

Beyond series-parallel concurrent systems: the case of arch processes*

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Abstract

In this paper we focus on concurrent processes built on *synchronization* by means of *futures*. This concept is an abstraction for processes based on a main execution thread but allowing to delay some computations. The structure of a general concurrent process with futures is more or less a directed acyclic graph. Since the quantitative study of such increasingly labeled graphs (directly related to processes) seems out of reach, we restrict ourselves to the study of *arch processes*, a simplistic model of processes with futures. They are based on two parameters related to their sizes and their numbers of arches. The increasingly labeled structures seems not to be specifiable in the sense of Analytic Combinatorics, but we manage to derive a recurrence equation for the enumeration.

For this model we first exhibit an *exact* and an *asymptotic* formula for the number of runs of a given process. The second main contribution is composed of an *uniform random sampler* algorithm and an *unranking* one that allow efficient generation and exhaustive enumeration of the runs of a given arch process.

Keywords: Concurrency Theory; Future; Uniform Random Sampling; Unranking; Analytic Combinatorics.

1 Introduction

Our study consists in the increasing labeling of combinatorial structures, which tightly relates to the notion of behaviors of concurrent processes. We conduct this study by using tools of Analytic Combinatorics. This work is a part of a long time project to understand the so-called *combinatorial explosion* phenomenon about the number of runs (or executions) of concurrent processes. In previous works the authors studied tree-like processes [BGP16], tree-like processes with non-deterministic choice [BGP13] and Series-Parallel processes [BDF⁺16, BDGP17b].

The main common idea consists in modeling a concurrent process as a partial order over the atomic actions of the process. Thus some precedence relations describe the process. In this way the runs of the process correspond to linear extensions of the poset. Then, we reinterpret this modelization in term of combinatorial structures (trees, Series-Parallel graphs, directed acyclic graph, ...) where increasing labelings are in one-to-one correspondence with the runs of the process. Until the present work all the structures were *decomposable*, in the sense of [FS09], by recursive specifications. For each of these families the objectives have always been the same: understanding the growth of the number of runs for large concurrent processes, which means understanding the combinatorial explosion phenomenon; and tuning efficient algorithm for the

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uniform random generation of increasing labelings which is a practical way to circumvent the combinatorial explosion phenomenon (see for example [GS05]).

In the present work we focus on processes built on *synchronization* by means of *futures* (or *promises*). This concept is an abstraction for processes based on a main execution thread but allowing to delay some computations. These computations are run asynchronously and are represented by an object that can be queried in two ways: `finish?` to know if the computation has terminated and `get` to retrieve the result of the computation (and properly proceed the synchronization).

The structure of a general concurrent process with futures is more or less a directed acyclic graph. Since the quantitative study of such increasingly labeled graphs seems out of reach, we restrict ourselves to the study of *arch processes*, a simplistic model of processes with futures. An arch process is composed of a main *trunk* from which start several *arches* (modelizing futures). The general shape of such a process is given in Fig. 1. The arch processes are based on two parameters related to their sizes and their numbers of arches. To our knowledge the increasingly labeled structures are not specifiable in the sense of [FS09].

For this limited model we exhibit an *exact* and an *asymptotic formula* for the number of increasing labelings. The second main contribution of this paper is composed of two algorithms. The first one is a uniform random sampler for runs of a given arch process and the second one is an *unranking* algorithm which allows to obtain an exhaustive builder of runs.

The paper is organized as follows. The next section is devoted to the formal description of (n, k) -arch processes and gives the solution of the recurrence equation driving their numbers of runs. In Section 3 we prove the algebraicity of the bivariate generating function, we give a closed form formula for it and, we give the asymptotic behaviors of the diagonal coefficients of the functions. Section 4 carefully describes both algorithms.

