Technical report: A generating funtion for the Euler numbers of the second kind and it's application

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27 февраля 2018 г.

Аннотация

In the paper, 2 explicit formulas for the Euler numbers of the second kind are obtained. Based on those formulas a exponential generating function is deduced. Using the generating function some well-known and new identities for the Euler number of the second kind are obtained.

1 Introduction

The Euler numbers of the second kind is very useful in combinatorics, number theory, graph theory, analytic geometry and other areas [1, 2, 3]. The Euler numbers of the second kind is a numerical triangle. In the paper, we consider the triangle with an initianal term $\left\langle\!\left\langle {1 \atop 1} \right\rangle\!\right\rangle$. This variant of the triangle is submitted in the On-Line Encyclopedia of Integer Sequences by number of secuence A008517[4].

The triange of the Euler numbers of the second kind is defined by recurrence expression

$$\left\langle\!\left\langle {n \atop m} \right\rangle\!\right\rangle = m \left\langle\!\left\langle {n-1 \atop m} \right\rangle\!\right\rangle + (2\,n - m) \left\langle\!\left\langle {n-1 \atop m-1} \right\rangle\!\right\rangle. \tag{1}$$

A detailed review of the Euler numbers of the second kind was given by Knuth [2]. However, in the present time there is not a generating funtion for those numbers and there are a few explicit ways to defined those numbers. In this paper we give 2 explicit formulas and a generating function for the Euler numbers of the second kind. Based of those results we obtain new identities for the Euler numbers of the second kind.

2 Explicit formulas

Theorem For the Euler numbers of the second kind there hold the following formulas:

$$\left\langle\!\left\langle {n \atop m} \right\rangle\!\right\rangle = \sum_{k=0}^{m} (-1)^{m-k} \binom{2\,n+1}{m-k} \left\{ {n+k \atop k} \right\}, \tag{2}$$

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$$\left\langle\!\!\left\langle {n \atop m} \right\rangle\!\!\right\rangle = \sum_{k=0}^{m} (-1)^{m-k}\, k \binom{2\,n}{m-k} \left\{ {n+k-1 \atop k} \right\}. \tag{3}$$

Proof First we state the well-known results that we will use:

1) For the binomial coefficients we have

$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}, \tag{4}$$

2) For the Stirling numbers of the second kind we have

$$\left\{ {n \atop m} \right\} = m \left\{ {n-1 \atop m} \right\} + \left\{ {n-1 \atop m-1} \right\}. \tag{5}$$

For the proving the formula (2) we apply the mathmatical induction. For $n = 1$ and $m = 1$ both of formulas are equal to 1. Suppose the formula (2) is right for every n and $m - 1$. Using the identity (1) for (2), we get

$$\left\langle\!\!\left\langle {n \atop m} \right\rangle\!\!\right\rangle = m \sum_{k=0}^{m} (-1)^{m-k} \binom{2\,n-1}{m-k} \left\{ {n+k-1 \atop k} \right\} +$$

$$+ (2\,n - m) \sum_{k=0}^{m-1} (-1)^{m-k-1} \binom{2\,n-1}{m-k-1} \left\{ {n+k-1 \atop k} \right\}.$$

We note that the second kind of the expression for $k = m$ is equal to 0. Then we combine both of sums:

$$\left\langle\!\!\left\langle {n \atop m} \right\rangle\!\!\right\rangle = \sum_{k=0}^{m} (-1)^{m-k} \left(m \binom{2\,n-1}{m-k} - (2\,n-m) \binom{2\,n-1}{m-k-1} \right) \left\{ {n+k-1 \atop k} \right\}.$$

Next we consider the difference between coefficients inside the brackets.

