Atoms for signed permutations

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Abstract

There is a natural analogue of weak Bruhat order on the involutions in any Coxeter group, which was first considered by Richardson and Springer in the context of symmetric varieties. The saturated chains in this order from the identity to a given involution are in bijection with the reduced words for a certain set of group elements which we call atoms. We study the combinatorics of atoms for involutions in the group of signed permutations. This builds on prior work concerning atoms for involutions in the symmetric group, which was motivated by connections to the geometry of certain spherical varieties. We prove that the set of atoms for any signed involution naturally has the structure of a graded poset whose maximal elements are counted by Catalan numbers. We also characterize the signed involutions with exactly one atom and prove some enumerative results about reduced words for signed permutations.

1 Introduction

1.1 Atoms

Let (W,S) be a Coxeter system with length function ℓ. There is a unique associative product $\circ : W \times W \to W$ such that $s \circ s = s$ for $s \in S$ and $u \circ v = uv$ for $u, v \in W$ with $\ell(u) + \ell(v) = \ell(uv)$ [25, Theorem 7.1]. This is sometimes called the *Demazure product*, while $(W, \circ)$ is the 0-*Hecke monoid* of (W,S). Write $\mathcal{I}(W) = \{w \in W : w^{-1} = w\}$ for the set of involutions in W. The operation $w \mapsto w^{-1} \circ w$ is a surjection $W \to \mathcal{I}(W)$, and we let

$$\mathcal{A}_{\texttt{hecke}}(z) = \left\{ w \in W : w^{-1} \circ w = z \right\}$$

denote the (nonempty) preimage of $z \in \mathcal{I}(W)$ under this map. In turn, define $\mathcal{A}(z)$ as the subset of minimal length elements in $\mathcal{A}_{\texttt{hecke}}(z)$. These sets are the main objects of interest in this paper. Following [14], we refer to the elements of $\mathcal{A}_{\texttt{hecke}}(z)$ as the *Hecke atoms* of $z \in \mathcal{I}(W)$, and we call the elements of $\mathcal{A}(z)$ the *atoms* of z.

For involutions in symmetric groups, the sets $\mathcal{A}_{\texttt{hecke}}(z)$ and $\mathcal{A}(z)$ have some remarkable combinatorial properties which are explored in the recent papers [6, 7, 14, 21, 30]. For example, the inverse Hecke atoms $\mathcal{A}_{\texttt{hecke}}(z)^{-1} = \{w^{-1} : w \in \mathcal{A}_{\texttt{hecke}}(z)\}$ are precisely the equivalence classes in the symmetric group under the so-called *Chinese relation* studied in [8, 9]. The number of atoms for the reverse permutation $n \cdots 321$ in given by the double factorial $(n-1)!!$ [6]. In finite and affine symmetric groups, there is a natural partial order which makes $\mathcal{A}(z)$ into a bounded, graded poset (and conjecturally, a lattice) [14, 30]. The goal of this work to extend some of these properties to the atoms of signed permutations, that is, involutions in the finite Weyl group of type B.

1.2 Motivation

Atoms are central to certain enumerative problems related to reduced words. An expression $w = s_1 s_2 \cdots s_l$ is a *reduced word* for $w \in W$ if each $s_i \in S$ and the number of factors l is as small as possible. Let $\mathcal{R}(w)$ be the set of reduced words for $w \in W$ and define $\hat{\mathcal{R}}(z) = \bigsqcup_{w \in \mathcal{A}(z)} \mathcal{R}(w)$ for $z \in \mathcal{I}(W)$. When W is finite and $w_0 \in W$ is the unique longest element, there are striking formulas for $|\mathcal{R}(w_0)|$ and $|\hat{\mathcal{R}}(w_0)|$ in terms of tableaux.

The first result of this type dates to Stanley's paper [37]. Fix a positive integer n and write S_n for the symmetric group of permutations of $[n] = \{1, 2, \ldots, n\}$, viewed as the Coxeter group of type A_{n-1} relative to the simple generators s_1, s_2, $\ldots$, s_{n-1} where $s_i = (i, i+1)$. Let $w_0^{A_n} = (n+1)n \cdots 321$ denote the longest permutation in S_{n+1}. Given a partition λ of n, let f^λ denote the number of standard Young tableaux of shape λ, or equivalently the dimension of the corresponding irreducible representation of S_n.

Theorem 1.1 (Stanley [37]). It holds that $\left|\mathcal{R}\left(w_0^{A_n}\right)\right| = f^{(n,n-1,\ldots,2,1)} = \frac{\binom{n+1}{2}!}{\prod_{i=1}^n (2i-1)^i}$.

The longest element of a Coxeter group is always an involution, and there is an analogue of the previous theorem for $\hat{\mathcal{R}}\left(w_0^{A_n}\right)$. The following shows that the size of this set is the number of *standard marked shifted tableaux* (see [39, §6]) of shape $(n, n-2, n-4, \ldots)$.

Theorem 1.2 (Hamaker, Marberg, and Pawlowski [13]). Let $p = \lfloor \frac{n}{2} \rfloor$ and $q = \lceil \frac{n}{2} \rceil$. Then

$$\left|\hat{\mathcal{R}}\left(w_0^{A_n}\right)\right| = \binom{\binom{p}{2} + \binom{q}{2}}{\binom{p}{2}} f^{(p-1,p-2,\ldots,2,1)} f^{(q-1,q-2,\ldots,2,1)}.$$

Let $[\pm n] = [n] \sqcup -[n]$ and write W_n for the hyperoctahedral group of bijections $w : [\pm n] \to [\pm n]$ with $w(-i) = -w(i)$ for each $i \in [n]$. We refer to the elements of W_n as signed permutations. Define $t_0 = (-1, 1) \in W_n$ and $t_i = (-i, -i-1)(i, i+1) \in W_n$ for $i \in [n-1]$. With respect to these generators, W_n is a Coxeter group of type B (and type C). The longest element $w_0^{B_n} \in W_n$ is the signed permutation mapping $i \mapsto -i$ for all $i \in [\pm n]$. Our primary motivation to study the sets $\mathcal{A}(w)$ for $w \in \mathcal{I}(W_n)$ is the following theorem, which was conjectured in our first joint paper with Pawlowski [13]. This is proved in [31] using the results from the present work:

Theorem 1.3 (Marberg and Pawlowski [31]). If $n \geq 2$ then $\left|\hat{\mathcal{R}}\left(w_0^{B_n}\right)\right| = \left|\mathcal{R}\left(w_0^{A_n}\right)\right|$.

In type A, the study of atoms is also motivated by the geometry of symmetric varieties [33, 34]. Suppose G is a reductive algebraic group with Borel subgroup B. Assume G is defined over a field of characteristic not equal to two and let K be the fixed point subgroup of an automorphism of G of order two. The (not necessarily connected) reductive group K acts with finitely many orbits on G/B, as does the opposite Borel subgroup, whose orbit closures are the Schubert varieties X_w indexed by the elements of the Weyl group W.

When $G = \mathrm{GL}_n$, the subgroup K can be O_n, Sp_n (when n is even), or $\mathrm{GL}_p \times \mathrm{GL}_q$ (when $p + q = n$). Here, the K-orbit closures Y_z in G/B are respectively indexed by either the involutions in the symmetric group, the fixed-point-free involutions in the symmetric group, or certain objects called *clans* (essentially, involutions with signed fixed points) [7, 41]. Atoms arise in a formula of Brion [5] relating the cohomology classes of Y_z and X_w. In the O_n-case, for example, we have

$$[Y_z] = 2^{c(z)} \sum_{w \in \mathcal{A}(z)} [X_w] \tag{1.1}$$

where $c(z)$ is the number of 2-cycles in z [5]. Similar results hold for the Sp_n- and $\mathrm{GL}_p \times \mathrm{GL}_q$-cases. The combinatorics of Hecke atoms likewise informs the K-theory of symmetric varieties, though the connections are less direct; see [18, 42].

The hyperoctahedral group W_n is the Weyl group for both $G = \mathrm{SO}_{2n+1}$ (type B) and $G = \mathrm{Sp}_{2n}$ (type C). In these types, the possibilities for (G, K) are $(\mathrm{SO}_{2n+1}, \mathrm{SO}_p \times \mathrm{SO}_q)$ when $p+q = 2n+1$, $(\mathrm{Sp}_{2n}, \mathrm{Sp}_{2p} \times \mathrm{Sp}_{2q})$ when $p+q = n$, and $(\mathrm{Sp}_{2n}, \mathrm{GL}_n)$. For each of these cases, the K-orbits in G/B are indexed by clan-like objects [32, 41] which may be interpreted as involutions in W_n with some auxiliary data. In type A, the combinatorial properties of clans and involutions are quite similar, and we expect that our present study will serve as a useful first step towards better understanding the combinatorics of K-orbit closures in types B and C.

1.3 Outline of results

Throughout, we use the term *word* to refer a finite sequence of integers. The *one-line representation* of a permutation w in S_n or W_n is the word $w_1 w_2 \cdots w_n$ where $w_i = w(i)$. We sometimes write $\overline{m}$ in place of $-m$ so that, for example, the 8 elements of W_2 are 12, $\overline{1}2$, $1\overline{2}$, $\overline{1}\,\overline{2}$, 21, $\overline{2}1$, $2\overline{1}$, and $\overline{2}\,\overline{1}$. Define $\lessdot_A$ as the relation on n-letter words with

$$\cdots cab \cdots \lessdot_A \cdots bca \cdots \tag{1.2}$$

whenever $a < b < c$ and the corresponding ellipses mask identical subsequences. We apply $\lessdot_A$ to permutations via their one-line representations.

A poset is *graded* if all maximal chains have the same length, and *bounded* if it has a unique minimum and a unique maximum. The transitive closure $<_A$ of $\lessdot_A$ is a partial order, which we call the *atomic order* of type A on account of the following:

Theorem 1.4 (See [14]). If $z \in \mathcal{I}(S_n)$ then $\mathcal{A}(z)^{-1}$ is a bounded, graded poset under $<_A$.

Here, we write $\mathcal{A}(z)^{-1}$ for the sets $\{w^{-1} : w \in \mathcal{A}(z)\}$, which are denoted $\mathcal{W}(z)$ in [6, 7]. Our preference for stating results in terms of $\mathcal{A}(z)$, despite the frequent need to invert this set, comes from formulas like (1.1).

As will be reviewed in Section 3, we can give an explicit construction for the minimal and maximal elements in the posets $(\mathcal{A}(z)^{-1}, <_A)$ for $z \in \mathcal{I}(S_n)$, and there is a simple algorithm to recover z from the one-line representation of $w \in \mathcal{A}(z)^{-1}$. Such properties play a key technical role in [15, 16, 17] and are what we aim to generalize to type B.

Over the course of this paper, we will discuss three partial orders which serve as natural type B analogues of $<_A$. For the purposes of this introduction, it is only necessary to define the weakest of the three. Write $\lessdot_B$ for the relation on n-letter words with

$$\overline{b}\overline{a} \cdots \lessdot_B a\overline{b} \cdots \qquad \text{and} \qquad \overline{c}\overline{b}\overline{a} \cdots \lessdot_B \overline{c}a\overline{b} \cdots \tag{1.3}$$

whenever $0 < a < b < c$ and the corresponding ellipses mask identical subwords. Unlike $\lessdot_A$, this relation only changes the letters at the start of a word. The *(weak) atomic order* $<_B$ of type B is the transitive closure of *both* $\lessdot_A$ and $\lessdot_B$. We apply these relations on words to elements of W_n via their one-line representations. Let $\mathrm{Neg}(z) = \{i \in [n] : z(i) = -i\}$ be the set of *negated points* of $z \in \mathcal{I}(W_n)$. We can summarize several of our main results with the following theorem, which combines Corollaries 3.7, 4.8, 5.9, and 6.3 and Theorems 5.5 and 5.6.

Theorem 1.5. Let $z \in \mathcal{I}(W_n)$ be an involution in the hyperoctahedral group.

(a) The set $\mathcal{A}(z)^{-1}$ is preserved by $\lhd_A$ and $\lhd_B$ and is a graded poset relative to $<_A$ and $<_B$.

(b) Let $m = |\operatorname{Neg}(z)|$. The poset $(\mathcal{A}(z)^{-1}, <_A)$ has $\binom{m}{\lfloor m/2 \rfloor}$ connected components, which are naturally in bijection with the matchings on $\operatorname{Neg}(z) \sqcup -\operatorname{Neg}(z) = \{i \in [\pm n] : z(i) = -i\}$ that are perfect, noncrossing, and symmetric with respect to negation.

(c) Moreover, each component in $(\mathcal{A}(z)^{-1}, <_A)$ is isomorphic to $(\mathcal{A}(\zeta)^{-1}, <_A)$ for some $\zeta \in \mathcal{I}(S_n)$.

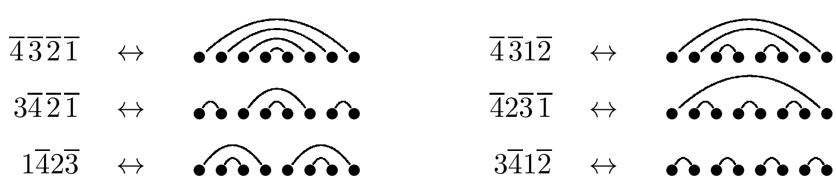
(d) Let $k = \lceil m/2 \rceil$. The poset $(\mathcal{A}(z)^{-1}, <_B)$ is connected and has $\frac{1}{k+1}\binom{2k}{k}$ maximal elements.

It is interesting to note the appearance of the Catalan numbers in part (d) and the "type B" Catalan numbers in part (b). Our proof of the theorem will establish more than what is stated here. In particular, we will completely describe the minimal and maximal elements in $(\mathcal{A}(z)^{-1}, <_A)$ for $z \in \mathcal{I}(W_n)$, and give an explicit construction of the bijection mentioned in part (b). The assertion that $(\mathcal{A}(z)^{-1}, <_B)$ is connected for all $z \in \mathcal{I}(W_n)$ is equivalent to a recent result of Hu and Zhang [22, Theorem 4.8]. Our methods do not rely on [22] and therefore provide a conceptually simpler alternate approach to the proof of Hu and Zhang's theorem.

Example 1.6. The Hasse diagram of $(\mathcal{A}(z)^{-1}, <_B)$ for $z = \overline{1}\,\overline{2}\,\overline{3}\,\overline{4} \in \mathcal{I}(W_4)$ is

$$\begin{tikzcd}
 & 1\overline{2}3\overline{4} & \\
13\overline{4}\,\overline{2} \arrow[ur] & \overline{2}\,\overline{1}3\overline{4} \arrow[u, dashed] & 1\overline{4}2\overline{3} \\
3\overline{4}1\overline{2} \arrow[u] & \overline{2}3\overline{4}\,\overline{1} \arrow[u] & \overline{4}\,\overline{1}2\overline{3} \arrow[u, dashed] \\
\overline{4}\,\overline{3}1\overline{2} \arrow[u, dashed] & 3\overline{4}\,\overline{2}\,\overline{1} \arrow[u] & \overline{4}2\overline{3}\,\overline{1} \arrow[u] \\
 & \overline{4}\,\overline{3}\,\overline{2}\,\overline{1} \arrow[u, dashed] \arrow[ur, dashed] &
\end{tikzcd}$$

The solid arrows indicate the relations $\lhd_A$ while the dashed arrows indicate $\lhd_B$. The six minimal elements relative to $<_A$ are in bijection with the six perfect noncrossing symmetric matchings on the set $[\pm 4]$ via the following correspondence, which is described in general in Section 5:



The partial orders $<_A$ and $<_B$ extend to relations whose equivalence classes are the sets of Hecke atoms for involutions in S_n and W_n. To be precise, let $\approx_A$ be the weakest equivalence relation on words with $\cdots bca \cdots \approx_A \cdots cab \cdots \approx_A \cdots cba \cdots$ whenever $a < b < c$, where the corresponding ellipses indicate identical entries. This relation was studied independently in [8, 9].

Theorem 1.7 (See [14])**.** The $\approx_A$-equivalence classes in S_n are the sets $\mathcal{A}_{\texttt{hecke}}(z)^{-1}$ for $z \in \mathcal{I}(S_n)$.

A similar result holds for the affine symmetric group [29, Proposition 1.10]. To adapt these statements to type B, we define $\approx_B$ as the weakest equivalence relation on words that extends $\approx_A$ and has $\overline{b}\overline{a}\cdots \approx_B a\overline{b}\cdots \approx_B \overline{a}\overline{b}\cdots$ and $\overline{c}\overline{b}\overline{a}\cdots \approx_B \overline{c}a\overline{b}\cdots \approx_B \overline{c}\overline{a}\overline{b}\cdots$ whenever $0 < a < b < c$, with ellipses again indicating identical entries. In the latter equivalences, only the initial letters of each word vary. We prove the following in Section 7:

Theorem 1.8. The $\approx_B$-equivalence classes in W_n are the sets $\mathcal{A}_{\texttt{hecke}}(z)^{-1}$ for $z \in \mathcal{I}(W_n)$.

Here is a brief outline of the paper. After some preliminaries in Section 2, we spend Sections 3, 4, 5, and 6 proving Theorem 1.5. Our discussion of Hecke atoms is mostly limited to Section 7. In Section 8, as an application of Theorem 1.5, we classify and enumerate the *atomic involutions* in W_n, that is, the elements $z \in \mathcal{I}(W_n)$ with $|\mathcal{A}(z)| = 1$. Section 9 discusses some open questions and conjectures of interest. An index of symbols appears in Appendix A.

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2 Preliminaries

We write $\mathbb{Z}$ for the integers, $\mathbb{N}$ for the nonnegative integers, and set $[n] = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$. Let (W, S) be a Coxeter system with length function $\ell : W \to \mathbb{N}$ and Demazure product $\circ$, as described at the start of the introduction. We begin with a few remarks about how to compute with $\circ$. If $w \in W$ and $s \in S$ then $w \circ s$ is either w or ws, while $s \circ w$ is either w or sw. If $w \in W$ and $w = s_1 s_2 \cdots s_l$ is a reduced word then $w = s_1 \circ s_2 \circ \cdots \circ s_l$. It follows from the exchange principle that if $s \in S$ and $z \in \mathcal{I}(W) = \{w \in W : w^2 = 1\}$ have $\ell(szs) = \ell(z)$ then $szs = z$ [24, Lemma 3.4]. Conjugation using $\circ$ therefore take the following form:

Lemma 2.1. If $s \in S$ and $z \in \mathcal{I}(W)$ then $s \circ z \circ s = \begin{cases} szs & \text{if } zs \neq sz \text{ and } \ell(zs) > \ell(z) \\ zs & \text{if } zs = sz \text{ and } \ell(zs) > \ell(z) \\ z & \text{if } \ell(zs) < \ell(z). \end{cases}$

It follows by an easy inductive argument that $\mathcal{A}_{\texttt{hecke}}(z) = \{w \in W : w^{-1} \circ w = z\}$ is nonempty for all $z \in \mathcal{I}(W)$. Denote the left and right descent sets of $w \in W$ by

$$\mathrm{Des}_L(w) = \{s \in S : \ell(sw) < \ell(w)\} \qquad \text{and} \qquad \mathrm{Des}_R(w) = \{s \in S : \ell(ws) < \ell(w)\}.$$

A finite Coxeter group has a unique longest element with no right or left descents. In S_n this element is the reverse permutation $n \cdots 321$, while in W_n it is the central element $\overline{1}\overline{2}\overline{3}\cdots\overline{n}$. Write $<_L$ and $<_R$ for the left and right weak orders on W. Recall that $\mathcal{A}(z)$ for $z \in \mathcal{I}(W)$ is the set of minimal length elements in $\mathcal{A}_{\texttt{hecke}}(z)$. The following property is sometimes useful:

Proposition 2.2. Suppose $z \in \mathcal{I}(W)$ and $w \in \mathcal{A}(z)$. Then $\mathrm{Des}_R(w) \subset \mathrm{Des}_R(z)$. Moreover, if $v \in W$ has $v <_R w$ then $v \in \mathcal{A}(y)$ for some $y \in \mathcal{I}(W)$.

