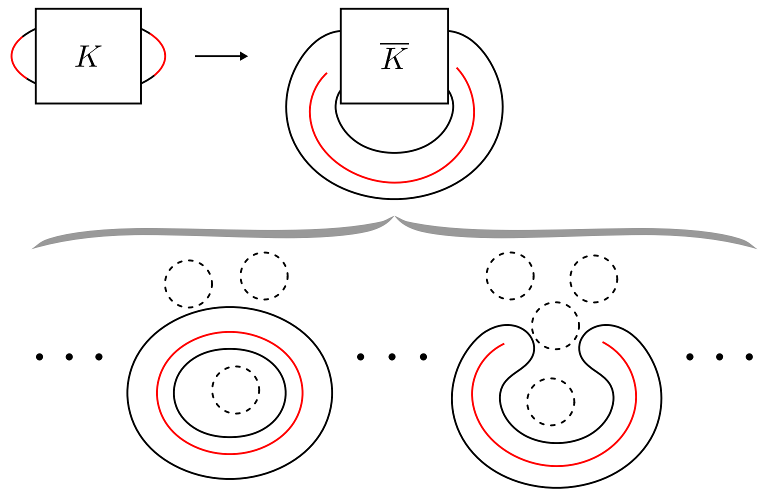

Figure 17: The state diagrams of a closure of the knot K.

the generating polynomial of the knot $\overline{K}$ as

$$\overline{K}(x) = \alpha_K(x)x^2 + \beta_K(x)x. \tag{39}$$

Conversely, if we "cancel" the connection, then we have the additional identity

$$K(x) = \beta_K(x)x^2 + \alpha_K(x)x. \tag{40}$$

Let us refer to $\alpha_K(x)$ and $\beta_K(x)$ as the components of the polynomial $K(x)$. We can then extend (39) as follows.

Proposition 36. *Let K, K' be two knot diagrams, and let $K(x)$, $K'(x)$ be respectively their generating polynomials. If $\alpha_K(x)$ and $\beta_K(x)$ are the components of the polynomial $K(x)$, then we obtain*

$$\overline{(K\#K')}(x) = \alpha_K(x)\overline{K'}(x) + \beta_K(x)K'(x).$$

We should note that if $K' = U$, then we get back to the definition of the closure (40). Indeed we have

$$\begin{aligned}\overline{K}(x) = \overline{(K\#U)}(x) &= \alpha_K(x)\overline{U}(x) + \beta_K(x)U(x)\\ &= \alpha_K(x)x^2 + \beta_K(x)x.\end{aligned}$$

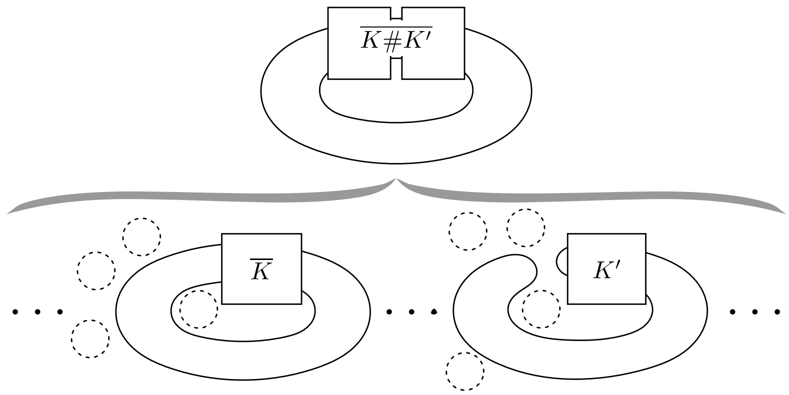

Figure 18: The states of a composition closure.

Proof of Proposition 36. We first need to compute the generating polynomial of the knot K. Referring to Figure 17, the "largest double circle" becomes the closure $\overline{K'}$, while the "largest single circle" is then connected with K'. See Figure 18 for the illustration. Next, we recover the same components $\alpha_K(x)$ and $\beta_K(x)$, and taking into account the previously mentioned substitution, we write

$$\overline{(K\#K')}(x) = \alpha_K(x)\overline{K'}(x) + \beta_K(x)K'(x).$$

□

Corollary 37. *Let K be an arbitrary knot whose components are $\alpha_K(x)$ and $\beta_K(x)$. For a nonnegative integer n, we have*

$$\begin{aligned}\overline{K_n}(x) &= \alpha_K(x)\overline{K_{n-1}}(x) + \beta_K(x)K_{n-1} \\ &= \alpha(x)^n x^2 + \beta_K(x)\sum_{k=0}^{n-1}\alpha_K(x)^{n-k-1}K_k(x).\end{aligned}$$

Proof. We write $\overline{K_n}(x) = \overline{(K\#K_{n-1})}(x)$, and we apply Proposition 36. It suffices next to expand the resulting recurrence. □

Lemma 38. *Let $\overline{K}(x;y) := \sum_{n\geq 0}\overline{K_n}(x)y^n$ and $K(x;y) := \sum_{n\geq 0}K_n(x)y^n$ respectively denote the generating function for the sequence $\{\overline{K_n}(x)\}_{n\geq 0}$ and $\{K_n(x)\}_{n\geq 0}$.*

Then we have

$$\overline{K}(x;y) = \frac{1}{1-y\alpha_K(x)}\Big(x^2 + y\beta_K(x)K(x;y)\Big), \tag{41}$$

where $\alpha_K(x)$ and $\beta_K(x)$ denote the components of $K(x)$.

Proof. Writing $\overline{K}(x;y) := \sum_{n\geq 0}\overline{K}_n(x)y^n$ implies

$$\overline{K}(x;y) = \sum_{n\geq 0}\left(\alpha_K(x)^n x^2 + \frac{\beta_K(x)}{\alpha_K(x)}\left(\sum_{p=0}^{n-1}\alpha_K(x)^{n-p}K_p(x) - K_n(x)\right)\right)y^n.$$

Then we apply the usual Cauchy product of two series [20, Rule 3, p. 36]:

$$\begin{aligned}\overline{K}(x;y) &= \frac{x^2}{1-y\alpha_K(x)} + \frac{\beta_K(x)}{\alpha_K(x)}\left(\frac{1}{1-y\alpha_K(x)}K(x;y) - K(x;y)\right) \\ &= \frac{1}{1-y\alpha_K(x)}\Big(x^2 + y\beta_K(x)K(x;y)\Big).\end{aligned}$$

□

The key to establishing the generating polynomial for the closure of a generated knot is then to identify the components of that of the generator. The next results are direct application of Corollary 37 and Lemma 38.

