# CAMBRIAN ACYCLIC DOMAINS: COUNTING c-SINGLETONS 

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#### Abstract

We study the size of certain acyclic domains that arise from geometric and combinatorial constructions. These acyclic domains consist of all permutations visited by commuting equivalence classes of maximal reduced decompositions if we consider the symmetric group and, more generally, of all $c$-singletons of a Cambrian lattice associated to the weak order of a finite Coxeter group. For this reason, we call these sets Cambrian acyclic domains. Extending a closed formula of Galambos-Reiner for a particular acyclic domain called Fishburn's alternating scheme, we provide explicit formulae for the size of any Cambrian acyclic domain and characterize the Cambrian acyclic domains of minimum or maximum size.


## 1. Introduction

Examples of $c$-singletons include certain acyclic domains in social choice theory, natural partial orders of crossings in pseudoline arrangements as well as certain vertices of particular convex polytopes called permutahedra and associahedra in discrete geometry. We first describe these objects and outline the relationship between these incarnations.

Acyclic domains are of great interest in social choice theory because of their importance for the following voting process: voters choose among a given collection of linear orders on $m$ candidates and the result of the ballot obeys the order imposed by the majority for each pair of candidates. As already mentioned by the Marquis de Condorcet in 1785 dC85, not every collection of linear orders yields a transitive order on the candidates in every election. Collections that do guarantee transitivity are called acyclic domains or Condorcet domains. According to Fishburn [Fis02, Introduction], the fundamental problem to determine the maximum cardinality of an acyclic domain for a given number of candidates is one of most fascinating and intractable combinatorial problems in social choice theory. Abello as well as Chameni-Nembua describe different constructions of "large" acyclic sets. They use maximal chains of the weak order on the symmetric group $\Sigma_{m}$ Abe91] and study covering distributive sublattices of the weak order on $\Sigma_{m}$ CN89.

Galambos and Reiner GR08 prove that maximal acyclic domains constructed by Abello coincide with those of Chameni-Nembua and describe them in terms of higher Bruhat orders. Moreover, they show that acyclic domains obtained from Fishburn's alternating scheme [Fis97] are a special case of Chameni-Nembua's construction and prove that the cardinality of Fishburn's acyclic domain is given by

$$
\mathrm{fb}(m)=2^{m-3}(m+3)- \begin{cases}\frac{m-1}{2}\binom{\frac{m-1}{2}}{\frac{m-1}{2}} & \text { for odd } m,  \tag{1}\\ \frac{2 m-3}{2}\binom{m-2}{\frac{m-2}{2}} & \text { for even } m .\end{cases}
$$

Weakening a conjecture of Fishburn Fis97, Conjecture 2], Galambos and Reiner conjecture that $\mathrm{fb}(m)$ is a tight upper bound on the cardinality of acyclic sets described in terms of higher Bruhat orders GR08, Conjecture 1]. We notice that Knuth had a conjecture related to the one of Galambos and Reiner discussing his Equation (9.8) Knu92, p. 39]. Felsner and Valtr as well as Danilov,

[^0]Karzanov and Koshevoy mention counterexamples to these conjectures FV11, DKK12. Galambos and Reiner base the formula for $\mathrm{fb}(m)$ and their conjecture on counting extensions of a certain pseudoline arrangement by adding a new pseudoline and relate these extensions to elementarily equivalent maximal chains in the weak order on $\Sigma_{m}$.

Planar pseudoline arrangements with contact points as well as pseudo- and multitriangulations are systematically studied by Pilaud and Pocchiola using the framework of networks PP12]. Subsequently, Pilaud and Santos construct polytopes from a given network and relate their combinatorics to the combinatorics of triangulations of point configurations PS12. For well-chosen networks, they construct associahedra (or Stasheff polytopes) which essentially coincide with a family of realizations obtained from the permutahedron by Hohlweg and Lange HL07. This family provides a geometric interpretation of Reading's Cambrian lattices Rea06. Cambrian lattices are remarkable as they generalize the Tamari lattice as lattice quotient of the weak order on $\Sigma_{m}$ in two ways. First, distinct lattice quotients are obtained by choosing different Coxeter elements $c$ and yield distinct realizations of the associahedron from the permutahedron. Second, the construction of distinct lattice quotients extends from the symmetric group $\Sigma_{m}$ to the weak order of any finite Coxeter group $W$. Hohlweg, Lange and Thomas then identify $c$-singletons as fundamental objects of Cambrian lattices and use them to derive distinct polytopal realizations of generalized associahedra from $W$-permutahedra HLT11. Generalized associahedra are CW-complexes defined in the context of cluster algebras of finite type [FZ03] that coincide with associahedra in type A. Finally, Pilaud and Stump extend the construction of polytopes from Pilaud and Santos to any finite Coxeter group and, analogous to type $A$, essentially reobtain realizations of generalized associahedra discovered by Hohlweg, Lange and Thomas PS15.

Two interpretations of $c$-singletons described in HLT11 are fundamental for our work. First, the geometric construction of generalized associahedra from $W$-permutahedra exhibits $c$-singletons as the common vertices of both polytopes and, second, $c$-singletons are combinatorially described as prefixes of a certain reduced expression for the longest element $w_{\circ} \in W$ up to commutations. For Coxeter groups of type $A$, the latter interpretation translates to higher Bruhat orders: the set of $c$-singletons for a fixed Coxeter element $c \in \Sigma_{m}$ is precisely the set of all elements $w \in \Sigma_{m}$ visited by the maximal chains contained in a certain equivalence class of elementarily equivalent maximal chains determined by $c$. Galambos and Reiner showed in type $A$ that these elements coincide with certain maximal acyclic domains and for this reason we define a Cambrian acyclic domain as the set of $c$-singletons for a given Coxeter element $c$ of a finite Coxeter group $W$. The main results of this article are

- Theorem 3.31that provides a combinatorial description for the cardinality of a Cambrian acyclic domain for any finite Coxeter system $(W, S)$ and any Coxeter element $c$.
- Theorem 5.7 that characterizes the possible choices of $c$ to minimize and maximize the cardinality of an Cambrian acyclic domain for any finite Coxeter system ( $W, S$ ).
These results solve Problem 3.1 of [MHPS12, Chapter 8]. Even though we mentioned above that the conjecture of Galambos and Reiner is not true in general, Theorem 5.7 proves that the conjecture holds if it is restricted to the large subclass of acyclic domains: Fishburn's alternating scheme yields the maximum cardinality for Cambrian acyclic domains of type $A$.

The article is organized as follows. In Section 2, we discuss the results in type $A$. Sections 2.12.3 provide a unified description of Cambrian acyclic domains as geometric entities in terms of vertices of convex polytopes, as pseudoline arrangements, and as certain order ideals for type $A$. Moreover, we derive formulae for the cardinality of Cambrian acyclic domains and give a new proof of Equation (11) using hypergeometric series in Section 2.4. Section 3 generalizes the discussion from type $A$ to other finite types. We introduce and discuss relevant notions in Sections 3.13 .5 before proving Theorem 3.31 in Section 3.6. More precisely, a poset called natural partial order by Galambos and Reiner GR08 as well as heap by Viennot Vie86 and Stembridge Ste96 is introduced in Section 3.1. In Section 3.2 we introduce $c$-singletons of a finite Coxeter system $(W, S)$ and show that the weak order on $c$-singletons is isomorphic to the lattice of order ideals of a well-chosen natural partial order. In Section 3.3, Hasse diagrams of natural partial orders are
embedded in a cylindrical oriented graph that we call 2-cover. The 2-cover replaces the network used in type $A$ as framework to count $c$-singletons in arbitrary type. The extension of a pseudoline arrangements considered by Galambos and Reiner in type $A$ is replaced by cut paths introduced in Section 3.3. It turns out that the total number of cut paths in the 2-cover exceeds the size $\mathrm{S}_{c}$ of Cambrian domains and the difference can be expressed in terms of "crossing" cut paths discussed in Section 3.4 In Section 4, we illustrate Theorem 3.31, we explicitly compute the cardinality of Cambrian acyclic domains for various finite types and different choices of Coxeter elements. In Section 5, we finally derive lower and upper bounds for the cardinality of Cambrian acyclic domains. The examples discussed in Section 4 cover all possibilities to minimize and maximize the size $S_{c}$ of Cambrian domains.

We assume familiarity with basic notions of convex polytopes and of Coxeter group theory and refer to Zie95] as well as Hum92 for details.

## 2. Associahedra, pseudoline arrangements and $c$-singletons in type $A$

2.1. Associahedra and $c$-singletons. An associahedron is a simple convex polytope of a particular combinatorial type. The underlying combinatorial structure relates to various branches of mathematics as mentioned in Tam51, MHPS12] or [Sta63]. We follow Lee and consider triangulations of a convex $(n+3)$-gon to define the combinatorics of an $n$-dimensional associahedron Lee89. A plethora of distinct polytopal realizations is known for the associahedron, e.g. Hai84, BFS90, GKZ94, HL07, Dev09, PS12, we focus on a family of realizations described by Hohlweg and Lange [HL07, LP18] that generalizes Lod04] and relates directly to triangulations of a labeled $(n+3)$-gon $P$ and to the symmetric group $\Sigma_{n+1}$. The resulting $n$-dimensional associahedra and the labelings of $P$ are parametrized by Coxeter elements $c \in \Sigma_{n+1}$. We refer to the labeled polygons as $P_{c}$ and to the various polytopal realizations of associahedra as Asso ${ }_{c}$.

Assume that $P$ is a convex $(n+3)$-gon in the plane with no two vertices on a vertical line. To obtain the labeled polygon $P_{c}$, we label the vertices of $P$ from smallest to greatest $x$-coordinate using the integers 0 to $n+2$. Without loss of generality, we assume that the vertices labeled 0 and $n+2$ lie on a horizontal line. This induces a partition of $[n+1]:=\{1,2, \ldots, n+1\}$ into a down set $D_{c}=\left\{d_{1}<d_{2}<\cdots<d_{k}\right\}$ and an up set $U_{c}=\left\{u_{1}<u_{2}<\cdots<u_{\ell}\right\}$ where the vertices in $U_{c}$ lie in the upper hull of $P_{c}$ and the vertices in $D_{c}$ in the lower hull. We have special notation in the following two situations. If $U_{c}=\varnothing$ then we replace the subscript $c$ by Lod to remind of Loday who gave a combinatorial interpretation of the vertex coordinates of Asso ${ }_{\text {Lod }}$ Lod04]. If $U_{c}=\{d \in \mathbb{N} \mid 0<d<n+2$ and $d$ even $\}$ then we replace the subscript $c$ by alt. This reminds of alternating (or bipartite) Coxeter elements and relates to Fishburn's alternating scheme.

The labeled $(n+3)$-gons $P_{c}$ are characterized by certain permutations $\pi_{c}$ with one peak which describe a relabeling to obtain $P_{c}$ from $P_{\text {Lod }}$, see Figure 1 for two examples.


Figure 1. Two examples of labeled heptagons $P_{c}$.
Moreover, the set of labeled $(n+3)$-gons $P_{c}$ is in bijection with orientations of Coxeter graphs of type $A$ and all Coxeter elements $c$ of $\Sigma_{n+1}$ Shi97. This bijection is crucial to extend the construction of associahedra to generalized associahedra for arbitrary finite Coxeter groups [HLT11].

Any proper diagonal $\delta$ of $P_{c}$ yields a facet-defining inequality $H_{\geq}^{\delta}$ for Asso ${ }_{c}$ as follows. Let $B_{\delta}$ be the label set of vertices of $P_{c}$ which lie strictly below the line supporting $\delta$ and include the endpoints of $\delta$ which are in $U_{c}$ and set $H_{\geq}^{\delta}:=\left\{x \in \mathbb{R}^{n+1} \left\lvert\, \sum_{i \in B_{\delta}} x_{i} \geq\binom{ n+1}{\left|B_{\delta}\right|}\right.\right\}$.
Theorem 2.1 ([HL07, Proposition 1.3],[LP18, Corollary 7]).
For every labeled $(n+3)$-gon $P_{c}$, the polytope

$$
\text { Asso }_{c}=\left\{\begin{array}{l|l}
x \in \mathbb{R}^{n+1} & \begin{array}{c}
\sum_{i \in[n+1]} x_{i}=\frac{(n+1)(n+2)}{2} \text { and } \\
x \in H_{\geq}^{\delta} \text { for any proper diagonal } \delta \text { of } P_{c}
\end{array}
\end{array}\right\}
$$

is a particular realization of an n-dimensional associahedron.
Each associahedron Asso $_{c}$ is an instance of a generalized permutahedron, introduced by Postnikov, as it is obtained from the classical permutahedron

$$
\begin{aligned}
\operatorname{Perm}_{n} & =\operatorname{conv}\left\{(\pi(1), \ldots, \pi(n+1))^{\top} \in \mathbb{R}^{n+1} \mid \pi \in \Sigma_{n+1}\right\} \\
& =\left\{\begin{array}{l|c}
x \in \mathbb{R}^{n+1} & \sum_{i \in[n+1]} x_{i}=\frac{(n+1)(n+2)}{2} \text { and } \\
\sum_{i \in I} x_{i} \geq\binom{ n+1}{|I|} \text { for any nonempty } I \subset[n+1]
\end{array}\right\}
\end{aligned}
$$

by discarding some facet-defining inequalities PRW08, Pos09. Following HLT11, Section 2.3], a $c$-singleton is a common vertex of Perm $n$ and of $\mathrm{Asso}_{c}$.


Figure 2. The permutahedron Perm ${ }_{4}$ and the associahedra Asso $_{\text {alt }}$ and Asso $_{\text {Lod }}$. Red vertices of the associahedra indicate $c$-singletons and maximal paths from $(1,2,3,4)^{\top}$ to $(4,3,2,1)^{\top}$ along red zig-zag edges of associahedra correspond to elementarily equivalent maximal chains in the weak order.

From Figure 2 where the 3 -dimensional polytopes Perm $_{4}$, Asso $_{\text {Lod }}$ and Asso ${ }_{\text {alt }}$ are shown, it is immediate that the number of $c$-singletons as well as the number of paths from $(1,2,3,4)^{\top}$ to $(4,3,2,1)^{\top}$ in the 1 -skeleton of $\mathrm{Asso}_{c}$ visiting only $c$-singletons depends on $c$. For later use, we remark that the realization of Asso $_{c}$ is completely determined by $U_{c} \cap\{2,3, \ldots, n+1\}$ and assume without loss of generality

$$
D_{c}=\left\{d_{1}=1<d_{2}<\cdots<d_{k}\right\} \quad \text { and } \quad U_{c}=\left\{u_{1}<u_{2}<\cdots<u_{\ell}\right\} .
$$

2.2. Pseudoline arrangements and $c$-singletons. Using the duality of points and lines in the Euclidean plane, we now describe $\mathrm{Asso}_{c}$ and $c$-singletons in terms of pseudoline arrangements, see PP12] and PS12] for details. We visualize pseudoline arrangements on an alternating and sorting network $\mathcal{N}_{c}$ that encodes the combinatorics of the point configuration of $P_{c}: \mathcal{N}_{c}$ consists of $n+3$ horizontal lines and $\binom{n+3}{2}=\frac{(n+3)(n+2)}{2}$ commutators which are vertical line segments connecting consecutive horizontal lines in an alternating way. A commutator is at level $i$ if it connects the horizontal lines $i$ and $i+1$ of $\mathcal{N}_{c}$ (counted from bottom to top starting with 0 ). Additionally, we label the ends of horizontal lines at the left end of $\mathcal{N}_{c}$ from 0 to $n+2$ bottom to top and from 0
to $n+2$ top to bottom at the right end. Figure 3 illustrates these notions for $\mathcal{N}_{\text {Lod }}$ and $\mathcal{N}_{\text {alt }}$ which correspond to $P_{\text {Lod }}$ and $P_{\text {alt }}$ of Figure 1 .


Figure 3. Two networks for pseudoline arrangements with labeled commutators: $\mathcal{N}_{\text {Lod }}$ (left) and $\mathcal{N}_{\text {alt }}$ (right) correspond to the labeled heptagons of Figure 1 , The 1-kernels $\mathcal{K}_{\text {Lod }}$ and $\mathcal{K}_{\text {alt }}$ are obtained by deletion of the dotted line segments.