Proof. We have $z = vw$ for some $v \in W$ with $\ell(z) = \ell(v) + \ell(w)$. $\square$

The *involution length function* $\hat{\ell} : \mathcal{I}(W) \to \mathbb{N}$ is the map which assigns to $z \in \mathcal{I}(W)$ the common value of $\ell(w)$ for $w \in \mathcal{A}(z)$. The *absolute length function* $\ell' : W \to \mathbb{N}$ is the map which assigns to $w \in W$ the minimum number of factors l needed to express $w = t_1 t_2 \cdots t_l$ as a product of *reflections* $t_i \in T = \{wsw^{-1} : (w, s) \in W \times S\}$. Hultman [23] proves that these functions are related as follows:

Proposition 2.3. If $z \in \mathcal{I}(W)$ then $\hat{\ell}(z) = \frac{1}{2}(\ell(z) + \ell'(z))$.

Remark 2.4. These sorts of general results concerning $\mathcal{I}(W)$, $\circ$, $\mathcal{A}_{\texttt{hecke}}(z)$, and $\mathcal{A}(z)$ have been noted several times in the literature, with widely varying terminology and notation. Important references include the papers of Richardson and Springer [33, 34] and Hultman [23, 24].

We show how to interpret these constructions for permutations and signed permutations. Continue to let $s_i = (i, i+1) \in S_n$ for $i \in [n-1]$. Relative to these simple generators, S_n is the Coxeter group of type A_{n-1}. Write $\mathrm{Inv}(w)$ for the set of *inversions* of $w \in S_n$ in $[n]$, that is, pairs $(i, j) \in [n] \times [n]$ with $i < j$ and $w(i) > w(j)$, and define $\mathrm{inv}(w) = |\mathrm{Inv}(w)|$. When $w \in S_n$, it is well-known that $\ell(w) = \mathrm{inv}(w)$ and that $s_i \in \mathrm{Des}_R(w)$ if and only if $w(i) > w(i+1)$.

The reflections in S_n are the transpositions (i, j) for $i \neq j$ in $[n]$. A *cycle* of $w \in S_n$ is an orbit in $[n]$ under the action of the cyclic group $\langle w \rangle$. If $w \in S_n$ has k cycles in $[n]$ then $\ell'(w) = n - k$, and if $z \in \mathcal{I}(S_n)$ then $\ell'(z)$ is the number of *2-cycles* of z, that is, cycles of size two.

If $w \in S_n$ and $i \in [n-1]$ then the Demazure product for S_n satisfies $w \circ s_i = ws_i$ if and only if $w(i) < w(i+1)$. The operation $z \mapsto s_i \circ z \circ s_i$ for an involution $z \in \mathcal{I}(S_n)$ has the following concrete interpretation. The cycles of z all have size at most two, so we can visualize z as a matching in $[n]$. If i and $i+1$ are isolated points in this matching, then $s_i \circ z \circ s_i$ is formed by adding the new edge $\{i, i+1\}$. Otherwise, if $z(i) < z(i+1)$, then we form $s_i \circ z \circ s_i$ by interchanging the positions of vertices i and $i+1$ and then reversing their labels.

Example 2.5. The atoms of $321 = s_2 \circ s_1 \circ s_1 \circ s_2 = s_1 \circ s_2 \circ s_2 \circ s_1 \in \mathcal{I}(S_3)$ are $231 = s_1 s_2$ and $312 = s_2 s_1$, while the Hecke atoms are these elements plus $321 = s_1 s_2 s_1 = s_2 s_1 s_2$.

As in the introduction, let $t_0 = (-1, 1) \in W_n$ and $t_i = (-i, -i-1)(i, i+1) \in W_n$ for $i \in [n-1]$. With respect to these generators, W_n is a Coxeter group of type B. If $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in W_n$ then $\sigma t_0 = \overline{\sigma_1} \sigma_2 \cdots \sigma_n$ and $\sigma t_i = \sigma_1 \cdots \sigma_{i+1} \sigma_i \cdots \sigma_n$ for $i \in [n-1]$. Let $\ell_0(\sigma) = |\{i \in [n] : \sigma(i) < 0\}|$. One can show that t_0 appears exactly $\ell_0(\sigma)$ times in any reduced word for $\sigma \in W_n$.

It is well-known that if $\sigma \in W_n$ then $\ell(\sigma) = \frac{1}{2}(\mathrm{inv}(\sigma) + \ell_0(\sigma))$, where $\mathrm{inv}(\sigma)$ denotes the number of inversions of σ in the set $[\pm n]$. It follows that if we let $\sigma_0 = 0$ and $i \in \{0\} \sqcup [n]$, then $t_i \in \mathrm{Des}_R(\sigma)$ if and only if $\sigma_i > \sigma_{i+1}$.

The reflections in W_n are the elements $s_{ii} = (i, -i)$, $s_{ij} = (i, -j)(j, -i)$, and $t_{ij} = (i, j)(-i, -j)$ for $i, j \in [n]$ with $i \neq j$. Let $\ell'_0(\sigma)$ denote the number of cycles of $\sigma \in W_n$ in $[\pm n] = [n] \sqcup -[n]$ which are preserved by the negation map. One can show that $\ell'(\sigma) = n - \frac{1}{2}(k - \ell'_0(\sigma))$ where k is the number of cycles of σ in $[\pm n]$. Let $z \in \mathcal{I}(W_n)$. Define $\mathrm{Pair}(z) = \{(a, b) \in [\pm n] \times [n] : |a| < z(a) = b\}$ and $\mathrm{pair}(z) = |\mathrm{Pair}(z)|$, and let $\mathrm{neg}(z) = |\mathrm{Neg}(z)|$ where $\mathrm{Neg}(z) = \{i \in [n] : z(i) = -i\}$. Then $\ell'(z) = \mathrm{neg}(z) + \mathrm{pair}(z)$.

Let $z \in \mathcal{I}(W_n)$ and consider the symmetric matching on $[\pm n]$ whose edges are the cycles of z. The operation $z \mapsto t_i \circ z \circ t_i$ may be described in terms of this matching as follows. If -1 and 1 are isolated points, then $t_0 \circ z \circ t_0$ is formed by adding the edge $\{-1, 1\}$, while if $z(-1) < z(1)$ then $t_0 \circ z \circ t_0$ is formed by interchanging vertices -1 and 1. Assume $i \in [n-1]$ and $z(i) < z(i+1)$. There are then three possibilities. We obtain $t_i \circ z \circ t_i$ by adding the edges $\{i, i+1\}$ and $\{-i, -i-1\}$ when i and $i+1$ are isolated points, by interchanging the vertices i and $i+1$ when $z(i+1) = -i$, or else by interchanging vertices i and $i+1$ and then also $-i$ and $-i-1$.

Example 2.6. The permutations $\overline{2}\,\overline{1} = t_0t_1t_0$ and $1\overline{2} = t_1t_0t_1$ belong to $\mathcal{A}(\overline{1}\,\overline{2})$, while $\overline{3}\,\overline{2}\,\overline{1} = t_0t_1t_2t_0t_1t_0$ and $2\overline{3}\,\overline{1} = t_0t_1t_2t_1t_0t_1$ belong to $\mathcal{A}(\overline{1}\,\overline{2}\,\overline{3})$, and $\overline{2}\,\overline{3}\,\overline{1} = t_0t_1t_2t_0t_1t_0t_1 \in \mathcal{A}_{\texttt{hecke}}(\overline{1}\,\overline{2}\,\overline{3})$.

The Hecke atoms of $w_0 = \overline{1}\,\overline{2}\,\overline{3}\cdots\overline{n} \in W_n$ have a nontrivial symmetry which is not apparent from our new results but worth noting. The following is a special case of [14, Corollary 4.10].

Proposition 2.7 (See [14])**.** It holds that $\mathcal{A}_{\texttt{hecke}}(w_0) = \mathcal{A}_{\texttt{hecke}}(w_0)^{-1}$ and $\mathcal{A}(w_0) = \mathcal{A}(w_0)^{-1}$.

There is a useful embedding of W_n in S_{2n}. Define $\Psi_n : W_n \to S_{2n}$ by

$$\Psi_n(\sigma) = \psi \circ \sigma \circ \psi^{-1} \qquad \text{for } \sigma \in W_n, \tag{2.1}$$

where ψ is the order-preserving bijection $[\pm n] \to [2n]$. The map Ψ_n is evidently an injective group homomorphism. Moreover, Ψ_n is the unique monoid homomorphism $(W_n, \circ) \to (S_{2n}, \circ)$ under which $t_0 \mapsto s_n$ and $t_i \mapsto s_{n+i}s_{n-i}$ for $i \in [n-1]$. As a consequence, we have

$$\ell\left(\Psi_n(\sigma)\right) = 2\ell(\sigma) - \ell_0(\sigma) \qquad \text{and} \qquad \ell'\left(\Psi_n(\sigma)\right) = 2\ell'(\sigma) - \ell'_0(\sigma) \qquad \text{for } \sigma \in W_n. \tag{2.2}$$

As a group homomorphism, Ψ_n restricts to a map $\mathcal{I}(W_n) \to \mathcal{I}(S_{2n})$, so we have

$$\hat{\ell}\left(\Psi_n(z)\right) = 2\hat{\ell}(z) - \tfrac{1}{2}\left(\ell_0(z) + \operatorname{neg}(z)\right) \qquad \text{for } z \in \mathcal{I}(W_n). \tag{2.3}$$

The following criteria will be of use later.

Lemma 2.8. Suppose $\sigma \in W_n$ and $z \in \mathcal{I}(W_n)$. Then $\sigma \in \mathcal{A}(z)$ if and only if it holds that $\Psi_n(\sigma) \in \mathcal{A}_{\texttt{hecke}}\left(\Psi_n(z)\right)$ and $\ell\left(\Psi_n(\sigma)\right) - \ell\left(\Psi_n(z)\right) = \frac{1}{2}\left(\ell_0(z) + \operatorname{neg}(z)\right) - \ell_0(\sigma)$.

Proof. Since Ψ_n is injective, $\sigma^{-1} \circ \sigma = z$ if and only if $\Psi_n(\sigma)^{-1} \circ \Psi_n(\sigma) = \Psi_n(z)$. By Equations 2.2 and 2.3, $\ell(\sigma) = \hat{\ell}(z)$ if and only if the given length condition holds. $\square$

Remark 2.9. There is an injective homomorphism $S_n \hookrightarrow W_n$ with $s_i \mapsto t_i$ for $i \in [n-1]$, which also defines a homomorphism of monoids $(S_n, \circ) \to (W_n, \circ)$. Via this embedding, everything we prove about atoms for involutions in W_n can be viewed as generalizations of results in [13, 14, 15] about atoms for elements of S_n. We rarely need to reference this map directly, however.

3 Nested descents

A *word* is a finite sequence of integers. A *(one-line) descent* of a word $w = w_1w_2\cdots w_n$ is a pair (w_i, w_{i+1}) with $w_i > w_{i+1}$. Let $\operatorname{Des}(w)$ be the set of descents of w. A *subword* of w is any (not necessarily consecutive) subsequence. Define $\operatorname{sort}_L(w)$ (respectively, $\operatorname{sort}_R(w)$) as the subword of w formed by omitting w_{i+1} (respectively, w_i) whenever $(w_i, w_{i+1}) \in \operatorname{Des}(w)$. Adapt these definitions to elements of S_n or W_n by identifying permutations with their one-line representations. For example, if $w = 2134765 \in S_7$ then $\operatorname{Des}(w) = \{(2,1), (7,6), (6,5)\}$, so $\operatorname{sort}_L(w) = 2347$ and $\operatorname{sort}_R(w) = 1345$. On the other hand, if $\sigma = \overline{2}1\overline{3}4\overline{7}6\overline{5} \in W_7$ then $\operatorname{Des}(\sigma) = \{(1,-3), (4,-7), (6,-5)\}$, so $\operatorname{sort}_L(\sigma) = \overline{2}146$ and $\operatorname{sort}_R(\sigma) = \overline{2}\,\overline{3}\,\overline{7}\,\overline{5}$.

Suppose X is a set of n integers $x_1 < x_2 < \cdots < x_n$. Let S_X denote the group of permutations of X, viewed as a Coxeter group relative to the generators (x_i, x_{i+1}) for $i \in [n-1]$. The *one-line representation* of $\sigma \in S_X$ is the word $\sigma(x_1)\sigma(x_2)\ldots\sigma(x_n)$. As a Coxeter group, S_X has a

Demazure product $\circ$, which gives us a set of atoms $\mathcal{A}(z)$ for each involution $z \in \mathcal{I}(S_X)$. For example, if $X = \{1, 3, 5\}$ then $531 \in \mathcal{I}(S_X)$ are $\mathcal{A}(531) = \{513, 351\}$.

If $w = w_1 w_2 \cdots w_n$ is a word, then we write $[[w]]$ for the subword formed by omitting each repeated letter after its first appearance, going left to right. For $z \in \mathcal{I}(S_X)$, define

$$\mathrm{Cyc}_A(z) = \{(a, b) \in X \times X : a \le b = z(a)\}.$$

Suppose we have $\mathrm{Cyc}_A(z) = \{(a_1, b_1), (a_2, b_2), \ldots, (a_l, b_l)\} = \{(c_1, d_1), (c_2, d_2), \ldots, (c_l, d_l)\}$ where $a_1 < a_2 < \cdots < a_l$ and $d_1 < d_2 < \cdots < d_l$. We define the permutations $0_A(z), 1_A(z) \in S_X$ by

$$0_A(z) = [[b_1 a_1 b_2 a_2 \cdots b_l a_l]] \qquad \text{and} \qquad 1_A(z) = [[d_1 c_1 d_2 c_2 \cdots d_l c_l]]. \tag{3.1}$$

Alternatively, $0_A(z)$ and $1_A(z)$ are the unique elements of S_X for which $\mathrm{sort}_R(0_A(z))$ and $\mathrm{sort}_L(1_A(z))$ are increasing and $\mathrm{Des}(0_A(z)) = \mathrm{Des}(1_A(z)) = \{(b, a) \in X \times X : a < b = z(a)\}$. Thus if $z = (1, 2)(4, 7)(5, 6) \in \mathcal{I}(S_7)$ then $0_A(z) = 2137465$ and $1_A(z) = 2136574$, while if $z = (1, 2)(4, 7)(5, 6) \in \mathcal{I}\left(S_{\{1,2,4,5,6,7\}}\right)$ then $0_A(z) = 217465$ and $1_A(z) = 216574$.

The symbol $\lessdot_A$ defined by (1.2) is the relation on n-letter words with $v \lessdot_A w$ if for some $i \in [n-2]$ and some numbers $a < b < c$, it holds that $v_i v_{i+1} v_{i+2} = cab$ and $w_i w_{i+1} w_{i+2} = bca$ while $v_j = w_j$ for $j \notin \{i, i+1, i+2\}$. Again let $<_A$ be the transitive closure of $\lessdot_A$, and write $\sim_A$ for the symmetric closure of $<_A$. Theorem 1.4 is a corollary of the following more explicit statement.

Theorem 3.1 (See [14])**.** Suppose $X \subset \mathbb{Z}$ is a finite set and $z \in \mathcal{I}(S_X)$. Then

$$\mathcal{A}(z)^{-1} = \{w \in S_X : 0_A(z) \le_A w\} = \{w \in S_X : w \le_A 1_A(z)\}.$$

Proof. It suffices to assume $X = [n]$; the result is then [14, Theorem 6.10 and Proposition 6.14]. $\square$

Remark 3.2. The theorem gives an efficient algorithm for generating the atoms of any $z \in \mathcal{I}(S_n)$: simply read off the permutations $0_A(z)$ and $1_A(z)$ from the cycle structure of z, then find all elements spanned from these by the covering relation $\lessdot_A$ and take inverses.

We can apply $\lessdot_A$, $<_A$, and $\sim_A$ to signed permutations in one-line notation. These relations preserve (but no longer span) all of the sets $\mathcal{A}(z)^{-1}$ for $z \in \mathcal{I}(W_n)$ in the following sense.

Lemma 3.3. If $w \in W_n$, $z \in \mathcal{I}(W_n)$, $v \in \mathcal{A}(z)^{-1}$, and $v \sim_A w$, then $w \in \mathcal{A}(z)^{-1}$.

Proof. If $v, w \in W_n$ then $v \lessdot_A w$ iff $v = u t_{i+1} t_i$ and $w = u t_i t_{i+1}$ for $(i, u) \in [n-1] \times W$ with $\ell(v) = \ell(w) = \ell(u) + 2$. The lemma follows since $(t_{i+1} t_i)^{-1} \circ (t_{i+1} t_i) = (t_i t_{i+1})^{-1} \circ (t_i t_{i+1}) = t_i t_{i+1} t_i$. $\square$

We will say that a word $w_1 w_2 \cdots w_n$ has a *consecutive 321-pattern* if for some $i \in [n-2]$ it holds that $w_i w_{i+1} w_{i+2} = cba$ where $a < b < c$. Define *consecutive* 312- and 231-*patterns* similarly. A permutation in S_n or W_n has a consecutive 321-pattern if its one-line representation does.

Proposition 3.4. If $w \in \mathcal{A}(z)^{-1}$ for z in $\mathcal{I}(S_n)$ or $\mathcal{I}(W_n)$, then w has no consecutive 321-patterns.

Proof. If $w \in W_n$ has $w(i) > w(i+1) > w(i+2)$, then we can write $w = v t_i t_{i+1} t_i = v t_{i+1} t_i t_{i+1}$ where $v \in W_n$ has $\ell(w) = \ell(v) + 3$, in which case $\ell(w t_i) < \ell(w)$ and $w^{-1} \circ w = (w t_i)^{-1} \circ (w t_i)$, so w^{-1} is not an atom of any involution. The same conclusion holds when $w \in S_n$ by Remark 2.9. $\square$

Lemma 3.5. Assume a word w has no consecutive 321-patterns. Then w is minimal (respectively, maximal) relative to $<_A$ if and only if $\mathrm{sort}_R(w)$ (respectively, $\mathrm{sort}_L(w)$) is increasing.

Proof. The word $\mathrm{sort}_R(w)$ (respectively, $\mathrm{sort}_L(w)$) fails to be increasing precisely when w has a consecutive 321- or 312-pattern (respectively, 321- or 231-pattern). ☐

Corollary 3.6. Suppose $\mathcal{E}$ is the $\sim_A$-equivalence class of a word with distinct letters which is minimal under $<_A$ and has no consecutive 321-patterns. If X is the set of letters in this word then $\mathcal{E} = \mathcal{A}(z)^{-1}$ for some $z \in \mathcal{I}(S_X)$.