4.2 Foil knot

Let $\overline{T_n}(x) := \sum_{k\geq 0} f(n,k)x^k$ denote the generating polynomial of the n-foil knot.

Theorem 39. *The generating polynomial of the n-foil knot is given by the recurrence relation*

$$\overline{T_n}(x) = \overline{T_{n-1}}(x) + T_{n-1}(x), \tag{42}$$

and is expressed by the closed form formula

$$\overline{T_n}(x) = (x+1)^n + x^2 - 1. \tag{43}$$

Proof. By Figure 19, we have $\alpha_{T_1}(x) = 1$ and $\beta_{T_1}(x) = 1$. Next apply Corollary 37. □

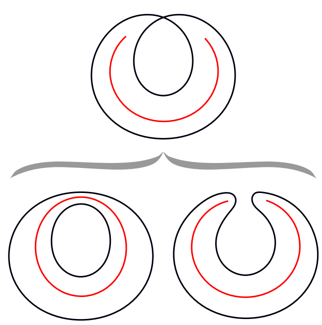

Figure 19: The states of the 1-foil, $\overline{T_1}(x) = x^2 + x$.

Corollary 40. *The generating function for the sequence $\left\{\overline{T_n}(x)\right\}_{n\geq 0}$ is given by*

$$\overline{T}(x;y) := \frac{1}{1-y}\left(x^2 + \frac{yx}{1-y(x+1)}\right). \tag{44}$$

By (42) and (43), the coefficients $f(n,k)$ must satisfy the following recurrence relation:

$$\begin{cases} f(0,2)=1; & \\ f(n,0)=0,\ f(n,1)=n, & n\geq 0; \\ f(n,k)=f(n-1,k)+t(n-1,k), & k\geq 1,\ n\geq 0. \end{cases} \tag{45}$$

Hence we have Table 8 for $0 \leq n \leq 12$ and $0 \leq k \leq 12$. Also, referring back to (17), we obtain the following sequences:

- $f(n,1)=n$, the nonnegative integers [18, A001477];
- $f(n,2)=\binom{n}{2}+1$ [18, A152947];
- if $k \geq 3$, then we obtain the usual Pascal's triangle [18, A007318].

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	1										
1	0	1	1										
2	0	2	2										
3	0	3	4	1									
4	0	4	7	4	1								
5	0	5	11	10	5	1							
6	0	6	16	20	15	6	1						
7	0	7	22	35	35	21	7	1					
8	0	8	29	56	70	56	28	8	1				
9	0	9	37	84	126	126	84	36	9	1			
10	0	10	46	120	210	252	210	120	45	10	1		
11	0	11	56	165	330	462	462	330	165	55	11	1	
12	0	12	67	220	495	792	924	792	495	220	66	12	1

Table 8: Values of $f(n,k)$ for $0 \leq n \leq 12$ and $0 \leq k \leq 12$.

Remark 41. From Table 2 and Table 8 we have

$$T_1(x) = \overline{T_1}(x) = x^2 + x,$$

from Table 5 and Table 8 we read

$$L_1(x) = \overline{T_2}(x) = 2x^2 + 2x,$$

and finally from Table 7 and Table 8 we read

$$H_1(x) = \overline{T_3}(x) = x^3 + 4x^2 + 3x.$$

Remark 42. Following Remark 26, we have the corresponding results on the $2n$-foil knot and the $3n$-foil knot, namely $\overline{(T_2)_n}(x) = \overline{(T_{2n})}(x)$ and $\overline{(T_3)_n}(x) = \overline{(T_{3n})}(x)$.

$2n$-foil knot: let $\overline{T_{2n}}(x) := \sum_{k\geq 0} f_2(n,k)x^k$.

1. Generating polynomial:
$$\overline{T_{2n}}(x) = (x+1)^{2n} + x^2 - 1. \tag{46}$$

2. Generating function:
$$\overline{T_2}(x;y) := \frac{1}{1-y}\left(x^2 + \frac{yx(x+2)}{1-y(x^2+2x+1)}\right). \tag{47}$$

3. Distribution of $f_2(n,k)$: see Table 9.
$$\begin{cases} f_2(0,2) = 1; \\ f_2(n,0) = 0,\, f_2(n,1) = 2n, & n \geq 0; \\ f_2(n,k) = f_2(n-1,k) + t_2(n-1,k-1) + 2t_2(n-1,k), & k \geq 1,\ n \geq 0. \end{cases} \tag{48}$$

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	0	1												
1	0	2	2												
2	0	4	7	4	1										
3	0	6	16	20	15	6	1								
4	0	8	29	56	70	56	28	8	1						
5	0	10	46	120	210	252	210	120	45	10	1				
6	0	12	67	220	495	792	924	792	495	220	66	12	1		
7	0	14	92	364	1001	2002	3003	3432	3003	2002	1001	364	91	14	1

Table 9: Values of $f_2(n,k)$ for $0 \leq n \leq 7$ and $0 \leq k \leq 14$.

- $f_2(n,1) = 2n$, the nonnegative even numbers, [18, A005843];
- $f_2(n,2) = 2n^2 - n + 1$, the maximum number of regions determined by n bent lines [5, 18, A130883];
- if $k \geq 3$, then we obtain $f_2(n,k) = \binom{2n}{k}$, the even-numbered rows of Pascal's triangle [18, A034870].

$3n$-foil knot: let $\overline{T_{3n}}(x) := \sum_{k\geq 0} f_3(n,k)x^k$.

1. Generating polynomial:
$$\overline{T_{3n}}(x) = (x+1)^{3n} + x^2 - 1. \tag{49}$$

2. Generating function:

$$\overline{T_3}(x;y) := \frac{1}{1-y}\left(x^2 + \frac{yx(x^2+3x+3)}{1-y(x^3+3x^2+3x+1)}\right). \tag{50}$$

3. Distribution of $f_3(n,k)$: see Table 10.

$$\begin{cases} f_3(0,2) = 1; \\ f_3(n,0) = 0,\ f_3(n,1) = 3n,\ f_3(n,2) = \dfrac{9n^2-3n+2}{2}, & n \geq 0; \\ f_3(n,k) = f_3(n-1,k) + t_3(n-1,k-2) \\ \qquad + 3t_3(n-1,k-1) + 3t_3(n-1,k), & k \geq 2,\ n \geq 0. \end{cases} \tag{51}$$

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	1													
1	0	3	4	1												
2	0	6	16	20	15	6	1									
3	0	9	37	84	126	126	84	36	9	1						
4	0	12	67	220	495	792	924	792	495	220	66	12	1			
5	0	15	106	455	1365	3003	5005	6435	6435	5005	3003	1365	455	105	15	1

Table 10: Values of $f_3(n,k)$ for $0 \leq n \leq 5$ and $0 \leq k \leq 15$.