Now, pseudoline $i$ (supported by $\mathcal{N}_{c}$ ) is an abscissa monotone path on $\mathcal{N}_{c}$ starting on the left at label $i$ and ending right at label $i$ and a pseudoline arrangement on $\mathcal{N}_{c}$ is a collection of pseudolines such that any pair intersects precisely along one commutator called crossing. A commutator that is touched by two pseudolines and traversed by none is called contact. There is a unique pseudoline arrangement with $n+3$ pseudolines on $\mathcal{N}_{c}$ which induces a labeling of all commutators by the two unique pseudolines that traverse along it, see Figure 3 The reader may prove the following facts:
i) labeled commutators of $\mathcal{N}_{c}$ at levels 0 and $n+1$ are in bijection to boundary diagonals of $P_{c}$.
ii) $\mathcal{N}_{c}$ is determined by $\mathcal{N}_{\text {Lod }}$ and $\pi_{c}$. The inversions of $\pi_{c}$ label commutators in the bottom right of $\mathcal{N}_{\text {Lod }}$ which can be moved to the upper left part if we temporarily consider $\mathcal{N}_{\text {Lod }}$ as Möbius strip by identifying its sides. Relabeling the commutators by $\pi_{c}$ yields $\mathcal{N}_{c}$.
The 1-kernel $\mathcal{K}_{c}$ of the network $\mathcal{N}_{c}$ is the network obtained from $\mathcal{N}_{c}$ by deletion of the horizontal lines 0 and $n+2$ as well as all commutators touching these lines. On $\mathcal{K}_{c}$, we use notions induced by $\mathcal{N}_{c}$, for example, the level of a commutator or its label are inherited from $\mathcal{N}_{c}$. Triangulations of $P_{c}$ are now in bijection to pseudoline arrangements with $n+1$ pseudolines supported by $\mathcal{K}_{c}$ : diagonals of a triangulation correspond to the contacts of a unique pseudoline arrangement on $\mathcal{K}_{c}$ (PP12, Theorem 23]). The simple fact that a commutator labeled by the endpoints of a proper diagonal $\delta$ of $P_{c}$ is at level $\left|B_{\delta}\right|$ extends [HL07, Proposition 1.4] and [LP18, Proposition 20] by statement iii) below:

Proposition 2.2. Let $v$ be a vertex of Asso $_{c}$ with corresponding triangulation $T_{v}$ of $P_{c}$ and let $C_{v}$ be the set of commutators of $\mathcal{N}_{c}$ labeled by proper diagonals of $T_{v}$. The following statements are equivalent:
i) $v$ is a vertex of Perm $_{n}$.
ii) The proper diagonals $\delta_{i}$ of $T_{v}$ can be ordered such that $\varnothing \subset B_{\delta_{1}} \subset \ldots \subset B_{\delta_{n}} \subset[n+1]$.
iii) $C_{v}$ contains one commutator from each level of $\mathcal{K}_{c}$ and commutators from consecutive levels are adjacent.

Proposition 2.2 shows that a $c$-singleton for Asso $_{c}$ corresponds to a path which traverses the 1 -kernel $\mathcal{K}_{c}$ from bottom to top and ascents whenever possible. We call such a path a greedy ordinate monotone path on $\mathcal{K}_{c}$. In the theory of pseudoline arrangements, a greedy ordinate monotone path is known as a pseudoline from the south pole to the north pole which extends the original arrangement by a new pseudoline.
2.3. Order ideals and $c$-singletons. It is possible to give another description of $c$-singletons that uses neighbouring transpositions $s_{i}=\left(\begin{array}{ll}i & i+1\end{array}\right), 1 \leq i \leq n$, if we combine Proposition 2.2 with [HLT11, Theorem 2.2]: a permutation $\pi \in \Sigma_{n+1}$ is a $c$-singleton of Asso ${ }_{c}$ if and only if there is a reduced word for $\pi$ in $s_{1}, \ldots, s_{n}$ that is a prefix up to commutation of a particular reduced
expression $\boldsymbol{w}_{\circ}^{\mathbf{c}}$ of the reverse permutation $w_{\circ}=[n+1, n, \ldots, 1]$ (given here in one-line notation). Although this point of view will be used in Section 3 to define $c$-singletons for arbitrary irreducible finite Coxeter systems ( $W, S$ ), we directly describe prefixes up to commutations of $\boldsymbol{w}_{\circ}^{\text {c }}$ in type $A$ using order ideals of a post $\left(\mathcal{S}, \prec_{c}\right)$ associated to $\mathcal{K}_{c}$.

Let $\mathcal{S}$ be the set that contains one copy of the transposition $s_{i}$ for each bounded region of $\mathcal{K}_{c}$ at level $i$ and distinguish copies of the same transposition by their associated region. Define the partial order $\prec_{c}$ on $\mathcal{S}$ as the transitive closure of the covering relation $s_{i} \rightarrow s_{j}$ that satisfies
i) $|i-j|=1$;
ii) the bounded regions associated to $s_{i}$ and $s_{j}$ intersect in a (nonempty) horizontal line segment;
iii) the commutator bounding the region of $s_{i}$ to the left is left of the region associated to $s_{j}$.

Two Hesse diagrams for $\left(\mathcal{S}, \prec_{c}\right)$ associated to $\mathcal{K}_{a l t}$ are illustrated in Figure 4 Each greedy ordinate monotone path $\mathfrak{p}$ in $\mathcal{K}_{c}$ is a cut of the $\operatorname{Hasse} \operatorname{diagram}\left(\mathcal{S}, \prec_{c}\right)$ that partitions the vertex set $\mathcal{S}$ of this oriented graph in two sets and the set $\mathcal{S}_{\mathfrak{p}} \subseteq \mathcal{S}$ below $\mathfrak{p}$ is an order ideal of $\left(\mathcal{S}, \prec_{c}\right)$. Therefore $\mathcal{S}_{\mathfrak{p}}$ corresponds to a $c$-singleton.

$\left(\mathcal{S}, \prec_{\text {alt }}\right)$ associated to $\mathcal{K}_{\text {alt }}$ for $n=4$

$\left(\mathcal{S}, \prec_{\text {alt }}\right)$ associated to $\mathcal{K}_{\text {alt }}$ for $n=5$

Figure 4. The Case diagram of $\left(\mathcal{S}, \prec_{a l t}\right)$. An ordinate monotone path $\mathfrak{p}$ (dashed red line) determines an order ideal $\mathcal{S}_{\mathfrak{p}}$ of $\left(\mathcal{S}, \prec_{\text {alt }}\right)$ (shaded region).

We briefly indicate the relation to the work of Galambos and Reiner as well as Mann and Schechtmann. For a given Coxeter element $c$, each $c$-singleton $w$ determines a unique greedy ordinate monotone path $\mathfrak{p}_{w}$ in the 1 -kernel $\mathcal{K}_{c}$ as well as a unique pseudoline arrangement $\mathcal{A}_{w}$ supported by $\mathcal{K}_{c}$ (where contacts are the commutators traversed by $\mathfrak{p}_{w}$ ). The arrangements $\mathcal{A}_{w}$ differ only by their embedding in $\mathcal{K}_{c}$. They describe a unique pseudoline arrangement $\mathcal{A}_{c}$ that depends only on $c$. Galambos and Reiner consider the "natural partial order" $\mathcal{P}_{\mathcal{A}}$ on the crossings of $\mathcal{A}_{c}$ which coincides with $\left(\mathcal{S}, \prec_{c}\right)$ described above. Moreover, they show that the order ideals of $\mathcal{P}_{\mathcal{A}}$ encode an equivalence class of elementarily equivalent maximal chains in the weak order on $\Sigma_{n+1}$ as defined by Manin and Schechtmann MS89. Hence the set of $c$-singletons for a given Coxeter element $c$ corresponds to an equivalence class of elementarily equivalent admissible permutations of $\binom{[n+1]}{2}$ also studied by Ziegler [Zie93].
2.4. Counting $c$-singletons. We first derive a general formula for the number $S_{c}$ of $c$-singletons by enumeration of greedy ordinate monotone paths of $\mathcal{K}_{c}$ with $n+1$ horizontal lines. To this respect, we define the trapeze network $\mathcal{T}_{c}$ associated to $\mathcal{K}_{c}$ as the maximal sorting network with $n+1$ horizontal lines with the following properties:
i) $\mathcal{T}_{c}$ contains $\mathcal{K}_{c}$,
ii) $\mathcal{T}_{c}$ is alternating,
iii) every commutator of $\mathcal{T}_{c}$ is included in a greedy ordinate monotone path in $\mathcal{T}_{c}$ that contains a commutator of $\mathcal{K}_{c}$ at level $n$.
A commutator in $\mathcal{T}_{c}$ which is not in $\mathcal{K}_{c}$ is called a trapeze commutator. If $\mathcal{K}_{c}=\mathcal{K}_{\text {Lod }}$ then $\mathcal{T}_{\text {tod }}=\mathcal{K}_{\text {Lod }}$, so $\mathcal{T}_{\text {tod }}$ contains no trapeze commutators. When $\mathcal{K}_{c}=\mathcal{K}_{\text {alt }}$, the trapeze network $\mathcal{T}_{\text {alt }}$ contains 8 trapeze commutators for $n=4$ and contains 10 trapeze commutators for $n=5$, see Figure 5

$\mathcal{T}_{\text {alt }}$ associated to $\mathcal{K}_{\text {alt }}$ for $n=4$

$\mathcal{T}_{\text {alt }}$ associated to $\mathcal{K}_{\text {alt }}$ for $n=5$

Figure 5. The trapeze network $\mathcal{T}_{\text {alt }}$ with trapeze commutators drawn dashed or zig-zag. The trapeze commutators in $\Theta_{\text {alt }}$ are drawn zig-zag.

Clearly, any greedy ordinate monotone path in $\mathcal{T}_{c}$ is either a greedy ordinate monotone path that traverses only commutators of $\mathcal{K}_{c}$ or a greedy ordinate monotone path in $\mathcal{T}_{c}$ that traverses at least one trapeze commutator. As there are $\left|U_{c}\right|+2$ commutators of $\mathcal{K}_{c}$ at level $n$ and since there are $2^{n-1}$ distinct ordinate monotone paths in $\mathcal{T}_{c}$ that end at each commutator at level $n$, we conclude that the number of greedy ordinate monotone paths in $\mathcal{T}_{c}$ equals $\left(\left|U_{c}\right|+2\right) 2^{n-1}$. It remains to count the ordinate monotone paths that traverse at least one trapeze commutator. Clearly, these paths are naturally partitioned by the last trapeze commutator traversed. Let $\Theta_{c}$ denote the set of trapeze commutators that appear as last trapeze commutator of some greedy ordinate monotone path in $\mathcal{T}_{c}$ and let $\gamma_{t}$ denote the number of ordinate monotone paths in $\mathcal{T}_{c}$ that stay in $\mathcal{K}_{c}$ after traversing $t \in \Theta_{c}$ at level $\ell_{t}$. Then there are $\gamma_{t} 2^{\ell_{t}-1}$ greedy ordinate monotone paths in $\mathcal{T}_{c}$ with $t \in \Theta_{c}$ as last trapeze commutator. We conclude

$$
\mathrm{S}_{c}=\left(\left|U_{c}\right|+2\right) 2^{n-1}-\sum_{t \in \Theta_{c}} \gamma_{t} 2^{\ell_{t}-1}
$$

A trivial consequence is $S_{L o d}=2^{n}$ as $\Theta_{L o d}=U_{L o d}=\varnothing$ and $D_{L o d}=[n+1]$.
For a less trivial example, we prove $\mathrm{S}_{\text {alt }}=\mathrm{fb}(n+1)$. Assume first that $n=2 k$. Then

$$
\left|U_{a l t}\right|=\frac{n}{2} \quad \text { and } \quad\left(\left|U_{a l t}\right|+2\right) 2^{n-1}=(n+4) 2^{n-2}
$$

For $0 \leq r \leq k-1$ there are exactly two distinct commutators in $\Theta_{\text {alt }}$ at odd level $\ell_{t}=2 r+1$ and no commutator at even level $\ell_{t}=2 r+2$, see Figure For $t \in \Theta_{\text {alt }}$ to the left of $\mathcal{K}_{\text {alt }}$ with $\ell_{t}=2 r+1$, any greedy ordinate monotone path $\mathfrak{p}$ with last trapeze commutator $t$ contains a greedy ordinate monotone path $\tilde{\mathfrak{p}}$ from $t$ to a commutator of $\mathcal{K}_{\text {alt }}$ at level $n$ that uses only commutators in $\mathcal{K}_{\text {alt }}$. The path $\tilde{\mathfrak{p}}$ traverses $n-\ell_{t}$ commutators, where the first one is determined since $\tilde{\mathfrak{p}}$ must take an "east" step after $t$. Moreover, at any position $\tilde{\mathfrak{p}}$ must have taken strictly more "east" steps than "west" steps. The number of such paths $\tilde{\mathfrak{p}}$ is $\binom{2(k-r-1)}{k-r-1}$, see Nil12, Corollary 6] for details.

A similar argument applies if $t \in \Theta_{\text {alt }}$ with $\ell_{t}=2 r+1$ is located the right of $\mathcal{K}_{\text {alt }}$, so $\gamma_{t}=\binom{2(k-r-1)}{k-r-1}=\binom{n-\ell_{t}-1}{\left(n-\ell_{t}-1\right) / 2}$ for all $t \in \Theta_{\text {alt }}$ relates to the sequence of central binomial coefficients A000984 of NJAS17. Setting $p:=k-r-1$ and using Gosper's algorithm to obtain a closed form for the hypergeometric sum [PWZ96, Chapter 5], we conclude

$$
\sum_{t \in \Theta_{a l t}} \gamma_{t} 2^{\ell_{t}-1}=2 \sum_{r=0}^{k-1}\binom{2(k-r-1)}{k-r-1} 2^{2 r}=2^{n-1} \sum_{p=0}^{k-1}\binom{2 p}{p} 2^{-2 p}=k\binom{2 k}{k}=\frac{n}{2}\binom{n}{\frac{n}{2}} .
$$

If we now assume $n=2 k-1$ then

$$
\left|U_{a l t}\right|=\frac{n-1}{2} \quad \text { and } \quad\left(\left|U_{a l t}\right|+2\right) 2^{n-1}=(n+3) 2^{n-2}
$$

For $0 \leq r \leq k-2$ there is precisely one commutator in $\Theta_{a l t}$ at even level $\ell_{t}=2 r+2$. Since $n-\ell_{t}$ is odd, we get $\gamma_{t}=\binom{n-\ell_{t}-1}{\left(n-\ell_{t}-1\right) / 2}=\binom{2(k-r-2)}{k-r-2}$. Further there is precisely one commutator in $\Theta_{\text {alt }}$ at odd level $\ell_{t}=2 r+1$, a similar argument yields $\gamma_{t}=\binom{2(k-r-2)+1}{(k-r-2)+1}$. Thus $\gamma_{t}=\binom{n-\ell_{t}-1}{\left\lfloor\frac{n-\ell_{t}-1}{2}\right\rfloor}$ relates
to sequence A001405 of NJAS17. Setting $p:=k-r-2$ and again using Gosper's algorithm we conclude

$$
\sum_{t \in \Theta_{\text {alt }}} \gamma_{t} 2^{\ell_{t}-1}=2^{n-3} \sum_{p=0}^{k-2} \frac{(4 p+3)}{p+1}\binom{2 p}{p} 2^{-2 p}=-2^{n-2}+\frac{2 n-1}{2}\binom{n-1}{\frac{n-1}{2}}
$$

We now set $n=m-1$ and this proves the claim $\mathrm{S}_{\text {alt }}=\mathrm{fb}(n+1)$. This provides a new proof of Formula (1).

## 3. Enumeration of $c$-singletons - General Case

In order to provide formulae to enumerate singletons in the general case of arbitrary finite Coxeter systems, we first generalize the poset $\left(\mathcal{S}, \prec_{c}\right)$ to general type, and present a planar embedding of its Hasse diagram in Section 3.1. In Section 3.2, we describe two equivalence relations on words and class representatives indexed by Coxeter elements. In Section 3.3, we present a graph called the 2-cover that we embed on a cylinder. In Section 3.4, we count cut paths in this embedding and provide a correspondence to Coxeter elements. In Section 3.5, we define when two cut paths are crossing. Finally, in Section [3.6, starting with a Coxeter element $c$, we obtain a formula for the cardinality $S_{c}$ of a Cambrian acyclic domain by counting certain cut path that do not cross the cut path corresponding to $c$.