Proof. Suppose $w = w_1 w_2 \cdots w_n$ is a word with distinct letters which is minimal under $<_A$ and has no consecutive 321-patterns. If $X = \{w_1, w_2, \dots, w_n\}$ then there is a unique involution $z \in \mathcal{I}(S_X)$ with $w = 0_A(z)$, and the $\sim_A$-equivalence class of w is $\mathcal{A}(z)^{-1}$ by Theorem 3.1. ☐

Suppose X is a set with $[\pm n] = X \sqcup -X$. The one-line representation of each $\sigma \in S_X$ is also the one-line representation of an element of W_n. Define $\mathcal{W}_{\mathtt{embed}}(z) \subset W_n$ for $z \in \mathcal{I}(S_X)$ as the image of $\mathcal{A}(z)^{-1}$ under this inclusion $S_X \hookrightarrow W_n$. The following is equivalent to Theorem 1.5(c):

Corollary 3.7. Suppose $z \in \mathcal{I}(W_n)$ and $\mathcal{E}$ is an equivalence class in $\mathcal{A}(z)^{-1}$ under $\sim_A$. Then $\mathcal{E} = \mathcal{W}_{\mathtt{embed}}(\zeta)$ where X is some subset with $[\pm n] = X \sqcup -X$ and ζ is some involution in S_X. Consequently, $\mathcal{E}$ has a unique minimal element and a unique maximal element under $<_A$.

Proof. Choose an element $w \in \mathcal{E}$ which is minimal under $<_A$. By Proposition 3.4, w has no consecutive 321-patterns, so the result follows by Corollary 3.6. ☐

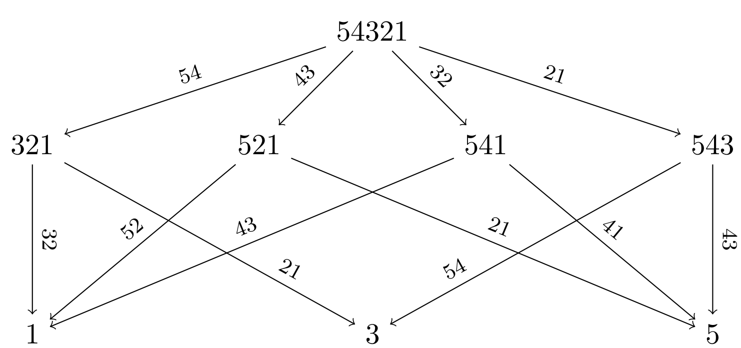
Example 3.8. The Hasse diagram of $(\mathcal{A}(z)^{-1}, <_A)$ for $z = \overline{1}\,\overline{2}\,\overline{4}\,\overline{3} \in \mathcal{I}(W_4)$ is

$$\begin{array}{cc}
\overline{2}\,\overline{1}4\overline{3} & 1\overline{2}4\overline{3} \\
\uparrow & \uparrow \\
\overline{2}4\overline{3}\,\overline{1} & 14\overline{3}\,\overline{2} \\
\uparrow & \uparrow \\
4\overline{3}\,\overline{2}\,\overline{1} & 4\overline{3}1\overline{2}
\end{array}$$

and $\mathcal{A}(z)^{-1} = \mathcal{W}_{\mathtt{embed}}(\zeta_1) \sqcup \mathcal{W}_{\mathtt{embed}}(\zeta_2)$ for $\zeta_1 = 4\overline{2}13 = (\overline{3},4)(\overline{2})(\overline{1})$ and $\zeta_2 = 41\overline{2}\,\overline{3} = (\overline{3},4)(\overline{2},1)$.

We introduce the following terminology to associate a certain directed graph to any word. Define the *children* of a word w to be the subwords formed by removing a single descent. Inductively define the *descendants* of w to consist of w along with the descendants of each of its children. Now construct the *nested descent graph* of w as the directed graph on the set of descendants of w with a directed edge $u \to v$ whenever v is a child of u. Label each edge in this graph by the unique descent (b, a) which is removed from the source to get the target. As usual, we adapt this definition to a permutation w in S_n or W_n by identifying w with its one-line representation.

Example 3.9. The nested descent graph of $w = 54321$ is shown below:



By construction, a word w is the unique global source in its nested descent graph.

Theorem-Definition 3.10. Suppose $z \in \mathcal{I}(W_n)$ and $w \in \mathcal{A}(z)^{-1}$. The nested descent graph of w then has a unique global sink. Choose a path from w to the global sink and suppose $(b_1, a_1), (b_2, a_2), \dots, (b_l, a_l)$ is the corresponding sequence of edge labels. The set

$$\mathrm{NDes}(w) = \{(b_1, a_1), (b_2, a_2), \dots, (b_l, a_l)\} \tag{3.2}$$

is then independent of the choice of path. Moreover, if $X = \{w(1), w(2), \dots, w(n)\}$ then it holds that $w \in \mathcal{W}_{\texttt{embed}}(\zeta)$ for $\zeta = (a_1, b_1)(a_2, b_2) \cdots (a_l, b_l) \in \mathcal{I}(S_X)$.

Proof. By Corollary 3.7, there exists a set X with $[\pm n] = X \sqcup -X$ and an involution $\zeta \in \mathcal{I}(S_X)$ such that $w \in \mathcal{W}_{\texttt{embed}}(\zeta)$. Let $\sigma \in \mathcal{A}(\zeta)^{-1}$ be the preimage of w under the map $\mathcal{A}(\zeta)^{-1} \to \mathcal{W}_{\texttt{embed}}(\zeta)$. To prove this result, it suffices to show that (a) the nested descent graph of σ has a unique global sink ξ, (b) the set of edge labels in the nested descent graph of σ is the same for all paths from σ to ξ, and (c) the set $\mathrm{NDes}(\zeta)$ of edge labels described in (b) is precisely $\{(b, a) \in X \times X : a < b = \zeta(a)\}$. These claims are a special case of [30, Theorem 7.3]. Alternatively, one can check (a), (b), and (c) directly for $\sigma = 0_A(\zeta)$, and then deduce by induction that the desired properties hold in general. $\square$

Let $z \in \mathcal{I}(W_n)$ and $w \in \mathcal{A}(z)^{-1}$. We call $\mathrm{NDes}(w)$ the set of *nested descents* of w. Write $\xi(w)$ for the unique global sink in the nested descent graph of $w \in \mathcal{A}(z)^{-1}$. Define $\mathrm{NFix}(w)$ as the set of letters in $\xi(w)$ that are positive, and define $\mathrm{NNeg}(w)$ as the set of absolute values of the letters in $\xi(w)$ that are negative. We call elements of these sets *nested negated points* and *nested fixed points* of w. If $w = w_1 w_2 \cdots w_n$ and $\mathrm{NDes}(w) = \{(b_1, a_1), (b_2, a_2), \dots, (b_l, a_l)\}$, then

$$\{w_1, w_2, \dots, w_n\} = \{a_1, a_2, \dots, a_l\} \sqcup \{b_1, b_2, \dots, b_l\} \sqcup \mathrm{NFix}(w) \sqcup -\mathrm{NNeg}(w).$$

Example 3.11. The nested descent graph of $w = \overline{1}67\overline{2}348\overline{9}5 \in W_9$ is shown below:

$$\begin{tikzcd}
 & \overline{1}67\overline{2}348\overline{9}5 \arrow[dl, "7\overline{2}"'] \arrow[dr, "8\overline{9}"] & \\
\overline{1}6348\overline{9}5 \arrow[d, "63"] \arrow[drr, "8\overline{9}"] & & \overline{1}67\overline{2}345 \arrow[d, "7\overline{2}"] \\
\overline{1}48\overline{9}5 \arrow[dr, "8\overline{9}"] & & \overline{1}6345 \arrow[dl, "63"'] \\
 & \overline{1}45 &
\end{tikzcd}$$

We have $w \in \mathcal{A}(z)^{-1}$ for $z = (1, \overline{1})(2, \overline{7})(\overline{2}, 7)(3, 6)(\overline{3}, \overline{6})(8, \overline{8})(9, \overline{9}) \in \mathcal{I}(W_9)$. As predicted by Theorem-Definition 3.10, the nested descent graph of w has a unique global sink $\xi(w) = \overline{1}45$, and all paths from the source to the sink have edge labels $(8, -9), (7, -2), (6, 3)$ in some order. We therefore have $\mathrm{NDes}(w) = \{(8, -9), (7, -2), (6, 3)\}$, $\mathrm{NNeg}(w) = \{1\}$, and $\mathrm{NFix}(w) = \{4, 5\}$.

Corollary 3.12. Let $z \in \mathcal{I}(W_n)$ and suppose $v, w \in \mathcal{A}(z)^{-1}$.

(a) No word in the nested descent graph of w has a consecutive 321-pattern.

(b) If $v \sim_A w$ then $\mathrm{NDes}(v) = \mathrm{NDes}(w)$, $\mathrm{NFix}(v) = \mathrm{NFix}(w)$, and $\mathrm{NNeg}(v) = \mathrm{NNeg}(w)$.

Proof. Part (a) is necessary for $\mathrm{NDes}(w)$ to be well-defined. Part (b) is an immediate consequence of Corollary 3.7 and Theorem-Definition 3.10. $\square$

Can, Joyce, and Wyser's results in type A [7] have this consequence for signed involutions.

Lemma 3.13. Let $z \in \mathcal{I}(W_n)$ and $w \in \mathcal{A}(z)^{-1}$. Suppose $e, e' \in \mathrm{NFix}(w) \sqcup -\mathrm{NNeg}(w)$ and $(a, b), (a', b') \in \mathrm{NDes}(w)$. The following properties then hold: (1) If $e < e'$ then ee' is a subword of $w = w_1 w_2 \cdots w_n$. (2) If $e < a < b$ (respectively, $a < b < e$) then eba (respectively, bae) is a subword of w. (3) Finally, if $a < a'$ and $b < b'$ then $bab'a'$ is a subword of w.

Proof. Because $w \in \mathcal{W}_{\mathtt{embed}}(\zeta)$ for the involution $\zeta \in \mathcal{I}(S_{\{w_1, w_2, \ldots, w_n\}})$ whose nontrivial cycles are the pairs in $\mathrm{NDes}(w)$, these properties are equivalent to [7, Theorem 2.5]. It is also an instructive exercise to derive the lemma by considering the nested descent graph of w. For each hypothesis, one can check that the desired conclusion fails only if an extraneous descent appears in $\mathrm{NDes}(w)$ or if we can relate w via $\sim_A$ to an element $v \in W_n$ with a consecutive 321-pattern. $\square$

A word $w_1 w_2 \cdots w_n$ has a *consecutive* $\overline{1}\,\overline{2}$*-pattern* if for some $i \in [n-1]$ it holds that $0 > w_i > w_{i+1}$. A permutation in S_X or W_n has a consecutive $\overline{1}\,\overline{2}$-pattern if its one-line representation does. We conclude this section with a type B analogue of Proposition 3.4:

Proposition 3.14. If $w \in \mathcal{A}(z)^{-1}$ for some $z \in \mathcal{I}(W_n)$, then neither w nor any other word in w's nested descent graph has a consecutive $\overline{1}\,\overline{2}$-pattern.

The proof of Proposition 3.14 requires the following technical lemma.

Lemma 3.15. Let $z \in \mathcal{I}(W_n)$, $w \in \mathcal{A}(z)$, and $i \in [n]$. Suppose $w(1) = i$ and $w(2) = -(i+1)$. Then $z(1) = -1$ and $z(2) = -2$.

Proof. Let $u = wt_1t_0t_1$ and $y = u^{-1} \circ u \in \mathcal{I}(W_n)$. Then $\ell(w) = \ell(u) + 3$, so $u \in \mathcal{A}(y)$ and $t_1 \notin \mathrm{Des}_R(y)$. Since $yt_1 = u^{-1} \circ t_i \circ u = t_1 y$, we have either $y(1) = 1$ and $y(2) = 2$, or $y(1) = -2$ and $y(2) = -1$. The second case is impossible since $y < t_1 \circ y \circ t_1 = yt_1 < t_0 \circ (yt_1) \circ t_0$, so it follows that $z = t_1 \circ t_0 \circ t_1 \circ y \circ t_1 \circ t_0 \circ t_1 = yt_0t_1t_0t_1$ and consequently that $z(1) = -1$ and $z(2) = -2$. $\square$

Proof of Proposition 3.14. Suppose that $w \in W_n$ has $w^{-1}(1) = -i$ and $w^{-1}(2) = -(i+1)$ where $i \in [n-1]$, and assume $w \in \mathcal{A}(z)^{-1}$ for some $z \in \mathcal{I}(W_n)$. We produce a contradiction. Since $t_0 w <_L w$, we have $t_0 w \in \mathcal{A}(y)^{-1}$ for some $y \in \mathcal{I}(W_n)$ by Proposition 2.2. As $(t_0 w)^{-1}(1) = i$ and $(t_0 w)^{-1}(2) = -(i+1)$, it follows from Lemma 3.15 that $y(1) = -1$. But this means that $y = (t_0 w) \circ (t_0 w)^{-1} = w \circ w^{-1} = z$, which is impossible since $\ell(t_0 w) < \ell(w)$ and $w^{-1} \in \mathcal{A}(z)$.

Next suppose $w \in W_n$ is an arbitrary permutation with a consecutive $\overline{1}\,\overline{2}$-pattern, so that for some $a < b$ in $[n]$ it holds that $w^{-1}(a) = -i$ and $w^{-1}(b) = -(i+1)$. It is an exercise to construct an element $v \in W_n$ with $v^{-1} <_R w^{-1}$ and $v^{-1}(1) = -i$ and $v^{-1}(2) = -(i+1)$. By the previous paragraph, v^{-1} is not an atom for any involution, so by Proposition 2.2 neither is w^{-1}. This shows that no inverse atom of a signed involution has a consecutive $\overline{1}\,\overline{2}$-pattern. By Corollary 3.12(a), the same holds for all words in the nested descent graph of an inverse atom. $\square$

4 Partial orders

The symbol $\lhd_B$ defined by (1.3) is the relation on n-letter words with $v \lhd_B w$ if either $v_j = w_j$ for $j \notin \{1, 2\}$ while $v_1 = w_2 < v_2 = w_1 < 0$, or if $v_j = w_j$ for $j \notin \{1, 2, 3\}$ while $v_1 = w_1 < v_2 = w_3 < v_3 = -w_2 < 0$. As in the introduction, define $<_B$ as the transitive closure of *both* $\lhd_A$ and $\lhd_B$, and let $\sim_B$ be the symmetric closure of $<_B$. We apply $<_B$, $\lhd_B$ and $\sim_B$ to elements of W_n via their one-line representations.

Lemma 4.1. If $w \in W_n$, $z \in \mathcal{I}(W_n)$, $v \in \mathcal{A}(z)^{-1}$, and $v \sim_B w$, then $w \in \mathcal{A}(z)^{-1}$.

Proof. Suppose $v, w \in W_n$ are such that $v \lhd_B w$. If $v_1 v_2 = \overline{b}\overline{a}$ and $w_1 w_2 = a\overline{b}$ where $0 < a < b$, then there exists $\sigma \in W_n$ such that $v = \sigma \circ \overline{2}\,\overline{1}$, $w = \sigma \circ 1\overline{2}$, and $\ell(v) = \ell(w) = \ell(\sigma)+3$. If $v_1v_2v_3 = \overline{c}\overline{b}\overline{a}$ and $w_1w_2w_3 = \overline{c}a\overline{b}$ where $0 < a < b < c$, then there exists $\sigma \in W_n$ such that $v = \sigma \circ \overline{3}\,\overline{2}\,\overline{1}$, $w = \sigma \circ \overline{3}1\overline{2}$, and $\ell(v) = \ell(w) = \ell(\sigma) + 6$. The lemma follows by checking that $\overline{2}\,\overline{1} = t_0t_1t_0$ and $1\overline{2} = t_1t_0t_1$ are inverse atoms of $\overline{1}\,\overline{2}$, while $\overline{3}\,\overline{2}\,\overline{1} = t_0t_1t_0t_2t_1t_0$ and $\overline{3}1\overline{2} = t_1t_0t_1t_2t_1t_0$ are inverse atoms of $\overline{1}\,\overline{2}\,\overline{3}$. $\square$

Define $\lhd_B^+$ as the "extended" relation on n-letter words with $v \lhd_B^+ w$ if for some $i \in [n-1]$

$$v_1 < v_2 < \cdots < v_i = w_{i+1} < v_{i+1} = -w_i < 0 \qquad \text{and} \qquad v_j = w_j \text{ if } j \notin \{i, i+1\}. \tag{4.1}$$

Thus $\overline{z}\cdots\overline{c}\overline{b}\overline{a}\cdots \lhd_B^+ \overline{z}\cdots\overline{c}a\overline{b}\cdots$ if $0 < a < b < c < \cdots < z$. We have $\lhd_B \Rightarrow \lhd_B^+$. Conversely:

Lemma 4.2. If v and w are words with n letters and $v \lhd_B^+ w$, then $v \sim_B w$.

Proof. Assume v, w, and i are as in (4.1). Form v' from v by replacing $v_{j-1}v_j$ by $\overline{v_j}v_{j-1}$ for each even index $j < i$. If i is odd (respectively, even), then define v'' by removing the subword v_iv_{i+1} (respectively, $v_{i-1}v_iv_{i+1}$) from v' and placing it at the start of the word. Define w' from w and w'' from w' analogously. By induction on i, we can assume that $v \sim_B v'$ and $w' \sim_B w$. It is an exercise to check that $v' <_A v''$ and $w' <_A w''$, and it holds by definition that $v'' \lhd_B w''$, so $v \sim_B w$. $\square$

Next define $\lll_B$ as the relation on n-letter words with $v \lll_B w$ if for some $i \in [n-1]$ and some positive numbers a, b it holds that $v_j = w_j$ for $j \notin \{i, i+1\}$ while

$$v_iv_{i+1} = \overline{b}\overline{a}, \qquad w_iw_{i+1} = a\overline{b}, \qquad \text{and} \qquad 0 < a < b = \min\{|v_1|, |v_2|, \ldots, |v_i|\}. \tag{4.2}$$

When $v_1 < v_2 < \cdots < v_i$ these conditions are equivalent to (4.1), so $\lll_B \Rightarrow \lhd_B^+$.

Lemma 4.3. Let $z \in \mathcal{I}(W_n)$ $v \in W_n$, and $w \in \mathcal{A}(z)^{-1}$. If $v \lll_B w$ then $w \sim_B v \in \mathcal{A}(z)^{-1}$.

The converse does not hold: when $v \lll_B w$ and $v \in \mathcal{A}(z)^{-1}$ it may occur that $w \notin \mathcal{A}(z)^{-1}$.

Proof. Assume v, w, and i are as in (4.2) and $w \in \mathcal{A}(z)^{-1}$. Let j be the maximal index in $\{1, 2, \ldots, i\}$ such that $w_j > 0$. If $i = j$ then the numbers $w_1, w_2, \ldots, w_{i-1}$ are all negative, so it follows from Proposition 3.14 that $v \lhd_B^+ w$ whence $v \sim_B w$ by Lemma 4.2.

Suppose $j < i$. Proposition 3.4 then implies that $j < i-1$. First consider the case when $j < i-2$, and let $e = v_j = w_j$, $c = v_{j+1} = w_{j+1}$, and $d = v_{j+2} = w_{j+2}$. Since c and d are both negative, it follows by Proposition 3.14 that $c < d < 0 < e$ so we have $ecd \lhd_A dec$. Form v' and w' from the one-line representations of v and w by replacing the subword $ecd = v_jv_{j+1}v_{j+2} = w_jw_{j+1}w_{j+2}$ by dec. Then $v \lhd_A v' \lll_B w'$ and $w \lhd_A w' \in \mathcal{A}(z)^{-1}$, so by induction $v \sim_B v' \sim_B w' \sim_B w$.