- $f_3(n,1) = 3n$, the multiples of 3 [18, A008585];
- $f_3(n,2) = \dfrac{9n^2-3n+2}{2}$, the generalized polygonal numbers [18, A080855];
- $f_3(n,3) = \dfrac{n(3n-1)(3n-2)}{2}$, the dodecahedral numbers [18, A006566];
- $f_3(n,2n+1) = \dbinom{3n}{n-1}$ [18, A004319];
- in fact, if $k \geq 3$, then $f_3(n,k) = t_3(n,k+1) = \dbinom{3n}{k}$ [18, A007318$(3n,k)$].

4.3 Chain Link

Let $\overline{L_n}(x) := \sum_{k\geq 0} c(n,k)x^k$ denote the generating polynomial of the n-chain link.

Theorem 43. *The generating polynomial of the n-chain link is given by the recurrence relation*

$$\overline{L_n}(x) = (x+2)\overline{L_{n-1}}(x) + L_{n-1}(x), \tag{52}$$

and is expressed by the closed form formula

$$\overline{L_n}(x) = x^2(x+2)^n + x\sum_{k=0}^{n-1}(x+2)^{n-k-1}(2x+2)^k. \tag{53}$$

Proof. By Figure 20 we get $\alpha_{L_1}(x) = x + 2$ and $\beta_{L_1}(x) = 1$. □

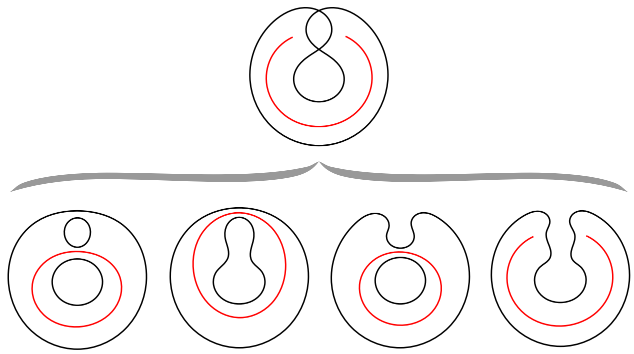

Figure 20: The states of the 1-chain link, $\overline{L_1}(x) = x^3 + 2x^2 + x$.

Corollary 44. *The generating function for the sequence* $\left\{\overline{L_n}(x)\right\}_{n\geq 0}$ *is given by*

$$\overline{L}(x;y) := \frac{1}{1-y(x+2)}\left(x^2 + \frac{yx}{1-y(x+2)}\right). \tag{54}$$

Combining (52) and (53), we have

$$\begin{cases} c(0,2) = 1; & \\ c(n,0) = 0, \;\; c(n,1) = n2^{n-1}, & n \geq 0; \\ c(n,k) = c(n-1,k-1) + 2c(n-1,k) + \ell(n-1,k), & k \geq 1, \;\; n \geq 0. \end{cases} \tag{55}$$

Table 11 gives the values of $c(n,k)$ for $0 \leq n \leq 9$ and $0 \leq k \leq 11$. Next, we identify the following integer sequences:

- $c(n,1) = n2^{n-1}$ [18, A001787];
- $c(n,n) = 2n(n-1) + 2^n - 1$ [18, A295077];
- $c(n,n+1) = 2n$, the nonnegative even numbers [18, A005843];
- $c(n,n+2) = 1$, the all 1's sequence [18, A000012].

Remark 45. From Table 2 and Table 11 we have

$$T_2(x) = \overline{L_1}(x) = x^2 + 2x + x,$$

and from Table 8 and Table 11 we read

$$\overline{T_4}(x) = \overline{L_2}(x) = x^4 + 4x^3 + 7x^2 + 4x.$$

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	1									
1	0	1	2	1								
2	0	4	7	4	1							
3	0	12	26	19	6	1						
4	0	32	88	88	39	8	1					
5	0	80	272	360	1230	71	10	1				
6	0	192	784	1312	1140	532	123	12	1			
7	0	448	2144	4368	4872	3164	1162	211	14	1		
8	0	1024	5632	13568	18592	15680	8176	2480	367	16	1	
9	0	2304	14336	39936	65088	67872	46368	20304	5262	655	18	1

Table 11: Values of $c(n,k)$ for $0 \leq n \leq 9$ and $0 \leq k \leq 11$.

4.4 Twist bracelet

Let $\overline{W_n}(x) := \sum_{k\geq 0} b(n,k)x^k$ denote the generating polynomial of the n-twist bracelet.

Theorem 46. *The generating polynomial of the n-twist bracelet is given by the recurrence relation*

$$\overline{W_n}(x) = (x+2)\overline{W_{n-1}}(x) + (2x+3)W_{n-1}(x) \tag{56}$$

and is expressed by the closed form formula

$$\overline{W_n}(x) = x^2(x+2)^n + x(2x+3)\sum_{k=0}^{n-1}(x+2)^{n-k-1}(2x^2+4x+2)^k. \tag{57}$$

Proof. We have $\alpha_{W_1}(x) = x+2$ and $\beta_{W_1}(x) = 2x+3$, see Figure 21. □

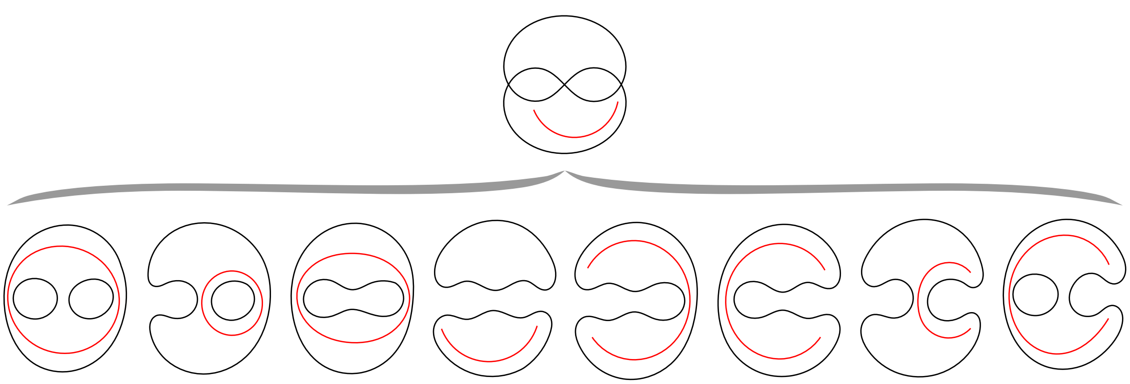

Figure 21: The states of the 1-twist bracelet, $\overline{W_1}(x) = x^3 + 4x^2 + 3x$.