Consider an irreducible finite Coxeter system $(W, S)$ of rank $n$ with generators $s_{1}, \ldots, s_{n}$ and length function $\ell$. A Coxeter element $c \in W$ is the product of $n$ distinct generators of $S$ in some order and $\operatorname{Cox}(W, S)$ is the set of all Coxeter elements of $(W, S)$. The Coxeter number $h$ is the smallest positive integer such that $c^{h}$ is the identity of $W$ and is independent of the choice of $c$. As proposed by Shi Shi97, we identify Coxeter elements $c \in W$ and orientations $\Gamma_{c}$ of the Coxeter graph $\Gamma$ associated to $W$ : an edge $\{s, t\}$ of $\Gamma$ is directed from $s$ to $t$ if and only if $s, t \in S$ do not commute and $s$ comes before $t$ in (any reduced expression of) $c$. A word $\mathbf{w}$ in $S$ is a concatenation $\sigma_{1} \ldots \sigma_{k}$ for some nonnegative integer $k$ and $\sigma_{i} \in S$, a subword of $\mathbf{w}=\sigma_{1} \ldots \sigma_{k}$ is a word $\sigma_{i_{1}} \ldots \sigma_{i_{r}}$ with $1 \leq i_{1}<\ldots<i_{r} \leq k$ and the support $\operatorname{supp}(\mathbf{w})$ of $\mathbf{w}$ is the set of generators that appear in $\mathbf{w}$. Of particular interest is the unique element $w_{\circ} \in W$ of maximum length $\ell\left(w_{\circ}\right)=N:=\frac{n h}{2}$ which is called longest element. A reduced expression $\mathbf{w}_{\circ}=\sigma_{1} \ldots \sigma_{N}$ is called longest word.
3.1. Natural partial order. A Coxeter triple ( $W, S, \mathbf{w}$ ) is an irreducible finite Coxeter system $(W, S)$ together with a word $\mathbf{w}$ in $S$ such that $\operatorname{supp}(\mathbf{w})=S$. Any Coxeter triple $(W, S, \mathbf{w})$ induces a unique reduced expression $\mathbf{c}_{\mathbf{w}}$ of a Coxeter element $c_{\mathbf{w}}$ where the elements of $S$ appear according to their first appearance in $\mathbf{w}$. In particular, $\mathbf{w}$ induces a canonical orientation on $\Gamma$. We first define the natural partial order $\prec_{\mathbf{w}}$ on the disjoint union $\mathcal{L}_{\mathbf{w}}:=\left\{\sigma_{1}, \ldots, \sigma_{k}\right\}$ of letters of the word $\mathbf{w}=\sigma_{1} \ldots \sigma_{k}$. The map $\mathrm{g}: \mathcal{L}_{\mathbf{w}} \rightarrow S$ assigns to each letter $\sigma_{i}$ at position $i$ the corresponding generator. The natural partial order appeared in Vie86, Section 2], Ste96, Section 2.2], GR08, Definition 6].

Definition 3.1 (Natural partial order).
The natural partial order $\prec_{\mathbf{w}}$ on $\mathcal{L}_{\mathbf{w}}$ is defined for any Coxeter triple ( $W, S, \mathbf{w}$ ) as follows: $\sigma_{r} \prec_{\mathbf{w}} \sigma_{s}$ if and only if there is a subword $\sigma_{i_{1}} \ldots \sigma_{i_{k}}$ of $\mathbf{w}$ such that $\sigma_{i_{1}}=\sigma_{r}, \sigma_{i_{k}}=\sigma_{s}$ and the elements $\mathrm{g}\left(\sigma_{i_{j}}\right)$ and $\mathrm{g}\left(\sigma_{i_{j+1}}\right)$ do not commute for all $1 \leq j \leq k-1$. The Hasse diagram of $\left(\mathcal{L}_{\mathbf{w}}, \prec_{\mathbf{w}}\right)$ is an oriented graph denoted by $\mathcal{G}_{\mathbf{w}}$.

Example 3.2. Let $\left(\Sigma_{5}, S, \mathbf{w}\right)$ be the Coxeter triple with generators $s_{i}=(i \quad i+1)$ and

$$
\mathbf{w}:=s_{3} s_{2} s_{1} s_{2} s_{3} s_{4} s_{2} s_{3} s_{2} s_{1} s_{2} s_{3} s_{4} s_{3} s_{2} s_{1} s_{3} s_{2} s_{3} s_{4}
$$

The induced reduced word for the Coxeter element $c_{\mathbf{w}}$ is $\mathbf{c}_{\mathbf{w}}=s_{3} s_{2} s_{1} s_{4}$ and we have $\left|\mathcal{L}_{\mathbf{w}}\right|=20$. Moreover, $\left(\mathcal{L}_{\mathbf{w}}, \prec_{\mathbf{w}}\right)$ consists of the following 21 covering relations:

$$
\begin{array}{rrrrrrr}
\sigma_{1} \prec_{\mathbf{w}} \sigma_{2}, & \sigma_{2} \prec_{\mathbf{w}} \sigma_{3}, & \sigma_{3} \prec_{\mathbf{w}} \sigma_{4}, & \sigma_{4} \prec_{\mathbf{w}} \sigma_{5}, & \sigma_{5} \prec_{\mathbf{w}} \sigma_{6}, & \sigma_{5} \prec_{\mathbf{w}} \sigma_{7}, & \sigma_{6} \prec_{\mathbf{w}} \sigma_{8}, \\
\sigma_{7} \prec_{\mathbf{w}} \sigma_{8}, & \sigma_{8} \prec_{\mathbf{w}} \sigma_{9}, & \sigma_{9} \prec_{\mathbf{w}} \sigma_{10}, & \sigma_{10} \prec_{\mathbf{w}} \sigma_{11}, & \sigma_{11} \prec_{\mathbf{w}} \sigma_{12}, & \sigma_{12} \prec_{\mathbf{w}} \sigma_{13}, & \sigma_{13} \prec_{\mathbf{w}} \sigma_{14}, \\
\sigma_{14} \prec_{\mathbf{w}} \sigma_{15}, & \sigma_{15} \prec_{\mathbf{w}} \sigma_{16}, & \sigma_{15} \prec_{\mathbf{w}} \sigma_{17}, & \sigma_{16} \prec_{\mathbf{w}} \sigma_{18}, & \sigma_{17} \prec_{\mathbf{w}} \sigma_{18}, & \sigma_{18} \prec_{\mathbf{w}} \sigma_{19}, & \sigma_{19} \prec_{\mathbf{w}} \sigma_{20} .
\end{array}
$$

Example 3.3. Let $\left(W, S, \mathbf{w}_{1}\right)$ and $\left(W, S, \mathbf{w}_{2}\right)$ be Coxeter triples such that $\mathbf{w}_{1} \neq \mathbf{w}_{2}$, but $\mathbf{w}_{1}$ is obtained from $\mathbf{w}_{2}$ by a sequence of braid relations of length 2 (and no deletions). Then $\mathbf{c}_{\mathbf{w}_{1}} \neq \mathbf{c}_{\mathbf{w}_{2}}$ are reduced expressions for the same Coxeter element $c \in W$ and the posets ( $\mathcal{L}_{\mathbf{w}_{1}}, \prec_{\mathbf{w}_{1}}$ ) and ( $\mathcal{L}_{\mathbf{w}_{2}}, \prec_{\mathbf{w}_{2}}$ ) are isomorphic.

The next result gives a crossing-free straight-line planar embedding of the Hasse diagram $\mathcal{G}_{\mathbf{w}}$ for $(W, S, \mathbf{w})$.

Proposition 3.4. Let $k$ be a positive integer and $(W, S, \mathbf{c})$ be a Coxeter triple where $\mathbf{c}$ is a reduced expression for $c \in \operatorname{Cox}(W, S)$. The graph $\mathcal{G}_{\mathbf{c}^{k}}$ is connected, planar, and has a crossing-free straight-line planar embedding using integer vertex coordinates such that the $x$-coordinate is strictly increasing in direction of every oriented edge.

Proof. If $k=1$ then $\mathcal{G}_{\mathbf{c}}$ is isomorphic (as oriented graph) to the Coxeter graph $\Gamma$ oriented according to $\mathbf{c}$. By the classification of finite Coxeter groups, $\Gamma$ is a tree which is connected and planar. Now label the vertices of $\Gamma$ such that $s_{1}, \ldots, s_{p}$ are successive vertices of $\Gamma$ along a path of maximum length. We have $p=n-1$ if $(W, S)$ is of type $D_{n}, E_{6}, E_{7}$ or $E_{8}$ and $p=n$ otherwise. If $p=n-1$ we label the path such that the remaining vertex $s_{n}$ is connected to $s_{r}$ where $r=n-2$ (type $D_{n}$ ) or $r=n-3$ (otherwise). To obtain the claimed drawing, locate $s_{1}$ at $(0,0)$ and determine coordinates $\left(x_{j}, y_{j}\right)$ for $s_{j}$ with $j \leq p$ inductively from $\left(x_{j-1}, y_{j-1}\right)$ for $s_{j-1}$ via $x_{j}:=x_{j-1} \pm 1$ and $y_{j}:=y_{j-1}+1$ where the sign depends on the orientation of $\left\{s_{j-1}, s_{j}\right\}$. If $p=n-1$ then coordinates for the remaining point $s_{n}$ are $x_{n}:=x_{r} \pm 1$ and $y_{n}:=y_{r}$.

We now inductively construct a planar drawing of $\mathcal{G}_{\mathbf{c}^{k+1}}$ from a drawing of $\mathcal{G}_{\mathbf{c}^{k}}$. Observe that
i) $\mathcal{L}_{\mathbf{c}^{k}}$ and $\mathcal{L}_{\mathbf{c}^{k+1}}$ differ by a copy of $S$. Denote the vertices of $\mathcal{L}_{\mathbf{c}^{k+1}} \backslash \mathcal{L}_{\mathbf{c}^{k}}$ by $s_{i}^{k+1}$ for $1 \leq i \leq n$.
ii) The new covering relations of $\left(\mathcal{L}_{\mathbf{c}^{k+1}}, \prec_{\mathbf{c}^{k+1}}\right)$ are of two types:

- $s_{j}^{k+1} \prec_{\mathrm{c}^{k+1}} s_{j^{\prime}}^{k+1}$ if and only if $s_{j^{\prime}} \rightarrow s_{j}$ in $\Gamma$;
- $s_{j}^{k+1} \prec_{\mathrm{c}^{k+1}} s_{j^{\prime}}^{k}$ if and only if $s_{j} \rightarrow s_{j}$ in $\Gamma$.

Now set $x_{j}^{k+1}:=x_{j}^{k}+2$ and $y_{j}^{k+1}=y_{j}^{k}$ to obtain valid coordinates for $s_{j}^{k+1}$ and include oriented edges according to the covering relation of $\prec_{c^{k+1}}$.

Definition 3.5 (Tiles and their boundary). Let ( $W, S, \mathbf{w}$ ) be a Coxeter triple. An (open) tile of $\mathcal{G}_{\mathbf{w}}$ is a bounded connected component of $\mathbb{R}^{2} \backslash \mathcal{G}_{\mathbf{w}}$. The boundary of the closure of $T$ is denoted by $\partial T$.

To simplify notation, the closure of a tile $T$ is also denoted by $T$.
Corollary 3.6. Let $T$ be a tile of $\mathcal{G}_{\mathbf{c}^{k}}$ for a Coxeter triple $\left(W, S, \mathbf{c}^{k}\right)$. The boundary $\partial T$ defines an induced subgraph of $\mathcal{G}_{\mathbf{c}^{k}}$ with four vertices; one vertex is a source of out-degree 2 and one vertex is a sink of in-degree 2. In particular, the source and sink of this subgraph are letters of $\mathbf{c}^{k}$ of consecutive copies of $\mathbf{c}$ that represent the same generator of $S$.
3.2. Equivalence classes and c-sorting words. As in [Ste96], we now define the equivalence relations $\approx$ and $\sim$ on words in $S$ as well as representatives for the equivalence classes $[\mathbf{w}] \approx$ and $[\mathbf{w}]_{\sim}$ that are determined by a reduced expression $\mathbf{c}$ for $c \in \operatorname{Cox}(W, S)$.

First, we write $\mathbf{u} \approx \mathbf{v}$ if and only if $\mathbf{u}, \mathbf{v}$ are reduced words that represent the same element $w \in W$. The equivalence class $[\mathbf{u}]_{\approx}$ depends only on $w$, so we often write $[w] \approx$ instead of $[\mathbf{w}]_{\approx}$. Following Reading Rea07, we define the $\mathbf{c}$-sorting word $\boldsymbol{w}^{\mathbf{c}}$ of $w$ as the lexicographically first subword of the infinite word $\mathbf{c}^{\infty}=\mathbf{c c c} \ldots$ (as a sequence of positions) which belongs to $[w]_{\approx}$.

Second, $\mathbf{u} \sim \mathbf{v}$ if and only if $\mathbf{u}, \mathbf{v}$ are words that coincide up to commutations, that is, one is obtained from the other by a sequence of braid relations of length 2 (and no deletions). The $\mathbf{c}$-sorting word $\mathbf{w}^{\mathbf{c}}$ of $\mathbf{w}$ is defined as the element of $[\mathbf{w}]_{\sim}$ that appears first lexicographically as a subword of the infinite word $\mathbf{c}^{\infty}=\mathbf{c c c} \ldots$ (as a sequence of positions).

Due to the similar definition, $\boldsymbol{w}^{\mathbf{c}}$ and $\mathbf{w}^{\mathbf{c}}$ are both called $\mathbf{c}$-sorting word. We emphasize that $\boldsymbol{w}^{\mathbf{c}}$ represents $[w] \approx$ while $\mathbf{w}^{\mathbf{c}}$ represents $[\mathbf{w}]_{\sim}$ and, by definition, $\ell(\mathbf{u})=\ell(\mathbf{v})$ if $\mathbf{u} \sim \mathbf{v}$ but $\mathbf{u}$ and $\mathbf{v}$ are not necessarily reduced. Although the definition of $\boldsymbol{w}^{\mathbf{c}}$ depends on a reduced expression $\mathbf{c}$ for $c$, we have $\boldsymbol{w}^{\mathbf{c}_{1}} \sim \boldsymbol{w}^{\mathbf{c}_{2}}$ if $\mathbf{c}_{1} \sim \mathbf{c}_{2}$.

Example 3.7. The words $\mathbf{u}=s_{1} s_{2} s_{1}$ and $\mathbf{v}=s_{2} s_{1} s_{2}$ are reduced words for the longest element $w_{\circ} \in \Sigma_{3}$ of the Coxeter system $\left(\Sigma_{3}, S\right)$ with generators $s_{i}=(i \quad i+1)$ for $i \in\{1,2\}$. Thus $\mathbf{u} \approx \mathbf{v}$. As both words do not coincide up to commutations, we have $\mathbf{u} \nsim \mathbf{v}$. More precisely, we have

$$
\boldsymbol{w}_{\circ}^{\mathbf{c}}=s_{1} s_{2}\left|s_{1}, \quad \mathbf{u}^{\mathbf{c}}=s_{1} s_{2}\right| s_{1} \quad \text { and } \quad \mathbf{v}^{\mathbf{c}}=s_{2} \mid s_{1} s_{2}
$$

if $\mathbf{c}=s_{1} s_{2}$ and

$$
\boldsymbol{w}_{\circ}^{\mathbf{c}}=s_{2} s_{1}\left|s_{2}, \quad \mathbf{u}^{\mathbf{c}}=s_{1}\right| s_{2} s_{1}, \quad \text { and } \quad \mathbf{v}^{\mathbf{c}}=s_{2} s_{1} \mid s_{2},
$$

if $\mathbf{c}=s_{2} s_{1}$. We write $\mid$ to distinguish between copies of $\mathbf{c}$ in $\mathbf{c}^{\infty}$.
Example 3.8 (Example 3.2 continued).
The $\mathbf{c}_{\mathbf{w}}$-sorting word of $\mathbf{w}$ is $\mathbf{w}^{\mathbf{c}_{\mathbf{w}}}=s_{3} s_{2} s_{1}\left|s_{2}\right| s_{3} s_{2} s_{4}\left|s_{3} s_{2} s_{1}\right| s_{2}\left|s_{3} s_{4}\right| s_{3} s_{2} s_{1}\left|s_{3} s_{2}\right| s_{3} s_{4}$.
Lemma 3.9. Let $(W, S, \mathbf{w})$ be a Coxeter triple. The oriented graph $\mathcal{G}_{\mathbf{w}}$ is an induced oriented subgraph of $\mathcal{G}_{\mathbf{c}_{\mathbf{w}}^{m}}$ for some positive integer $m$.
Proof. Let $\widetilde{\mathbf{w}}$ be the subword of $\mathbf{c}_{\mathbf{w}} \mathbf{c}_{\mathbf{w}} \mathbf{c}_{\mathbf{w}} \ldots$ that is lexicographically first (as a sequence of positions) among all subword of $\mathbf{c}_{\mathbf{w}} \mathbf{c}_{\mathbf{w}} \mathbf{c}_{\mathbf{w}} \ldots$ that coincide with $\mathbf{w}$ up to commutations and let $m$ be the minimum integer such that $\widetilde{\mathbf{w}}$ is a subword of $\mathbf{c}_{\mathbf{w}}^{m}=\left(\mathbf{c}_{\mathbf{w}}\right)^{m}$. Then $\left(\mathcal{L}_{\widetilde{\mathbf{w}}}, \prec_{\widetilde{\mathbf{w}}}\right)$ and $\left(\mathcal{L}_{\mathbf{w}}, \prec_{\mathbf{w}}\right)$ are isomorphic, so their Hasse diagrams coincide and $\mathcal{G}_{\mathbf{w}}$ is an induced subgraph of $\mathcal{G}_{\mathbf{c}_{\mathbf{w}}^{m}}$.

We often write $\mathcal{G}_{\mathbf{w}}$ for the graph $\mathcal{G}_{\mathbf{w}}$ embedded according to Lemma 3.9,
Example 3.10 (Example 3.2 continued).
As $\mathbf{w}$ is a subword of $\mathbf{c}_{\mathbf{w}}^{9}$, we obtain a planar drawing of $\mathcal{G}_{\mathbf{w}}$ induced from the planar drawing of $\mathcal{G}_{\mathbf{C}_{\mathrm{w}}^{9}}$ as shown in Figure 6 .