$$m \binom{2\,n-1}{m-k} - (2\,n-m) \binom{2\,n-1}{m-k-1} =$$

$$= m \binom{2\,n-1}{m-k} + m \binom{2\,n-1}{m-k-1} - 2\,n \binom{2\,n-1}{m-k-1} =$$

$$= m \binom{2\,n}{m-k} - (m-k) \binom{2\,n}{m-k} = k \binom{2\,n}{m-k}.$$

Hence, we get the formula (3)

$$\left\langle\!\!\left\langle {n \atop m} \right\rangle\!\!\right\rangle = \sum_{k=0}^{m} (-1)^{m-k}\, k \binom{2\,n}{m-k} \left\{ {n+k-1 \atop k} \right\}.$$

Applying the identity for the Stirling number of the second kind for the formula (2) we get

$$\left\langle\!\!\left\langle {n \atop m} \right\rangle\!\!\right\rangle = \sum_{k=0}^{m}(-1)^{m-k}\binom{2n+1}{m-k}k\left\{{n+k-1 \atop k}\right\} + \tag{6}$$

$$+\sum_{k=0}^{m-1}(-1)^{m-k}\binom{2n+1}{m-k}\left\{{n+k-1 \atop k-1}\right\} \tag{7}$$

Consider the second sum in the above expression we note that for $k = 0$ the expression inside sum is equal to 0, because of the Stirling numbers of the second kind have negative second parameter.

$$\sum_{k=1}^{m}(-1)^{m-k}\binom{2n+1}{m-k}\left\{{n+k-1 \atop k-1}\right\}.$$

Substituting $k+1$ for k, we get

$$\sum_{k=0}^{m-1}(-1)^{m-k-1}\binom{2n+1}{m-k-1}\left\{{n+k \atop k}\right\}$$

Therefore, we obtain the formula for $\left\langle\!\!\left\langle {n \atop m-1} \right\rangle\!\!\right\rangle$.

Next we consider sum in (6) and apply the identity for the binomial coefficients

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{2n+1}{m-k}k\left\{{n+k-1 \atop k}\right\} =$$

$$= \sum_{k=0}^{m}(-1)^{m-k}\binom{2n}{m-k}k\left\{{n+k-1 \atop k}\right\} +$$

$$+\sum_{k=0}^{m}(-1)^{m-k}\binom{2n}{m-k-1}k\left\{{n+k-1 \atop k}\right\}.$$

The first sum is the formula (3) that obtained based on the identity for the Euler numbers of the second kind. For $k = m$ the expression inside the second sum is equal to 0, because the binomial coefficients is equal to 0. Since that we have

$$\sum_{k=0}^{m-1}(-1)^{m-k}\binom{2n}{m-k-1}k\left\{{n+k-1 \atop k}\right\}$$

multiplying on (-1) we get

$$-\sum_{k=0}^{m-1}(-1)^{m-k-1}\binom{2n}{m-k-1}k\left\{{n+k-1 \atop k}\right\}.$$

Using (3), we obtain

$$\sum_{k=0}^{m-1}(-1)^{m-k-1}\binom{2n}{m-k-1}k\left\{{n+k-1 \atop k}\right\} = -\left\langle\!\!\left\langle {n \atop m-1} \right\rangle\!\!\right\rangle.$$

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Since that, we arrive to desired result

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{2\,n+1}{m-k}\left\{\begin{matrix}n+k\\k\end{matrix}\right\} = \left\langle\!\!\left\langle\begin{matrix}n\\m\end{matrix}\right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle\begin{matrix}n\\m-1\end{matrix}\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle\begin{matrix}n\\m-1\end{matrix}\right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle\begin{matrix}n\\m\end{matrix}\right\rangle\!\!\right\rangle.$$

3 Generating function for the Euler numbers of the second kind

Using the obtained explicit formulas, now we can find an exponential generating function for the Euler numbers of the second kind.

Theorem 2 The exponential generating function for the Euler numbers of the second kind is defined by the following expression:

$$\sum_{n\geqslant 0}\sum_{m\geqslant 0}\left\langle\!\!\left\langle\begin{matrix}n\\m\end{matrix}\right\rangle\!\!\right\rangle\frac{x^n}{n!}t^m = \frac{1-t}{W\left(-t\,e^{(1-t)^2\,x-t}\right)+1}, \tag{8}$$

where $W(x)$ – Lambert function and $x>0$.