Now consider the case when $j = i-2$, and let $d = v_{i-2} = w_{i-2}$ and $c = -v_{i-1} = -w_{i-1}$. By construction both c and d are positive and greater than $b > a$, so we have $d\overline{c}a\overline{b} \lhd_A ad\overline{c}\overline{b}$. It follows by Proposition 3.14 that $b < c$, so we also have $ad\overline{c}\overline{b} \lhd_A a\overline{b}d\overline{c}$ and similarly $d\overline{c}\overline{b}\overline{a} \lhd_A \overline{b}d\overline{c}\,\overline{a} \lhd_A \overline{b}\overline{a}d\overline{c}$. Let v'' and w'' be the signed permutations formed from the one-line representations of v and w by replacing the subword $d\overline{c}\overline{b}\overline{a}$ by $\overline{b}\overline{a}d\overline{c}$ and the subword $d\overline{c}a\overline{b}$ by $a\overline{b}d\overline{c}$. Then $v <_A v'' \lll_B w''$ and $w <_A w'' \in \mathcal{A}(z)^{-1}$, so again by induction $v \sim_B v'' \sim_B w'' \sim_B w$. □

The following lemma concerns the sets $\mathrm{NDes}(w)$, $\mathrm{NFix}(w)$, and $\mathrm{NNeg}(w)$ from Section 3.

Lemma 4.4. Suppose $v, w \in W_n$ and $i \in [n-1]$ are as in (4.2) so that $v \lll_B w$. Assume $v, w \in \mathcal{A}(z)^{-1}$ for some $z \in \mathcal{I}(W_n)$, and set $a = -v_{i+1} = w_i < b = -v_i = -w_{i+1}$. Then $\mathrm{NDes}(w) = \mathrm{NDes}(v) \sqcup \{(a, -b)\}$, $\mathrm{NNeg}(v) = \mathrm{NNeg}(w) \sqcup \{a, b\}$, and $\mathrm{NFix}(v) = \mathrm{NFix}(w)$.

Proof. By Corollary 3.12(a) and Proposition 3.14, we know that (1) none of the vertices in the nested descent graphs of v or w have 321- or $\overline{1}\,\overline{2}$-patterns, and we assume by hypothesis that (2) $0 < a < b < \min\{|v_1|, |v_2|, \dots, |v_{i-1}|\} = \min\{|w_1|, |w_2|, \dots, |w_{i-1}|\}$. It follows that we may choose a path from w to the global sink in its nested descent graph whose last edge is the unique one labeled by the descent $a\overline{b}$. Replacing the subword $a\overline{b}$ by $\overline{b}\overline{a}$ in all but the last vertex in this path produces a path from the source to some vertex in the nested descent graph of v. In view of (1) and (2), this vertex must be the global sink. The lemma now follows from Theorem-Definition 3.10. □

The *(weak) atomic order* of type B is the transitive closure $<_B$ of $\lhd_A$ and $\lhd_B$. Define the *strong atomic order* $\lll_B$ of type B as the transitive closure of the relations $\lhd_A$ and $\lll_B$.

Corollary 4.5. If $z \in \mathcal{I}(W_n)$ then $<_B$ and $\lll_B$ restrict to partial orders on $\mathcal{A}(z)^{-1}$.

Proof. Define $h(u) = |\{(a, -b) \in \mathrm{NDes}(u) : 0 < a < b\}|$ for $u \in \mathcal{A}(z)^{-1}$. It suffices to show that $\lll_B$ is antisymmetric. This follows since if $v, w \in \mathcal{A}(z)^{-1}$ have $v \lhd_A w$ or $v \lhd_B w$, then either w exceeds v in reverse lexicographic order while $h(w) = h(v)$, or $h(w) = h(v) + 1$ by Lemma 4.4. □

In the example shown in Figure 1, the orders $<_B$ and $\lll_B$ restricted to $\mathcal{A}(z)^{-1}$ are graded and connected, and $(\mathcal{A}(z)^{-1}, \lll_B)$ has a unique minimal element. We will show that these properties are general phenomena. Fix $z \in \mathcal{I}(W_n)$. Let $\mathrm{Fix}(z) = \{i \in [n] : z(i) = i\}$ and define $\mathrm{Pair}(z)$ and $\mathrm{Neg}(z)$ as in Section 2. Let

$$\mathrm{Cyc}_B(z) = \mathrm{Pair}(z) \sqcup \{(-a, -a) : a \in \mathrm{Neg}(z)\} \sqcup \{(a, a) : a \in \mathrm{Fix}(z)\}. \tag{4.3}$$

If $\mathrm{Cyc}_B(z) = \{(a_1, b_1), (a_2, b_2), \dots, (a_l, b_l)\}$ where $a_1 < a_2 < \cdots < a_l$, then we finally define

$$0_B(z) = [[b_1 a_1 b_2 a_2 \cdots b_l a_l]]. \tag{4.4}$$

Recall that $[[w]]$ denotes the subword of $w = w_1 w_2 \cdots w_n$ formed by omitting all repeated letters after their first appearance. Thus $0_B(z)$ is a word with distinct letters by construction, and it is straightforward to check that $0_B(z)$ is in fact the one-line representation of an element of W_n. If $z = (1, \overline{1})(2, \overline{7})(\overline{2}, 7)(3, 6)(\overline{3}, \overline{6})(8, \overline{8})(9, \overline{9}) \in \mathcal{I}(W_9)$ as in Example 3.11, then $\mathrm{Neg}(z) = \{1, 8, 9\}$, $\mathrm{Fix}(z) = \{4, 5\}$, and $\mathrm{Pair}(z) = \{(-2, 7), (3, 6)\}$, so $0_B(z) = \overline{9}\,\overline{8}7\overline{2}\,\overline{1}6345$.

Lemma 4.6. If $z \in \mathcal{I}(W_n)$ then $0_B(z) \in \mathcal{A}(z)^{-1}$.

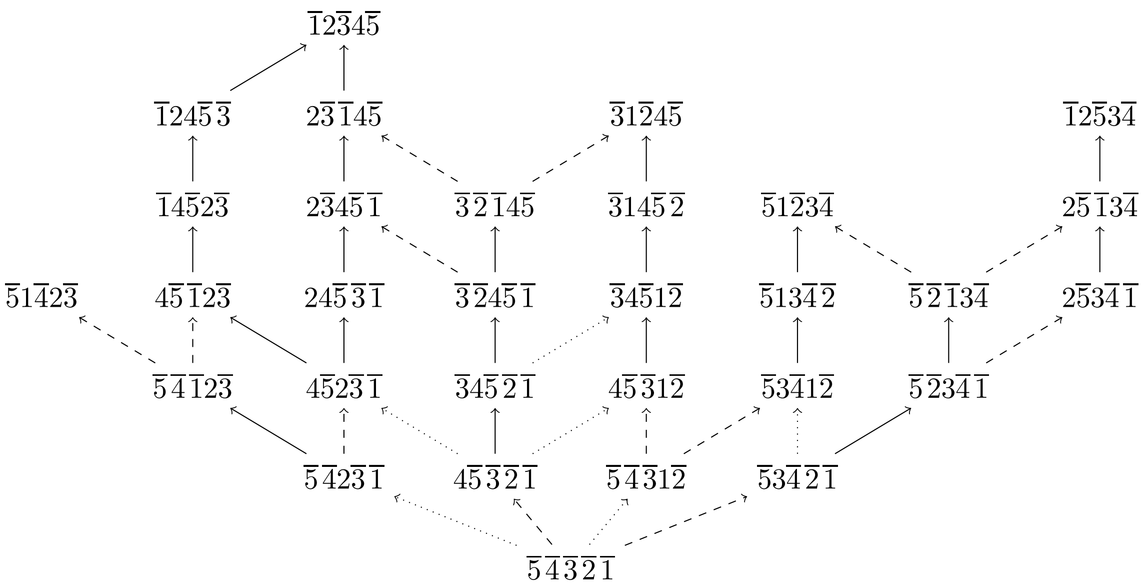


Figure 1: The Hasse diagram of the poset $(\mathcal{A}(z)^{-1}, \lll_B)$ for $z = \overline{1}\,\overline{2}\,\overline{3}\,\overline{4}\,\overline{5} \in \mathcal{I}(W_5)$. The solid arrows correspond to the covering relations $\lessdot_A$, the dashed arrows correspond to $\lessdot_B$, and the dotted arrows correspond to the remaining relations $\lll_B$.

Proof. Fix $z \in \mathcal{I}(W_n)$ and recall the definition of $\Psi_n : W_n \to S_{2n}$ from Section 2. Since $\frac{1}{2}(\ell_0(z) + \mathrm{neg}(z)) = \ell_0(0_B(z)^{-1})$ by definition, it suffices by Lemma 2.8 to check that $\Psi_n(0_B(z))^{-1} \in \mathcal{A}(\Psi_n(z))$. This follows by applying [7, Theorem 2.5], which is just Lemma 3.13 restricted to $S_n \hookrightarrow W_n$. In detail, if $v \in S_n$, $\zeta \in \mathcal{I}(S_n)$, and $\mathrm{Cyc}_A(\zeta) = \{(a,b) \in [n] \times [n] : a \leq b = \zeta(a)\}$, then we have $v \in \mathcal{A}(\zeta)^{-1}$ precisely when (1) if $(a,b) \in \mathrm{Cyc}_A(\zeta)$ then b is weakly left of a in the one-line representation of v, and no number $e \in [n]$ with $a < e < b$ appears between a and b, and (2) if $(a,b), (a',b') \in \mathrm{Cyc}_A(\zeta)$ are such that $a < a'$ and $b < b'$ then ba' is a subword of v. It is straightforward to check that (1) and (2) hold for $\zeta = \Psi_n(z)$ and $v = \Psi_n(0_B(z))$. $\square$

Putting things together leads to a short proof of the following theorem.

Theorem 4.7. If $z \in \mathcal{I}(W_n)$ then $0_B(z)$ is the unique minimum in $(\mathcal{A}(z)^{-1}, \lll_B)$.

Proof. Fix $z \in \mathcal{I}(W_n)$ and $w \in \mathcal{A}(z)^{-1}$. Corollary 3.7 and Proposition 3.14 imply that $v \leq_A w$ for a unique element of the form $v = [[b_1 a_1 b_2 a_2 \cdots b_l a_l]] \in W_n$ where $a_i = b_i$ or $a_i < b_i > 0$ for each $i \in [l]$ and $a_1 < a_2 < \cdots < a_l$. We have $\mathrm{NDes}(w) = \mathrm{NDes}(v) = \mathrm{Des}(v) = \{(b_i, a_i) : i \in [l],\ a_i < b_i\}$ by Corollary 3.12(b). As in the proof of Corollary 4.5, let $h(w) = |\{(a, -b) \in \mathrm{NDes}(w) : 0 < a < b\}|$. If $h(w) = 0$, then evidently $v = 0_B(y)$ for some $y \in \mathcal{I}(W_n)$, in which case we must have $y = z$ by Lemma 4.6, so $0_B(z) = v \leq_A w$. Assume $h(w) > 0$ and let $j \in [l]$ be the smallest index such that $a_j < 0 < b_j < -a_j$. Then whenever $i < j$ and $a_i < b_i$, it must hold that $a_i < a_j < 0$ and $-a_j < b_j$, so we have $a_i < a_j < b_j < b_i$. Therefore $u \lll_B v$ for the signed permutation $u = [[b_1 a_1 \cdots a_j \overline{b_j} \cdots b_l a_l]] \in W_n$. Since $h(u) + 1 = h(v) = h(w)$ by Lemma 4.4, we may assume by induction that $0_B(z) = u$ or $0_B(z) \lll_B u$, so $0_B(z) \lll_B w$ since $u \lll_B v \leq_A w$. $\square$

Corollary 4.8. If $z \in \mathcal{I}(W_n)$ then $\mathcal{A}(z)^{-1}$ is a single equivalence class under $\sim_B$.

Proof. This is immediate from the preceding theorem and Lemma 4.3. $\square$

Let $\mathcal{R}(w)$ denote the set of reduced words for $w \in W_n$ and define $\hat{\mathcal{R}}(z) = \bigsqcup_{w\in\mathcal{A}(z)} \mathcal{R}(w)$ when $z \in \mathcal{I}(W_n)$. It is well-known that $\mathcal{R}(w)$ is spanned and preserved by the *braid relations* $\cdots t_0t_1t_0t_1 \cdots \sim \cdots t_1t_0t_1t_0 \cdots$ and $\cdots t_it_{i+1}t_i \cdots \sim \cdots t_{i+1}t_it_{i+1} \cdots$ for $i \in [n-1]$, where as usual the corresponding ellipses are required to mask identical subwords. The previous corollary is equivalent to the following result of Hu and Zhang [22, Theorem 4.8].

Corollary 4.9 (Hu and Zhang [22])**.** If $z \in \mathcal{I}(W_n)$ then $\hat{\mathcal{R}}(z)$ is spanned and preserved by the usual set of braid relations for W_n plus the extra "initial" relations $t_it_{i+1}\cdots \sim t_{i+1}t_i\cdots$ for $i \in [n-1]$ and $t_0t_1t_0\cdots \sim t_1t_0t_1\cdots$ and $t_0t_1t_2t_0t_1t_0\cdots \sim t_0t_1t_2t_1t_0t_1\cdots$.

Proof. To deduce this from Corollary 4.8 or vice versa, it suffices to check that $z \in \mathcal{I}(W_n)$ has atoms $v, w \in \mathcal{A}(z)$ with $v^{-1} \lhd_A w^{-1}$ or $v^{-1} \lhd_B w^{-1}$ if and only if v and w have reduced words connected by the given relations. This holds by a simple calculation using Proposition 3.14. $\square$

Hansson and Hultman, extending this result, have found a general description of the relations needed to span the sets $\hat{\mathcal{R}}(z)$ for involutions z in any (twisted) Coxeter group [20].

5 Noncrossing shapes

The connected posets $(\mathcal{A}(z)^{-1}, <_B)$ and $(\mathcal{A}(z)^{-1}, \ll_B)$ for $z \in \mathcal{I}(W_n)$ are no longer intervals as in type A. In this section, we study these posets' extremal elements and characterize the components of the disconnected poset $(\mathcal{A}(z)^{-1}, <_A)$.

Consider a subset $X \subset [\pm n]$ with $X = -X$. A *matching* in (the complete graph on) X is a set M of pairwise disjoint 2-element subsets of X. A matching M is *symmetric* if $\{-i,-j\} \in M$ whenever $\{i,j\} \in M$; *perfect* if for each $i \in X$ there exists a unique $j \in X$ with $\{i,j\} \in M$; and *noncrossing* if no two subsets $\{i,k\},\{j,l\} \in M$ have $i < j < k < l$. The 3 perfect noncrossing symmetric matchings in $[\pm 3]$ are $\{\{1,\overline{1}\},\{2,\overline{2}\},\{3,\overline{3}\}\}$, $\{\{1,2\},\{\overline{1},\overline{2}\},\{3,\overline{3}\}\}$, and $\{\{1,\overline{1}\},\{2,3\},\{\overline{2},\overline{3}\}\}$. In general, there are $\binom{n}{\lfloor n/2 \rfloor}$ symmetric noncrossing perfect matchings in $[\pm n]$, and such matchings are in bijection with many other combinatorially defined objects (see [36, A001405]).

Let $z \in \mathcal{I}(W_n)$ and recall the sets $\mathrm{Neg}(z)$, $\mathrm{Fix}(z)$, and $\mathrm{Pair}(z)$ introduced before Lemma 4.6. Define $\mathsf{NCSP}(z)$ as the set of noncrossing, symmetric, perfect matchings in $\mathrm{Neg}(z) \sqcup -\mathrm{Neg}(z)$. For each matching $M \in \mathsf{NCSP}(z)$, we define three related sets:

$$\begin{aligned} \mathrm{Neg}(z,M) &= \{i \in \mathrm{Neg}(z) : \{i,-i\} \in M\}, \\ \mathrm{Pair}(z,M) &= \mathrm{Pair}(z) \sqcup \{(-b,a) : \{a,b\} \in M \text{ and } 0 < a < b\}, \\ \mathrm{Cyc}_B(z,M) &= \mathrm{Pair}(z,M) \sqcup \{(-a,-a) : a \in \mathrm{Neg}(z,M)\} \sqcup \{(a,a) : a \in \mathrm{Fix}(z)\}. \end{aligned} \tag{5.1}$$

Suppose we have $\mathrm{Cyc}_B(z,M) = \{(a_1,b_1),(a_2,b_2),\dots,(a_l,b_l)\} = \{(c_1,d_1),(c_2,d_2),\dots,(c_l,d_l)\}$ where $a_1 < a_2 < \cdots < a_l$ and $d_1 < d_2 < \cdots < d_l$. We then define $0_B(z,M)$ and $1_B(z,M)$ as the words

$$0_B(z,M) = [[b_1a_1b_2a_2\cdots b_la_l]] \qquad \text{and} \qquad 1_B(z,M) = [[d_1c_1d_2c_2\cdots d_lc_l]]. \tag{5.2}$$

Example 5.1. Let $z = (1,\overline{1})(2,\overline{7})(\overline{2},7)(3,6)(\overline{3},\overline{6})(8,\overline{8})(9,\overline{9}) \in \mathcal{I}(W_9)$ as in Example 3.11, so that $\mathrm{Neg}(z) = \{1,8,9\}$. The three elements of $\mathsf{NCSP}(z)$ are

$$M^1 = \{\{\overline{9},9\},\{\overline{8},8\},\{\overline{1},1\}\}, \quad M^2 = \{\{\overline{9},9\},\{\overline{1},\overline{8}\},\{1,8\}\}, \quad M^3 = \{\{\overline{8},\overline{9}\},\{\overline{1},1\},\{8,9\}\}.$$

We have $\mathrm{Neg}(z, M^1) = \mathrm{Neg}(z) = \{1, 8, 9\}$ and $\mathrm{Pair}(z, M) = \mathrm{Pair}(z) = \{(\overline{2}, 7), (3, 6)\}$, so

$$0_B(z, M^1) = \overline{9}\,\overline{8}7\overline{2}\,\overline{1}6345 \qquad \text{and} \qquad 1_B(z, M^1) = \overline{9}\,\overline{8}\,\overline{1}45637\overline{2}.$$

Similarly $\mathrm{Neg}(z, M^2) = \{9\}$ and $\mathrm{Pair}(z, M) = \{(\overline{2}, 7), (3, 6), (1, \overline{8})\}$, so

$$0_B(z, M^2) = \overline{9}1\overline{8}7\overline{2}6345 \qquad \text{and} \qquad 1_B(z, M^2) = \overline{9}1\overline{8}45637\overline{2}.$$

Finally $\mathrm{Neg}(z, M^3) = \{1\}$ and $\mathrm{Pair}(z, M) = \{(\overline{2}, 7), (3, 6), (8, \overline{9})\}$, so

$$0_B(z, M^3) = 8\overline{9}7\overline{2}\,\overline{1}6345 \qquad \text{and} \qquad 1_B(z, M^3) = \overline{1}45637\overline{2}8\overline{9}.$$

Proposition 5.2. Let $z \in \mathcal{I}(W_n)$ and $M \in \mathsf{NCSP}(z)$. The words $0_B(z, M)$ and $1_B(z, M)$ may be interpreted as elements of W_n written in one-line notation. Under $<_A$, the permutation $0_B(z, M)$ is minimal while $1_B(z, M)$ is maximal, and it holds that $0_B(z, M) \leq_A 1_B(z, M)$.