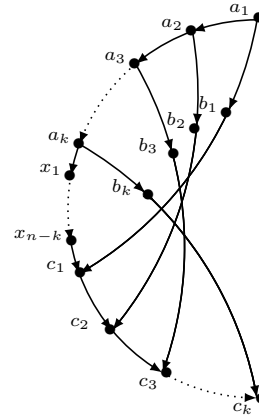


Figure 1: The (n, k) -arch process

2 The arch processes and their runs

Definition 1. Let n and k be two positive integers with $k \leq n + 1$. The (n, k) -arch process, denoted by $A_{n,k}$, is built in the following way:

- the trunk of the process: a sequence of $n+k$ actions $a_1, a_2, \dots, a_k, x_1, x_2, \dots, x_{n-k}, c_1, c_2, \dots, c_k$ represented in Fig. 1 on a semicircle;
- for all $i \in \{1, \dots, k\}$, the actions a_i and c_i are directly linked by an arch containing a single action b_i .

We remark the value k corresponds to the number of arches in the process, and n is the length (in the trunk) between both extremities of each arch. There are two extreme cases: when $k = n$, it corresponds to the arch processes that do not contain any node x_i in the trunk, and the case $k = n + 1$ that corresponds to the case where both the nodes a_k and c_1 are merged into a single node (and thus there is no node x_i).

In Fig. 1 representing the (n, k) -arch process, the precedence constraints are encoded with the directed edges such that $a \rightarrow b$ means that the action a precedes b . We remark that the (n, k) -arch process contains exactly $n + 2k$ actions.

Due to the intertwining of the arches, we immediately observe when k is larger than 1 then the arch processes are not Series-Parallel processes. Hence the results we exhibited in our papers [BDGP17a, BDGP17b] cannot be applied in this context.

Definition 2. For a concurrent process a run is a total order of the actions that is compatible with the precedence constraints describing the process.

Definition 3. An increasing labeling for a concurrent process containing ℓ actions is a bijection between the integers $\{1, \dots, \ell\}$ and the actions of the process, satisfying the following constraint: if an action a precedes an action b then the label associated to a is smaller than the one related to b .

In Fig. 2 we have represented an increasing labeling of the $(5, 4)$ -arch process $A_{5,4}$ corresponding to the run $\langle a_1, b_1, a_2, a_3, b_3, a_4, x_1, b_4, c_1, b_2, c_2, c_3, c_4 \rangle$. As one can see, every directed path (induced by the precedence relation) is increasingly labeled. Our quantitative goal is to calculate the number of runs for a given arch process.

Proposition 4. The number of runs of a concurrent process is the number of increasing labelings of the actions of the process.

While there is the classical hook-length formula for tree-processes [Knu98, BGP16] and its generalization for Series-Parallel processes [BDGP17a], to the best of our knowledge, no closed form formula is known for more general classes of processes. In the rest of the paper, for a given process A , we denote by $\sigma(A)$ its number of runs.

First, let us easily exhibit a lower bound and an upper bound (in the case $k < n + 1$) in order to obtain a first idea for the growth of the numbers of runs for the arch processes. Remark that a similar approach could be used for the case when $k = n + 1$. We first enumerate the runs where all the nodes b_i 's are preceded by a_k , and all of them precede the node c_1 . This imposes new precedence constraints for the process, and thus its number of runs is a lower bound for the total number of runs. In this case the b_i 's permute without any constraint, i.e. $k!$ possibilities and then each permutation of the b_i 's shuffles with the sequence x_1, \dots, x_{n-k} . Thus we get the following lower bounds for the number of runs of $A_{n,k}$:

$$\sigma(A_{n,k}) \geq k! \binom{k+n-k}{k} = \frac{n!}{(n-k)!}.$$

We now focus on an upper bound for the number of runs of $A_{n,k}$. Here again we suppose that all the permutations of the b_i 's are possible, but we allow each b_i to appear everywhere between a_1 and c_k . This constraint is satisfied by all the runs, but some possibilities are not valid runs: thus we are computing an upper bound. Once the permutation of the b_i 's is calculated, we shuffle it into the trunk (containing $n + k$ nodes):

$$\sigma(A_{n,k}) \leq k! \binom{k+n+k-1}{n+k-1} = \frac{(n+2k-1)!}{(n+k-1)!}.$$

A refinement of these ideas for the bounds computation allows to exhibit a recurrence formula for the value $\sigma(A_{n,k})$.