Corollary 47. *The generating function for the sequence* $\{\overline{W_n}(x)\}_{n\geq 0}$ *is given by*

$$\overline{W}(x;y) := \frac{1}{1-y(x+2)}\left(x^2 + \frac{yx(2x+3)}{1-y(2x^2+4x+2)}\right).$$

By (32), (56) and (57) we have the following recurrence

$$\begin{cases} b(0,2) = 1; & \\ b(n,0) = 0, \; b(n,1) = 3n2^{n-1}, & n \geq 0; \\ b(n,k) = b(n-1,k-1) + 2b(n-1,k) & \\ \qquad\qquad + 2w(n-1,k-1) + 3w(n-1,k), & k \geq 1, \; n \geq 0. \end{cases} \tag{58}$$

Table 12 gives some of the values of $b(n,k)$. We collect the following sequences:

- $b(n,1) = 3n2^{n-1}$ [18, A167667];
- $b(n,2n) = 2^n$ except $b(1,2) = 4$ and $b(2,4) = 5$, the independence number of Keller graphs [18, A258935].

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	1										
1	0	3	4	1									
2	0	12	27	20	5								
3	0	36	122	171	126	49	8						
5	0	96	440	920	1143	904	449	128	16				
6	0	240	1392	3880	6790	8103	6730	3841	1440	320	32		
7	0	576	4048	14112	31860	50836	59195	50700	31681	14080	4224	768	64

Table 12: Values of $b(n,k)$ for $0 \leq n \leq 7$ and $0 \leq k \leq 12$.

Remark 48. From Table 7, Table 8 and Table 12 we have

$$H_1(x) = \overline{T_3}(x) = \overline{W_1}(x) = x^3 + 4x^2 + 3x.$$

4.5 Ringbolt hitching

Let $\overline{H_n}(x) := \sum_{k\geq 0} r(n,k)x^k$ denote the generating polynomial of the n-ringbolt hitching.

Theorem 49. *The generating polynomial of the* n*-ringbolt hitching knot is given by the recurrence relation*

$$\overline{H_n}(x) = (2x+3)\overline{H_{n-1}}(x) + (x+2)H_{n-1}(x). \tag{59}$$

and is expressed by the closed form formula

$$\overline{H_n}(x) = x^2(2x+3)^n + x(x+2)\sum_{k=0}^{n-1}(2x+3)^{n-k-1}\left(x^2+4x+3\right)^k. \tag{60}$$

Proof. Proceeding with the usual fashion we obtain $\alpha_{H_1}(x) = 2x + 3$ and $\beta_{H_1}(x) = x + 2$, see Figure 22. □

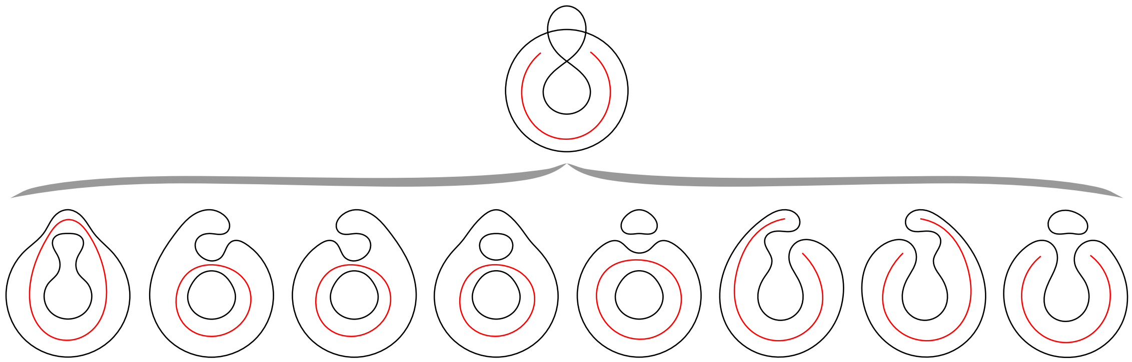

Figure 22: The states of the 1-ringbolt hitching, $\overline{H_1}(x) = 2x^3 + 4x^2 + 2x$.

Corollary 50. *The generating function for the sequence* $\{\overline{H_n}(x)\}_{n\geq 0}$ *is given by*

$$\overline{H}(x;y) := \frac{1}{1-y(2x+3)}\left(x^2 + \frac{yx(x+2)}{1-y(x^2+4x+3)}\right) \tag{61}$$

Combining (59) with (60), we obtain the recurrence relation

$$\begin{cases} r(0,2) = 1; \\ r(n,0) = 0,\ r(n,1) = 2n3^{n-1}, & n \geq 0; \\ r(n,k) = 2r(n-1,k-1) + 3r(n-1,k) \\ \qquad\qquad + h(n-1,k-1) + 2h(n-1,k), & k \geq 1,\ n \geq 0. \end{cases} \tag{62}$$

We then obtain the values of $r(n,k)$ for $0 \leq n \leq 6$ and $0 \leq k \leq 12$ as given in Table 13. We only recognize here the sequence defined by $r(n,1) = 2n3^{n-1}$ [18, A212697].

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	1										
1	0	2	4	2									
2	0	12	27	20	5								
3	0	54	162	182	93	20	1						
4	0	216	837	1320	1086	496	124	16	1				
5	0	810	3888	8010	9270	6632	3050	912	175	20	1		
6	0	2916	16767	42876	64395	63216	42732	20400	6919	1640	258	24	1

Table 13: Values of $r(n,k)$ for $0 \leq n \leq 6$ and $0 \leq k \leq 12$.

Remark 51. Reading Table 6 and Table 13 yields

$$W_3(x) = \overline{H_1}(x) = 2x^3 + 4x^2 + 2x.$$

4.6 Sinnet of square knotting

Let $\overline{O_n}(x) := \sum_{k\geq 0} s(n,k)x^k$ denote the generating polynomial of the n-sinnet of square knotting.