Proof. Let X be the set of numbers a and b occurring in pairs $(a, b) \in \mathrm{Cyc}_B(z, M)$. Check that $[\pm n] = X \sqcup -X$, and conclude that $0_B(z, M)$ and $1_B(z, M)$ belong to W_n. Define $\zeta \in \mathcal{I}(S_X)$ as the involution with $a < b = z(a)$ for $a, b \in X$ if and only if $(a, b) \in \mathrm{Pair}(z, M)$. Then we have $0_B(z, M) = 0_A(\zeta)$ and $1_B(z, M) = 1_A(\zeta)$ as words, so the result follows from Theorem 3.1. $\square$

We have $\mathrm{Neg}(z) = \mathrm{Neg}(z, M_{\min})$, $\mathrm{Pair}(z) = \mathrm{Pair}(z, M_{\min})$, and $0_B(z) = 0_B(z, M_{\min})$ for $M_{\min} = \{\{i, -i\} : i \in \mathrm{Neg}(z)\}$. The following corollary refers to the map $\Psi_n : W_n \to S_{2n}$ from Section 2.

Corollary 5.3. If $z \in \mathcal{I}(W_n)$ and $w \in W_n$ are such that $0_B(z) \leq_A w$, then $\Psi_n(w)^{-1} \in \mathcal{A}(\Psi_n(z))$.

Proof. Let $z \in \mathcal{I}(W_n)$. We know that $0_B(z) \in \mathcal{A}(z)^{-1}$ by Lemma 4.6. If $u, v \in W_n$ and $u \lessdot_A v$, then $\Psi_n(u) \lessdot_A w \gtrdot_A \Psi_n(v)$ for some $w \in S_{2n}$. Hence if $u, v \in W_n$ and $u \sim_A v$ then $\Psi_n(u) \sim_A \Psi_n(v)$, so the corollary follows from Theorem 3.1. $\square$

If $w \in \mathcal{A}(z)^{-1}$ for $z \in \mathcal{I}(W_n)$, then we define $\mathrm{NDes}_B(w)$ as the subset of $\mathrm{NDes}(w)$ given by removing all pairs of the form $(a, -b)$ where $0 < a < b$, and we define $\mathrm{NNeg}_B(w)$ as the set given by adding to $\mathrm{NNeg}(w)$ both a and b for each pair $(a, -b) \in \mathrm{NDes}(w)$ with $0 < a < b$. For example, if $w = \overline{1}67\overline{2}348\overline{9}5$ then $\mathrm{NDes}(w) = \{(8, -9), (7, -2), (6, 3)\}$ and $\mathrm{NNeg}(w) = \{1\}$, so $\mathrm{NDes}_B(w) = \{(7, -2), (6, 3)\}$ and $\mathrm{NNeg}_B(w) = \{1, 8, 9\}$.

Given $w \in \mathcal{A}(z)$, one can recover z by finding a reduced word $w = t_{i_1} t_{i_2} \cdots t_{i_l}$ and then calculating $z = t_{i_l} \circ \cdots \circ t_{i_2} \circ t_{i_1} \circ t_{i_1} \circ t_{i_2} \circ \cdots \circ t_{i_l}$. This naive algorithm is very inefficient. The following result shows that z is in fact determined by the nested descent set of w^{-1}.

Lemma 5.4. Let $z \in \mathcal{I}(W_n)$ and $w \in \mathcal{A}(z)^{-1}$. Then $\mathrm{Fix}(z) = \mathrm{NFix}(w)$, $\mathrm{Neg}(z) = \mathrm{NNeg}_B(w)$ and $\mathrm{Pair}(z) = \{(a, b) : (b, a) \in \mathrm{NDes}_B(w)\}$.

Proof. It straightforward to check that each claim holds if $w = 0_B(z)$ by inspection and when $w \sim_B 0_B(z)$ by Corollary 3.12(b) and Lemma 4.4, so is true for all $w \in \mathcal{A}(z)^{-1}$ by Corollary 4.8. $\square$

When M is a symmetric matching we call $\{i, j\} \in M$ a *trivial block* if $i + j = 0$. Suppose $z \in \mathcal{I}(W_n)$ and $w \in \mathcal{A}(z)^{-1}$. Define the *shape* of w to be the symmetric perfect matching $\mathrm{sh}(w)$ whose nontrivial blocks are the subsets $\{a, b\}$ and $\{-a, -b\}$ for each pair $(a, -b) \in \mathrm{NDes}(w)$ with $0 < a < b$, and whose trivial blocks are the subsets $\{e, -e\}$ for each $e \in \mathrm{NNeg}(w)$. By the previous theorem, $\mathrm{sh}(w)$ is a matching in the set $\mathrm{Neg}(z) \sqcup -\mathrm{Neg}(z) = \{i \in [\pm n] : z(i) = -i\}$. For example, if $w = \overline{1}67\overline{2}348\overline{9}5$ then $\mathrm{sh}(w) = \{\{\overline{8}, \overline{9}\}, \{\overline{1}, 1\}, \{8, 9\}\}$.

The following shows that $w \mapsto \mathrm{sh}(w)$ is a well-defined map $\mathcal{A}(z)^{-1} \to \mathsf{NCSP}(z)$.

Theorem 5.5. If $z \in \mathcal{I}(W_n)$ and $w \in \mathcal{A}(z)^{-1}$ then $\mathrm{sh}(w) \in \mathsf{NCSP}(z)$, i.e., the shape of w is a perfect matching which is symmetric and noncrossing.

Proof. Let $z \in \mathcal{I}(W_n)$. The value of $\mathrm{sh}(\cdot)$ is constant on $\sim_A$-equivalence classes by Corollary 3.12(b), and $\mathrm{sh}(0_B(z)) = \{\{i, -i\} : i \in \mathrm{Neg}(z)\} \in \mathsf{NCSP}(z)$. Suppose $v, w \in \mathcal{A}(z)^{-1}$ and $i \in [n-1]$ are as in (4.1) so that $v \lessdot_B^+ w$ and $v_1 = w_1 < v_2 = w_2 < \cdots < v_{i-1} = w_{i-1} < v_i < 0$. Set $a = -v_{i+1} = w_i$ and $b = -v_i = -w_{i+1}$ so that $0 < a < b$. Since $\sim_B$ is the transitive, symmetric closure $\lessdot_A$ and $\lessdot_B^+$ and since $0_B(z) \in \mathcal{A}(z)^{-1}$, it suffices by Corollary 4.8, to show that $\mathrm{sh}(v)$ is noncrossing if and only if $\mathrm{sh}(w)$ is noncrossing.

We have $\mathrm{sh}(v) \backslash \mathrm{sh}(w) = \{\{a, -a\}, \{b, -b\}\}$ and $\mathrm{sh}(w) \backslash \mathrm{sh}(v) = \{\{a, b\}, \{-a, -b\}\}$ by Lemma 4.4. If $\mathrm{sh}(w)$ is noncrossing, then the only way $\mathrm{sh}(v)$ can fail to be noncrossing is if there exists a nontrivial block $\{c, d\} \in \mathrm{sh}(v) \cap \mathrm{sh}(w)$ with $0 < c < a < b < d$. But this would imply that both $(a, -b)$ and $(c, -d)$ were elements of $\mathrm{NDes}(w)$, contradicting Lemma 3.13(3) since $c\overline{d}a\overline{b}$ is not a subword of w.

Conversely, if $\mathrm{sh}(v)$ is noncrossing, then $\mathrm{sh}(w)$ can fail to be noncrossing only is if there exists a trivial block $\{e, -e\} \in \mathrm{sh}(v) \cap \mathrm{sh}(w)$ with $a < e < b$. But then we would have $\{a, b, e\} \subset \mathrm{NNeg}(v)$, so Lemma 3.13(1) would imply that $\overline{b}\overline{e}\overline{a}$ is a subword of v, which is impossible as $\overline{b}$ and $\overline{a}$ are consecutive in v. Thus $\mathrm{sh}(v)$ is noncrossing if and only if $\mathrm{sh}(w)$ is also. $\square$

Let $z \in \mathcal{I}(W_n)$ and $M \in \mathsf{NCSP}(z)$. If $0_B(z, M)$ and $1_B(z, M)$ are contained in $\mathcal{A}(z)^{-1}$, then they evidently have shape $\mathrm{sh}(0_B(z, M)) = \mathrm{sh}(1_B(z, M)) = M$, as do all elements $w \in \mathcal{A}(z)^{-1}$ with $0_B(z, M) \leq_A w \leq_A 1_B(z, M)$ by Corollary 3.12(b). The previous theorem shows that only noncrossing shapes are possible for inverse atoms; the following confirms that all such shapes occur. The map $\mathrm{sh} : \mathcal{A}(z)^{-1} \to \mathsf{NCSP}(z)$ therefore provides the bijection mentioned in Theorem 1.5(b).

Theorem 5.6. Let $z \in \mathcal{I}(W_n)$. If $M \in \mathsf{NCSP}(z)$ then $0_B(z, M)$ and $1_B(z, M)$ are minimal and maximal elements of $(\mathcal{A}(z)^{-1}, <_A)$, respectively. Moreover, all minimal (respectively, maximal) elements in $(\mathcal{A}(z)^{-1}, <_A)$ have the form $0_B(z, M)$ (respectively, $1_B(z, M)$) for some $M \in \mathsf{NCSP}(z)$.

Proof. Suppose $w \in \mathcal{A}(z)^{-1}$ is minimal under $<_A$ and $M = \mathrm{sh}(w) \in \mathsf{NCSP}(z)$. As $\mathrm{sort}_R(w)$ is then increasing by Lemma 3.5, it follows that $\mathrm{NDes}(w) = \mathrm{Des}(w)$. From this observation and Lemma 5.4, it is an exercise to deduce that w must be equal to $0_B(z, M)$. If $w \in \mathcal{A}(z)^{-1}$ is maximal under $<_A$, then it follows similarly that $w = 1_B(z, \mathrm{sh}(w))$.

Choose an arbitrary matching $M \in \mathsf{NCSP}(z)$. It remains to show that $0_B(z, M)$ and $1_B(z, M)$ in fact belong to $\mathcal{A}(z)^{-1}$. From Lemma 3.3, Corollary 3.7, and the previous paragraph, it is enough to construct a single element $w \in \mathcal{A}(z)^{-1}$ with $\mathrm{sh}(w) = M$. We prove this by induction on the number of nontrivial blocks in M. If M has no nontrivial blocks then $0_B(z, M) = 0_B(z)$ has shape M and belongs to $\mathcal{A}(z)^{-1}$ by Corollary 5.3. Otherwise, we can find a nontrivial block $\{a, b\} \in M$ with $0 < a < b$ such that no $\{a', b'\} \in M$ has $0 < a' < a < b < b'$. Replacing $\{a, b\}$ and $\{-a, -b\}$ in M by $\{a, -a\}$ and $\{b, -b\}$ yields another noncrossing matching $M' \in \mathsf{NCSP}(z)$ with strictly fewer nontrivial blocks. Let $v = 1_B(z, M')$. By induction, we may assume that $v \in \mathcal{A}(z)^{-1}$. Since M is noncrossing, we must have $\{a, b\} = \{a, a+1, \ldots, b\} \cap \mathrm{Neg}(z, M')$, so $v_1 < v_2 < \cdots < v_i = -b < v_{i+1} = -a < 0$ for some $i \in [n-1]$. Replacing the subword $v_i v_{i+1} = \overline{b}\overline{a}$ in the one-line representation of v by $a\overline{b}$ gives a signed permutation w with $v \sim_B w \in \mathcal{A}(z)^{-1}$ by Lemma 4.2, and it follows by Lemma 4.4 that $\mathrm{sh}(w) = M$. $\square$

Corollary 5.7. If $z \in \mathcal{I}(W_n)$ then $(\mathcal{A}(z)^{-1}, <_A)$ is connected if and only if $\mathrm{neg}(z) \leq 1$.

Proof. The number of components in $(\mathcal{A}(z)^{-1}, <_A)$ is $|\mathsf{NCSP}(z)|$, which is 1 iff $\operatorname{neg}(z) \leq 1$. $\square$

Let $\mathsf{NCSP}^k(z)$ for $z \in \mathcal{I}(W_n)$ be the set of matchings in $\mathsf{NCSP}(z)$ with at most k trivial blocks. We have $\mathsf{NCSP}^0(z) = \mathsf{NCSP}^1(z)$ if $\operatorname{neg}(z)$ is even and $\mathsf{NCSP}^0(z) = \varnothing$ if $\operatorname{neg}(z)$ is odd.

Corollary 5.8. Let $z \in \mathcal{I}(W_n)$. The permutations $1_B(z, M)$ for $M \in \mathsf{NCSP}^1(z)$ are the maximal elements in $\mathcal{A}(z)^{-1}$ under both atomic orders $<_B$ and $\ll_B$. Moreover, $\mathcal{A}(z)^{-1}$ is the union of the lower intervals in $(W_n, \ll_B)$ bounded above by these elements.

Proof. Each maximal element in $\mathcal{A}(z)^{-1}$ under either atomic order is necessarily of the form $1_B(z, M)$ for some $M \in \mathsf{NCSP}(z)$ by Theorem 5.6. If $M \in \mathsf{NCSP}(z)$ has k trivial blocks, then we can write $1_B(z, M) = \overline{a_k} \cdots \overline{a_2}\,\overline{a_1} b_1 b_2 \cdots b_{n-k}$ where $0 < a_1 < a_2 < \cdots < a_k$, $0 < b_1$, and $b_1 b_2 \cdots b_{n-k}$ contains no consecutive negative numbers. Evidently $1_B(z, M)$ is maximal under $<_B$ (and also $\ll_B$) if and only if $k < 2$. The last assertion in the corollary holds by Lemma 4.3. $\square$

This result has the following amusing consequence.

Corollary 5.9. If $z \in \mathcal{I}(W_n)$ and $m = \lceil \frac{1}{2} \operatorname{neg}(z) \rceil$, then the number of elements in $\mathcal{A}(z)^{-1}$ which are maximal under $<_B$ (equivalently, $\ll_B$) is the mth Catalan number $C_m = \frac{1}{m+1}\binom{2m}{n}$.

Proof. There is a simple bijection from $\mathsf{NCSP}^1(z)$ to the set of noncrossing perfect matchings in $[2m]$, whose enumeration by C_m is well-known: remove all blocks without positive elements from $M \in \mathsf{NCSP}^1(z)$ and standardize the numbers in the remaining blocks to be $1, 2, \ldots, 2m$. $\square$

6 Rank functions

In this section we show that the atomic orders $<_B$ and $\ll_B$ are graded. Fix $z \in \mathcal{I}(W_n)$ and $w \in \mathcal{A}(z)^{-1}$. Define $\operatorname{offset}_A(w)$ as the number of pairs $((b_1, a_1), (b_2, a_2)) \in \operatorname{NDes}(w) \times \operatorname{NDes}(w)$ with $a_1 < a_2 < b_2 < b_1$. Let

$$L = \{b : (b, a) \in \operatorname{NDes}(w) \text{ for some } a\} \qquad \text{and} \qquad R = \{w_1, w_2, \ldots, w_n\} \setminus L$$

and define $\operatorname{rank}_A : \mathcal{A}(z)^{-1} \to \mathbb{Z}$ as the function with

$$\operatorname{rank}_A(w) = \operatorname{inv}(w|_R) - \operatorname{inv}(w|_L) + \operatorname{offset}_A(w) \in \mathbb{Z},$$

where $w|_R$ and $w|_L$ are the words formed from $w_1 w_2 \cdots w_n$ by omitting all entries not in R and L, respectively, and $\operatorname{inv}(v) = |\{(i, j) \in [k] \times [k] : i < j \text{ and } v_i > v_j\}|$ for a word $v = v_1 v_2 \cdots v_k$. If $z = \overline{5}\,\overline{4}\,\overline{3}\,\overline{2}\,\overline{1}$ and $w = \overline{3}4\overline{5}1\overline{2} \in \mathcal{A}(z)^{-1}$, for example, then $\operatorname{NDes}(w) = \{(1, \overline{2}), (4, \overline{5})\}$, $L = \{1, 4\}$, $R = \{\overline{2}, \overline{3}, \overline{5}\}$, $w|_R = \overline{3}\,\overline{5}\,\overline{2}$, $w|_L = 41$, and $\operatorname{inv}(w|_R) = \operatorname{inv}(w|_L) = \operatorname{offset}_A(w) = \operatorname{rank}_A(w) = 1$. As a consequence of Corollary 3.7, the following result is equivalent to [14, Lemma 6.13].

Proposition 6.1. Let $z \in \mathcal{I}(W_n)$. If $v, w \in \mathcal{A}(z)^{-1}$ and $v \lessdot_A w$, then $\operatorname{rank}_A(w) = \operatorname{rank}_A(v) + 1$, and an element $w \in \mathcal{A}(z)^{-1}$ is minimal relative to $<_A$ if and only if $\operatorname{rank}_A(w) = 0$.

Proof. The first claim is immediate from the way we define $\lessdot_A$ and $\operatorname{NDes}(w)$. By Lemma 3.5, $w \in \mathcal{A}(z)^{-1}$ is minimal relative to $<_A$ if and only if $\operatorname{inv}(w|_R) = 0$ and $\operatorname{inv}(w|_L) = \operatorname{offset}_A(w)$. $\square$

Still with $z \in \mathcal{I}(W_n)$ and $w \in \mathcal{A}(z)^{-1}$, define $\mathrm{offset}_B(w)$ as the number of pairs of nested descents $((b_1, a_1), (b_2, a_2)) \in \mathrm{NDes}(w) \times \mathrm{NDes}(w)$ satisfying $a_1 \leq a_2 < -b_1 < 0 < b_1 \leq b_2$. Set

$$\mathrm{rank}_B(w) = \mathrm{rank}_A(w) + \mathrm{offset}_B(w) \in \mathbb{N}.$$

The value of $\mathrm{offset}_B(w)$ is the sum of three quantities: the number of descents $(a, -b) \in \mathrm{NDes}(w)$ with $a < b$, the number of pairs $(a_1, -b_1), (a_2, -b_2) \in \mathrm{NDes}(w)$ with $a_1 < a_2 < b_2 < b_1$, and the number of pairs $(a_1, -b_1), (b_2, -a_2) \in \mathrm{NDes}(w)$ with $a_1 < a_2 < \min\{b_1, b_2\}$. For example, if $w = 1\overline{5}2\overline{3}6\overline{4}$ then $\mathrm{NDes}(w) = \{(1, \overline{5}), (2, \overline{3}), (6, \overline{4})\}$, $w|_L = 126$, $w|_R = \overline{5}\overline{3}\overline{4}$, $\mathrm{inv}(w|_L) = 0$, $\mathrm{inv}(w|_R) = \mathrm{offset}_A(w) = 1$, $\mathrm{rank}_A(w) = 2$, $\mathrm{offset}_B(w) = 4$, and $\mathrm{rank}_B(w) = 6$.