Theorem 5. Let n and k be two integers such that $0 \leq k \leq n + 1$. The number $\sigma(A_{n,k})$ of runs of the process $A_{n,k}$ is equal to $t_{n,k}$ with:

$$t_{n,k} = \frac{n+2k-1}{2} t_{n,k-1} + \frac{n-k}{2} t_{n+1,k-1} \quad \text{and} \quad t_{n,0} = 1. \quad (1)$$

In order to provide the proof, we first introduce the four processes in Fig.3. Notice that they are not arch processes. From the left handside to right handside, the first process, denoted by $D_{n,k}$, is almost the process $A_{n,k}$. In fact, the single difference is that $D_{n,k}$ contains exactly one more action, denoted by c'_1 , that is preceded by all the other actions. The second process $\bar{D}_{n,k}$ is related to $D_{n,k}$ in the following way: the precedence relation starting at b_1 is replaced, instead of having $b_1 \rightarrow c_1$, it is $b_1 \rightarrow c'_1$. Finally, for the two last processes $\bar{D}_{n,k}^1$ and $\bar{D}_{n,k}^2$, it is also the relations $a_1 \rightarrow b_1 \rightarrow c_1$ which are modified.

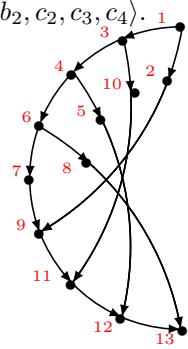


Figure 2: A run of the $(5, 4)$ -arch process

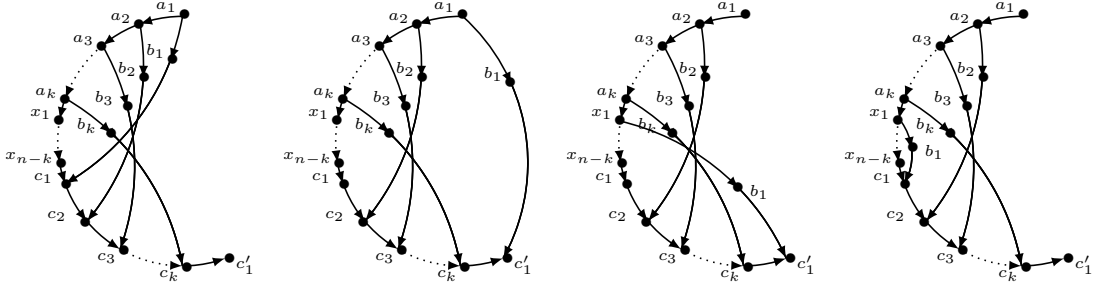


Figure 3: From left to right, the processes denoted $D_{n,k}$, $\bar{D}_{n,k}$, $\bar{D}_{n,k}^1$ and $\bar{D}_{n,k}^2$

Proof. The extreme case $A_{n,0}$ corresponds to a process without any arch: just a trunk. Obviously it admits a single increasingly labeling: it has a single run.

Suppose first that $k < n + 1$. The number $\sigma(A_{n,k})$ is equal to the number of runs $\sigma(D_{n,k})$ because for all runs, the integer associated to c'_1 is inevitably the largest one: $2k + n + 1$. Then we compute with some inclusion/exclusion rule the number $\sigma(D_{n,k})$:

$$\sigma(D_{n,k}) = \sigma(\bar{D}_{n,k}) - \left(\sigma(\bar{D}_{n,k}^1) - \sigma(\bar{D}_{n,k}^2) \right). \quad (2)$$

In fact we are focusing on the action preceded by b_1 . In $D_{n,k}$ it corresponds to c_1 . By modifying it by c'_1 in $\bar{D}_{n,k}$ we allow runs where b_1 appears after c_1 , thus that are not valid for $D_{n,k}$. We remove this number of non-valid runs with $\sigma(\bar{D}_{n,k}^1) - \sigma(\bar{D}_{n,k}^2)$, by playing with both actions x_1 and c_1 . To compute $\sigma(\bar{D}_{n,k})$, first omit the action b_1 (and its incoming and outgoing edges); the remaining process is a $(n, k - 1)$ -arch process, up to renaming, with added top and bottom actions (a_1 and c'_1) which do not modify the number of runs of $A_{n,k-1}$. It remains to insert b_1 in this “almost” $A_{n,k-1}$, somewhere between a_1 and c'_1 : there is $(2 \cdot (k - 1) + n - 1) + 2 = 2k + n - 1$ possibilities. The term $(2 \cdot (k - 1) + n - 1)$ are the cases where b_1 is put between a_2 and c_k and the term 2 corresponds to the cases where b_1 is either before a_2 or after c_k . The process $\bar{D}_{n,k}^1$ is similar to the arch process $A_{n,k}$, there is only an action a_1 that precedes it, so $\sigma(\bar{D}_{n,k}^1) = t_{n,k}$. Lastly, for the process $\sigma(\bar{D}_{n,k}^2)$, forgetting b_1 we recognize $A_{n+1,k-1}$ up to renaming, so b_1 can be inserted between x_1 and c_1 : there are $n - k$ possibilities. Finally we obtain the following equation