Figure 6. The crossing-free straight-line embedding of $\mathcal{G}_{\mathbf{c}_{\mathrm{w}}^{9}}$ described in Proposition 3.4 together with $\mathcal{G}_{\mathbf{w}}$ as subgraph according to Lemma 3.9,

## Remark 3.11.

a) If $\mathbf{u} \sim \mathbf{v}$ then $\left(\mathcal{L}_{\mathbf{u}}, \prec_{\mathbf{u}}\right)$ and $\left(\mathcal{L}_{\mathbf{v}}, \prec_{\mathbf{v}}\right)$ are isomorphic posets and $\mathcal{G}_{\mathbf{u}}$ and $\mathcal{G}_{\mathbf{v}}$ are isomorphic directed graphs.
b) Let $(W, S, \mathbf{c})$ be a Coxeter triple of type $A$ and $\mathbf{w} \in\left[\boldsymbol{w}_{\mathbf{o}}^{\mathbf{c}}\right]_{\sim}$. Then $\left(\mathcal{L}_{\mathbf{w}}, \prec_{\mathbf{w}}\right)$ is isomorphic to $\left(\mathcal{S}, \prec_{c}\right)$ described in Section 2.3
c) The graph $\mathcal{G}_{\boldsymbol{w}_{\circ}^{\mathrm{c}}}$ is isomorphic to the Auslander-Reiten quiver associated to $c \in \operatorname{Cox}(W, S)$ and $\mathcal{G}_{\mathbf{c}^{k}}$ is a finite truncation of the repetition quiver described by Keller [Kel10, Section 2.2] for all positive integers $k$.

A word $\sigma_{1} \ldots \sigma_{r}$ is a prefix up to commutations of a word $\mathbf{w}$ if and only if there is a word $\mathbf{w}^{\prime} \sim \mathbf{w}$ such that the first $r$ letters of $\mathbf{w}^{\prime}$ are $\sigma_{1} \ldots \sigma_{r}$. The following characterization of $c$-singletons serves as definition and does not depend on $\mathbf{c}$ but on $c \in \operatorname{Cox}(W, S)$.

Definition 3.12 (c-singletons HLT11, Theorem 2.2]).
Let $(W, S, \mathbf{c})$ be a Coxeter triple. An element $w \in W$ is a c-singleton if and only if some reduced expression of $w$ is a prefix of $\boldsymbol{w}_{\circ}^{\mathbf{c}}$ up to commutations. The number of $\mathbf{c}$-singletons is denoted by $\mathrm{S}_{\mathbf{c}}$.

Definition 3.13 (Cambrian acyclic domains).
Let $(W, S, \mathbf{c})$ be a Coxeter triple. The set Acyc $_{\mathbf{c}}$ of $c$-singletons is called Cambrian acyclic domain and its cardinality is $S_{c}$.

The set of c-singletons, endowed with the weak order inherited from $(W, S)$, forms a distributive lattice $L_{\mathbf{c}}$, HLT11, Proposition 2.5]. Any distributive lattice $L$ is isomorphic to the lattice of order ideals of a poset $(P, \leq)$ which is unique up to isomorphism, Sta12, Theorem 3.4.1]. Before we show that $\left(\mathcal{L}_{\boldsymbol{w}_{\mathrm{o}}}, \prec_{\boldsymbol{w}_{\mathrm{o}}}\right)$ is such a poset $(P, \leq)$ for $L_{\mathbf{c}}$, we recall that an order ideal (or down-set or semi-ideal) of $(P, \leq)$ is a subset $I \subseteq P$ such that $t \in I$ and $s \leq t$ implies $s \in I$ and that antichains of a finite poset $P$ are in bijection with order ideals of $P$ Sta12, Section 3.1]. A generator $s \in S$ is called initial (resp. final) in $w \in W$ if and only if $\ell(s w)<\ell(w)$ (resp. $\ell(w s)<\ell(w)$ ).

Proposition 3.14. Let $(W, S, \mathbf{c})$ be Coxeter triple. The lattice of order ideals of $\left(\mathcal{L}_{\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}}, \prec_{\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}}\right)$ is isomorphic to the poset of c-singletons ordered by the weak order.
Proof. If $w \in W$ is a $\mathbf{c}$-singleton then $w$ is represented by a prefix $\sigma_{i_{1}} \ldots \sigma_{i_{k}}$ of $\boldsymbol{w}_{\mathrm{o}}^{\mathbf{c}}=\sigma_{1} \ldots \sigma_{N}$ up to commutations and the set $F_{w} \subseteq\left\{\sigma_{i_{1}}, \ldots, \sigma_{i_{k}}\right\}$ of final letters for $w$ is an antichain of $\left(\mathcal{L}_{\boldsymbol{w}_{\mathrm{o}}}, \prec_{\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}}\right)$. Conversely, if $\left\{a_{1}, \ldots, a_{k}\right\}$ is an antichain of $\left(\mathcal{L}_{\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}}, \prec_{\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}}\right)$ then let $I_{j}$ be the order ideal of $\left(\mathcal{L}_{\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}}, \prec_{\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}}\right)$ generated by $a_{j}$ for $1 \leq j \leq k$ and consider the order ideal $I:=\bigcup_{j=1}^{k} I_{j}$ of $\left(\mathcal{L}_{\boldsymbol{w}_{\circ}^{\mathrm{c}}}, \prec_{\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}}\right)$ together with some linear extension of $\prec_{\boldsymbol{w}}$ c on $I$. The product of elements of $I$ with respect to this linear order yields a prefix up to commutations of $\boldsymbol{w}_{\circ}^{\mathbf{c}}=\sigma_{1} \ldots \sigma_{N}$ with final letters $\left\{a_{1}, \ldots, a_{k}\right\}$.
3.3. 2-covers. The longest element $w_{\circ}$ of $(W, S)$ defines an automorphism $\psi: W \rightarrow W$ defined via $w \mapsto w_{\circ}^{-1} w w_{\circ}$ that preserves length and adapts to words $\mathbf{w}=\sigma_{1} \ldots \sigma_{r}$ via $\psi(\mathbf{w}):=\psi\left(\sigma_{1}\right) \ldots \psi\left(\sigma_{r}\right)$. Let $\operatorname{rev}(\mathbf{w}):=\sigma_{r} \ldots \sigma_{1}$ denote the reverse word of $\mathbf{w}$ and let $\boldsymbol{w}_{\circ}^{\mathbf{c}} \psi\left(\boldsymbol{w}_{\circ}^{\mathbf{c}}\right)$ denote the concatenation of $\boldsymbol{w}_{\circ}^{\mathbf{c}}$ and $\psi\left(\boldsymbol{w}_{\mathbf{\circ}}^{\mathbf{c}}\right)$. By Remark 7.6 of [CLS14],

$$
\boldsymbol{w}_{\mathrm{o}}^{\psi(\mathbf{c})} \sim \operatorname{rev}\left(\boldsymbol{w}_{\circ}^{\mathrm{rev}(\mathbf{c})}\right) \quad \text { and } \quad \mathbf{c}^{h} \sim \boldsymbol{w}_{\mathrm{o}}^{\mathbf{c}} \boldsymbol{w}_{\mathrm{o}}^{\psi(\mathbf{c})}=\boldsymbol{w}_{\mathrm{o}}^{\mathbf{c}} \psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathbf{c}}\right) .
$$

In combination with Remark 3.11, we obtain the next lemma.
Lemma 3.15. Let $(W, S, \mathbf{c})$ be a Coxeter triple. The graphs $\mathcal{G}_{\mathbf{c}^{h}}$ and $\mathcal{G}_{\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}} \psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}\right)}$ are isomorphic as oriented graphs. In particular, $\mathcal{G}_{\boldsymbol{w}_{\circ}^{\mathrm{c}} \psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}\right)}$ depends on $c \in \operatorname{Cox}(W, S)$ but not on the reduced expression $\mathbf{c}$.
Example 3.16 (Example 3.2 continued).
Consider the longest word $\widetilde{\mathbf{w}_{\circ}}:=s_{3} s_{2} s_{1} s_{2} s_{3} s_{4} s_{2} s_{3} s_{2} s_{1}$ of $\left(\Sigma_{5}, S\right)$. Then $\mathbf{w}$ is the concatenation $\widetilde{\mathbf{w}_{\circ}} \psi\left(\widetilde{\mathbf{w}_{\circ}}\right)$ and a direct computation yields

$$
\widetilde{\mathbf{w}}_{\mathbf{o}}^{\mathbf{c}_{\mathbf{w}}}=s_{3} s_{2} s_{1}\left|s_{2}\right| s_{3} s_{2} s_{4} \mid s_{3} s_{2} s_{1} \quad \text { and } \quad \psi\left(\widetilde{\mathbf{w}}^{\mathbf{c}_{\mathbf{w}}}\right)=s_{2} s_{3} s_{4}\left|s_{3}\right| s_{2} s_{3} s_{1} \mid s_{2} s_{3} s_{4}
$$

Hence, $\mathcal{G}_{\mathbf{c}^{h}}$ and $\mathcal{G}_{\widetilde{\mathbf{w}}^{\mathbf{c}}{ }^{\mathbf{w}} \psi\left(\widetilde{\mathbf{w}}^{\mathbf{c} w}\right)}$ are not isomorphic as oriented graphs. On the other hand,

$$
\boldsymbol{w}_{\mathrm{o}}^{\mathbf{c}_{\mathbf{w}}}=s_{3} s_{2} s_{1} s_{4}\left|s_{3} s_{2} s_{1} s_{4}\right| s_{3} s_{4} \quad \text { and } \quad \psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathbf{c}_{\mathbf{w}}}\right)=s_{2} s_{3} s_{4} s_{1}\left|s_{2} s_{3} s_{4} s_{1}\right| s_{2} s_{1}
$$

show that $\mathbf{c}^{5} \sim \boldsymbol{w}_{\mathrm{o}}^{\mathbf{c}_{\mathrm{w}}} \psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathbf{c}_{\mathrm{w}}}\right)$. Thus $\mathcal{G}_{\mathbf{c}^{5}}$ and $\mathcal{G}_{\boldsymbol{w}_{\mathrm{o}}}^{\mathrm{c}_{\boldsymbol{w}}} \psi\left(\boldsymbol{w}_{\mathrm{o}}^{\boldsymbol{c}_{\mathrm{w}}}\right)$ are isomorphic as oriented graphs.

## Remark 3.17.

a) The constructions of $\left(\mathcal{L}_{\mathbf{w}}, \prec_{\mathbf{w}}\right)$ and of $\mathcal{G}_{\mathbf{c}^{k}}$ extend to the infinite word $\mathbf{c}^{\infty}$. Thus, there is a canonical 'left-most' embedding of $\mathcal{G}_{\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}} \psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}\right)}$ inside $\mathcal{G}_{\mathbf{c}^{\infty}}$ by Lemma 3.15 (using $\mathcal{G}_{\mathbf{c}^{h}}$ ). Moreover, there is a natural projection $p r_{\mathbf{c}^{h}}: \mathcal{L}_{\mathbf{c}^{\infty}} \rightarrow \mathcal{L}_{\mathbf{c}^{h}}$ that maps a letter of copy $\mathbf{c}^{h k+i}$ in $\mathbf{c}^{\infty}$ to the corresponding letter of copy $\mathbf{c}^{i}$ in $\mathbf{c}^{h}$ for $0 \leq i \leq h-1$ and non-negative integers $k$.
b) There are obvious 'bi-infinite' versions $\mathcal{L}_{\mathbf{c}^{\mathbb{Z}}}$ of $\overline{\mathcal{L}}_{\mathbf{c}^{\infty}}$ and $\mathcal{G}_{\mathbf{c}^{\mathbb{Z}}}$ of $\mathcal{G}_{\mathbf{c}^{\infty}}$ with infinitely many copies of $\mathbf{c}$ in the two possible directions and a projection $p r_{\mathbf{c}^{h}}: \mathcal{L}_{\mathbf{c}^{\mathbb{Z}}} \rightarrow \mathcal{L}_{\mathbf{c}^{h}}$ that we use in Section 5

Definition 3.18 (2-cover).
Let $(W, S, \mathbf{c})$ be a Coxeter triple and $\boldsymbol{w}_{\circ}^{\mathbf{c}}$ the $\mathbf{c}$-sorting word of $w_{\circ}$.
a) The 2 -cover $\mathcal{C}_{\mathbf{c}}^{2}$ is a graph with vertices $\mathcal{L}_{\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}} \psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}\right)}$ and directed edges induced by $\mathrm{pr}_{\mathbf{c}^{h}}$ from $\mathcal{G}_{\mathbf{c}^{\infty}}$.
b) Let $V_{\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}}$ be the vertices of $\mathcal{C}_{\mathbf{c}}^{2}$ that correspond to the first $\frac{n h}{2}$ letters of $\boldsymbol{w}_{\circ}^{\mathbf{c}} \psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathbf{c}}\right)$ and $V_{\psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}\right)}$ be the vertices of $\mathcal{C}_{\mathbf{c}}^{2}$ that correspond to the last $\frac{n h}{2}$ letters of $\boldsymbol{w}_{\mathrm{o}}^{\mathbf{c}} \psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathbf{c}}\right)$. Then $\mathcal{C}_{\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}}$ is the subgraph of $\mathcal{C}_{\mathbf{c}}^{2}$ induced by $V_{\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}}$ and $\mathcal{C}_{\psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}\right)}$ is the subgraph of $\mathcal{C}_{\mathbf{c}}^{2}$ induced by $V_{\psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}\right)}$.
The planar drawing for $\mathcal{G}_{\mathbf{c}^{h}}$ described in Proposition 3.4 and the simple observation that $\mathcal{G}_{\mathbf{c}^{h}}$ is isomorphic to some induced subgraph of $\mathcal{G}_{\tilde{\mathbf{c}} \infty}$ for all $\mathbf{c}, \tilde{\mathbf{c}} \in \operatorname{Cox}(W, S)$ imply the following lemma.

Lemma 3.19. Let $(W, S, \mathbf{c})$ and $(W, S, \tilde{\mathbf{c}})$ be Coxeter triples.
i) The 2 -cover $\mathcal{C}_{\mathbf{c}}^{2}$ has a crossing-free drawing on the open cylinder $S^{1} \times \mathbb{R}$.
ii) The 2 -covers $\mathcal{C}_{\mathbf{c}}^{2}$ and $\mathcal{C}_{\tilde{\mathbf{c}}}^{2}$ are isomorphic as directed graphs.

We refer to a particular embedding of $\mathcal{C}_{\mathbf{c}}^{2} \subset S^{1} \times \mathbb{R} \subset \mathbb{R}^{2} \times \mathbb{R}$ induced by $p r_{\mathbf{c}^{h}}$ which can be visualized as 'wrapping the plane drawing of $\mathcal{G}_{\mathbf{c}^{k}}$ around a cylinder'. Without loss of generality, we assume that the $y$-direction for the plane drawing of $\mathcal{G}_{\mathbf{c}^{k}}$ is parallel to the $z$-axis of $\mathbb{R}^{3}=\mathbb{R}^{2} \times \mathbb{R}$ which we assume to coincide with the axis of the cylinder $S^{1} \times \mathbb{R} \subset \mathbb{R}^{3}$. For this reason, we identify $y$-coordinates in the plane drawing of $\mathcal{G}_{\mathbf{c}^{k}}$ with third coordinates in $\mathbb{R}^{3}$. This embedding $\mathcal{C}_{\mathbf{c}}^{2} \subset \mathbb{R}^{3}$ has the following obvious properties:
i) copies of $s_{i}$ have strictly smaller third coordinate than copies of $s_{j}$ for $1 \leq i<j \leq p$ (where $p$ is defined in the proof of Proposition (3.4);
ii) for fixed $i \in[n]$, the third coordinate of all copies of $s_{i}$ coincide;
iii) if $p<n$ then the third coordinate of $s_{r}$ and $s_{n}$ coincide.

We say that copies of the generator $s_{1}$ are located at the bottom of $\mathcal{C}_{\mathbf{c}}^{2}$ while the copies of the generator $s_{p}$ are located at the top of $\mathcal{C}_{\mathbf{c}}^{2}$. The definition of a tile as well as the proof of Corollary 3.6 extend to this embedding of $\mathcal{C}_{\mathbf{c}}^{2}$.