The function $\mathrm{offset}_B(\cdot)$ is constant on $\sim_A$-equivalence classes by Corollary 3.12(b), so we have $\mathrm{rank}_B(w) = \mathrm{rank}_B(v) + 1$ for $v, w \in \mathcal{A}(z)^{-1}$ with $v \lessdot_A w$ by Proposition 6.1. In addition:

Proposition 6.2. Let $z \in \mathcal{I}(W_n)$. If $v, w \in \mathcal{A}(z)^{-1}$ and $v \lll_B w$, then $\mathrm{rank}_B(w) = \mathrm{rank}_B(v) + 1$, and an element $w \in \mathcal{A}(z)^{-1}$ is minimal relative to $\ll_B$ if and only if $\mathrm{rank}_B(w) = 0$.

Proof. Fix $v, w \in \mathcal{A}(z)^{-1}$. First assume $v \lessdot_B^+ w$ and let $i \in [n-1]$ be as in (4.1). Let $a = -v_{i+1} = w_i$ and $b = -v_i = -w_{i+1}$ so that $0 < a < b$ and $\mathrm{NDes}(w) = \mathrm{NDes}(v) \sqcup \{(a, -b)\}$. Then $v_j = w_j \in \mathrm{NNeg}(v) \cap \mathrm{NNeg}(w)$ for $1 \leq j < i$. From Theorem-Definition 3.10 and Lemma 3.13, we deduce that the difference $A = \mathrm{inv}(v|_R) - \mathrm{inv}(w|_R)$ is the number of pairs $(y, x) \in \mathrm{NDes}(v)$ with $x < -a$, the difference $B = \mathrm{inv}(w|_L) - \mathrm{inv}(v|_L)$ is the number of pairs $(y, x) \in \mathrm{NDes}(v)$ with $y < a$, and the difference $C = \mathrm{offset}_A(w) - \mathrm{offset}_A(v)$ is the number of pairs $(y, x) \in \mathrm{NDes}(v)$ with either $x < -b < a < y$ or $-b < x < y < a$. On the other hand, the difference $D = \mathrm{offset}_B(w) - \mathrm{offset}_B(v) - 1$ is the number of pairs $(y, x) \in \mathrm{NDes}(v)$ with $-b < x < -a < 0 < a < y$. To prove that $\mathrm{rank}_B(w) = \mathrm{rank}_B(v) + 1$, it suffices to show that $A + B = C + D$. This is straightforward on noting that $\mathrm{NDes}(w)$ contains no elements (y, x) with $x < -b < y < a$ by Lemma 3.13(3), or with $-b < x < -a < 0 < y < a$ since $\mathrm{sh}(w)$ is noncrossing.

Next suppose that $v \lll_B w$ and let $i \in [n-1]$ be as in (4.2). Since we have $|v_{i+1}| < |v_i| = \max\{|v_1|, |v_2|, \dots, |v_i|\}$, and since inverse atoms do not have consecutive 321- or $\overline{1}\overline{2}$-patterns, it follows by Lemma 3.13 that there are chains of elements $v = v^0 \lessdot_A v^1 \lessdot_A \cdots \lessdot_A v^k$ and $w = w^0 \lessdot_A w^1 \lessdot_A \cdots \lessdot_A w^k$ with $v^k \lessdot_B^+ w^k$. By Proposition 6.1 and the previous paragraph, we deduce that $\mathrm{rank}_B(w) = \mathrm{rank}_B(w^k) - k = \mathrm{rank}_B(v^k) + 1 - k = \mathrm{rank}_B(v) + 1$.

The last assertion follows from Proposition 6.1 since $\mathrm{offset}_B(w) = 0$ only if $\mathrm{sh}(w)$ is trivial. $\square$

Propositions 6.1 and 6.2 let us conclude the following:

Corollary 6.3. If $z \in \mathcal{I}(W_n)$ then $(\mathcal{A}(z)^{-1}, <_A)$, $(\mathcal{A}(z)^{-1}, <_B)$, and $(\mathcal{A}(z)^{-1}, \ll_B)$ are graded.

A notable property of $(\mathcal{A}(z)^{-1}, <_B)$ and $(\mathcal{A}(z)^{-1}, \ll_B)$, apparent in Example 1.6 and Figure 1, is that these connected, graded posets have unique elements of maximal rank. To prove that this holds in general, we are lead to consider a third variation of the covering relation $\lessdot_B$.

Define $\blacktriangleleft\!\!\blacktriangleleft_B$ as the relation on n-letters words which has $v \blacktriangleleft\!\!\blacktriangleleft_B w$ if for some indices $1 \leq i < j < n$ and some positive numbers a, b, c it holds that $v_k = w_k$ for $k \notin \{i, j, j+1\}$ while

$$v_i v_j v_{j+1} = \overline{c}a\overline{b}, \quad w_i w_j w_{j+1} = \overline{a}b\overline{c}, \quad \text{and} \quad \max\{|w_1|, |w_2|, \dots, |w_{j-1}|\} = a < b < c. \tag{6.1}$$

Equivalently, we have $\cdots\overline{c}\cdots a\overline{b}\cdots \blacktriangleleft\!\!\blacktriangleleft_B \cdots\overline{a}\cdots b\overline{c}\cdots$ whenever the corresponding ellipses mask identical subsequences and $0 < a < b < c$ and all hidden letters to the left of a in the first word (equivalently, to the left of b in the second word) have absolute value less than a. As usual, we apply this relation to signed permutations via their one-line representations.

Proposition 6.4. Let $z \in \mathcal{I}(W_n)$. Suppose $v, w \in W_n$ are such that $v \mathrel{<\!\!\!\blacktriangleleft}_B w$. Then $v \in \mathcal{A}(z)^{-1}$ if and only if $w \in \mathcal{A}(z)^{-1}$, and if this holds then $\mathrm{rank}_B(w) = \mathrm{rank}_B(v) + 1$.

Proof. Let $1 \leq i < j < n$ be such that (6.1) holds. We prove the result by induction on $i + j$. Our argument relies on two base cases. When $i = j - 1 = 1$, the result follows from Lemma 4.1 and Proposition 6.2 since $\overline{c}\overline{b}\overline{a} \lhd_B \overline{c}a\overline{b}$ and $\overline{c}\overline{b}\overline{a} \lhd_B b\overline{c}\overline{a} \lhd_A \overline{a}b\overline{c}$ for any $0 < a < b < c$. When $i = j - 1 = 2$ the lemma follows similarly from the fact that if $0 < a < b < c < d$ then $\overline{d}\overline{c}\overline{b}\overline{a} \lhd_B \overline{d}b\overline{c}\overline{a} \lhd_A \overline{d}\overline{a}b\overline{c} \lhd_B a\overline{d}b\overline{c}$ and $\overline{d}\overline{c}\overline{b}\overline{a} \lhd_B c\overline{d}\,\overline{b}\overline{a} \lhd_A \overline{b}c\overline{d}\overline{a} \lhd_A \overline{b}\overline{a}c\overline{d} \lhd_B a\overline{b}c\overline{d}$.

For the inductive step, let $0 < a = v_j = -w_i < b = -v_{j+1} = w_j < c = -v_i = -w_{j+1}$. First suppose $i < j - 1$. Define v' from v by replacing the subword $v_{j-1}v_jv_{j+1}$ by $v_jv_{j+1}v_{j-1}$, and form w' from w similarly. Since $v_{j-1} = w_{j-1}$ is less than a in absolute value, it follows that $v' \lhd_A v$ and $w' \lhd_A w$ and $v' \mathrel{<\!\!\!\blacktriangleleft}_B w'$. By induction, the proposition holds with v and w replaced by v' and w', so by Lemma 4.1 and Proposition 6.2, the result also holds for v and w.

Suppose alternatively that $2 < i = j - 1$. We may assume that at least one of v or w belongs to $\mathcal{A}(z)^{-1}$. Since inverse atoms do not have consecutive 321- or $\overline{1}\,\overline{2}$-patterns and since all numbers in the subwords $v_1v_2 \cdots v_{i-1} = w_1w_2 \cdots w_{i-1}$ have absolute value less than a, it must hold that $v_i < v_{i-2} < v_{i-1}$ and $w_i < w_{i-2} < w_{i-1}$. Define v' from v by replacing the subword $v_{i-2}v_{i-1}v_i$ by $v_{i-1}v_iv_{i-2}$, and form w' from w similarly. We once again have $v' \lhd_A v$ and $w' \lhd_A w$ and $v' \mathrel{<\!\!\!\blacktriangleleft}_B w'$, and may deduce that the proposition holds by induction.

If $i = j - 1 \in \{1, 2\}$, finally, then we are in one of the base cases already considered. ☐

Fix $z \in \mathcal{I}(W_n)$ with $m = \mathrm{neg}(z)$, and suppose $\{i \in [\pm n] : z(i) = -i\} = \{a_1 < a_2 < \cdots < a_{2m}\}$. Let $M_{\max} = \{\{a_1, a_2\}, \{a_3, a_4\}, \ldots, \{a_{2m-1}, a_{2m}\}\} \in \mathsf{NCSP}(z)$ and define

$$1_B(z) = 1_B(z, M_{\max}). \tag{6.2}$$

For example, if $z = \overline{1}\,\overline{2}\,\overline{3}\,\overline{4}$ then $1_B(z) = 1\overline{2}3\overline{4}$ while if $z = \overline{1}\,\overline{2}\,\overline{3}\,\overline{4}\,\overline{5}$ then $1_B(z) = \overline{1}2\overline{3}4\overline{5}$. Theorem 5.6 implies that $1_B(z) \in \mathcal{A}(z)^{-1}$. We have $0_B(z) = 0_B(z, M_{\min})$ for $M_{\min} = \{\{i, -i\} : i \in \mathrm{Neg}(z)\}$. Define $\lll_B$ as the transitive closure of the three relations $\lhd_A$, $\mathrel{<\!\!\!\lhd}_B$, and $\mathrel{<\!\!\!\blacktriangleleft}_B$. For lack of a better term, we refer to $\lll_B$ as the *very strong atomic order* of type B.

Proposition 6.5. Restricted to $\mathcal{A}(z)^{-1}$ for any $z \in \mathcal{I}(W_n)$, the relation $\lll_B$ is a bounded, graded partial order, whose unique minimum is $0_B(z)$ and whose unique maximum is $1_B(z)$.

This result is reminiscent of Stembridge's discussion of the top and bottom classes of a permutation in [40, §4]. Beyond formal analogies, however, there does not seem to be a direct connection between our methods and the results in [40].

Proof. Let $z \in \mathcal{I}(W_n)$. The claim that $\lll_B$ is a graded partial order on $\mathcal{A}(z)^{-1}$ with $0_B(z)$ as its unique minimum is immediate from Propositions 6.2 and 6.4. Let $M \in \mathsf{NCSP}^1(z)$. By Corollary 5.8, it is enough to show that $1_B(z, M)$ is not maximal under $\lll_B$ if $M \neq M_{\max}$.

To this end, assume $M \neq M_{\max}$ and write $\{i \in [\pm n] : z(i) = -i\} = \{a_1 < a_2 < \cdots < a_{2m}\}$ as above. Since M is noncrossing and symmetric with at most one trivial block, there must exist a pair of nesting blocks $\{a_i, a_{k+1}\}, \{a_j, a_k\} \in M$ with $i < j < k$ and $m < j$. If possible, choose these blocks such that $i = 2m - k$ so that $\{a_i, a_{k+1}\}$ is trivial; this is always possible if M has a trivial block distinct from $\{a_m, a_{m+1}\}$. Let $a = a_j$, $b = a_k$, and $c = a_{k+1}$ so that $0 < a < b < c$. Since $\{a_i, a_{k+1}\}$ is then the only block $\{x, y\} \in M$ with $x < a < b < y$, the one-line representation of $1_B(z, M)$ has the form $\cdots \overline{c} \cdots a\overline{b} \cdots$ and all letters to the left of a in this word have absolute value less than a. Hence $1_B(z, M)$ is not maximal with respect to $\lll_B$, as needed. ☐

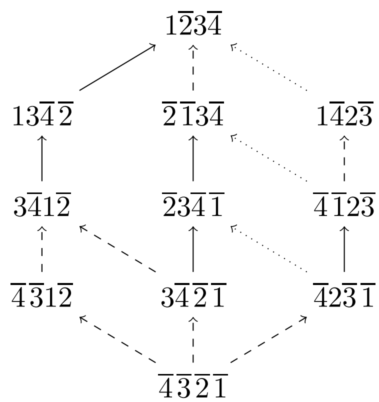


Figure 2: The Hasse diagram of $(\mathcal{A}(z)^{-1}, \lll_B)$ for $z = \overline{1}\,\overline{2}\,\overline{3}\,\overline{4} \in \mathcal{I}(W_4)$; compare with Example 1.6. The solid, dashed, and dotted arrows correspond to $\lhd_A$, $\ll\!\lhd_B$, and $\ll\!\blacktriangleleft_B$, respectively.

Corollary 6.6. If $z \in \mathcal{I}(W_n)$ then $1_B(z)$ is the unique element at which $\mathrm{rank}_B : \mathcal{A}(z)^{-1} \to \mathbb{N}$ attains its maximum value.

7 Hecke atoms

In this brief section, we convert a few of the preceding results into statements about the sets of Hecke atoms $\mathcal{A}_{\texttt{hecke}}(z) = \{w \in W_n : w^{-1} \circ w = z\}$. The symbol $\approx_A$ defined in the introduction is the weakest equivalence relation on words $w = w_1 w_2 \cdots w_n$ with $u \approx_A v \approx_A w$ if, for some $i \in [n-2]$, it holds that $u_j = v_j = w_j$ for $j \notin \{i, i+1, i+2\}$ while

$$u_i u_{i+1} u_{i+2} = cba, \qquad v_i v_{i+1} v_{i+2} = cab, \qquad \text{and} \qquad w_i w_{i+1} w_{i+2} = bca \tag{7.1}$$

for some numbers $a < b < c$. We apply $\approx_A$ to signed permutations via their one-line representations.

Lemma 7.1. If $v, w \in W_n$ are such that $v \approx_A w$, then $v \circ v^{-1} = w \circ w^{-1}$.

Proof. If $u, v, w \in W_n$ and $i \in [n-2]$ are as in (7.1) then we can write $u = \sigma t_i t_{i+1} t_i = \sigma t_{i+1} t_i t_{i+1}$, $v = \sigma t_{i+1} t_i$, and $w = \sigma t_i t_{i+1}$ for some $\sigma \in W_n$ with $\ell(u) = \ell(\sigma) + 3$ and $\ell(w) = \ell(v) = \ell(\sigma) + 2$. It follows that $u \circ u^{-1} = v \circ v^{-1} = w \circ w^{-1}$ since $t_i t_{i+1} t_i \circ t_i t_{i+1} t_i = t_{i+1} t_i \circ t_i t_{i+1} = t_i t_{i+1} \circ t_{i+1} t_i$. $\square$

Define $\approx_B$, as in the introduction, as the weakest equivalence relation on n-letter words which has $v \approx_B w$ when $v \approx_A w$, and which has $u \approx_B v \approx_B w$ either if $u_j = v_j = w_j$ for $j \notin \{1, 2\}$ while

$$u_1 u_2 = \overline{a}\overline{b}, \qquad v_1 v_2 = \overline{b}\overline{a}, \qquad \text{and} \qquad w_1 w_2 = a\overline{b} \tag{7.2}$$

for some numbers $0 < a < b$, or if $u_j = v_j = w_j$ for $j \notin \{1, 2, 3\}$ while

$$u_1 u_2 u_3 = \overline{c}\,\overline{a}\overline{b}, \qquad v_1 v_2 v_3 = \overline{c}\overline{b}\overline{a}, \qquad \text{and} \qquad w_1 w_2 w_3 = \overline{c}a\overline{b} \tag{7.3}$$

for some numbers $0 < a < b < c$. This relation includes $<_A$, $\sim_A$, $\approx_A$, $<_B$, and $\sim_B$ as subrelations.

Lemma 7.2. If $v, w \in W_n$ are such that $v \approx_B w$, then $v \circ v^{-1} = w \circ w^{-1}$.

Proof. If u, v, w are as in (7.2) then there exists a common element $\sigma \in W_n$ such that $u = \sigma \circ \overline{1}\,\overline{2}$, $v = \sigma \circ \overline{2}\,\overline{1}$, $w = \sigma \circ 1\overline{2}$, and $\ell(u) - 1 = \ell(v) = \ell(w) = \ell(\sigma) + 3$. If u, v, w are as in (7.3) then there exists $\sigma \in W_n$ such that $u = \sigma\circ\overline{3}\,\overline{1}\,\overline{2}$, $v = \sigma\circ\overline{3}\,\overline{2}\,\overline{1}$, $w = \sigma\circ\overline{3}1\overline{2}$, and $\ell(u)-1 = \ell(v) = \ell(w) = \ell(\sigma) = 6$. The lemma follows by checking that the inverses of $\overline{1}\,\overline{2} = t_1t_0t_1t_0$, $\overline{2}\,\overline{1} = t_0t_1t_0$, and $1\overline{2} = t_1t_0t_1$ are all Hecke atoms of $\overline{1}\,\overline{2}$, while the inverses of $\overline{3}\,\overline{1}\,\overline{2} = t_1t_0t_1t_0t_2t_1t_0$, $\overline{3}\,\overline{2}\,\overline{1} = t_0t_1t_0t_2t_1t_0$, and $\overline{3}1\overline{2} = t_1t_0t_1t_2t_1t_0$ are all Hecke atoms of $\overline{1}\,\overline{2}\,\overline{3}$. ☐

Lemma 7.3. Let u, v, w be words with n letters. Suppose, for some $i \in [n-1]$, that $v_1 < v_2 < \cdots < v_i = u_{i+1} = w_{i+1} < v_{i+1} = u_i = -w_i < 0$ and $u_j = v_j = w_j$ if $j \notin \{i, i+1\}$. Then $u \approx_B v \approx_B w$.

In other words, $\overline{z}\cdots\overline{c}\,\overline{a}\,\overline{b}\cdots \approx_B \overline{z}\cdots\overline{c}\,\overline{b}\,\overline{a}\cdots \approx_B \overline{z}\cdots\overline{c}a\overline{b}\cdots$ if $0 < a < b < c < \cdots < z$.

Proof. Define v' and v'' from v as in the proof of Lemma 4.2. That result already shows that $v \sim_B w$, which implies $v \approx_B w$, so we only need to check that $u \approx_B v$. Define u' and u'' from u analogously: in other words, form u' from u by replacing $u_{j-1}u_j$ by $\overline{u_j}u_{j-1}$ for each even index $j < i$; then, if i is odd (respectively, even), define u'' by removing the subword u_iu_{i+1} (respectively, $u_{i-1}u_iu_{i+1}$) from u' and placing it at the start of the word. Lemma 4.2 shows that that $u \sim_B u'$ and $v' \sim_B v$, it is an exercise to check that $u' <_A u''$ and $v' <_A v''$, and by definition $u'' \approx_B v''$. ☐

The following repeats Theorem 1.8 from the introduction.

Theorem 7.4. The $\approx_B$-equivalence classes in W_n are the sets $\mathcal{A}_{\texttt{hecke}}(z)^{-1}$ for $z \in \mathcal{I}(W_n)$.