Theorem 52. *The generating polynomial of the n-sinnet of square knotting is given by the recurrence relation*

$$\overline{O_n}(x) = \left(x^2+3x+3\right)\overline{O_{n-1}}(x) + O_{n-1}(x), \tag{63}$$

and is expressed by the closed form formula

$$\overline{O_n}(x) = x^2\left(x^2+3x+3\right)^n + x\sum_{k=0}^{n-1}\left(x^2+3x+3\right)^{n-k-1}\left(x^2+4x+3\right)^k. \tag{64}$$

Proof. Again, the usual routine allows us to give both expressions of $O_n(x)$. By Figure 23, we have $\alpha_{O_1}(x) = x^2+3x+3$ and $\beta_{O_1}(x) = 1$. We conclude by Corollary 37. ☐

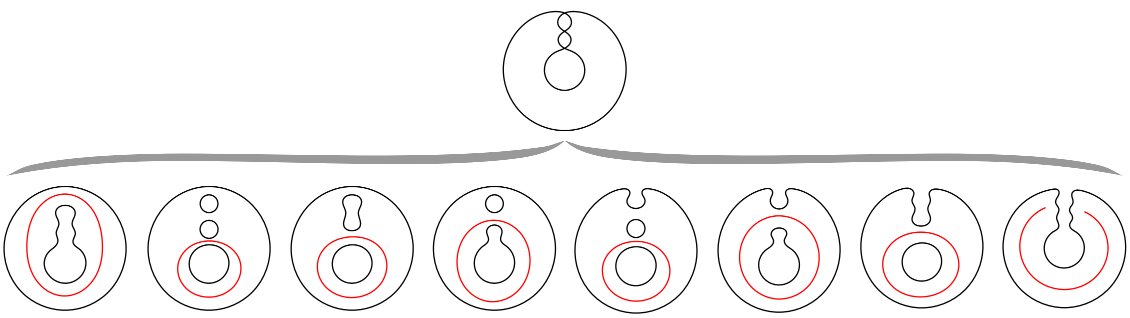

Figure 23: The states of the 1-sinnet of square knotting, $\overline{O_1}(x) = x^4+3x^3+3x^2+x$.

Corollary 53. *The generating function for the sequence $\left\{\overline{O_n}(x)\right\}_{n\geq 0}$ is given by*

$$\overline{O}(x;y) := \frac{1}{1-y(x^2+3x+3)}\left(x^2+\frac{yx}{1-y(x^2+4x+3)}\right). \tag{65}$$

Combining (35), (63) and (64) yields

$$\begin{cases} s(0,2)=1; \\ s(n,0)=0,\ \ s(n,1)=n3^{n-1},\ \ s(n,2)=3^n+7\frac{n(n-1)}{2}3^{n-2}, & n\geq 0; \\ s(n,k)=s(n-1,k-2)+3s(n-1,k-1)+3s(n-1,k) & \\ \qquad\qquad +h(n-1,k), & k\geq 2,\ \ n\geq 0. \end{cases} \tag{66}$$

In Table 14, we list the values of $s(n,k)$ for small n and k. We recognize the following sequences:

- $s(n,1) = n3^{n-1}$ [18, A027471];
- $s(n, 2n+1) = 3n$, the multiples of 3 [18, A008585];
- $s(n, 2n-1) = \dfrac{n(3n-1)(3n-2)}{2}$, the dodecahedral numbers [18, A006566];
- $s(n, 2n) = \dfrac{3n(3n-1)}{2}$, three times pentagonal numbers [18, A062741].

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	1										
1	0	1	3	3	1								
2	0	6	16	20	15	6	1						
3	0	27	90	136	129	84	36	9	1				
4	0	108	459	876	1021	832	501	220	66	12	1		
5	0	405	2133	5085	7350	7321	5420	3103	1375	455	105	15	1

Table 14: Values of $s(n,k)$ for $0 \le n \le 5$ and $0 \le k \le 12$.

Remark 54. From Table 2 and Table 14 we have

$$T_3(x) = \overline{O_1}(x) = x^4 + 3x^3 + 3x^2 + x,$$

and from Table 8 and Table 14 we have

$$\overline{T_6}(x) = \overline{O_2}(x) = x^6 + 6x^5 + 15x^4 + 20x^3 + 16x^2 + 6x.$$

4.7 Twist knot

A *twist knot* is a knot obtained by repeatedly twisting a closed loop, and then linking the ends together [9]. We let τ_n denote a *n-twist knot*, i.e., a twist knot of n half twists [16]. Examples of twist knots are given in Figure 24.

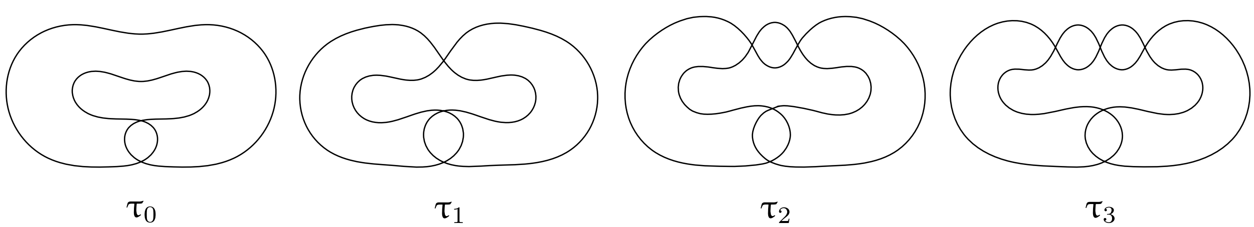

Figure 24: n-twist knots, $n = 0, 1, 2, 3$

Let then $\tau_n(x) := \sum_{k\ge 0} \tau(n,k)x^k$ denote the generating polynomial of the n-twist knot.

Theorem 55. *The generating polynomial of the n-twist knot is given by the relation*

$$\tau_n(x) = (x+2)\overline{T_n}(x) + T_n(x), \tag{67}$$

and has the following closed form

$$\tau_n(x) = 2(x+1)^{n+1} + x^3 + 2x^2 - x - 2. \tag{68}$$

Proof. The n-twist knot can be decomposed into the closure of the connected sum of the Hopf Link and the n-twist loop, i.e., $\tau_n = \overline{L_1 \# T_n}$, see Figure 25.

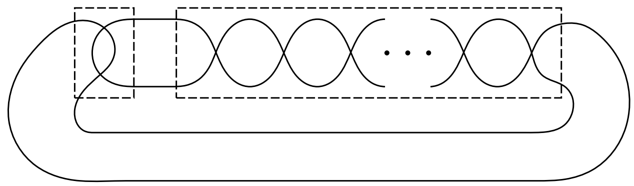

Figure 25: The twist knot as the closed connected sum of the Hopf Link and the twist loop.