### 3.4. Cut paths.

Definition 3.20 (Cut paths, primary and secondary cut paths).
Let $(W, S, \mathbf{c})$ be a Coxeter triple and $\mathcal{C}_{\mathbf{c}}^{2}$ the associated 2-cover.
i) A cut path $\kappa$ of $\mathcal{C}_{\mathbf{c}}^{2}$ is a set of edges of $\mathcal{C}_{\mathbf{c}}^{2}$ such that every directed cycle of $\mathcal{C}_{\mathbf{c}}^{2}$ contains precisely one edge of $\kappa$. The set of all cut paths of $\mathcal{C}_{\mathbf{c}}^{2}$ is denoted by $\operatorname{CP}\left(\mathcal{C}_{\mathbf{c}}^{2}\right)$.
ii) The primary cut path $\kappa_{\mathbf{c}}$ of $\mathcal{C}_{\mathbf{c}}^{2}$ is the cut path that consists of all edges of $\mathcal{C}_{\mathbf{c}}^{2}$ that are not (projections of) edges of $\mathcal{G}_{\boldsymbol{w}_{\mathrm{o}}^{\mathbf{c}} \psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}\right)}$. The secondary cut path $\kappa_{\mathbf{c}}^{*}$ is the cut path that consists of all edges of $\mathcal{C}_{\mathbf{c}}^{2} \backslash \kappa_{c}$ that are neither edges of $\mathcal{C}_{\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}}$ nor edges of $\mathcal{C}_{\psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}\right)}$.
The primary and secondary cut paths are disjoint because $\operatorname{supp}\left(\mathbf{w}_{\circ}^{\mathbf{c}}\right)=\operatorname{supp}\left(\psi\left(\mathbf{w}_{\circ}^{\mathbf{c}}\right)\right)=S$. Primary and secondary cut paths relate to cuts of a graph as $\kappa_{\mathbf{c}} \cup \kappa_{\mathbf{c}}^{*}$ partitions the 2-cover $\mathcal{C}_{\mathbf{c}}^{2}$ into two connected components. Every cut path $\kappa$ induces a sequence ('path') of tiles that cuts the 2cover $\mathcal{C}_{\mathbf{c}}^{2}$ from bottom to top: consider the set of tiles such that consecutive tiles have at least one common edge in $\kappa$.

Example 3.21. The notions of Definition 3.20 are illustrated in Figure 7 for $\mathbf{c}_{\text {Lod }}$ and $\mathbf{c}_{\text {alt }}$ in $\left(\Sigma_{5}, S\right)$. The 2 -covers $\mathcal{C}_{\mathbf{c}}^{2}$ coincide in both situations and are shown in a planar drawing where vertices $\sigma \in \mathcal{L}_{\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}} \psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}\right)}$ of $\mathcal{C}_{\mathbf{c}}^{2}$ are labeled by corresponding generators $\mathrm{g}(\sigma) \in S$ and oriented edges of $\mathcal{C}_{\mathbf{c}}^{2}$ contained in a cut path are indicated by $\rightsquigarrow$. The primary and secondary cut paths $\kappa_{i}$ and $\kappa_{i}^{*}$ for $i \in\{1,2\}$ are edges $\rightsquigarrow$ intersected by a dashed red line.

Since every cut path of $\mathcal{C}_{\mathbf{c}}^{2}$ that avoids edges of $\mathcal{C}_{\psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}\right)}$ (or, equivalently, every cut path of $\mathcal{C}_{\mathbf{c}}^{2}$ that avoids edges of the Hasse diagram of $\left.\left(\mathcal{L}_{\psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}\right)}, \prec_{\psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}\right)}\right)\right)$ defines an antichain of $\left(\mathcal{L}_{\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}}, \prec_{\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}}\right)$, we obtain the following characterization of order ideals of $\left(\mathcal{L}_{\boldsymbol{w}_{\circ}^{\mathrm{c}}}, \prec_{\boldsymbol{w}_{\mathrm{o}}}\right)$.
Lemma 3.22. Let $(W, S, \mathbf{c})$ be a Coxeter triple. The set of c-singletons is in bijection with the set of cut paths of $\mathcal{C}_{\mathbf{c}}^{2}$ that avoid edges of $\mathcal{C}_{\psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}\right)}$.

In particular, if the number of all cut paths and the number of cut paths that contain edges of $\mathcal{C}_{\psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}\right)}$ are known, Lemma 3.22 implies a formula for the cardinality $\mathrm{S}_{c}$. A formula for $\left|\operatorname{CP}\left(\mathcal{C}_{\mathbf{c}}^{2}\right)\right|$ is obtained from the next theorem and a formula for $S_{c}$ will be derived in Section 3.5


Figure 7. The primary and secondary cut paths $\kappa_{i}, \kappa_{i}^{*}$ depicted in a planar drawing of the 2-cover $\mathcal{C}_{\mathbf{c}}^{2}$ for $\mathbf{c}_{1}=s_{1} s_{2} s_{3} s_{4}$ and $\mathbf{c}_{2}=s_{2} s_{4} s_{1} s_{3}$.

Theorem 3.23. Let $(W, S, \mathbf{c})$ be a Coxeter triple, with Coxeter graph $\Gamma$, and $\kappa \in \operatorname{CP}\left(\mathcal{C}_{\mathbf{c}}^{2}\right)$.
i) For each edge $e=\left\{s_{i}, s_{j}\right\}$ of $\Gamma$, there exists a unique directed edge $e_{\kappa}=(\sigma, \tau) \in \kappa$ with $\{\mathrm{g}(\sigma), \mathrm{g}(\tau)\}=\left\{s_{i}, s_{j}\right\}$.
ii) Let $s_{j}$ be a vertex of degree 2 of $\Gamma$ with incident edges $e=\left\{s_{i}, s_{j}\right\}, \widetilde{e}=\left\{s_{j}, s_{k}\right\}$. There is a unique tile $T$ that contains the corresponding directed edges $e_{\kappa}=(\sigma, \tau)$, and $\widetilde{e}_{\kappa}=(\widetilde{\sigma}, \widetilde{\tau})$ from i).
iii) Let $s_{r}$ be a vertex of degree 3 of $\Gamma$ with incident edges

$$
e=\left\{s_{r-1}, s_{r}\right\}, \quad \widetilde{e}=\left\{s_{r}, s_{r+1}\right\} \text { and } \quad \bar{e}=\left\{s_{r}, s_{n}\right\}
$$

and corresponding directed edges of $\kappa$ from $i$ )

$$
e_{\kappa}=(\sigma, \tau), \quad \widetilde{e}_{\kappa}=(\widetilde{\sigma}, \widetilde{\tau}) \text { and } \quad \bar{e}_{\kappa}=(\bar{\sigma}, \bar{\tau})
$$

There are two unique tiles $T_{1}, T_{2}$ such that $e_{\kappa}, \bar{e}_{\kappa} \in \partial T_{1}, \widetilde{e}_{\kappa}, \bar{e}_{\kappa} \in \partial T_{2}$ and $\partial T_{1} \cap \partial T_{2}$ consists of two edges of $\mathcal{C}_{\mathbf{c}}^{2}$.

Proof.
i) For every edge $\left\{s_{i}, s_{j}\right\}$ of the Coxeter graph $\Gamma$ there exists a directed cycle in $\mathcal{C}_{\mathbf{c}}^{2}$ that visits only vertices corresponding to $s_{i}$ and $s_{j}$. This implies for every edge $\left\{s_{i}, s_{j}\right\}$ of $\Gamma$ that a cut path $\kappa$ must contain precisely one oriented edge $(\sigma, \tau)$ of $\mathcal{C}_{\mathbf{c}}^{2}$ such that $\{\mathrm{g}(\sigma), \mathrm{g}(\tau)\}=\left\{s_{i}, s_{j}\right\}$.
ii) By (il), there are unique oriented edges $e_{\kappa}=(\sigma, \tau), \widetilde{e}_{\kappa}=(\widetilde{\sigma}, \widetilde{\tau}) \in \kappa$ with $\{\mathrm{g}(\sigma), \mathrm{g}(\tau)\}=\left\{s_{i}, s_{j}\right\}$ and $\{\operatorname{g}(\widetilde{\sigma}), \mathrm{g}(\widetilde{\tau})\}=\left\{s_{j}, s_{k}\right\}$. Suppose there is no tile $T$ with $e_{\kappa}, \widetilde{e}_{\kappa} \in \partial T$. Without loss of generality, let $T$ be the unique tile such that $e_{\kappa} \in \partial T$ and the two other vertices of $T$ correspond to the generators $s_{j}$ and $s_{k}$. Then consider the directed cycle that only uses edges $(\sigma, \tau)$ of $\mathcal{C}_{\mathbf{c}}^{2}$ with $\{\mathrm{g}(\sigma), \mathrm{g}(\tau)\}=\left\{s_{i}, s_{j}\right\}$ where the two edges of $\partial T$ are replaced by the other two edges of $\partial T$. Clearly, no edge of this cycle is an edge of $\kappa$. This contradicts the assumption that $\kappa$ is a cut path.
iii) The argument to proveiii) can be used to show that there are unique tiles $T_{1}$ and $T_{2}$ such that $e_{\kappa}, \bar{e}_{\kappa} \in \partial T_{1}$ and $\widetilde{e}_{\kappa}, \bar{e}_{\kappa} \in \partial T_{2}$. But $\partial T_{1}$ and $\partial T_{2}$ clearly share a directed edge $(\alpha, \beta)$ of $\mathcal{C}_{\mathbf{c}}^{2}$ with $\{\mathrm{g}(\alpha), \mathrm{g}(\beta)\}=\left\{s_{r}, s_{n}\right\}$ that is distinct from $\bar{e}_{\kappa}$.

Corollary 3.24. Each cut path $\kappa \in \mathrm{CP}\left(\mathcal{C}_{\mathbf{c}}^{2}\right)$ determines a unique set of $n-2$ tiles:

$$
\text { tile }(\kappa)=\left\{\begin{array}{l|l}
T_{1}, \ldots, T_{k} & \begin{array}{c}
\partial T_{i} \text { contains } \\
2 \text { edges of } \kappa
\end{array}
\end{array}\right\} .
$$

We tacitly order the tiles $T_{i} \in \operatorname{tile}(\kappa)$ from bottom to top in $\mathcal{C}_{\mathbf{c}}^{2} \subset S^{1} \times \mathbb{R}$ : if $z_{i}$ denotes the smallest third coordinate of all points in $T_{i}$ then $1 \leq i<j \leq n-2$ implies $z_{i} \leq z_{j}$.

Lemma 3.19 states that $\mathcal{C}_{\mathbf{c}}^{2}$ is isomorphic to $\mathcal{C}_{\tilde{\mathbf{c}}}^{2}$ for all $\mathbf{c}, \tilde{\mathbf{c}}$, so it is impossible to recover $c \in$ $\operatorname{Cox}(W, S)$ from $\mathcal{C}_{\mathbf{c}}^{2}$. As any cut path $\kappa$ provides one oriented edge for every edge of $\Gamma$, we have an induced Coxeter element $c_{\kappa}$, its reduced expressions are uniquely determined up to commutations.

Corollary 3.25. Let $(W, S, \mathbf{c})$ be a Coxeter triple, $\kappa_{c}$ be the associated primary cut path and $c_{\kappa_{\mathbf{c}}}$ be the Coxeter element obtained from $\kappa_{\mathbf{c}}$. For any reduced expression $\mathbf{w}$ of $c_{\kappa}$, we have $\mathbf{w} \sim \operatorname{rev}(\mathbf{c})$.
Proof. The edges of $\mathcal{C}_{\mathbf{c}}^{2}$ that are not projections of edges of $\mathcal{G}_{\boldsymbol{w}_{\mathrm{o}}} \psi\left(\boldsymbol{w}_{\mathrm{o}}^{\mathrm{c}}\right)$ define Coxeter element that correspond to the equivalence class $[\operatorname{rev}(\mathbf{c})]_{\sim}$.
Example 3.26 (Example 3.21 continued).
Figure 7 also illustrates Corollaries 3.24 and 3.25 . First, the set of tiles tile $\left(\kappa_{1}\right)$ and tile $\left(\kappa_{2}\right)$ associated to $\kappa_{1}$ and $\kappa_{2}$ according to Corollary 3.24 are illustrated. This example shows that the set of tiles can coincide even if $\kappa_{1} \neq \kappa_{2}$. Moreover, the Coxeter element $c_{\kappa_{1}}$ is represented by $s_{4} s_{3} s_{2} s_{1}$ and $s_{4} s_{3} s_{2} s_{1} \in\left[\operatorname{rev}\left(\mathbf{c}_{\text {Lod }}\right)\right]_{\sim}$. Similarly, $c_{\kappa_{2}}$ is represented by $s_{1} s_{3} s_{2} s_{4}$ and $s_{1} s_{3} s_{2} s_{4} \in\left[\operatorname{rev}\left(\mathbf{c}_{\text {alt }}\right)\right]_{\sim}$.
Corollary 3.27. The map $\Phi: \operatorname{CP}\left(\mathcal{C}_{\mathbf{c}}^{2}\right) \rightarrow \operatorname{Cox}(W, S)$ sending a cut path $\kappa$ to its corresponding Coxeter element $c_{\kappa}$ is surjective and satisfies $\left|\Phi^{-1}(c)\right|=h$ for each $c \in \operatorname{Cox}(W, S)$. In particular, $\left|\mathrm{CP}\left(\mathcal{C}_{\mathbf{c}}^{2}\right)\right|=2^{n-1} h$.
Proof. We only prove $\left|\Phi^{-1}(c)\right|=h$. There are $h$ choices in $\mathcal{C}_{\mathbf{c}}^{2}$ to pick a vertex $\sigma_{1}$ with $\mathrm{g}\left(\sigma_{1}\right)=s_{1}$. Now $\sigma_{1}$ determines a unique tile $T_{1}$ with $\sigma_{1} \in \partial T_{1}$ and there is a unique directed edge $e_{1}^{\kappa}$ of $\partial T_{1}$ that reflects the order of $s_{1}$ and $s_{2}$ in $c$. Now consider the unique tile $T_{2}$ whose vertices map to $s_{2}, s_{3}$ and $s_{4}$ under $g$ such that the orientation of $T_{1} \cap T_{2}$ reflects the order of $s_{2}$ and $s_{3}$ in $c$ and proceed similarly with the following generators until all generators have been considered. This process determines a unique cut path $\kappa$ with $\Phi(\kappa)=c$ after choosing one of the $h$ possible initial vertices $\sigma$ at the bottom of $\mathcal{C}_{\mathbf{c}}^{2}$.

### 3.5. Crossings of cut paths.

Definition 3.28 (crossing of cut paths, initial and final side, crossing tile).
Let ( $W, S, \mathbf{c}$ ) be a Coxeter triple and $\kappa_{\mathbf{c}} \in \mathrm{CP}\left(\mathcal{C}_{\mathbf{c}}^{2}\right)$ be the associated primary cut path.
i) A cut path $\kappa$ crosses $\kappa_{\mathbf{c}}$ if tile $(\kappa) \cap$ tile $\left(\kappa_{\mathbf{c}}\right) \neq \varnothing$ and there are edges $e_{1}, e_{2} \in \kappa$ with $e_{1} \in \mathcal{C}_{w_{o}^{c}}$ and $e_{2} \in \mathcal{C}_{\psi\left(w_{\circ}^{\mathrm{c}}\right)}$.
ii) Let $\kappa$ be a cut path that crosses $\kappa_{\mathbf{c}}$. The initial side of $\kappa$ is the connected component of $\mathcal{C}_{\mathbf{c}}^{2} \backslash\left(\kappa_{\mathbf{c}} \cup \kappa_{\mathbf{c}}^{*}\right)$ that contains the edge of $\kappa \backslash\left(\kappa_{\mathbf{c}} \cup \kappa_{\mathbf{c}}^{*}\right)$ whose midpoint has minimal third coordinate. The final side of $\kappa$ is the connected component of $\mathcal{C}_{\mathbf{c}}^{2} \backslash\left(\kappa_{\mathbf{c}} \cup \kappa_{\mathbf{c}}^{*}\right)$ that is not the initial side of $\kappa$.
iii) Let $\kappa$ be a cut path that crosses $\kappa_{\mathbf{c}}$. The crossing tile $T^{\kappa, \mathbf{c}}$ of $\kappa$ in $\mathcal{C}_{\mathbf{c}}^{2}$ is the first tile of tile $(\kappa)$ (with respect to the bottom-to-top order) that contains an edge of $\kappa$ in the final side of $\kappa$.
Definition 3.29 (Initial and final segments).
Let $(W, S, \mathbf{c})$ be a Coxeter triple, $\kappa_{\mathbf{c}} \in \mathrm{CP}\left(\mathcal{C}_{\mathbf{c}}^{2}\right)$ be the associated primary cut path, $T^{\mathbf{c}} \in \operatorname{tile}\left(\kappa_{\mathbf{c}}\right)$ and $\kappa \in \mathrm{CP}\left(\mathcal{C}_{\mathrm{c}}^{2}\right)$ with tile $(\kappa)=\left\{T_{1}, \ldots, T_{n-2}\right\}$.
i) Let $i \in[n-1]$. The initial segment of $\kappa$ up to $T_{i}$ is defined as

$$
\left\{e \in \kappa \mid e \in \partial T_{j} \text { for } j \in[i-1]\right\} \cup\left\{\{u, v\} \in \kappa \mid \mathrm{g}(u)=s_{1} \text { or } \mathrm{g}(v)=s_{1}\right\}
$$

and the final segment of $\kappa$ starting at $T_{i}$ is defined as

$$
\left\{e \in \kappa \mid e \in \partial T_{j} \text { with } j>i\right\} \cup\left\{\{u, v\} \in \kappa \mid \mathrm{g}(u)=s_{p} \text { or } \mathrm{g}(v)=s_{p}\right\}
$$

where $s_{1}, \ldots, s_{p}$ are successive vertices of $\Gamma$ along a path of maximum length.
ii) Let $e_{1}=\left(u_{1}, v_{1}\right)$ and $e_{2}=\left(u_{2}, v_{2}\right)$ be the distinct edges of $\partial T^{\mathbf{c}} \backslash \kappa_{\mathbf{c}}$ such that the midpoint of $e_{1}$ has smaller third coordinate than the midpoint of $e_{2}$. The connected component of $\mathcal{C}_{\mathbf{c}}^{2} \backslash\left(\kappa_{\mathbf{c}} \cup \kappa_{\mathbf{c}}^{*}\right)$ that contains $e_{2}$ is denoted by out $\left(T^{\mathbf{c}}\right)$ and the other component is denoted by in $\left(T^{\mathbf{c}}\right)$.
iii) Let $I\left(T^{\mathbf{c}}\right)$ be the number of distinct initial segments of cut paths $\kappa$ up to $T^{\mathbf{c}}$ with edges contained in $\mathcal{C}_{\mathbf{c}}^{2} \backslash \operatorname{out}\left(T^{\mathbf{c}}\right)$.
iv) Let $F\left(T^{\mathbf{c}}\right)$ be the number of distinct final segments of cut paths $\kappa$ starting at $T^{\mathbf{c}}$ that contain $e_{2}$.