Proof. Lemma 7.2 implies that each set $\mathcal{A}_{\texttt{hecke}}(z)^{-1}$ for $z \in \mathcal{I}(W_n)$ is preserved by $\approx_B$. Let $w \in W_n$. It suffices to show that w is equivalent under $\approx_B$ to an element of $\mathcal{A}(z)^{-1}$ for some $z \in \mathcal{I}(W_n)$. By Theorems 3.1 and 1.7, we have $v \approx_A w$ for a signed permutation $v \in W_n$ of the form $v = [[b_1a_1b_2a_2\cdots b_la_l]]$ where $a_i \leq b_i$ for each $i \in [l]$ and $a_1 < a_2 < \cdots < a_l$. Consider the minimal index j with $a_j < b_j < |a_j|$, if such an index exists. Use the relation $\sim_A$ to move all descents b_ia_i with $i < j$ and $a_j < 0 < |a_i| < b_i$ to the right of b_ja_j, then apply Lemma 7.3 to change b_ja_j to $a_j\overline{c}$ where $c = |b_j| < -a_j$, then use $\sim_A$ to move the descents b_ia_i back to their original positions, and finally use $\approx_A$ to transform the subword $[[\overline{c}b_{j+1}a_{j+1}\cdots b_la_l]]$ to a word of the form $[[b'_1a'_1\cdots b'_ka'_k]]$ where $a'_i \leq b'_i$ for each $i \in [k]$ and $a_1 < \cdots < a_j < a'_1 < \cdots < a'_k$. This results in an element equivalent to v under $\approx_B$ and of the same form, but in which the first occurrence of a one-line descent ba with $a < b < |a|$, if one exists, is farther to the right than before. By repeating this process and replacing v with the result, we may assume that $w \approx_B v$ where v, defined as above, has no descents b_ja_j with $a_j < b_j < |a_j|$. This element $v \in W_n$ is equal to $0_B(z) \in \mathcal{A}(z)^{-1}$ for the involution $z \in \mathcal{I}(W_n)$ whose cycles in $[\pm n]$ consist of $\{-a_j, a_j\}$ for each $j \in [l]$ with $a_j = b_j$ together with $\{a_j, b_j\}$ and $\{-a_j, -b_j\}$ for each $j \in [l]$ with $a_j < b_j$. ☐

8 Atomic elements

In this section, let n be a fixed positive integer. An involution z in a Coxeter group is *atomic* if it has exactly one atom, i.e., if $|\mathcal{A}(z)| = 1$. An element of a Coxeter group is *fully commutative* if each of its reduced words can be transformed to any other by a sequence of moves interchanging adjacent commuting simple factors.

Theorem 8.1 (See [14])**.** Every fully commutative involution in a Coxeter group is atomic.

In type A, there is an equivalence between these notions.

Theorem 8.2 (See [14]). If $z \in \mathcal{I}(S_n)$ then the following are equivalent: (a) z is atomic; (b) the permutations $0_A(z)$ and $1_A(z)$ are equal; (c) z has no cycles $\{a,d\},\{b,c\}$ with $a < b \leq c < d$; (d) z is 321-avoiding; (e) z is fully commutative.

Corollary 8.3. There are $\binom{n}{\lfloor n/2 \rfloor}$ atomic involutions in S_n.

Proof. It is well-known (see [35]) that this is the number of 321-avoiding involutions in S_n. $\square$

Every atomic involution in the affine symmetric group $\tilde{S}_n$ is also fully commutative [30, Corollary 6.17]. This property does not extend to W_n, however. The following is evident from Theorem 5.6:

Proposition 8.4. Let $z \in \mathcal{I}(W_n)$ and define $M_{\min} = \{\{-i,i\} : i \in \mathrm{Neg}(z)\}$. Then z is atomic if and only if $\mathrm{neg}(z) \leq 1$ and $0_B(z) = 0_B(z, M_{\min}) = 1_B(z, M_{\min})$.

The cycles of a signed permutation $w \in W_n$ are the orbits of the cyclic group $\langle w \rangle$ acting on the set $[\pm n]$. Each cycle of an involution $z \in \mathcal{I}(W_n)$ has one or two elements.

Proposition 8.5. An involution in W_n is atomic if and only if it has at most one negated point and it does not have two cycles $\{a,d\},\{b,c\}$ in $[\pm n]$ with $-d \neq a < b \leq c < d$.

Thus, an atomic involution can have nesting cycles, but only if the outer cycle is symmetric.

Proof. Suppose $z \in \mathcal{I}(W_n)$ has $\mathrm{neg}(z) \leq 1$ and $M_{\min} = \{\{-i,i\} : i \in \mathrm{Neg}(z)\}$. If the given condition holds then evidently no pairs $(a,d),(b,c) \in \mathrm{Cyc}_B(z)$ can have $a < b \leq c < d$, so $0_B(z, M_{\min}) = 1_B(z, M_{\min})$. Conversely, suppose z has two cycles $\{a,d\},\{b,c\}$ in $[\pm n]$ with $-d \neq a < b \leq c < d$. Since the set of cycles of z is symmetric under the map induced by $i \mapsto -i$, we may assume that $|a| < d$. By invoking this symmetry a second time, we may further assume that either $0 < b = c$ or $|b| < c$. But then (a,d) and (b,c) are both in $\mathrm{Cyc}_B(z)$, so $0_B(z, M_{\min}) \neq 1_B(z, M_{\min})$. $\square$

We can describe the atomic elements of $\mathcal{I}(W_n)$ more precisely. Let $\mathcal{X}_n^0$ and $\mathcal{X}_n^1$ be the sets of atomic involutions in W_n with 0 and 1 negated points, respectively, and let $\mathcal{X}_n = \mathcal{X}_n^0 \sqcup \mathcal{X}_n^1$. Define the *radius* of $z \in \mathcal{X}_n$ to be the largest integer $r \in [n]$ such that $z(r) < -r$, or 0 if no such r exists. Denote the radius of z by $\rho(z)$, and let $\mathcal{X}_{n,r} = \{z \in \mathcal{X}_n : \rho(z) = r\}$ and $\mathcal{X}_{n,r}^i = \mathcal{X}_n^i \cap \mathcal{X}_{n,r}$. The set $\mathcal{X}_{n,0}$ consists of the atomic involutions $z = z_1 z_2 \cdots z_n \in \mathcal{I}(W_n)$ with $z_i \in [n]$ for all i. These elements may be identified with the atomic involutions in S_n, so $|\mathcal{X}_{n,0}| = \binom{n}{\lfloor n/2 \rfloor}$.

Lemma 8.6. If $z \in \mathcal{X}_n$ then $\rho(z) \leq \lfloor n/2 \rfloor$.

Proof. Suppose $n < 2r$ and $z \in \mathcal{I}(W_n)$ has $z(r) < -r$. Since $[\pm n] \setminus [\pm r]$ has $2n - 2r < 2r$ elements, z must have a cycle $\{i,j\}$ with $z(r) < -r < i < j < r$, so z is not atomic by Proposition 8.5. $\square$

Let $r = \lfloor n/2 \rfloor$. When $n \geq 2$, define $\eta : \{\pm 1\}^{r-1} \to \mathcal{I}(W_n)$ as the map given as follows: for $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{r-1}) \in \{\pm 1\}^{r-1}$, let $a_1 < a_2 < \cdots < a_{r-1}$ be the numbers $i\epsilon_i$ for $i \in [r-1]$ listed in order, and let $\eta(\epsilon)$ be the unique involution $z \in \mathcal{I}(W_n)$ with $z(r+1) = -r$, with $z(r+1+i) = a_i$ for $i \in [r-1]$, and with $z(n) = n$ if $n = 2r+1$ is odd. The map η is clearly injective. For example, if $n = 7$ so $r = 3$ and $\epsilon = (-1, +1)$, then $a_1 = -1$ and $a_2 = 2$ so

$$\eta(\epsilon) = \overline{5}6\overline{4}\,\overline{3}\,\overline{1}27 = (3,\overline{4})(\overline{3},4)(5,\overline{1})(\overline{5},1)(6,2)(\overline{6},\overline{2}) \in \mathcal{I}(W_7).$$

Define $\mathcal{Y}_n = \left\{z \in \mathcal{X}_n^0 : z(\lfloor n/2 \rfloor) < -\lfloor n/2 \rfloor\right\}$ when $n \geq 2$, and set $\mathcal{Y}_0 = \mathcal{Y}_1 = \{1\} \subset W_n$.

Lemma 8.7. Let $r = \lfloor n/2 \rfloor$. If $n \geq 2$ then $\mathcal{X}^0_{n,r} = \mathcal{Y}_n$ and $\eta : \{\pm 1\}^{r-1} \to \mathcal{Y}_n$ is a bijection.

Proof. Assume $n \geq 2$ and let $z \in \mathcal{Y}_n$. Since z is atomic with no negated points, every number $i \in [\pm r]$ must have $z(i) < -r$ or $r < z(i)$. It follows that there are $2r$ numbers $i \in [\pm n] \setminus [\pm r]$ with $z(i) \in [\pm r]$. Since $[\pm n] \setminus [\pm r]$ has at most $2r+2$ elements, we deduce that $z(n) = n$ if n is odd, and that every $i \in [\pm 2r] \setminus [\pm r]$ has $z(i) \in [\pm r]$. Thus $\mathcal{X}^0_{n,r} \supset \mathcal{Y}_n$, so we have $\mathcal{X}^0_{n,r} = \mathcal{Y}_n$ since the reverse containment holds by definition. As $z(r) < -r$ and $r < z(-r)$, it must hold that $-r = z(r+1)$ since otherwise we would have $-r < z(r+1) < r+1 < z(r)$, contradicting Proposition 8.5. By the same lemma, it follows that $-r = z(r+1) < z(r+2) < \cdots < z(2r) < r$. We conclude that if $\epsilon \in \{\pm 1\}^{r-1}$ is the sequence of signs of $z(r+2), z(r+3), \ldots, z(2r)$ then $z = \eta(\epsilon)$.

Consider an arbitrary sequence $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{r-1}) \in \{\pm 1\}^{r-1}$. The involution $\eta(\epsilon)$ has no negated points and satisfies $\eta(\epsilon)(r) < -r$, so to finish the proof of the lemma it suffices to check that $\eta(\epsilon)$ is atomic, whence contained in $\mathcal{Y}_n$. This is easy to deduce from Proposition 8.5. $\square$

Let $\mathcal{Z}_n$ be the set of atomic involutions in S_n. For each $r \in \mathbb{N}$, define $\mathcal{Z}_{n,r}$ as the subset of involutions $w \in \mathcal{Z}_n$ with $i < w(i)$ for $i \in [r]$, so that $\mathcal{Z}_{n,0} = \mathcal{Z}_n$ and $\mathcal{Z}_{n,r} = \varnothing$ if $2r > n$.

Lemma 8.8. If $0 \leq r \leq \lfloor n/2 \rfloor$ then $|\mathcal{Z}_{n,r}| = \binom{n-r}{\lceil n/2 \rceil}$.

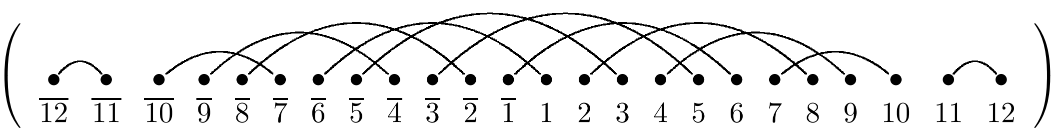
Proof. The formula for $|\mathcal{Z}_{n,0}|$ holds by Corollary 8.3. If $z \in \mathcal{Z}_n$ then either $z \in \mathcal{Z}_{n,1}$ or $z(1) = 1$, so $|\mathcal{Z}_n| = |\mathcal{Z}_{n-1}| + |\mathcal{Z}_{n,1}|$ and $|\mathcal{Z}_{n,1}| = |\mathcal{Z}_n| - |\mathcal{Z}_{n-1}| = \binom{n}{\lceil n/2 \rceil} - \binom{n-1}{\lceil (n-1)/2 \rceil} = \binom{n-1}{\lceil n/2 \rceil}$. Assume $0 < r \leq \lfloor n/2 \rfloor$. If $z \in \mathcal{Z}_{n,r}$ and $z(r+1) < r+1$ then necessarily $z(r+1) = 1$. In this case, removing 1 and $r+1$ from the one-line representation of z and standardizing what remains produces an arbitrary element of $\mathcal{Z}_{n-2,r-1}$. It is not possible for $z \in \mathcal{Z}_{n,r}$ to have $z(r+1) = r+1$ and the set of elements $z \in \mathcal{Z}_{n,r}$ with $r+1 < z(r+1)$ is precisely $\mathcal{Z}_{n,r+1}$. We conclude that $|\mathcal{Z}_{n,r}| = |\mathcal{Z}_{n-2,r-1}| + |\mathcal{Z}_{n,r+1}|$, so by induction $|\mathcal{Z}_{n,r+1}| = |\mathcal{Z}_{n,r}| - |\mathcal{Z}_{n-2,r-1}| = \binom{n-r}{\lceil n/2 \rceil} - \binom{n-r-1}{\lceil n/2 \rceil - 1} = \binom{n-r-1}{\lceil n/2 \rceil}$. $\square$

Fix $0 < r \leq \lfloor n/2 \rfloor$ and $x \in \mathcal{X}^0_{n,r}$. Let $I = [\pm r] \sqcup x([\pm r])$ and $J = x([\pm r]) \cap [n]$. Since $x([\pm r]) \cap [\pm r] = \varnothing$, and we have $|I| = 4r$ and $|J| = r$. Define $y = \phi^{-1} \circ x \circ \phi \in \mathcal{I}(W_{2r})$ where ϕ is the unique order-preserving bijection $[\pm 2r] \to I$. Now let $j_1 < j_2 < \cdots < j_r$ be the distinct elements of J, set $w = (1, j_1)(2, j_2) \cdots (r, j_r) \in S_n$, and define $z \in \mathcal{I}(S_n)$ as the involution with $z(i) = w(i)$ if $i \in [r] \sqcup J$ and with $z(i) = x(i)$ for all other $i \in [n]$. We write

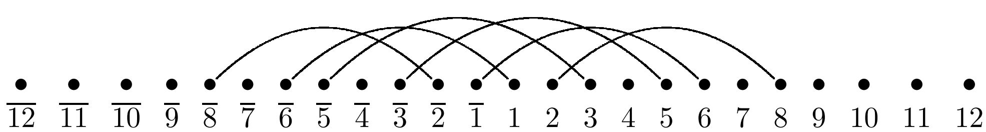
$$\pi^0 : \mathcal{X}^0_{n,r} \to \mathcal{I}(W_{2r}) \times \mathcal{I}(S_n)$$

for the map with $\pi^0(x) = (y, z)$. When $r = 0$ and $x \in \mathcal{X}^0_{n,0}$, we set $y = 1$ and $z = x|_{[n]} \in \mathcal{I}(S_n)$.

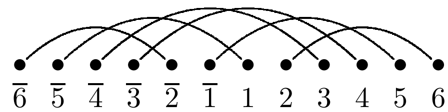
Example 8.9. This map may be understood in terms of the symmetric matchings on $[\pm n]$ which we draw to represent involutions in W_n. For example, if $n = 12$, $r = 3$, and $x \in \mathcal{X}^0_{n,r}$ is

$x =$ 

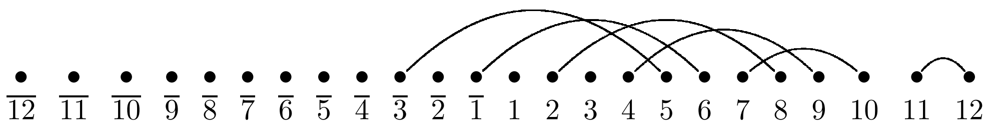
then y is obtained by first removing all edges which do not have an endpoint in $[\pm r]$ to get



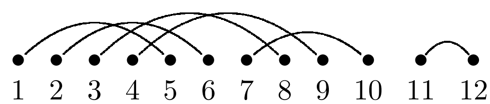
We then remove the isolated vertices from this picture and standardize what remains:

$y = \Big($  $\Big) \in \mathcal{I}(W_{2r})$.

To construct the involution z, we remove from the diagram of x all edges which do not have an endpoint in $[n] \setminus [r] = \{4, 5, \dots, 12\}$ to get



We then remove all isolated vertices up to $r = 3$ and relabel the endpoints $\overline{3}$, $\overline{1}$, 2 as 1, 2, 3:

$z = \Big($  $\Big) \in \mathcal{I}(S_n)$.

We also have a simpler map $\pi^1 : \mathcal{X}^1_{n+1,r} \to \mathcal{I}(W_n) \times \{r+1, r+2, \dots, n+1\}$ given as follows. If $x \in \mathcal{X}^1_{n+1,r}$ has $\mathrm{Neg}(x) = \{m\}$, and ψ is the unique order-preserving bijection $[\pm n] \to [\pm(n+1)] \setminus \{\pm m\}$, then we set $y = \psi^{-1} \circ x \circ \psi$ and $\pi^1(x) = (y, m)$. In terms of matchings, y is obtained from x by removing the single symmetric edge $\{-m, m\}$ and standardizing the remaining vertices.

Lemma 8.10. Suppose $0 \leq r \leq \lfloor n/2 \rfloor$.

(a) The map π^0 is a bijection $\mathcal{X}^0_{n,r} \to \mathcal{Y}_{2r} \times \mathcal{Z}_{n,r}$.

(b) The map π^1 is a bijection $\mathcal{X}^1_{n+1,r} \to \mathcal{X}^0_{n,r} \times \{r+1, r+2, \dots, n+1\}$.

Proof. One can verify the lemma directly when $r = 0$, so assume $0 < r \leq \lfloor n/2 \rfloor$. Let $x \in \mathcal{X}^0_{n,r}$ and $(y, z) = \pi^0(x) \in \mathcal{I}(W_{2r}) \times \mathcal{I}(S_n)$. By construction y has no negated points and satisfies $y(r) < -r$. It follows from Proposition 8.5 that y and z are also atomic, so we have $y \in \mathcal{Y}_{2r}$ and $z \in \mathcal{Z}_{n,r}$. To show that π^0 is a bijection, consider the inverse map defined as follows. Given $(y, z) \in \mathcal{Y}_{2r} \times \mathcal{Z}_{n,r}$, let $E = [\pm r] \sqcup z([r]) \sqcup -z([r])$, write θ for the unique order preserving bijection $[\pm 2r] \to E$, and define $x \in W_n$ as the permutation with $x(i) = (\theta \circ y \circ \theta^{-1})(i)$ for $i \in E$ and with $x(i) = z(i)$ and $x(-i) = -z(i)$ for $i \in [n] \setminus E$. Since y and z are both atomic and since y has no negated points, it follows from Proposition 8.5 that $x \in \mathcal{X}^0_{n,r}$, and it is easy to see that $(y, z) \mapsto x$ is the inverse of π^0, which is therefore a bijection.

For part (b), suppose $x \in \mathcal{X}^1_{n+1,r}$ and $(y, m) = \pi^1(x)$. Proposition 8.5 implies that the single negated point $m \in \mathrm{Neg}(x)$ is greater than r, so $y \in \mathcal{X}^0_{n,r}$. It is straightforward to construct an inverse map $\mathcal{X}^0_{n,r} \times \{r+1, r+2, \dots, n+1\} \to \mathcal{X}^1_{n+1,r}$, and we conclude that π^1 is also a bijection. $\square$

Theorem 8.11. Suppose $r \in \mathbb{N}$. The following identities hold:

(a) It holds that $|\mathcal{X}^0_{n,r}| = \lceil 2^{r-1} \rceil \binom{n-r}{\lceil n/2 \rceil}$ and $|\mathcal{X}^1_{n+1,r}| = (\lceil n/2 \rceil + 1) \lceil 2^{r-1} \rceil \binom{n-r+1}{\lceil n/2 \rceil + 1}$.