$$\sigma(A_{n,k}) = (n + 2k - 1) \cdot \sigma(A_{n,k-1}) - \sigma(A_{n,k}) + (n - k) \cdot \sigma(A_{n+1,k-1}).$$

Induction principle let us conclude that equation (1) is proved.

Suppose now that $k = n + 1$. Here there is no action x_i and both the nodes a_k and c_1 are merged into a single node. We can adapt equation (2) and obtain the same recurrence, but via a small difference in the computation:

$$\sigma(A_{k-1,k}) = 3k \cdot \sigma(A_{k-1,k-1}) - \sigma(A_{k-1,k}) - \sigma(A_{k,k-1}).$$

But since $k = n + 1$, this recurrence is equal to equation (1) too. \square

Notice when $k > n + 1$, for our model, it consists to merge the last actions a_{n-i} 's with the first actions c_i 's. But the recursive formula (1) does not apply to such models: once $k > n + 1$ the recurrence loses its combinatorial meaning.

The next result exhibits a closed form formula for the number of runs of the arch processes.

Theorem 6. *Let n and k be integers such that $0 < k \leq n + 1$. The number¹ of runs of the (n, k) -arch process is*

$$\sigma(A_{n,k}) = \frac{(2k + n - 1)!!}{2^{k-1}} \sum_{s=0}^{k-1} \frac{(n + s) \text{par}(n, s)}{(n + s + 1)!!} \sum_{1 \leq i_1 < i_2 < \dots < i_s \leq k} \prod_{j=1}^s (i_j + j + n - k - 1) \frac{\Gamma\left(\frac{2k + n - 2i_j + j}{2} + 1\right)}{\Gamma\left(\frac{2k + n - 2i_j + j + 1}{2} + 1\right)},$$

¹In Theorem 6 we use the convention that the sum over the sequence of i_j 's is equal to 1 when $s = 0$.

with the following function

$$\text{par}(n, s) = \begin{cases} \frac{1}{2^{s/2}} & \text{if } s \text{ is even} \\ \frac{\sqrt{\pi}}{2^{(s+1)/2}} & \text{if } s \text{ is odd and } n \text{ is even} \\ \frac{1}{2^{(s-1)/2}\sqrt{\pi}} & \text{if } s \text{ is odd and } n \text{ is odd.} \end{cases}$$

Let us recall the double factorial notation: for $n \in \mathbb{N}$, $n!! = n \cdot (n-2)!!$ with $0!! = 1!! = 1$. We remark that the ratio of the two Γ -function is related to the central binomial coefficient. The asymptotic behavior of the sequence does not seem immediate to obtain using this formula.

key-ideas. The formula for $\sigma(A_{n,k})$ is obtained by resolving the recurrence stated in equation (1). First remark that the calculation of $\sigma(A_{n,k})$ requires the values of $\sigma(A_{i,j})$ in the triangle such that $n \leq i \leq n+k$ and $0 \leq j \leq k - (i-n)$. The formula is computed by unrolling k times the recurrence. In particular, the index s in the formula corresponds to the number of times we have used the second term of equation (1), to reach the final term $\sigma(A_{n+s,0})$. The i_j 's values indicate in which iteration the second terms of equation (1) have been chosen. They describe the path from (n, k) to $(n+s, 0)$. The brute formula obtained in this way is composed of a product of truncated double factorials that can be written as ratios of double factorial numbers. Finally, by coupling the adequate numerators and denominators in the product we exhibit several Wallis's ratios [AS64] that are easily simplified by using the Γ function: $\frac{(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}$. \square