In Definitions 3.28 and 3.29, the primary cut path $\kappa_{\mathbf{c}}$ can be replaced by any cut path $\kappa \in \mathrm{CP}\left(\mathcal{C}_{\mathbf{c}}^{2}\right)$. Moreover, concatenation of an initial segment counting towards $I\left(T^{\mathbf{c}}\right)$ that differs from the initial segment of $\kappa_{\mathbf{c}}$ and a final segment counting towards $F\left(T^{\mathbf{c}}\right)$ yields a cut path that crosses $\kappa_{\mathbf{c}}$.

Example 3.30 (Example 3.21 continued).
We illustrate Definitions 3.28 and 3.29 in Figure 8 . The cut path $\widetilde{\kappa}_{1}$ crosses $\kappa_{1}$ with crossing tile $T_{2}^{\kappa_{1}}$ while $\widetilde{\kappa}_{2}$ does not cross $\kappa_{2}$. A straightforward counting of crossing cut paths verifies

$$
I\left(T_{1}^{\kappa_{1}}\right)=2 \quad \text { and } \quad I\left(T_{2}^{\kappa_{2}}\right)=4 \quad \text { as well as } \quad I\left(T_{1}^{\kappa_{2}}\right)=1 \quad \text { and } \quad I\left(T_{2}^{\kappa_{2}}\right)=2
$$

The reasoning of Section 2.4 yields formulae for all positive integers $i$ :

$$
I\left(T_{i}^{c_{L o d}}\right)=I\left(T_{i}^{\mathrm{rev}\left(c_{L o d}\right)}\right)=2^{i} \quad \text { and } \quad I\left(T_{i}^{c_{a l t}}\right)= \begin{cases}\binom{2 j}{j}, & \text { if } i=2 j, \\ \frac{1}{2}\binom{2 j}{j}, & \text { if } i=2 j-1\end{cases}
$$



Figure 8. Two planar drawings of the 2 -cover for $\Sigma_{5}$. The cut path $\widetilde{\kappa}_{1}$ crosses the primary cut path $\kappa_{1}$ (left) and the cut path $\widetilde{\kappa}_{2}$ does not cross the primary cut path $\kappa_{2}$ (right).

### 3.6. Enumerating $c$-singletons.

Theorem 3.31. Let $(W, S, \mathbf{c})$ be Coxeter triple with associated primary cut path $\kappa_{\mathbf{c}}$ and set of tiles tile $\left(\kappa_{\mathbf{c}}\right)=\left\{T_{1}^{c}, \ldots, T_{n-2}^{c}\right\}$. The cardinality of the Cambrian acyclic domain $\mathrm{Acyc}_{\mathbf{c}}$ is

$$
\mathrm{S}_{c}=2^{n-2}(h+1)-\sum_{i \in[n-2]} 2^{n-2-i} I\left(T_{i}^{c}\right) .
$$

Proof. We count the cut paths of $\mathrm{CP}\left(\mathcal{C}_{\mathbf{c}}^{2}\right)$ twice. By Corollary 3.27, the cardinality of $\mathrm{CP}\left(\mathcal{C}_{\mathbf{c}}^{2}\right)$ equals $2^{n-1} h$. On the other hand, $\kappa \in \mathrm{CP}\left(\mathcal{C}_{\mathbf{c}}^{2}\right)$ satisfies precisely one of the following statements:
i) $\kappa$ crosses $\kappa_{c}$ or $\kappa_{c}^{*}$, but not both;
ii) $\kappa \subseteq \mathcal{C}_{w_{\mathrm{o}}^{\mathrm{c}}} \cup \kappa_{c} \cup \kappa_{c}^{*}$ or $\kappa \subseteq \mathcal{C}_{\psi\left(w_{\mathrm{o}}^{\mathrm{c}}\right)} \cup \kappa_{c} \cup \kappa_{c}^{*}$ but $\kappa \notin\left\{\kappa_{c}, \kappa_{c}^{*}\right\}$;
iii) $\kappa \in\left\{\kappa_{c}, \kappa_{c}^{*}\right\}$.

We first claim that the number $Q_{c}$ of cut paths that cross $\kappa_{c}$ equals the number of cut paths that cross $\kappa_{c}^{*}$. Indeed, the automorphism $\psi$ maps $c$ to $\psi(c)$ and induces an involution $\varphi$ between tile $\left(\kappa_{c}\right)$ and tile $\left(\kappa_{c}^{*}\right)$ that extends to an involution between cut paths that cross $\kappa_{c}$ and cut paths that cross $\kappa_{c}^{*}$ (where $\varphi\left(T^{\kappa, c}\right)$ is the crossing tile of $\varphi(\kappa)$ ). Thus, the number of cut paths satisfying i) equals $2 \mathrm{Q}_{c}$. Moreover, this involution extends to an involution on $\mathrm{CP}\left(\mathcal{C}_{\mathbf{c}}^{2}\right)$ that maps cut paths contained in $\mathcal{C}_{w_{\mathrm{c}}^{\mathrm{c}}} \cup \kappa_{c} \cup \kappa_{c}^{*}$ to cut paths contained in $\mathcal{C}_{\psi\left(w_{\mathrm{o}}\right)} \cup \kappa_{c} \cup \kappa_{c}^{*}$. As $\kappa_{c}$ and $\kappa_{c}^{*}$ are the only cut paths that correspond simultaneously to a $c$-singleton and a $\psi(c)$-singleton, the number of cut paths satisfying ii) equals $2 \mathrm{~S}_{c}-4$ by Lemma 3.22. Thus we established

$$
\begin{equation*}
2 \mathrm{Q}_{c}+\left(2 \mathrm{~S}_{c}-4\right)+2=2^{n-1} h \quad \text { or equivalently } \quad \mathrm{S}_{c}=2^{n-1} h-\mathrm{Q}_{c}+1 \tag{2}
\end{equation*}
$$

To analyze $\mathrm{Q}_{c}$, we partition the cut paths that cross $\kappa_{c}$ according to their crossing tile $T_{i}^{c}$ and observe that $I\left(T_{i}^{c}\right)$ exceeds the number of cut paths with crossing tile $T_{i}^{c}$ by one (the initial segment of $\kappa_{c}$ is counted by $I\left(T_{i}^{c}\right)$ but it is not the initial segment of a cut path with crossing tile $T_{i}^{c}$. The number of final segments $F\left(T_{i}\right)$ of cut paths starting at $T_{i}^{c}$ satisfies $F\left(T_{i}\right)=2^{n-2-i}$
because each final segment consists of $n-2-i$ tiles (not counting $T_{i}^{c}$ ) and there are 2 valid choices to exit each tile. This gives

$$
\mathrm{Q}_{c}=\sum_{i \in[n-2]}\left(I\left(T_{i}^{c}\right)-1\right) F\left(T_{i}^{c}\right)=\sum_{i \in[n-2]} 2^{n-2-i} I\left(T_{i}^{c}\right)-\left(2^{n-2}-1\right)
$$

Substitution in Equation (2) yields the claim.
At this point, we could characterize the Coxeter elements of any irreducible finite Coxeter system ( $W, S$ ) that minimize the cardinality $\mathrm{S}_{\mathrm{c}}$ of the Cambrian acyclic domain $\mathrm{Acyc}_{\mathrm{c}}$. Instead, we first provide explicit formulae for $S_{\mathbf{c}}$ of two families of Cambrian acyclic domains in Section 4 and characterize the Coxeter elements of $(W, S)$ that minimize and maximize $\mathrm{S}_{\mathbf{c}}$ in Section 5 where we use the following result obtained in the previous proof.
Corollary 3.32. Let $(W, S)$ be an irreducible finite Coxeter system of rank $n$ and $c \in \operatorname{Cox}(W, S)$. Then the number of cut paths $\kappa$ that cross $\kappa_{c}$ is

$$
\mathrm{Q}_{c}=\sum_{i \in[n-2]} 2^{n-2-i} I\left(T_{i}^{c}\right)-\left(2^{n-2}-1\right)
$$

Remark 3.33. For reducible finite Coxeter groups, the cardinality of a Cambrian acyclic domain is the product of the cardinalities of the acyclic domains for each irreducible component with respect to the corresponding parabolic Coxeter elements.

## 4. Examples

In this section, we determine explicit formulae for the cardinality $S_{c}$ of a Cambrian acyclic domain $\mathrm{Acyc}_{\mathbf{c}}$ when $c$ is a Coxeter element that minimizes or maximizes the total number of sources and sinks of $\Gamma_{c}$. We denote the relevant subsets of $\operatorname{Cox}(W, S)$ by Cox min and Cox max , standard examples in type $A$ are $c_{L o d} \in \mathrm{Cox}_{\min }$ and $c_{a l t} \in \mathrm{Cox}_{\text {max }}$. We call a Coxeter system path-like if $\Gamma$ is a path.
4.1. Maximum total number of sources and sinks. If $c$ provides a bipartition of $\Gamma$ then every node of $\Gamma_{c}$ is a source or a sink and there are $n$ sources and sinks in total. Theorem 2.3 of [BHLT09] implies that $\mathrm{S}_{c}$ does not depend on $c$ as the associated associahedra Asso $_{c}$ are isometric. In particular, $\mathrm{S}_{c}$ depends only on the type and rank of $(W, S)$.

Proposition 4.1. Let $(W, S)$ be an irreducible Coxeter system of rank $n>1$ and $c \in$ Cox $_{\text {max }}$.
i) If $(W, S)$ is path-like then

$$
\mathrm{S}_{c}= \begin{cases}2^{n-2}(h+3)-n \cdot\binom{n-1}{\frac{n}{2}}, & n \text { even } \\ 2^{n-2}(h+3)-\frac{2 n-1}{2} \cdot\binom{n-1}{\frac{n-1}{2}}, & n \text { odd }\end{cases}
$$

ii) If $(W, S)$ is of type $D_{n}$ then

$$
\mathrm{S}_{c}= \begin{cases}2^{n-2}(h+3)-n \cdot\binom{n-1}{\frac{n}{2}}+\frac{1}{2} \cdot\binom{n-2}{\frac{n-2}{2}}, & n \text { even }, \\ 2^{n-2}(h+3)-(n-1)\binom{n-1}{\frac{n-1}{2}}-\binom{n-3}{\frac{n-3}{2}}, & n \text { odd } .\end{cases}
$$

iii) If $(W, S)$ is of type $E_{6}, E_{7}$ or $E_{8}$ then

$$
\mathrm{S}_{c}= \begin{cases}2^{n-2}(h+3)-2(n-2)\binom{n-2}{\frac{n-2}{2}}-2 \cdot\binom{n-4}{\frac{n-4}{2}}-(n-3)(n-4), & n \text { even } \\ 2^{n-2}(h+3)-(n-1)\binom{n-1}{\frac{n-1}{2}}+\binom{n-3}{\frac{n-3}{2}}-(n-3)(n-4), & n \text { odd }\end{cases}
$$

Proof. We aim for explicit formulae for $I\left(T_{i}^{c}\right), 1 \leq i \leq n-2$, apply Theorem 3.31 and simplify the result using the closed form of a hypergeometric sum used in Section 2.4.

Suppose that $(W, S)$ is path-like and recall from Example 3.30 that

$$
I\left(T_{i}^{c}\right)=\binom{2 j}{j} \quad \text { if } i=2 j, \quad \text { and } \quad I\left(T_{i}^{c}\right)=\frac{1}{2}\binom{2 j}{j} \quad \text { if } i=2 j-1
$$

We prove the claim for $n=2 k-1$, the proof is along the same lines if $n=2 k$. Theorem 3.31 and $2^{2 n+1} \sum_{i=0}^{n} 2^{-2 i}\left(\begin{array}{c}\binom{i}{i}\end{array}\right)=(n+1)\binom{2 n+2}{n+1}$ imply

$$
\begin{aligned}
\mathrm{S}_{c} & =2^{n-2}(h+1)-\sum_{i \in[n-2]} 2^{n-2-i} I\left(T_{i}^{c}\right) \\
& =2^{n-2}(h+1)-2^{n-1}\left(\sum_{j \in[k-2]}\left(2^{-2 j} I\left(T_{2 j-1}^{c}\right)+2^{-(2 j+1)} I\left(T_{2 j}^{c}\right)\right)+2^{-(n-1)} I\left(T_{n-2}^{c}\right)\right) \\
& =2^{n-2}(h+1)-2^{n-1}\left(\sum_{j \in[k-2]} 2^{-2 j}\binom{2 j}{j}+2^{-(2 k-1)}\binom{2(k-1)}{k-1}+1-1\right) \\
& =2^{n-2}(h+3)-2^{n-1}\left(2^{-(2 k-3)}(k-1)\binom{2(k-1)}{k-1}+2^{-(2 k-1)}\binom{2(k-1)}{k-1}\right) \\
& =2^{n-2}(h+3)-\frac{2 n-1}{2}\binom{n-1}{\frac{n-1}{2}} .
\end{aligned}
$$

Suppose that $(W, S)$ is of type $D_{n}$. Then $I\left(T_{n-2}^{c}\right)=I\left(T_{n-3}^{c}\right)$ as well as

$$
I\left(T_{i}^{c}\right)=\binom{2 j}{j} \quad \text { if } i=2 j, \quad \text { and } \quad I\left(T_{i}^{c}\right)=\frac{1}{2}\binom{2 j}{j} \quad \text { if } i=2 j-1,
$$

for $1 \leq i \leq n-3$. Substitution of $I\left(T_{i}^{c}\right)$ in the formula for $\mathrm{S}_{c}$ of Theorem 3.31 and a computation similar to the previous case yields the claim.

Finally, we assume that $(W, S)$ is of type $E_{n}$ for $n \in\{6,7,8\}$. If $i \in[n-4]$ then

$$
I\left(T_{i}^{c}\right)=\binom{2 j}{j} \quad \text { if } i=2 j, \quad \text { and } \quad I\left(T_{i}^{c}\right)=\frac{1}{2}\binom{2 j}{j} \quad \text { if } i=2 j-1,
$$

as well as $I\left(T_{n-3}^{c}\right)=I\left(T_{n-4}^{c}\right)$ and $I\left(T_{n-2}^{c}\right)=(n-3)(n-4)$. Theorem 3.31 implies the claim.

## Remark 4.2.

i) Notice that Proposition 4.1 yields Equation (11) of Galambos and Reiner if $(W, S)$ is of type $A$ as $W \cong \Sigma_{n+1}, c_{a l t} \in$ Cox $_{\text {max }}$ and $h=n+1=m$.
ii) For the Coxeter groups of type $I_{2}(m)$ we obtain $\mathrm{S}_{c}=m+1$ for $c \in$ Cox $_{\text {max }}$.
iii) Substitution of the relevant Coxeter numbers yields $S_{c}$ for exceptional finite Coxeter groups and $c \in$ Cox $_{\text {max }}$ :

| $(W, S)$ | $H_{3}$ | $H_{4}$ | $F_{4}$ | $E_{6}$ | $E_{7}$ | $E_{8}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{~S}_{c}$ | 21 | 120 | 48 | 182 | 546 | 1840 |.

4.2. Minimum total number of sources and sinks. Since $\Gamma$ is a tree with at most one branching point, the minimum number of sources and sinks of $\Gamma_{c}$ is two or three: if $\Gamma$ is a path then $\mid$ Cox $_{\text {min }} \mid=2$ while $\mid$ Cox $_{\text {min }} \mid=6$ if $\Gamma$ has a branching point. In the latter case, we partition $\mathrm{Cox}_{\text {min }}$ into $\mathrm{Cox}_{\mathrm{a}}, \mathrm{Cox}_{\mathrm{b}}$ and $\mathrm{Cox}_{\mathrm{c}}$ where each set consists of the Coxeter element shown in Figure 9 together with $\operatorname{rev}(c)$.