Applying Proposition 36, we have

$$\tau_n(x) = \alpha_{L_1}(x)\overline{T_n}(x) + \beta_{L_1}(x)T_n(x),$$

where $\alpha_{L_1}(x) = x+2$ and $\beta_{L_1}(x) = 1$ are the components of $L_1(x)$. The closed form immediately follows. □

Corollary 56. *The generating function for the sequence $\{\tau_n(x)\}_{n\geq 0}$ is given by*

$$\tau(x;y) = \frac{2x+2}{1-y(x+1)} + \frac{x^3+2x^2-x-2}{1-y}.$$

Now we can draw up the usual table of coefficients and the collect the OEIS A-records using the following recurrence:

$$\begin{cases} \tau(n,0) = 0, \ \tau(n,1) = 2n+1, & n \geq 0; \\ \tau(n,k) = f(n,k-1) + 2f(n,k) + t(n,k), & k \geq 1, \ n \geq 0. \end{cases}$$

- $\tau(n,1) = 2n+1$, the odd numbers [18, A005408];
- $\tau(n,2) = n^2+n+2$, the maximum number of regions into which the plane is divided by $n+1$ circles [10, A014206];
- $\tau(n,3) = \frac{1}{3}\left(n^3 - n + 3\right)$ [18, A064999];
- if $k \geq 4$, then $(\tau(n,k))_{n\geq 3}$ is a horizontal shifted twice Pascal's triangle [18, A028326].

Remark 57. From Table 2 and Table 15 we have

$$T_2(x) = \tau_0(x) = x^3 + 2x^2 + x,$$

and from Table 7, Table 8 and Table 15 we read

$$H_1(x) = \overline{T_3}(x) = \tau_1(x) = x^3 + 4x^2 + 3x.$$

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	2	1									
1	0	3	4	1									
2	0	5	8	3									
3	0	7	14	9	2								
4	0	9	22	21	10	2							
5	0	11	32	41	30	12	2						
6	0	13	44	71	70	42	14	2					
7	0	15	58	113	140	112	56	16	2				
8	0	17	74	169	252	252	168	72	18	2			
90	0	19	92	241	420	504	420	240	90	20	2		
10	0	21	112	331	660	924	924	660	330	110	22	2	
11	0	23	134	441	990	1584	1848	1584	990	440	132	24	2

Table 15: Values of $\tau(n,k)$ for $0 \leq n \leq 11$ and $0 \leq k \leq 12$.

4.8 The alternative closures

We first introduce the following notation.

Notation 58. Let

- $\mathcal{S}_1\#\mathcal{S}_1 := \{K\#K' \mid (K,K') \in \mathcal{S}_1 \times \mathcal{S}_1\}$;
- $\mathcal{S}_1\#\mathcal{S}_1\#\mathcal{S}_1 := \{K\#K'\#K'' \mid (K,K',K'') \in \mathcal{S}_1 \times \mathcal{S}_1 \times \mathcal{S}_1\}$;
- $\mathcal{S}_1\#\mathcal{S}_{2,1} := \{K\#K' \mid (K,K') \in \mathcal{S}_1 \times \mathcal{S}_{2,1}\}$.

Then we have

- $\mathcal{S}_{2,2} = \mathcal{S}_1\#\mathcal{S}_1$;
- $\mathcal{S}_{3,1} \cup \mathcal{S}_{3,2} = \mathcal{S}_1\#\mathcal{S}_1\#\mathcal{S}_1$;
- $\mathcal{S}_{3,3} = \mathcal{S}_1\#\mathcal{S}_{2,1}$.

Moreover, writing $K = U\#K$ allows us to assume that the elementary knot K can be decomposed in a way that one might disconnect some knot factors that are not taken into consideration when connecting with a copy of the actual knot. For the sake of clarity, let us use the asterisk sign $*$ to indicate that none of the arcs of the concerned knot are involved when generating K_n and $\overline{K_n}$. For example, $K = T_1\#L_1^*$ means that the connected sum is performed along some section of T_1, and L_1 might be disconnected. We distinguish the following excluding cases:

- if we cannot disconnect an elementary knot, then $\overline{K_n} \in \{\overline{T_n}, \overline{T_{2n}}, \overline{T_{3n}}, \overline{L_n}, \overline{W_n}, \overline{H_n}, \overline{O_n}\}$;

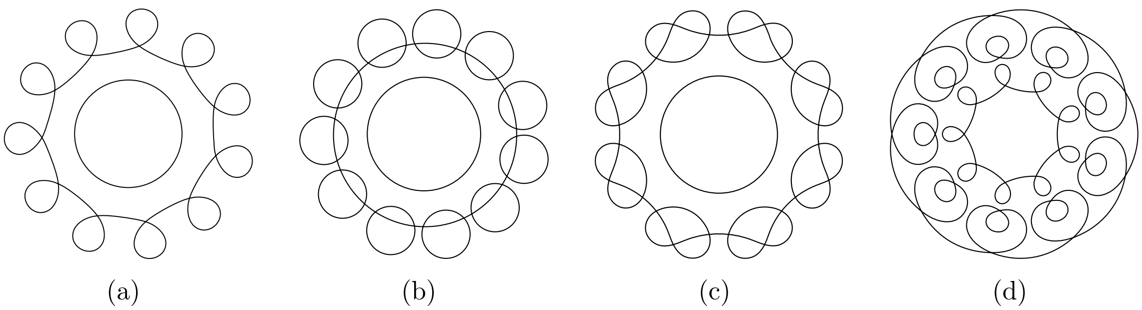

Figure 26: Examples of closed connected sum along the section of the unknot: (a) $\overline{(U\#T_1^*)_{11}}$; (b) $\overline{(U\#L_1^*)_{11}}$; (c) $\overline{(U\#H_1^*)_8}$; (d) $\overline{(U\#T_1^*\#T_1^*\#T_1^*)_9}$.