By using this closed form formula, or the bivariate recurrence (cf. equation (1)), we easily compute the first diagonals of the recurrence. The values of a given diagonal correspond to the class of arch processes with the same number of actions x_i 's in the trunk.

$$\begin{aligned} (\sigma(A_{k-1,k}))_{k \in \mathbb{N} \setminus \{0,1\}} &= (1, 12, 170, 2940, 60760, 1466640, 40566680, 1266064800, 44030186200, 1688858371200, \dots) \\ (\sigma(A_{k,k}))_{k \in \mathbb{N}^*} &= (1, 5, 44, 550, 8890, 176120, 4130000, 111856360, 3435632200, 117991273400, \dots) \\ (\sigma(A_{k+1,k}))_{k \in \mathbb{N}^*} &= (2, 11, 100, 1270, 20720, 413000, 9726640, 264279400, 8137329200, 280012733000, \dots) \\ (\sigma(A_{k+2,k}))_{k \in \mathbb{N}^*} &= (3, 19, 186, 2474, 41670, 850240, 20386800, 561863960, 17501627640, 608063465800, \dots) \end{aligned}$$

We remark that the first terms of the sequence $(\sigma(A_{k+1,k}))_{k \in \mathbb{N}^*}$ coincide with the first terms of the sequence A220433 (shifted by 2) in OEIS². This sequence is related to a specific Alia algebra and is exhibited in the paper of Khoroshkin and Piontkovski [KP15]. In their paper, the exponential univariate generating function naturally appears as an algebraic function. This motivates us to study in detail the bivariate generating function for $(t_{n,k})$ and in particular its diagonals.

3 Algebraic generating functions

Let us associate to the bivariate sequence $(t_{n,k})_{n,k}$ the generating function, denoted by $A(z, u)$, exponential in u and ordinary in z :

$$A(z, u) = \sum_{n \geq 0, k \geq 0} \frac{t_{n,k}}{k!} z^n u^k.$$

Recall this series enumerates the increasing labelings of the arch processes, when $k \leq n+1$, but has no combinatorial meaning beyond this bound.

Proposition 7. *The bivariate generating function $A(z, u)$ is holonomic and satisfies the following differential equation.*

$$(2zu - 2z - u) \frac{\partial}{\partial u} A(z, u) + (z - 2) A(z, u) + z(z + 1) \frac{\partial}{\partial z} A(z, u) + C(u) = 0.$$

The differential equation can be exhibited since the recursive behavior of $(t_{n,k})$ is not disturbed beyond the bound $k > n+1$.

²OEIS corresponds to the On-line Encyclopedia of Integer Sequences: <http://oeis.org/>.

key-ideas. The differential equation is directly obtained from the recurrence equation (1). The function $C(u)$ encodes the initial conditions of the equation. The differential equation satisfied by $A(z, u)$ ensures its holonomicity (cf. [Sta01, FS09]). \square

It is important to remark that $C(u)$ is holonomic. In fact we have $C(u) = u \frac{\partial}{\partial u} A(0, u) + 2A(0, u)$ and consequently $C(u)$ is holonomic as a specialization of an holonomic bivariate generating function. A direct computation for $C(u)$ exhibits the following differential equation

$$4(24u^2 + 3u + 1)C(u) - 4u(84u^2 - 3u + 1)\frac{d}{du}C(u) - 2u^2(216u^2 - 151u + 13)\frac{d^2}{du^2}C(u) - 2u^2(58u^3 - 75u^2 + 33u - 2)\frac{d^3}{du^3}C(u) - u^3(8u^3 - 15u^2 + 12u - 4)\frac{d^4}{du^4}C(u) - 8(3u + 1) = 0.$$

Note that we prove also that $C(u)$ is solution of an algebraic equation. This fact is really not obvious from a combinatorial point of view. But it is deduced through the fact that the function $A(0, u)$ is algebraic:

$$(8u^3 - 15u^2 + 12u - 4)A(0, u)^3 + (12u^2 - 12u + 6)A(0, u) - 2u^3 = 0. \quad (3)$$