Proof Present the generating function as dual series with shift $(n-1)$

$$E(x,t) = \sum_{n>0}\sum_{m>0}\left\langle\!\!\left\langle\begin{matrix}n-1\\m\end{matrix}\right\rangle\!\!\right\rangle\frac{x^n}{n!}t^m$$

Preset the formula(2) as dual series with shift as follows

$$E(x,t) = \sum_{n>0}P_n(t)\frac{x^n}{n!}\left(1-t\right)^{2n-1},$$

where

$$(1-t)^{2\,n-1} = \sum_{n\geqslant 0}(-1)^m\binom{2\,n-1}{m}t^m$$

and

$$P_n(t) = \sum_{m\geqslant 0}\left\{\begin{matrix}n+m-1\\m\end{matrix}\right\}t^m$$

Define the following generating function

$$u(x,t) = \sum_{n>0}\sum_{m\geqslant 0}\left\{\begin{matrix}n+m-1\\m\end{matrix}\right\}\frac{x^n}{n!}t^m$$

and show that this generating function is a compositional inverse generating function for

$$y(x,t) = (x - t(e^x - 1)).$$

Suppose the reciprocal generating function for $y(x,t)$ is

$$\frac{x}{y(x,t)} = \frac{1}{1-t\frac{e^x-1}{x}}.$$

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Then we find an expression for k powers. For that we write

$$\left(t\left(\frac{e^x-1}{x}\right)\right)^k = \sum_{n>0}\sum_{m>0} T(n,m,k)x^n t^n$$

Since for (e^x-1) there is the following identity

$$(e^x-1)^k = \sum_{n\geqslant 0} k!\left\{\begin{matrix}n\\k\end{matrix}\right\}\frac{x^n}{n!},$$

we get

$$T(n,m,k) = \delta(m,k)\left\{\begin{matrix}n+k\\k\end{matrix}\right\}\frac{k!}{(n+k)!},$$

where $\delta(m,k)$ is the Kroneker symbol.

Since for $\frac{1}{(1-x)}$ there is the following identity

$$\frac{1}{(1-x)^k} = \sum_{n\geqslant 0}\binom{n+k-1}{n}x^n.$$

Then for k powers of the composition of the two generating function

$$\left(\frac{1}{1-t\frac{e^x-1}{x}}\right)^k = \sum_{n\geqslant 0}\sum_{m\geqslant 0} D(n,m,k)x^n t^m,$$

by using the formula from [6], we have

$$D(n,m,k) = \sum_{i=0}^{n+m} T(n,m,i)\binom{i+k-1}{i} = \sum_{k=0}^{n+m}\delta(m,i)\left\{\begin{matrix}n+i\\i\end{matrix}\right\}\frac{i!}{(n+i)!}\binom{i+k-1}{i} =$$

$$= \left\{\begin{matrix}n+m\\m\end{matrix}\right\}\frac{m!}{(n+m)!}\binom{m+k-1}{m}.$$

According the Lagrange inverse theorem for power series $u(x,t)$ satisfied for the functional equation

$$u = x\,F(g,t)$$

where $F(x,t)$ - power series with $F(0,0)\neq 0$ there hold

$$[x^n]u(x,t) = \frac{k}{n}[x^{n-k}]F(x,t)^n.$$

Applying that on our case $F(x,t) = \frac{1}{1-t\frac{e^x-1}{x}}$ we get

$$u = \frac{x}{1-t\frac{e^u-1}{u}}.$$

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The solution of the equation is

$$u(x,t)^k = \sum_{n>0}\sum_{m\geqslant 0} \frac{k}{n} D(n-k,m,n) x^n \, t^m$$

where $D(n,m,k)$ is coefficients of $F(x,t)^k$.