- if $K = U\#K^*$ with $K^* \in \{T_1^*, T_1^*\#T_1^*, T_1^*\#T_1^*\#T_1^*, L_1^*\#T_1^*, L_1^*, H_1^*, O_1^*\}$, then $\overline{K_n}(x) = \left(U^2\#K_n\right)(x)$ and $\overline{K}(x;y) = xK(x;y)$, see Figure 26;
- if $K \in \mathcal{S}_{2,2}$ and $K = T_1\#T_1^*$, then $\overline{K_n}(x) = \left(\overline{T_n}\#T_n\right)(x)$, see Figure 27 (a);
- if $K \in \mathcal{S}_{3,1} \cup \mathcal{S}_{3,2}$, then $\overline{K_n}(x) = \begin{cases} \left(\overline{T_{2n}}\#T_n\right)(x), & \text{if } K = T_2\#T_1^*, \text{ see Figure 27 (b);} \\ \left(\overline{T_n}\#T_{2n}\right)(x), & \text{if } K = T_1\#T_2^* \text{ or } K = T_1\#T_1^*\#T_1^*, \\ & \text{see Figure 27 (c);} \end{cases}$
- if $K \in \mathcal{S}_{3,3}$, then $\overline{K_n}(x) = \begin{cases} \left(\overline{L_n}\#T_n\right)(x), & \text{if } K = L_1\#T_1^*, \text{ see Figure 27 (d);} \\ \left(\overline{T_n}\#L_n\right)(x), & \text{if } K = T_1\#L_1^*, \text{ see Figure 27 (e).} \end{cases}$

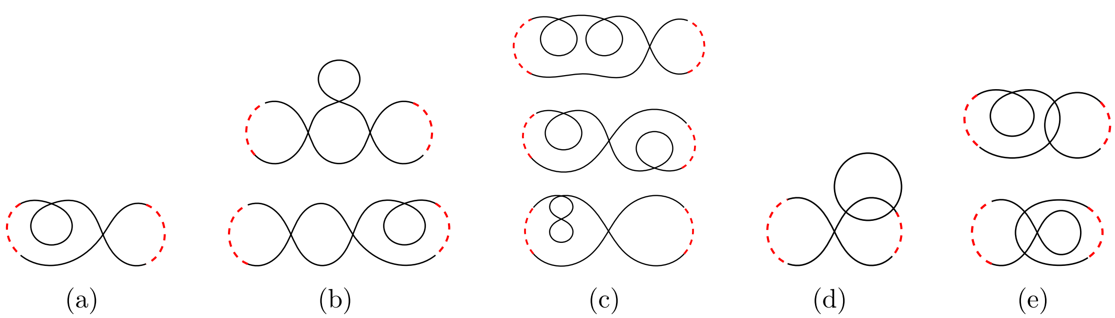

Figure 27: The arcs at which the connected sums are applied are indicated by the red dashed sections: (a) $T_1\#T_1^*$; (b) $T_1\#T_1^*\#T_1^*$ and $T_1\#T_2^*$; (c) $T_1\#T_1^*$; (d) $T_1\#L_1^*$; (e) $L_1\#T_1^*$.

Results on $\left(\overline{T_n}\#T_n\right)(x)$:

1. Generating polynomial:

$$\left(\overline{T_n}\#T_n\right)(x) = \left(x^2 + 2x + 1\right)^n + \left(x^2 - 1\right)(x+1)^n. \tag{69}$$

2. Generating function:

$$\frac{1}{1-y(x+1)}\left(x^2+\frac{yx(x+1)}{1-y(x^2+2x+1)}\right):=\sum_{n\geq 0}\left(\overline{T_n}\#T_n\right)(x)y^n. \tag{70}$$

3. Distribution of $\sigma_{\mathrm{a}}(n,k):=\left[x^k\right]\left(\overline{T_n}\#T_n\right)(x)$: see Table 9.

$$\begin{cases}\sigma_{\mathrm{a}}(0,2)=1; & \\ \sigma_{\mathrm{a}}(n,0)=0,\ \ \sigma_{\mathrm{a}}(n,1)=n, & n\geq 0;\\ \sigma_{\mathrm{a}}(n,k)=\sigma_{\mathrm{a}}(n-1,k-1)+\sigma_{\mathrm{a}}(n-1,k) & \\ \qquad\qquad +t_2(n-1,k-1)+t_2(n-1,k), & k\geq 1,\ \ n\geq 0.\end{cases} \tag{71}$$

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	0	1												
1	0	1	2	1											
2	0	2	6	6	2										
3	0	3	13	22	18	7	1								
4	0	4	23	56	75	60	29	8	1						
5	0	5	36	115	215	261	215	121	45	10	1				
6	0	6	52	206	495	806	938	798	496	220	66	12	1		
7	0	7	71	336	987	2016	3031	3452	3010	2003	1001	364	91	14	1

Table 16: Values of $\sigma_{\mathrm{a}}(n,k)$ for $0\leq n\leq 7$ and $0\leq k\leq 14$.

- $\sigma_{\mathrm{a}}(n,1)=n$, the nonnegative integers [18, A001477];
- $\sigma_{\mathrm{a}}(n,2)=\dfrac{3n^2-n+2}{2}$ [18, A143689];
- $\sigma_{\mathrm{a}}(n,n+3)=\dbinom{2n}{n-3}$ [18, A002696];
- $\sigma_{\mathrm{a}}(n,n+4)=\dbinom{2n}{n-4}$ [18, A004310];
- $\sigma_{\mathrm{a}}(n,n+5)=\dbinom{2n}{n-5}$ [18, A004311].

Results on $\left(\overline{T_{2n}}\#T_n\right)(x)$:

1. Generating polynomial:

$$\left(\overline{T_{2n}}\#T_n\right)(x)=\left(x^3+3x^2+3x+1\right)^n+\left(x^2-1\right)(x+1)^n. \tag{72}$$

2. Generating function:

$$\frac{1}{1-y(x+1)}\left(x^2+\frac{yx\left(x^2+3x+2\right)}{1-y\left(x^3+3x^2+3x+1\right)}\right):=\sum_{n\geq 0}\left(\overline{T_{2n}}\#T_n\right)(x)y^n. \tag{73}$$

3. Distribution of $\sigma_{\mathrm{b}}(n,k):=\left[x^k\right]\left(\overline{T_{2n}}\#T_n\right)(x)$: see Table 17.

$$\begin{cases}\sigma_{\mathrm{b}}(0,2)=1;\\ \sigma_{\mathrm{b}}(n,0)=0,\ \sigma_{\mathrm{b}}(n,1)=2n,\ \sigma_{\mathrm{b}}(n,2)=4n^2-n+1, & n\geq 0;\\ \sigma_{\mathrm{b}}(n,k)=\sigma_{\mathrm{b}}(n-1,k-1)+\sigma_{\mathrm{b}}(n-1,k)\\ \qquad +t_3(n-1,k-2)+3t_3(n-1,k-1)+2t_3(n-1,k), & k\geq 2,\ n\geq 0.\end{cases} \tag{74}$$

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	1													
1	0	2	4	2												
2	0	4	15	22	16	6	1									
3	0	6	34	86	129	127	84	36	9	1						
4	0	8	61	220	500	796	925	792	495	220	66	12	1			
5	0	10	96	450	1370	3012	5010	6436	6435	5005	3003	1365	455	105	15	1

Table 17: Values of $\sigma_{\mathrm{b}}(n,k)$ for $0\leq n\leq 5$ and $0\leq k\leq 15$.