(b) If n is odd then $|\mathcal{X}^1_{n,r}| = \frac{1}{2}(n+1)|\mathcal{X}^0_{n,r}|$.

(c) If n is even and $r > 0$ then $|\mathcal{X}^1_{n,0}| = \frac{1}{2}(n+2)|\mathcal{X}^0_{n+1,1}|$ and $|\mathcal{X}^1_{n,r}| = \frac{1}{4}(n+2)|\mathcal{X}^0_{n+1,r+1}|$.

Proof. Part (a) follows from Lemmas 8.7, 8.8, and 8.10. Parts (b) and (c) follow from (a). $\square$

The elements of $\mathcal{X}_n^0$ are also naturally partitioned by their absolute lengths. Let $\mathcal{X}_n^{0,k}$ be the set of atomic involutions $z \in \mathcal{I}(W_n)$ with zero negated points and absolute length $\ell'(z) = k$. Equivalently, $\mathcal{X}_n^{0,k}$ is the set of atomic involutions in W_n with $2k$ distinct 2-element cycles in $[\pm n]$. To count the elements in these sets, we relate them to lattice paths of the following type.

Define $\mathcal{D}_n$ as the set of n-step paths $(p_0, p_1, \dots, p_n)$ in the nonnegative quadrant $\mathbb{N}^2$ which being at $p_0 = (0,0)$ and end at a point $p_n \in \{(n, 2m) : m \in \mathbb{N}\}$, which have $p_i - p_{i-1} \in \{(1,1), (1,-1), (1,0)\}$ for each $i \in [n]$, but which have $p_i - p_{i-1} = (1,0)$ only if p_i is on the x-axis. Paths of this type terminating at $(n,0)$ are sometimes called *dispersed Dyck paths*. Each path in $\mathcal{D}_n$ must have an even number of steps not equal to $(1,0)$. For each $k \in \mathbb{N}$ let $\mathcal{D}_{n,k}$ denote the subset of paths in $\mathcal{D}_n$ which have $p_i - p_{i-1} = (1,0)$ for exactly $n - 2k$ values of $i \in [n]$.

Lemma 8.12. If $0 \leq k \leq \lfloor n/2 \rfloor$ then $|\mathcal{D}_{n,k}| = \binom{n}{k}$.

Proof. Showing that $|\mathcal{D}_{2k,k}| = \binom{2k}{k}$ for $k \in \mathbb{N}$ is a standard exercise using the reflection principle (see, e.g., [10, §III.1]). among the $2k$-step paths starting at the origin in $\mathbb{Z}^2$ using just the steps $(1,1)$ and $(1,-1)$, those which do not stay in $\mathbb{N}^2$ are in bijection with those which do not terminate at $(2k, 0)$; the number of paths of the latter type is evidently $\sum_{j \neq k} \binom{2k}{j}$, and subtracting this from 2^{2k} gives $|\mathcal{D}_{2k,k}| = \binom{2k}{k}$. It is also apparent that $|\mathcal{D}_{n,0}| = 1$ for all $n \in \mathbb{N}$.

Assume $0 < k < \lfloor n/2 \rfloor$. The subset of paths in $\mathcal{D}_{n,k}$ beginning with a horizontal step are clearly in bijection with $\mathcal{D}_{n-1,k}$, while the subset of paths in $\mathcal{D}_{n,k}$ beginning with an up step are in bijection with $\mathcal{D}_{n-1,k-1}$ via the following operation: given a path in $\mathcal{D}_{n,k}$, remove its initial up step and replace the first down step which returns to the x-axis with a horizontal step. Such a down step exists since a path in $\mathcal{D}_{n,k}$ contains $n - 2k > 0$ horizontal steps. We deduce that $|\mathcal{D}_{n,k}| = |\mathcal{D}_{n-1,k}| + |\mathcal{D}_{n-1,k-1}|$, so by induction $|\mathcal{D}_{n,k}| = \binom{n}{k}$ for all $k \in \mathbb{N}$. $\square$

Theorem 8.13. If $0 \leq k \leq \lfloor n/2 \rfloor$ then $|\mathcal{X}_n^{0,k}| = \binom{n}{k}$.

Proof. By the previous lemma, it suffices to construct a bijection $\mathcal{X}_n^{0,k} \to \mathcal{D}_{n,k}$. Given $z \in \mathcal{X}_n^{0,k}$, define $p = (p_0, p_1, \dots, p_n)$ as the path starting at $p_0 = (0,0)$ for which the step $p_i - p_{i-1}$ is given by $(1,0)$, $(1,1)$, or $(1,-1)$ according to whether $j = -n + i - 1$ has $z(j) = j$, $j < z(j)$, or $z(j) < j$, respectively. It follows from Proposition 8.5 that $p \in \mathcal{D}_{n,k}$, so $z \mapsto p$ gives a map $\mathcal{X}_n^{0,k} \to \mathcal{D}_{n,k}$.

One defines an inverse map as follows. Fix a path $p = (p_0, p_1, \dots, p_n) \in \mathcal{D}_{n,k}$ and let U and D be the respective set of indices $i \in [n]$ where $p_i - p_{i-1} = (1,-1)$ and $p_i - p_{i-1} = (1,-1)$. Write $a_0 < a_1 < \dots < a_{2k-1}$ for the numbers in $\{-n+i-1 : i \in U\} \sqcup \{n-i+1 : i \in D\}$ arranged in order, and define $z \in \mathcal{I}(W_n)$ as the unique involution which has $z(a_i) = -a_{2k-i}$ for $i = 0, 1, \dots, 2k-1$ and which fixes all numbers not equal to a_i or $-a_i$ for some i. Since the path p remains in $\mathbb{N}^2$, we have $a_i < -a_{2k-i}$ for each i. An index $i \in [n]$ corresponds to a horizontal step in p if and only if $-n+i-1$ and $n-i+1$ are fixed points of z; since these steps are all at height zero, z has no fixed points b with $a < b < z(a)$ for any $a \in [\pm n]$. This is enough to conclude by Proposition 8.5 that z is atomic. Since z has $2k$ left endpoints $i \in [\pm n]$ with $i < z(i)$, it follows that z has no negated points and belongs to $\mathcal{X}_n^{0,k}$. Moreover, it holds essentially by definition that $p \mapsto z$ is the inverse of the map $z \mapsto p$ described in the first paragraph. Thus $|\mathcal{X}_n^{0,k}| = |\mathcal{D}_{n,k}| = \binom{n}{k}$. $\square$

Corollary 8.14. The number $a_n^0 = |\mathcal{X}_n^0|$ of atomic involutions in W_n with no negated points is

$$a_n^0 = \binom{n}{\lceil n/2 \rceil} + \sum_{r=1}^{\lfloor n/2 \rfloor} 2^{r-1}\binom{n-r}{\lceil n/2 \rceil} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} = \begin{cases} 2^{n-1} & \text{if } n \text{ is odd} \\ 2^{n-1} + \frac{1}{2}\binom{n}{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Proof. Rewrite $|\mathcal{X}_n^0| = \sum_r |\mathcal{X}_{n,r}^0| = \sum_k |\mathcal{X}_n^{0,k}|$ using Theorems 8.11 and 8.13. □

The sequence $\{a_n^0\}_{n=0,1,2,\dots} = (1,1,3,4,11,16,42,64,\dots)$ is [36, A027306].

Corollary 8.15. Let $a_n^1 = |\mathcal{X}_n^1|$ be the number of atomic involutions in W_n with one negated point.

(a) If n is odd then $a_n^1 = (n+1)2^{n-2}$.

(b) If n is even then $a_n^1 = \frac{1}{4}(n+2)\left(2^n - \binom{n}{n/2}\right)$.

Proof. Theorem 8.11 implies that if n is odd then $|\mathcal{X}_n^1| = \sum_r |\mathcal{X}_{n,r}^1| = \frac{1}{2}(n+1)|\mathcal{X}_n^0|$ and if n is even then $|\mathcal{X}_n^1| = \frac{1}{4}(n+2)\left(|\mathcal{X}_{n+1}^0| - |\mathcal{X}_{n+1,0}^0| + |\mathcal{X}_{n+1,1}^0|\right) = \frac{1}{4}(n+2)\left(|\mathcal{X}_{n+1}^0| - \binom{n+1}{n/2+1} + \binom{n}{n/2+1}\right)$. The corollary follows by substituting Corollary 8.14 and the identity $\binom{n+1}{n/2+1} = \binom{n}{n/2+1} + \binom{n}{n/2}$. □

Combining Corollaries 8.14 and 8.15 finally gives the following:

Corollary 8.16. The number $a_n = a_n^0 + a_n^1$ of atomic involutions in W_n is as follows:

(a) If n is odd then $a_n = (n+3)2^{n-2}$.

(b) If n is even then $a_n = (n+4)2^{n-2} - \frac{n}{4}\binom{n}{n/2}$.

The even-indexed terms of $\{a_n\}_{n=0,1,2,\dots} = (1,2,5,12,26,64,130,320,\dots)$ form the sequence [36, A003583]. The odd-indexed terms are a subsequence of [36, A045623].

Let $\mathcal{R}(w)$ be the set of reduced words for w and define $\hat{\mathcal{R}}(z) = \bigsqcup_{w\in\mathcal{A}(z)} \mathcal{R}(w)$ as in the introduction. We mention an application of the preceding results to the enumeration of the latter sets. Suppose $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$ is an integer partition and $\mu = (\mu_1 > \mu_2 > \cdots > \mu_l > 0)$ is a strict partition of a number N. The diagram of λ is the set of positions (i,j) for $i \in [l]$ and $j \in [\lambda_i]$, oriented as in a matrix. The shifted diagram of μ is the set of positions $(i, i+j-1)$ for $i \in [l]$ and $j \in [\mu_i]$. A *standard tableaux* of shape λ (respectively, *standard shifted tableaux*) is an arrangement of the numbers $1,2,\dots,N$ in the diagram of λ (respectively, the shifted diagram of μ) such that rows and columns are increasing from left to right and top to bottom. Let f^λ be the number of standard tableaux of shape λ and let g^μ be the number of standard shifted tableaux of shape μ. For the well-known hook-length formulas for these quantities, see [38].

Proposition 8.17. Let $p = \lfloor \frac{n+1}{2} \rfloor$, $q = \lceil \frac{n+1}{2} \rceil$, and $\gamma_n = \overline{n}\cdots\overline{2}\,\overline{1} \in \mathcal{I}(W_n)$. Then

$$|\hat{\mathcal{R}}(\gamma_n)| = f^\lambda = g^\mu = \frac{0!\cdot 1!\cdot 2!\cdots(p-1)!\cdot(pq)!}{q!\cdot(q+1)!\cdots(q+p-1)!}$$

where $\lambda = p^q = (p,p,\dots,p)$ and $\mu = (n, n-2, n-4, \dots)$.

An element $w \in W_n$ is *Grassmannian* if $w(1) < w(2) < w(3) < \dots$. The only Grassmannian involutions in W_n apart from 1 are the permutations γ_m for $m \leq n$ defined in this proposition.

Proof. By Lemma 4.6 and Proposition 8.5, γ_n is atomic and its unique atom $v = 0_B(z)^{-1}$ is either $v = \overline{n}\,\overline{(n-2)}\cdots\overline{4}\,\overline{2}135\cdots(n-1)$ or $v = \overline{n}\,\overline{(n-2)}\cdots\overline{3}\,\overline{1}246\cdots(n-1)$. We have $|\mathcal{R}(v)| = g^\mu$ by [28, Corollaries 3.3 and 4.4] (see also [2, Proposition 3.14]); the second equality is a special case of [11, Proposition 8.11]; and the third equality holds by standard hook-length formulas. □

The sequence $\{|\hat{\mathcal{R}}(\gamma_n)|\}_{n=1,3,5,\dots} = (1, 2, 42, 24024, 701149020, \dots)$ is [36, A039622]. These numbers also count the reduced words for $\overline{1}\,\overline{2}\cdots\overline{q}$ where $q = (n+1)/2$ by [12, Theorem 5.12]. The numbers $\{|\hat{\mathcal{R}}(\gamma_n)|\}_{n=2,4,6,\dots} = (1, 5, 462, 1662804, \dots)$ are sequence [36, A060855], and count the *winnowed expressions* for $\overline{1}\,\overline{2}\cdots\overline{q}$ where $q = n/2$, that is, the expressions obtained by omitting all factors t_0 from a reduced word [12, Theorem 5.16].

9 Future directions

We include some comments about related conjectures and open problems.

It seems possible that the atomic orders $<_A$, $<_B$, $\ll_B$, and $\lll_B$ have even more structure than what is shown in this paper. The following is stated in [30] in a slightly more general form:

Conjecture 9.1. If $z \in \mathcal{I}(S_n)$ then $(\mathcal{A}(z)^{-1}, <_A)$ is a lattice.

The poset $(\mathcal{A}(z)^{-1}, \lll_B)$ is not always a lattice for $z \in \mathcal{I}(W_n)$; counterexamples exist when $n = 7$. Computations show that $(\mathcal{A}(z)^{-1}, \ll_B)$ is a lower semilattice for at least $n \leq 8$, however.

Question 9.2. Can one define a partial order on $\mathcal{A}(z)^{-1}$ extending $\lll_B$ so that it is a lattice?

More generally, it would be interesting to define a type-independent partial order on $\mathcal{A}(z)$ so that it is a lattice.

It is an open problem to efficiently determine the sets $\mathcal{A}(z)$ when z is an involution in an arbitrary Coxeter group. A combinatorial description of the atoms for involutions in affine symmetric groups appears in [30]. For finite Coxeter groups we have the following criterion [14, Theorem 4.12] which is conjectured to hold for arbitrary (twisted) Coxeter systems.

Theorem 9.3. Let W be a finite Coxeter group and $z \in \mathcal{I}(W)$. Then $\mathcal{A}(z)$ is the set of elements $w \in W$ of minimal possible length such that $wz \leq w$.

This succinct classification tends not to be very explicit in practice.

Problem 9.4. Give a type-independent combinatorial characterization of the sets $\mathcal{A}(z)$.

A natural first step towards solving this general problem is the following:

Problem 9.5. Give combinatorial descriptions of the atoms in all finite and affine Coxeter groups.

Suppose (W, S) is a Coxeter system with an involution $* \in \mathrm{Aut}(W)$ that preserves the set S. When $z \in \mathcal{I}_*(W) = \{w \in W : w^* = w^{-1}\}$, it is natural to study the sets of (twisted) atoms $\mathcal{A}_*(z)$, consisting of the minimal length elements $w \in W$ with $(w^*)^{-1} \circ w = z$. This is the perspective of [14], and the previous problem should be considered in this more general context. The present work only considers the case when $* = \mathrm{id}$ since this is the only possibility if (W, S) has type B.

A Index of symbols

The table below lists our non-standard notations, with references to definitions where relevant.

Symbol	Meaning	
S_X	The group of permutations of a finite set X	§3
Ψ_n	An injective group homomorphism $W_n \to S_{2n}$	(2.1)
s_i	The adjacent transposition $(i, i+1) \in S_n$	
t_i	Either $(-1, 1)$ for $i = 0$ or $(-i-1, -i)(i, i+1)$ for $i \in [n-1]$	
$\mathrm{Cyc}_A(z)$	The set $\{(a, b) \in X \times X : a \leq b = z(a)\}$ for $z \in \mathcal{I}(S_X)$	
$\mathrm{Pair}(z)$	The set $\{(a, b) \in [\pm n] \times [n] : \lvert a\rvert < z(a) = b\}$ for $z \in \mathcal{I}(W_n)$	
$\mathrm{Neg}(z)$	The set $\{i \in [n] : z(i) = -i\}$ for $z \in \mathcal{I}(W_n)$	
$\mathrm{Fix}(z)$	The set $\{i \in [n] : z(i) = i\}$ for $z \in \mathcal{I}(W_n)$	
$\mathrm{Des}(w)$	The set of pairs (w_i, w_{i+1}) with $w_i > w_{i+1}$ for a word w	§3
$\mathrm{NDes}(w)$	Nested descent set of an inverse atom w for $z \in \mathcal{I}(W_n)$	(3.2)
$\mathrm{NFix}(w)$	Nested fixed points of an inverse atom w for $z \in \mathcal{I}(W_n)$	Def. 3.10
$\mathrm{NNeg}(w)$	Nested negated points of an inverse atom w for $z \in \mathcal{I}(W_n)$	Def. 3.10
$\mathsf{NCSP}(z)$	Noncrossing symmetric perfect matchings on $\{i : z(i) = -i\}$	§5
$\mathrm{sh}(w)$	Shape of an inverse atom $w \in \mathcal{A}(z)^{-1}$ for $z \in \mathcal{I}(W_n)$	§5
$\mathrm{Pair}(z, M)$	Variant of $\mathrm{Pair}(z, M)$ for $z \in \mathcal{I}(W_n)$ and $M \in \mathsf{NCSP}(z)$	(5.1)
$\mathrm{Neg}(z, M)$	Variant of $\mathrm{Neg}(z, M)$ for $z \in \mathcal{I}(W_n)$ and $M \in \mathsf{NCSP}(z)$	(5.1)
$\mathrm{Cyc}_B(z, M)$	A certain set of pairs for $z \in \mathcal{I}(W_n)$ and $M \in \mathsf{NCSP}(z)$	(5.1)
$\lhd_A$	Covering relation with $\cdots cab \cdots \lhd_A \cdots bca \cdots$	(1.2)
$\lhd_B$	Covering relation with $\overline{b}\overline{a}\cdots \lhd_B a\overline{b}\cdots$ and $\overline{c}\overline{b}\overline{a}\cdots \lhd_B \overline{c}a\overline{b}\cdots$	(1.3)
$\lhd_B^+$	A stronger form of $\lhd_B$	(4.1)
$\lhd\!\!\lhd_B$	A stronger form of $\lhd_B^+$	(4.2)
$\blacktriangleleft\!\!\blacktriangleleft_B$	A stronger form of $\lhd\!\!\lhd_B$	(6.1)
$<_A$	The transitive closure of $\lhd_A$	
$<_B$	The transitive closure of $\lhd_A$ and $\lhd_B$	
$\ll_B$	The transitive closure of $\lhd_A$ and $\lhd\!\!\lhd_B$	
$\lll_B$	The transitive closure of $\lhd_A$, $\lhd\!\!\lhd_B$, and $\blacktriangleleft\!\!\blacktriangleleft_B$	
$\sim_A$	The symmetric closure of $<_A$	
$\sim_B$	The symmetric closure of $<_B$	
$\approx_A$	Equiv. relation with $\cdots cba \cdots \sim_A \cdots cab \cdots \sim_A \cdots bca \cdots$	(7.1)
$\approx_B$	Type B analogue of the equivalence relation $\approx_A$	(7.2)
$0_A(z)$, $1_A(z)$	Extremal inverse atoms for $z \in \mathcal{I}(S_n)$ under $<_A$	(3.1)
$0_B(z, M)$	Minimal inverse atom for $z \in \mathcal{I}(W_n)$ under $<_A$	(5.2)
$1_B(z, M)$	Maximal inverse atom for $z \in \mathcal{I}(W_n)$ under $<_A$	(5.2)
$0_B(z)$	Minimum inverse atom for $z \in \mathcal{I}(W_n)$ under $<_B$	(4.4)
$1_B(z)$	Maximum inverse atom for $\in \mathcal{I}(W_n)$ under $\lll_B$	(6.2)

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