Then

$$D(n-k,m,n) = \left\{ \begin{matrix} n+m-k \\ m \end{matrix} \right\} \frac{m!}{(n+m-k)!} \binom{m+n-1}{m}$$

For $k=1$ we get

$$u(x,t) = \sum_{n>0}\sum_{m\geqslant 0} \frac{1}{n} \left\{ \begin{matrix} n+m-1 \\ m \end{matrix} \right\} \frac{m!}{(n+m-1)!} \binom{m+n-1}{m} x^n \, t^m$$

or after simplification we have

$$u(x,t) = \sum_{n>0}\sum_{m\geqslant 0} \frac{1}{n!} \left\{ \begin{matrix} n+m-1 \\ m \end{matrix} \right\} x^n \, t^m.$$

Hence, funtion $u(x,t)$ is the compositional inverse function for $y(x,t) = (x - t(e^x - 1))$. The Lambert function is defined by [5]

$$x = W(x)e^{W(x)}.$$

It is well-known that for the equation

$$g(x) = f(x)e^{f(x)}$$

the solution is

$$f(x) = W(g(x)).$$

Next we apply that for our case

$$y = x(y,t) - t\, e^{x(t,y)} + t.$$

Replacing

$$z(y,t) = x(y,t) + t - y$$

our equation will be

$$z(y,t) = t\, e^{Z(y,t)-t+y}$$

or

$$z(y,t)e^{-Z(y,t)} = t\, e^{y-t}.$$

Then the solution for $z(y,t)$ is

$$z(y,t) = -W(-t\, e^{y-t})$$

Hence,

$$x(y,t) = y - t - W(-t\, e^{y-t})$$

Considering the following product

$$E(x,t) = \sum_{n>0} P_n(t)\frac{x^n}{n!}(1-t)^{2n-1} = \frac{1}{1-t}\sum_{n>0} P_n(t)\frac{(x(1-t)^2)^n}{n!}$$

we get

$$E(x,t) = \frac{E_2(x(1-t)^2,t)}{(1-t)} = \frac{x(1-t)^2 - t - W\left(-t\,e^{x(1-t)^2-t}\right)}{1-t}.$$

differentiating with respect to x the expression for $E(x,t)$ according properties of the derivative of the Lambert funtion $W(x)$ we arrive to the desired generating function

$$\sum_{n>0}\sum_{m>0} \left\langle\!\!\left\langle {n \atop m} \right\rangle\!\!\right\rangle \frac{x^n}{n!}t^m = \frac{1-t}{W\left(-t\,e^{(1-t)^2\,x-t}\right)+1}.$$

4   Identities

First we prove an identity for the Stirling number of the second kind that presented in [2] (see the formula 6.4.3):

$$u(x,t) = \sum_{n>0}\sum_{m\geqslant 0} \frac{1}{n!}\left\{ {n+m-1 \atop m} \right\} t^m$$

$$E(x,t) = \frac{u(x(1-t)^2,t)}{1-t} = \sum_{m\geqslant 0}\frac{1}{n!}\left\langle\!\!\left\langle {n-1 \atop m} \right\rangle\!\!\right\rangle x^n\,t^m$$

Then

$$(1-t)E\left(\frac{x}{(1-t)^2},t\right) = \frac{1}{n!}\left\{ {n+m-1 \atop m} \right\} t^m$$

For the coefficients of composition $E(x,t)$ and $\frac{x}{(1-t)^2}$ we find coefficients for $\left(\frac{x}{(1-t)^2}\right)^k$

$$\left(\frac{x}{(1-t)^2}\right)^k = \sum_{n>0}\sum_{m>0} p(n,m,k) = \delta(n,k)\binom{m+2k-1}{m}x^n\,t^m,$$

where

$$p(n,m,k) = \delta(n,k)\binom{m+2k-1}{m}.$$

$$E\left(\frac{x}{(1-t)^2},t\right) = \sum_{n}\sum_{m} e(n,m)x^n\,t^m,$$

where

$$e(n,m) = \sum_{i=0}^{m}\sum_{k=0}^{n+m-i}\frac{1}{n!}\left\langle\!\!\left\langle {k-1 \atop i} \right\rangle\!\!\right\rangle \delta_{k,n}\binom{m+2k-i-1}{m-i} = \sum_{i=0}^{m}\frac{1}{n!}\left\langle\!\!\left\langle {n-1 \atop i} \right\rangle\!\!\right\rangle \binom{m+2n-i-1}{m-i}$$