- $\sigma_{\mathrm{b}}(n,1)=2n$, the nonnegative even numbers [18, A005843];
- $\sigma_{\mathrm{b}}(n,2)=4n^2-n+1$ [18, A054556].

Results on $\left(\overline{T_n}\#T_{2n}\right)(x)$:

1. Generating polynomial:

$$\left(\overline{T_n}\#T_{2n}\right)=(x+1)^{3n}+(x^2-1)(x+1)^{2n}. \tag{75}$$

2. Generating function:

$$\frac{1}{1-y(x^2+2x+1)}\left(x^2+\frac{yx(x^2+2x+1)}{1-y(x^3+3x^2+3x+1)}\right):=\sum_{n\geq 0}\left(\overline{T_n}\#T_{2n}\right)(x)y^n. \tag{76}$$

3. Distribution of $\sigma_{\mathrm{c}}(n,k):=\left[x^k\right]\left(\overline{T_n}\#T_{2n}\right)(x)$: see Table 18.

$$\begin{cases}\sigma_{\mathrm{c}}(0,2)=1;\\ \sigma_{\mathrm{c}}(n,0)=0,\ \sigma_{\mathrm{c}}(n,1)=n,\ \sigma_{\mathrm{c}}(n,2)=\dfrac{5n^2-n+2}{2}, & n\geq 0;\\ \sigma_{\mathrm{c}}(n,k)=\sigma_{\mathrm{c}}(n-1,k-2)+2\sigma_{\mathrm{c}}(n-1,k-1)+\sigma_{\mathrm{c}}(n-1,k)\\ \qquad +t_3(n-1,k-2)+2t_3(n-1,k)+t_3(n-1,k), & k\geq 2,\ n\geq 0.\end{cases} \tag{77}$$

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	1													
1	0	1	3	3	1											
2	0	2	10	20	20	10	2									
3	0	3	22	70	126	140	98	42	10	1						
4	0	4	39	172	453	792	966	840	522	228	67	12	1			
5	0	5	61	345	1200	2871	5005	6567	660	5115	3047	1375	456	105	15	1

Table 18: Values of $\sigma_c(n,k)$ for $0 \le n \le 5$ and $0 \le k \le 15$.

- $\sigma_c(n,1) = n$, the nonnegative integers [18, A001477];
- $\sigma_c(n,2) = \dfrac{5n^2 - n + 2}{2}$ [18, A140066].

Results on $\left(\overline{L_n}\#T_n\right)(x)$:

1. Generating polynomial:

$$\left(\overline{L_n}\#T_n\right)(x) = x^2\left(x^2+3x+2\right)^n + x(x+1)^n \sum_{k=0}^{n-1}(x+2)^{n-k-1}(2x+2)^k. \tag{78}$$

2. Generating function:

$$\frac{1}{1-y\left(x^2+3x+2\right)}\left(x^2 + \frac{yx(x+1)}{1-y(2x^2+4x+2)}\right) := \sum_{n\ge 0}\left(\overline{L_n}\#T_n\right)(x)y^n. \tag{79}$$

3. Distribution of $\sigma_d(n,k) := \left[x^k\right]\left(\overline{L_n}\#T_n\right)(x)$: see Table 19.

$$\begin{cases} \sigma_d(0,2) = 1; \\ \sigma_d(n,0) = 0,\ \sigma_d(n,1) = n2^{n-1},\ \sigma_d(n,2) = 2^{n-3}\left(7n^2-3n+8\right), & n \ge 0; \\ \sigma_d(n,k) = \sigma_d(n-1,k-2) + 3\sigma_d(n-1,k-1) + 2\sigma_d(n-1,k) \\ \qquad\qquad + w(n-1,k-1) + w(n-1,k), & k \ge 2,\ n \ge 0. \end{cases} \tag{80}$$

- $\sigma_d(n,1) = n2^{n-1}$ [18, A001787];
- $\sigma_d(n,2n+1) = 3n$, the multiples of 3 [18, A008585].

Results on $\left(\overline{T_n}\#L_n\right)(x)$:

1. Generating polynomial:

$$\left(\overline{T_n}\#L_n\right)(x) = \left(2x^2+4x+2\right)^n + (x^2-1)(2x+2)^n. \tag{81}$$

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	0	1												
1	0	1	3	3	1										
2	0	4	15	22	16	6	1								
3	0	12	62	133	153	102	40	9	1						
4	0	32	216	632	1047	1076	707	296	77	12	1				
5	0	80	672	2520	5550	7941	7705	5133	2325	695	131	15	1		
6	0	192	1936	8896	24612	45612	59567	56106	38314	18778	6432	1470	210	18	1

Table 19: Values of $\sigma_{\mathrm{d}}(n,k)$ for $0 \le n \le 6$ and $0 \le k \le 14$.

2. Generating function:

$$\frac{1}{1-y(2x+2)}\left(x^2+\frac{yx(2x+2)}{1-y(2x^2+4x+2)}\right) := \sum_{n\ge 0}\left(\overline{T_n}\#L_n\right)(x)y^n. \tag{82}$$

3. Distribution of $\sigma_{\mathrm{e}}(n,k) := \left[x^k\right]\left(\overline{T_n}\#L_n\right)(x)$: see Table 20.

$$\begin{cases} \sigma_{\mathrm{e}}(0,2) = 1; \\ \sigma_{\mathrm{e}}(n,0) = 0,\ \sigma_{\mathrm{e}}(n,1) = n2^n \text{ [18, A036289]}, & n \ge 0; \\ \sigma_{\mathrm{e}}(n,k) = 2\sigma_{\mathrm{e}}(n-1,k-1) + 2\sigma_{\mathrm{e}}(n-1,k) & \\ \qquad\qquad + 2w(n-1,k-1) + 2w(n-1,k), & k \ge 1,\ n \ge 0. \end{cases} \tag{83}$$

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	1										
1	0	2	4	2									
2	0	8	24	24	8								
3	0	24	104	176	144	56	8						
4	0	64	368	896	1200	960	464	128	16				
5	0	160	1152	3680	6880	8352	6880	3872	1440	320	32		
6	0	384	3328	13184	31680	51584	60032	51072	31744	14080	4224	768	64

Table 20: Values of $\sigma_{\mathrm{e}}(n,k)$ for $0 \le n \le 6$ and $0 \le k \le 12$.

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