Figure 9. Oriented Coxeter graphs $\Gamma_{c}$ with branching point and minimum total number of sources and sinks ( $r=n-2$ in type $D$ and $r=n-3$ in type $E)$.

The characterization of isometric Asso $_{c}$ in BHLT09] implies that if $(W, S)$ is of type $D$ or $E$ then there are three distinct isometry classes of associahedra $\mathrm{Asso}_{c}$ that correspond to the sets Cox $_{a}$, Cox $_{b}$ and Cox ${ }_{c}$ unless

- $(W, S)$ is of type $D_{4}$, where Cox ${ }_{\text {min }}$ provides one isometry class;
- ( $W, S$ ) is of type $D_{n}$ with $n \geq 5$, where Cox $_{\mathrm{a}}$ and Cox $_{\mathrm{b}} \cup$ Cox $_{c}$ provide two isometry classes;
- $(W, S)$ is of type $E_{6}$, where the $\mathrm{Cox}_{\mathrm{a}} \cup \mathrm{Cox}_{\mathrm{b}}$ and Cox ${ }_{c}$ provide two isometry classes.

The next proposition provides explicit formulae for $S_{c}$ and the various situations of $\mathbf{c} \in \mathrm{Cox}_{\text {min }}$.
Proposition 4.3. Let $(W, S)$ be an irreducible Coxeter system of rank $n>1$ and $c \in C_{\text {Cox }}$.
i) Suppose that $(W, S)$ is path-like then

$$
\mathrm{S}_{c}=2^{n-2}(h-n+3)
$$

ii) Suppose $(W, S)$ is of type $D_{n}$ and $n \geq 4$ then

$$
\mathrm{S}_{c}= \begin{cases}2^{n-2}\left(h-n+\frac{7}{2}\right), & c \in \operatorname{Cox}_{\mathrm{a}} \\ 2^{n-2}(h-n+4)-2, & c \in \operatorname{Cox}_{\mathrm{b}} \cup \operatorname{Cox}_{\mathrm{c}}\end{cases}
$$

iii) Suppose $(W, S)$ is of type $E_{6}, E_{7}$ or $E_{8}$. Then

$$
\mathrm{S}_{c}= \begin{cases}2^{n-2}(h-n+4)-2^{n-4}, & c \in \operatorname{Cox}_{\mathrm{a}} \\ 2^{n-2}(h-n+4)-4, & c \in \operatorname{Cox}_{\mathrm{b}} \\ 2^{n-2}(h-n+4)+2^{n-2}-2 n, & c \in \operatorname{Cox}_{\mathrm{c}}\end{cases}
$$

Proof. Again, all claims follow from Theorem 3.31 together with the following formulae for $I\left(T_{i}^{c}\right)$.
First, if $(W, S)$ is path-like then $I\left(T_{i}^{\mathbf{c}}\right)=2^{i}$ for all $i \in[n-2]$ and all $c \in \mathrm{C}_{\text {min }}$.
Second, if $(W, S)$ is of type $D_{n}$ and $n \geq 4$. Then $I\left(T_{i}^{\mathbf{c}}\right)=2^{i}$ for all $i \leq[n-4]$ and $\mathbf{c} \in$ Cox $_{\text {min }}$ as well as
$I\left(T_{i}^{\mathbf{c}_{\mathbf{a}}}\right)=\left\{\begin{array}{ll}2^{n-3}, & i=n-3, \\ 2^{n-3}, & i=n-2,\end{array} \quad I\left(T_{i}^{\mathbf{c}_{\mathrm{b}}}\right)=\left\{\begin{array}{ll}1, & i=n-3, \\ 2^{n-2}, & i=n-2,\end{array} \quad\right.\right.$ and $\quad I\left(T_{i}^{\mathbf{c}_{\mathrm{c}}}\right)= \begin{cases}2^{n-3}, & i=n-3, \\ 2, & i=n-2\end{cases}$
where $c_{\mathrm{a}} \in \operatorname{Cox}_{\mathrm{a}}, c_{\mathrm{b}} \in \operatorname{Cox}_{\mathrm{b}}$ and $c_{\mathrm{c}} \in \operatorname{Cox}_{\mathrm{c}}$.
Third, if $(W, S)$ is of type $E_{n} \in\left\{E_{6}, E_{7}, E_{8}\right\}$. Then $I\left(T_{i}^{\mathbf{c}}\right)=2^{i}$ for $i \in[n-5]$ and all $\mathbf{c} \in$ Cox $_{\text {min }}$ as well as
$I\left(T_{i}^{c_{\mathrm{a}}}\right)=\left\{\begin{array}{ll}2^{n-4}, & i=n-4, \\ 2^{n-4}, & i=n-3, \\ 3 \cdot 2^{n-4}, & i=n-2\end{array} \quad I\left(T_{i}^{c_{\mathrm{b}}}\right)=\left\{\begin{array}{ll}1, & i=n-4, \\ 2^{n-3}, & i=n-3, \\ 2^{n-2}, & i=n-2\end{array} \quad\right.\right.$ and $\quad I\left(T_{i}^{c_{\mathrm{c}}}\right)= \begin{cases}2^{n-4}, & i=n-4, \\ 2, & i=n-3, \\ 2 n-4, & i=n-2,\end{cases}$
where $c_{\mathrm{a}} \in \operatorname{Cox}_{\mathrm{a}}, c_{\mathrm{b}} \in \operatorname{Cox}_{\mathrm{b}}$ and $c_{\mathrm{c}} \in \operatorname{Cox}_{\mathrm{c}}$.

## Remark 4.4.

i) If $(W, S)$ is of type $I_{2}(m)$ then Cox $_{\text {min }}=$ Cox $_{\text {max }}$. Notice that Proposition 4.1 and 4.3 yield $\mathrm{S}_{c}=m+1$ in both cases for all $m$.
ii) Let $(W, S)$ be of type $D_{n}$. If $n=4$ then both formulae of Proposition 4.3 yield $\mathrm{S}_{\mathbf{c}}=22$. Moreover, if $n \geq 5$ and $\mathbf{c} \in$ Cox $_{\text {min }}$ then $S_{\mathbf{c}}$ is minimal if and only if $\mathbf{c} \in$ Cox $_{a}$.
iii) Let $(W, S)$ be of type $E_{n}$.

If $n=6$ and $c \in \operatorname{Cox}_{\mathrm{a}} \cup \operatorname{Cox}_{\mathrm{b}}$ then $\mathrm{S}_{c}=156$ while $\mathrm{S}_{c}=164$ for $c \in \operatorname{Cox}_{\mathrm{c}}$.
If $n=7$ then $\mathrm{S}_{c}=472$ for $c \in \mathrm{Cox}_{\mathrm{a}}, \mathrm{S}_{c}=476$ for $c \in \mathrm{Cox}_{\mathrm{b}}$ and $\mathrm{S}_{c}=498$ for $c \in \mathrm{Cox}_{\mathrm{c}}$.
If $n=8$ then $\mathrm{S}_{c}=1648$ for $c \in \mathrm{Cox}_{\mathrm{a}}, \mathrm{S}_{c}=1660$ for $c \in \mathrm{Cox}_{\mathrm{b}}$ and $\mathrm{S}_{c}=1904$ for $c \in \mathrm{Cox}_{\mathrm{c}}$.
iv) The minimum numbers of $\mathrm{S}_{c}$ for all exceptional finite Coxeter groups and $c \in \mathrm{Cox}_{\text {min }}$ are:

| $(W, S)$ | $H_{3}$ | $H_{4}$ | $F_{4}$ | $E_{6}$ | $E_{7}$ | $E_{8}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{~S}_{c}$ | 20 | 116 | 44 | 156 | 472 | 1648 |

## 5. Lower and Upper bounds

In this section, we use Theorem 3.31 and Corollary 3.32 to prove upper and lower bounds for the cardinality $\mathrm{S}_{c}$ of a Cambrian acyclic domain $\mathrm{Acyc}_{\mathbf{c}}$ by identification of minimizers and maximizers for $\sum_{i=1}^{n-2} 2^{n-2-i} I\left(T_{i}^{c}\right)$. As done in the proof of Proposition 3.4 we fix a Coxeter triple ( $W, S, \mathbf{c}$ ) and label the generators of $S$ along a longest path of $\Gamma$ such that $s_{1}, \ldots, s_{p}$ with $p \in\{n-1, n\}$ are successive vertices and in types $D$ and $E$ we have $p=n-1$ and the vertex $s_{n}$ is connected to $s_{r}$ where $r=n-2\left(\right.$ type $\left.D_{n}\right)$ or $r=n-3($ type $E)$.

### 5.1. Cut functions.

Definition 5.1 (Cut function).
A function $f: S \rightarrow \mathbb{Z}$ is a cut function of $(W, S)$ if $f\left(s_{1}\right)$ is odd and $|f(s)-f(t)|=1$ for all non-commuting pairs $s, t \in S$. We write $\left(f\left(s_{1}\right), \ldots, f\left(s_{n}\right)\right)$ for the cut function $f$ and denote the set of all cut functions of $(W, S)$ by $\operatorname{CF}(W, S)$. A generator $s \in S$ is an extremum of the cut function $f$ if either $f(s)<f(t)$ or $f(s)>f(t)$ for all $t \in S$ that do not commute with $s$.

Since $|f(s)-f(t)|=1$ for all non-commuting pairs $s, t \in S$, any cut function $f$ determines a unique Coxeter element $c_{f} \in \operatorname{Cox}(W, S)$ such that $f(s)<f(t)$ if and only if $(s, t)$ is a directed edge of $\Gamma_{c_{f}}$ and every Coxeter element determines a cut function up to an even constant. Moreover, sources and sinks of $\Gamma_{c_{f}}$ correspond to extrema of $f$ and, among the $h$ cut paths of $\Phi^{-1}\left(\operatorname{rev}\left(c_{f}\right)\right)$ from Corollary 3.27, the cut function $f$ determines a unique cut path $\kappa_{f}$ as follows. Let $\sigma$ be the vertex of $\mathcal{C}_{\mathbf{c}}^{2}$ with $\mathrm{g}(\sigma)=s_{2}$ such that the $x$-coordinate $x_{v}$ of $v \in p r_{\mathbf{c}^{h}}^{-1}(\sigma)$ satisfies $x_{v} \equiv f\left(s_{1}\right)$ $\bmod 2 h$ and let $(\rho, \sigma)$ and $(\sigma, \tau)$ be the two edges of $\mathcal{C}_{\mathbf{c}}^{2}$ with $\mathrm{g}(\rho)=\mathrm{g}(\tau)=s_{1}$. Now $\kappa_{f}$ contains $(\rho, \sigma)$ if $f(\mathrm{~g}(\rho))>f(\mathrm{~g}(\sigma))$ and $(\sigma, \tau)$ if $f(\mathrm{~g}(\sigma))<f(\mathrm{~g}(\tau))$. We say that the cut path $\kappa_{f}$ represents the cut function $f$.
Example 5.2. Consider $(W, S)$ of type $A_{4}$ and $\mathbf{c}=s_{2} s_{1} s_{4} s_{3}$. The cut functions $f$ and $g$ with $f\left(s_{1}, s_{2}, s_{3}, s_{4}\right)=(-1,0,1,2)$ and $g\left(s_{1}, s_{2}, s_{3}, s_{4}\right)=(1,0,1,0)$ determine the Coxeter elements $c_{f}=s_{1} s_{2} s_{3} s_{4}$ and $c_{g}=s_{2} s_{1} s_{4} s_{3}$ indicated as shaded subgraphs in the planar drawing of $\mathcal{C}_{\mathbf{c}}^{2}$ in Figure 10. As $f\left(s_{1}\right)=-1 \equiv 9 \bmod 10$ and $f\left(s_{1}\right)<f\left(s_{2}\right)$ as well as $g\left(s_{1}\right)=1 \equiv 1 \bmod 10$ and $g\left(s_{1}\right)>g\left(s_{2}\right)$, we obtain the cut paths $\kappa_{f}$ and $\kappa_{g}$ that represent the cut functions $f$ and $g$ as indicated. The extrema of $f$ are $s_{1}$ and $s_{4}$, while all generators in $S$ are extrema of $g$.


Figure 10. The cut functions $f$ and $g$ and their associated cut paths $\kappa_{f}$ and $\kappa_{g}$ and Coxeter elements $c_{f}$ and $c_{g}$ in type $A_{4}$.

Definition 5.3 (equivalence and crossing of cut functions).
Let $f, g: S \rightarrow \mathbb{Z}$ be cut functions of the finite and irreducible Coxeter system $(W, S)$.
i) $f$ and $g$ are equivalent, $f \simeq g$, if and only if $f(s) \equiv g(s) \bmod 2 h$ for all $s \in S$.
ii) $f$ and $g$ cross if and only if there exist $s, t \in S$ and $\tilde{f} \simeq f$ such that $\widetilde{f}(s)<g(s)$ and $\widetilde{f}(t)>g(t)$.

Before showing that the notion of crossing cut functions $f$ and $g$ coincides with the crossing of cut paths $\kappa_{f}$ and $\kappa_{g}$ that represent $f$ and $g$ in the next lemma, we illustrate the definition with an example.

Example 5.4 (Example 5.2 continued).
Consider $f^{\prime}\left(s_{1}, s_{2}, s_{3}, s_{4}\right)=(19,20,21,22)$ and $f^{\prime \prime}\left(s_{1}, s_{2}, s_{3}, s_{4}\right)=(0,1,2,3)$. Then $f \simeq f^{\prime}$ and $f \not 千 f^{\prime \prime}$. Moreover, $f^{\prime}$ crosses $g$ as $f^{\prime} \simeq f$ and $f\left(s_{1}\right)<g\left(s_{1}\right)$ and $f\left(s_{4}\right)>g\left(s_{4}\right)$. Further, notice that the edges $a$ and $b$ required by Definition 3.28 show that $\kappa_{g}$ crosses $\kappa_{f}$ while the edges $c$ and $d$ show that $\kappa_{f}$ crosses $\kappa_{g}$.

Lemma 5.5. Let $(W, S, \mathbf{c})$ be a Coxeter triple with associated 2-cover $\mathcal{C}_{\mathbf{c}}^{2}$ as well as cut paths $\kappa_{f}$ and $\kappa_{g}$ representing cut functions $f$ and $g$. The cut paths $\kappa_{f}$ and $\kappa_{g}$ cross if and only if the cut functions $f$ and $g$ cross.
Proof. Assume that the cut path $\kappa_{f}$ crosses $\kappa_{g}$ on $\mathcal{C}_{\mathbf{c}}^{2}$. Since $\kappa_{f}$ and $\kappa_{g}$ are crossing, they have a common tile $T$ and there are edges $e_{1} \in \kappa_{g}$ in the initial side of $\kappa_{f}$ and $e_{2} \in \kappa_{g}$ in the final side of $\kappa_{f}$. These edges are also in $\mathcal{C}_{\mathbf{c}}^{2} \backslash \kappa_{f}$. The common tile $T$ allows us to get specific representatives $\widetilde{f}$ and $\widetilde{g}$ for $f$ and $g$ by going down to the first tiles of $\kappa_{f}$ and $\kappa_{g}$. Indeed, consider the (unoriented) edge $\{\rho, \sigma\} \in \kappa_{f}$ such that $\mathrm{g}(\rho)=s_{2}$ and $\mathrm{g}(\sigma)=s_{1}$ and use the $x$-coordinate $x_{\rho}$ of $\rho$ as the value $\widetilde{f}\left(s_{1}\right)=\widetilde{f}(\mathrm{~g}(\sigma)):=x_{\rho} \simeq f\left(s_{1}\right) \bmod 2 h$. This determines $\widetilde{f} \simeq f$. Proceed similarly to obtain $\widetilde{g}$. Further, since we have edges $e_{1}$ and $e_{2}$ on distinct connected components of $\mathcal{C}_{\mathbf{c}}^{2} \backslash\left(\kappa_{f} \cup \kappa_{f}^{*}\right)$, there exist $s, t \in S$ such that

$$
\widetilde{f}(s)<\widetilde{g}(s) \quad \text { and } \quad \widetilde{f}(t)>\widetilde{g}(t) \quad \text { or } \quad \widetilde{f}(s)>\widetilde{g}(s) \quad \text { and } \quad \widetilde{f}(t)<\widetilde{g}(t)
$$

depending if the initial side of $\kappa_{f}$ is "on the right" or "on the left" of $\kappa_{f}$.
Now suppose that $f$ and $g$ are crossing cut functions with equivalent cut functions $\tilde{f} \simeq f$ and $\widetilde{g} \simeq g$ such that $\widetilde{f}(s)<\widetilde{g}(s)$ and $\widetilde{f}(t)>\widetilde{g}(t)$ for some $s, t \in S$. Since the Coxeter graph $\Gamma$ is a tree and since a cut function $h$ satisfies $\mid h(a)-h(b)) \mid=1$ for non-commuting $a, b \in S$, every integral value between $\widetilde{f}(s)$ and $\widetilde{f}(t)$ is in the image of $\widetilde{f}$. Because of the latter inequalities for $\widetilde{f}(s)$ and $\widetilde{g}(s)$, this implies that there exists a generator $u \in S$ such that $\widetilde{f}(u)=\widetilde{g}(u)$. Following the procedure to obtain the cut path $\kappa_{h}$ from a cut function $h$, this implies that the representing cut paths $\kappa_{f}$ and $\kappa_{g}$ will have a common tile once we get to a tile with vertex label $u$. Further, the inequalities guarantee that there will be one edge in the initial side of $\kappa_{f}$ and one edge in the final side of $\kappa_{f}$ taken by $\kappa_{g}$.