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Then coefficients for $(1 - t)E\left(\frac{x}{(1-t)^2}, t\right)$ are equal to

$$e(n, m) - e(n, m-1) = \sum_{i=0}^{m} \frac{1}{n!} \left\langle\!\!\left\langle \begin{matrix} n-1 \\ i \end{matrix} \right\rangle\!\!\right\rangle \binom{m+2n-i-1}{m-i} - \sum_{i=0}^{m-1} \frac{1}{n!} \left\langle\!\!\left\langle \begin{matrix} n-1 \\ i \end{matrix} \right\rangle\!\!\right\rangle \binom{m+2n-i-2}{m-i-1}$$

Note that inner expression of the second sum for $i = m$ is equal to 0. Then both sums could be combined

$$e(n, m) - e(n, m-1) = \sum_{i=0}^{m} \frac{1}{n!} \left\langle\!\!\left\langle \begin{matrix} n-1 \\ i \end{matrix} \right\rangle\!\!\right\rangle \left(\binom{m+2n-i-1}{m-i} - \binom{m+2n-i-2}{m-i-1}\right) =$$

$$= \sum_{i=0}^{m} \frac{1}{n!} \left\langle\!\!\left\langle \begin{matrix} n-1 \\ i \end{matrix} \right\rangle\!\!\right\rangle \binom{m+2n-i-2}{m-i}$$

Then

$$\frac{1}{n!} \left\{ \begin{matrix} n+m-1 \\ m \end{matrix} \right\} = \sum_{i=0}^{m} \frac{1}{n!} \left\langle\!\!\left\langle \begin{matrix} n-1 \\ i \end{matrix} \right\rangle\!\!\right\rangle \binom{m+2n-i-2}{m-i}$$

$$\left\{ \begin{matrix} n+m \\ m \end{matrix} \right\} = \sum_{i=0}^{m} \left\langle\!\!\left\langle \begin{matrix} n \\ i \end{matrix} \right\rangle\!\!\right\rangle \binom{m+2n-i}{m-i}$$

Next we find new identities for the Euler numbers of the second kind based on the generating function $Eu(x, t)$. For that we consider the following composition of generating functions $Eu(x, t)$ and $\frac{t(x+1)}{(1-t)^2}$

$$Eu\left(\frac{t(x+1)}{(1-t)^2}, t\right) = \frac{1-t}{W\left(-t\, e^{x\, t}\right) + 1}$$

$$\left(\frac{t(x+1)}{(1-t)^2}\right)^k = \sum_{n \geqslant 0} \sum_{m > 0} \binom{k}{n} \binom{m+k-1}{m-k} x^n t^m$$

$$Eu\left(\frac{t(x+1)}{(1-t)^2}, t\right) = \sum_{n > 0} \sum_{m > 0} \sum_{i=0}^{m} \sum_{k=0}^{n+m-i} \frac{1}{k!} \left\langle\!\!\left\langle \begin{matrix} k \\ i \end{matrix} \right\rangle\!\!\right\rangle \binom{k}{n} \binom{m-i+k-1}{m-i-k} x^n\, t^m \qquad (9)$$

For $x = 0$ we have

$$\frac{1-t}{W(-t)+1}$$

For the Lambert function there hold

$$\ln(W(x)) = W(x) - \ln(x), x > 0$$

Differentiating, we get

$$\frac{W'(x)}{W(x)} = \frac{1}{x} - W'(x)$$

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or

$$W'(x) = \frac{W(x)}{x(1 + W(x))}.$$

Then

$$\frac{1}{x} - W'(x) = \frac{1}{x(1 + W(x))}$$

$$\frac{1}{(1 + W(x))} = 1 - x\,W'(x)$$

Substitute $x = -x$

$$\frac{1}{(1 + W(-x))} = 1 + x\,W'(-x)$$

$$W_0(-x) = \sum_{n>0} \frac{n^{n-1}}{n!} x^n$$

$$\frac{1}{(1 + W(-x))} = \sum_{n \geqslant 0} \frac{n^n}{n!} x^n$$

$$\frac{1-x}{(1 + W(-x))} = 1 + \sum_{n>0} \left(\frac{n^n}{n!} - \frac{(n-1)^{n-1}}{(n-1)!} \right) x^n = 1 + \sum_{n>0} \left(n^{n-1} - (n-1)^{n-1} \right) \frac{x^n}{(n-1)!}$$

On the other side for $x = 0$ the dual series will be the power series with one variable t with $n = 0$. The coefficients of the power series are defined by

$$\sum_{i=0}^{m} \sum_{k=0}^{m-i} \frac{1}{k!} \left\langle\!\!\left\langle \begin{matrix} k \\ i \end{matrix} \right\rangle\!\!\right\rangle \binom{k}{0} \binom{m-i+k-1}{m-i-k}$$

Then we have the following identity

$$\sum_{i=0}^{n} \sum_{k=0}^{n-i} \frac{1}{k!} \left\langle\!\!\left\langle \begin{matrix} k \\ i \end{matrix} \right\rangle\!\!\right\rangle \binom{n-i+k-1}{n-i-k} = \frac{1}{(n-1)!} \left(n^{n-1} - (n-1)^{n-1} \right), n > 0.$$

For $x = -1$ we have the following identity

$$\frac{1-t}{W(-t\,e^{-t}) + 1} = 1.$$

Considering (9), we get the explicit expression for $r(n,m)$

$$r(n,m) = \sum_{i=0}^{m} \sum_{k=0}^{n+m-i} \frac{1}{k!} \left\langle\!\!\left\langle \begin{matrix} k \\ i \end{matrix} \right\rangle\!\!\right\rangle \binom{k}{n} \binom{m-i+k-1}{m-i-k}.$$

Note that $k \geqslant 0$, otherwise $\binom{k}{n} = 0$. Then

$$r(n,m) = \sum_{i=0}^{m} \sum_{k=n}^{n+m-i} \frac{1}{k!} \left\langle\!\!\left\langle \begin{matrix} k \\ i \end{matrix} \right\rangle\!\!\right\rangle \binom{k}{n} \binom{m-i+k-1}{m-i-k}$$

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$$r(n,m) = \sum_{i=0}^{m} \sum_{k=0}^{m-i} \frac{1}{k!} \left\langle\!\!\left\langle {k \atop i} \right\rangle\!\!\right\rangle \binom{k+n}{n} \binom{m-i+k+n-1}{m-i-k-n}$$

Note that $r(n,m)$ is numeric triangle with $m \geqslant n$, because for $m < n$ there hold

$$\binom{m-i+k+n-1}{m-i-k-n} = 0,$$

Consider (9) for $x = -1$:

$$\sum_{m \geqslant 0} \sum_{n \geqslant 0} r(n,m)(-1)^n \, t^m$$

Since $n < m$

$$\sum_{m \geqslant 0} \sum_{n=0}^{m} r(n,m)(-1)^n \, t^m = 1$$

$$\sum_{n=0}^{m} r(n,m)(-1)^n = 0.$$

we obtain

$$\sum_{m=0}^{n} (-1)^m \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{m+k}{m} \left\langle\!\!\left\langle {m+k \atop n-i} \right\rangle\!\!\right\rangle \binom{m+k+i-1}{2\,m+2\,k-1} = 0$$

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