### 5.2. Upper and lower bounds for the cardinality of Cambrian acyclic domains.

To obtain lower and upper bounds for $S_{c}=\left|A c y c_{c}\right|$, the next lemma about the minimum and maximum number of extrema of a cut function $f$ is essential. Clearly, the maximum number of extrema is equal to $n$ while the minimum number of extrema is equal to 2 if $(W, S)$ is path-like and equal to 3 if $(W, S)$ is of type $D$ or $E$.

Lemma 5.6. Let $f$ be a cut function of the finite irreducible Coxeter system $(W, S)$.
a) The number of cut functions that cross $f$ is minimum if and only if the number of extrema of $f$ is maximum.
b) If $(W, S)$ is path-like, then the number of cut functions that cross $f$ is maximum if and only if the number of extrema of $f$ is minimum.
If $(W, S)$ is of type $D$ or $E$, then the number of cut functions that cross $f$ is maximum implies that the number of extrema of $f$ is minimum.

Proof. a) For $m \in \mathbb{Z}$, the $m$-reflection $\mathcal{R}_{m}: \operatorname{CF}(W, S) \rightarrow \operatorname{CF}(W, S)$ is the bijection

$$
\mathcal{R}_{m}(f)(m+\ell):=f(m-\ell) \quad \text { for all } \ell \in \mathbb{Z}
$$

If $f \in \mathrm{CF}(W, S)$ has less than $n$ extrema then set

$$
f^{\prime}: S \longrightarrow \mathbb{Z} \quad \text { via } \quad s \longmapsto \begin{cases}f(s), & \text { if } f(s)<\max _{t \in S} f(t) \\ f(s)-2, & \text { if } f(s)=\max _{t \in S} f(t)\end{cases}
$$

Clearly, every extremum of $f$ is an extremum of $f^{\prime}$ and there is at least one $s \in S$ that is extremal for $f^{\prime}$ but not for $f$, see Figure 11. Now define

$$
\mathcal{F}:=\{g \in \mathrm{CF}(W, S) \mid g \text { crosses } f\} \quad \text { and } \quad \mathcal{F}^{\prime}:=\left\{g \in \mathrm{CF}(W, S) \mid g \text { crosses } f^{\prime}\right\} .
$$

We first prove $\left|\mathcal{F}^{\prime}\right|<|\mathcal{F}|$ by showing that $\left|\mathcal{F}^{\prime} \backslash \mathcal{F}\right|<\left|\mathcal{F} \backslash \mathcal{F}^{\prime}\right|$. To see this, let $\mu:=\max _{t \in S} f(t)$ and notice that

$$
\mathcal{R}_{\mu-1}(g) \in \mathcal{F} \backslash \mathcal{F}^{\prime} \quad \text { for all } g \in \mathcal{F}^{\prime} \backslash \mathcal{F}
$$

so the reflection $\mathcal{R}_{\mu-1}$ is an injection $\mathcal{F}^{\prime} \backslash \mathcal{F} \hookrightarrow \mathcal{F} \backslash \mathcal{F}^{\prime}$ which is not surjective because $f \notin \mathcal{F}^{\prime} \backslash \mathcal{F}$ and $\mathcal{R}_{\mu-1}(f) \in \mathcal{F} \backslash \mathcal{F}^{\prime}$. Thus $\left|\mathcal{F}^{\prime}\right|<|\mathcal{F}|$ and iterating this process yields a cut function where every generator of $S$ is extremal. To complete the proof, we show that $|\mathcal{F}|=|\mathcal{G}|$ if $f$ and $g$ are cut functions where every $s \in S$ is extremal. This follows from the observation that two cut functions where every $s \in S$ is extremal differ only by translation and reflection.


Figure 11. $\mathcal{F} \supset \mathcal{F}^{\prime}$ for cut functions $f$ and $f^{\prime}$ as in the proof of Lemma 5.6, We have $R_{\mu-1}(f) \in \mathcal{F} \backslash \mathcal{F}^{\prime}$ as well as $g \in \mathcal{F}^{\prime} \backslash \mathcal{F}$ implies $R_{\mu-1}(g) \in \mathcal{F} \backslash \mathcal{F}^{\prime}$.
b) For $n=1$ and $n=2$ there is nothing to prove, as all generators are extremal for each cut function. We therefore assume $n \geq 3$ and prove the claim first if $(W, S)$ is path-like. A cut function $f$ determines the Coxeter element $\mathbf{c}_{f}$ and the number of cut functions that cross $f$ is equal to the number $\mathrm{Q}_{\mathrm{rev}\left(\mathbf{c}_{f}\right)}$ of cut paths that cross $\kappa_{\mathrm{rev}\left(\mathbf{c}_{f}\right)}$ in $\mathcal{C}_{\mathbf{c}}^{2}$ by Lemma 5.5. By Corollary 3.32, we have

$$
\mathrm{Q}_{\mathrm{rev}\left(\mathbf{c}_{f}\right)}=\sum_{i \in[n-2]} 2^{n-2-i} I\left(T_{i}^{\mathrm{rev}\left(\mathbf{c}_{f}\right)}\right)-\left(2^{n-2}-1\right)
$$

where $T_{i}^{\mathrm{rev}\left(\mathbf{c}_{f}\right)}$ is tile $i$ of $\kappa_{\mathrm{rev}\left(\mathbf{c}_{f}\right)}$ and $I\left(T_{i}^{\mathrm{rev}\left(\mathbf{c}_{f}\right)}\right)$ is the number of distinct initial segments of cut paths $\kappa$ up to $T_{i}^{\operatorname{rev}\left(\mathbf{c}_{f}\right)}$ with edges contained in $\mathcal{C}_{\mathbf{c}}^{2} \backslash \operatorname{out}\left(T_{i}^{\mathrm{rev}\left(\mathbf{c}_{f}\right)}\right)$. The reasoning of Example 3.30 shows

$$
\operatorname{rev}\left(\mathbf{c}_{f}\right) \in\left\{s_{1} s_{2} \cdots s_{n}, s_{n} s_{n-1} \cdots s_{1}\right\} \quad \Longrightarrow \quad I\left(T_{i}^{\operatorname{rev}\left(\mathbf{c}_{f}\right)}\right)=2^{i} \quad \text { for all } i \in[n-2]
$$

These are clearly the only Coxeter elements with $I\left(T_{i}^{\mathrm{rev}\left(\mathbf{c}_{f}\right)}\right)=2^{i}$ for all $i \in[n-2]$ and these values are maximum. Thus $I\left(T_{i}^{\operatorname{rev}\left(\mathbf{c}_{f}\right)}\right)$ attains its maximum value for each $i \leq n-2$ if and only if $f$ is strictly monotone on $[i+2]$. In particular, $\mathrm{Q}_{\mathrm{rev}\left(\mathbf{c}_{f}\right)}$ is maximum if and only if the cut function $f$ is strictly monotone. Thus, we conclude for any path-like Coxeter system $(W, S)$ that $f$ has precisely two extrema if and only if the number of cut functions that cross $f$ is maximum.

To analyze type $D$, we first consider $D_{4}$. Without loss of generality, we have to analyze the following four cases for $c_{f}$ and $\kappa_{\operatorname{rev}\left(c_{f}\right)}$ as they represent all cut functions $f$ in type $D_{4}$ :

$c_{f}=c_{a} \in \operatorname{Cox}_{\mathrm{a}}$

$c_{f}=c_{b} \in \operatorname{Cox}_{\mathrm{b}}$

$c_{f}=c_{c} \in \operatorname{Cox}_{\mathrm{c}} \quad c_{f}=c_{\max } \in \mathrm{Cox}_{\text {max }}$

Corollary 3.32 implies

$$
\mathrm{Q}_{c_{a}}=6-3 \quad \mathrm{Q}_{c_{b}}=6-3 \quad \mathrm{Q}_{c_{c}}=6-3 \quad \mathrm{Q}_{c_{\max }}=3-3
$$

This shows that if the number of crossing cut functions of $f \in \operatorname{CF}\left(D_{4}, S\right)$ is maximum then $f$ has three extrema which is the minimum number of extrema in this situation.

We now consider an extension from $\left(D_{n}, S\right)$ to $\left(D_{n+1}, \widetilde{S}\right)$ with $n \geq 4$ by adding a new vertex adjacent to the leaf $s_{1}$ of $\Gamma_{D_{n}}$ and appropriate relabeling of generators. Thus $\Gamma_{D_{n}}$ corresponds to the subgraph of $\Gamma_{D_{n+1}}$ induced by $\widetilde{S} \backslash\left\{s_{1}\right\}$ and every Coxeter element $c_{n} \in \operatorname{Cox}\left(D_{n}, S\right)$ can be extended in precisely two ways to a Coxeter element $c_{n+1} \in \operatorname{Cox}\left(D_{n+1}, \widetilde{S}\right)$. Clearly, we have

$$
I\left(T_{1}^{c_{n+1}}\right) \in\{1,2\} \quad \text { as well as } \quad I\left(T_{i}^{c_{n}}\right) \leq I\left(T_{i+1}^{c_{n+1}}\right) \leq 2 I\left(T_{i}^{c_{n}}\right) \text { for } i \in[n-2]
$$

Thus

$$
\begin{aligned}
\mathrm{Q}_{c_{n+1}} & =\sum_{i \in[n-1]} 2^{n-1-i} I\left(T_{i}^{c_{n+1}}\right)-\left(2^{n-1}-1\right) \\
& \leq 2^{n-1-1} \cdot 2+\sum_{i \in[n-2]} 2^{n-2-i} \cdot\left(2 I\left(T_{i}^{c_{n}}\right)\right)-2\left(2^{n-2}-1\right)-1 \\
& =2^{n-1}+2 \mathrm{Q}_{c_{n}}-1
\end{aligned}
$$

with equality if $I\left(T_{i+1}^{c_{n+1}}\right)=2 I\left(T_{i}^{c_{n}}\right)$ for all $i \in[n-2]$ and $I\left(T_{1}^{c_{n+1}}\right)=2$ which happens if and only if $\operatorname{out}\left(T_{k}^{c_{n+1}}\right)$ and out $\left(T_{\ell}^{c_{n+1}}\right)$ coincide for all $1 \leq k, \ell \leq n-1$. Thus, if $Q_{c_{n+1}}$ is maximum then $c_{n+1} \in \mathrm{Cox}_{\mathrm{a}} \subseteq \mathrm{Cox}_{\text {min }}$. In other words, if the number of cut functions that cross $f$ is maximum then $f$ has three extrema. This proves the claim if $(W, S)$ is of type $D$.

Finally, we prove the claim in type $E$. We first analyze $E_{6}$. Clearly, removing the vertex $s_{p}=s_{5}$ from $\Gamma_{E_{6}}$ yields a Coxeter graph of type $D_{5}$. Let $c$ be a Coxeter element for type $E_{6}$ and $\widetilde{c}$ be the corresponding Coxeter element for ( $\widetilde{W}, S \backslash\left\{s_{5}\right\}$ ) of type $D_{5}$. Since $I\left(T_{k}^{c}\right)=I\left(T_{k}^{\widetilde{c}}\right)$ for $1 \leq k \leq 3$, we obtain $\mathrm{Q}_{c}=2 \mathrm{Q}_{\tilde{c}}+I\left(T_{4}^{c}\right)-1$.

A case analysis reveals that $\mathrm{Q}_{c}$ is maximum in type $E_{6}$ if and only if $c \in \operatorname{Cox}_{\mathrm{a}} \cup \operatorname{Cox}_{\mathrm{b}}$. Thus, if $f$ is a cut function with the maximum number of crossing cut functions then $\mathrm{c}_{f} \in \operatorname{Cox}_{\mathrm{a}} \cup \operatorname{Cox}_{\mathrm{b}}$, that is, $f$ has three extrema. To solve the remaining cases $E_{7}$ and $E_{8}$ we extend from type $E_{6}$ to $E_{7}$ and from type $E_{7}$ to $E_{8}$ similarly to the induction step from $D_{n}$ to $D_{n+1}$. Again, we obtain $\mathrm{Q}_{c_{n+1}} \leq 2^{n-1}+2 \mathrm{Q}_{c_{n}}-1$ with equality if and only if $\operatorname{out}\left(T_{k}^{c_{n+1}}\right)$ and $\operatorname{out}\left(T_{\ell}^{c_{n+1}}\right)$ coincide for all $1 \leq k, \ell \leq n-1$. Therefore, if $Q_{c_{n+1}}$ is maximum then $c_{n+1} \in \operatorname{Cox}_{a} \subseteq \operatorname{Cox}_{\min }$. This proves the claim if $(W, S)$ is of type $E_{7}$ and $E_{8}$.

We now characterize the Coxeter elements $c$ that maximize and minimize the cardinality $\mathrm{S}_{c}$ of a Cambrian acyclic domain $\mathrm{Acyc}_{c}$.

Theorem 5.7. Let $(W, S)$ be a finite irreducible Coxeter system, $c \in \operatorname{Cox}(W, S)$ and $\mathrm{S}_{c}=\left|\mathrm{Acyc}_{\mathbf{c}}\right|$.
a) The cardinality $\mathrm{S}_{c}$ of $\mathrm{Acyc}_{c}$ is maximum if and only if $c \in \mathrm{Cox}_{\text {max }}$.
b) The cardinality $\mathrm{S}_{c}$ of $\mathrm{Acyc}_{c}$ is minimum if and only if
i) $c \in$ Cox $_{\min }$ and $(W, S)$ is path-like or of type $D_{4}$;
ii) $c \in \operatorname{Cox}_{\mathrm{a}} \cup \operatorname{Cox}_{\mathrm{b}}$ and $(W, S)$ is of type $E_{6}$;
iii) $c \in \operatorname{Cox}_{a}$ and $(W, S)$ is of type $E_{7}, E_{8}$ or $D_{n}$ for $n \geq 5$.

Proof. a) This is a consequence of Theorem 3.31 and Corollary 3.32 combined with Lemma 5.6
b) When $(W, S)$ is path-like or of type $D_{4}$, it follows immediately from Lemma [5.6. Otherwise, to decide the remaining cases for type $D$ and $E$, we use the relevant values for $I\left(T_{i}^{\mathbf{c}}\right)$ from the proof of Proposition 4.3. We have to analyze $\mathbf{c}_{\mathrm{a}} \in \operatorname{Cox}_{\mathrm{a}}, \mathbf{c}_{\mathrm{b}} \in \mathrm{Cox}_{\mathrm{b}}$ and $\mathbf{c}_{\mathrm{c}} \in \mathrm{Cox}_{\mathrm{c}}$. If $(W, S)$ is of type $D$, we obtain

$$
\mathbf{Q}_{\mathbf{c}_{\mathrm{a}}}=(n-4) 2^{n-1}+2^{n-3}+1, \quad \mathbf{Q}_{\mathbf{c}_{\mathrm{b}}}=(n-4) 2^{n-1}+3 \quad \text { and } \quad \mathbf{Q}_{\mathbf{c}_{\mathrm{c}}}=(n-4) 2^{n-1}+3 .
$$

The maximum is achieved by $\mathbf{c}_{\mathrm{a}}, \mathbf{c}_{\mathrm{b}}$ and $\mathbf{c}_{\mathrm{c}}$ if $n=4$ and only by $\mathbf{c}_{\mathrm{a}}$ if $n \geq 5$. If $(W, S)$ is of type $E$, we similarly obtain
$\mathbf{Q}_{\mathbf{c}_{\mathbf{a}}}=(n-5) 2^{n-2}+5 \cdot 2^{n-4}+1, \quad \mathbf{Q}_{\mathbf{c}_{\mathrm{b}}}=(n-5) 2^{n-2}+2^{n-2}+5 \quad$ and $\quad \mathbf{Q}_{\mathbf{c}_{\mathrm{c}}}=(n-5) 2^{n-2}+2 n+1$.
The maximum is achieved by $\mathbf{c}_{\mathrm{a}}$ and $\mathbf{c}_{\mathrm{b}}$ if $n=6$ and by $\mathbf{c}_{\mathrm{a}}$ if $n \in\{7,8\}$. In particular, this shows that the number of cut functions that cross $f$ is not always maximized if the number of extrema of $f$ is minimized.

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