The equation is obtained by a *guess and prove approach*. Once it has been guessed it remains to prove it by using the holonomic equation proven in Proposition 7. Thus we get

$$(8u^3 - 15u^2 + 12u - 4)^3 C(u)^3 + 48(36u^6 - 120u^5 + 202u^4 - 199u^3 + 123u^2 - 44u + 8)(u - 1)^2 C(u) + 32(9u^2 - 12u + 8)(u - 1)^3 = 0.$$

Theorem 8. *The function $A(z, u)$ is an algebraic function in $(z$ and $u)$ whose annihilating polynomial has degree 3:*

$$(8u^3 - 15u^2 + 12u - 4)(z^3 + 6zu + 3z^2 - 3z - 1)A(z, u)^3 + 6z^2(8u^3 - 15u^2 + 12u - 4)A(z, u)^2 + 6(12zu^3 - 18zu^2 - 2u^2 + 13zu + 2u - 3z - 1)A(z, u) + 2 = 0.$$

Note that the choice to use a doubly exponential generating function (in u and z) for $(t_{n,k})$ would have made sense and would be holonomic too (closure property of Borel transform). But it would not be algebraic because the inappropriate asymptotic expansion (cf. Theorem 10).

Proof. The fact that the initial conditions and a diagonal of $A(z, u)$ are algebraic suggests that it could also be algebraic as a function of z and u . Applying a bivariate guessing procedure, we observe that bivariate function $H(z, u) = (u + 1)(z^3 + 3z^2 + 6zu - 3z - 1)A(z, u)$ is such that $[z^n]H(z, u) = 0$ for $n > 2$. Furthermore $[z^j]H(z, u)$ is algebraic for $j = \{0, 1, 2\}$. So, let us calculate these z -extractions. First recall that $[z^0]A(z, u)$ satisfies the algebraic equation (3). In the same vein, $[z^1]A(z, u)$ verifies the algebraic equation

$$(8u^3 - 15u^2 + 12u - 4)f(u)^3 + 3(8u^3 - 15u^2 + 12u - 4)f(u)^2 + 3(8u^3 - 15u^2 + 10u - 2)f(u) + 8u^3 - 15u^2 + 6u = 0,$$

and finally $[z^2]A(z, u)$ verifies the algebraic equation

$$(8u^3 - 15u^2 + 12u - 4)f(u)^3 + (-24u^3 + 45u^2 - 36u + 12)f(u)^2 + (-72u^3 + 135u^2 - 84u + 18)f(u) - 40u^3 + 75u^2 - 36u = 0.$$

Thus we obtain

$$\begin{aligned} [z^0]H(z, u) &= -(1 + u)A(0, u) \\ [z^1]H(z, u) &= -1 + (u + 1)((6u - 3)A(0, u) - [z^1]A(z, u)) \\ [z^2]H(z, u) &= (u + 1)((6u - 3)[z^1]A(z, u) - [z^2]A(z, u) + 3A(0, u) + (6u - 4)). \end{aligned}$$

Finally we get

$$A(z, u) = \frac{[z^{\leq 2}]H(z, u)}{(u + 1)(z^3 + 3z^2 + 6uz - 3z - 1)}.$$

By using the elimination theory, we finally get a closed form algebraic equation for $A(z, u)$ of degree 27, that obviously cannot fit in the conference paper format. Nevertheless, this equation is

not minimal. By simplifying it, we finally get a minimal polynomial of degree 3 which annihilates $A(z, u)$:

$$(8u^3 - 15u^2 + 12u - 4)(z^3 + 3z^2 + 6zu - 3z - 1)A(z, u)^3 + 6z^2(8u^3 - 15u^2 + 12u - 4)A(z, u)^2 + 6(12zu^3 - 18zu^2 - 2u^2 + 13zu + 2u - 3z - 1)A(z, u) + 2 = 0.$$

A direct proof by recurrence confirms the validity of this equation. \square

We remark in the previous section that the diagonals of the function $A(z, u)$ are of particular interest because they define subclasses of arch processes with a fixed number of actions x_i 's covered by all the arches. In order to extract the generating functions of these subclass, we could use the Cauchy formula to compute $[u^0]A(z/u, u)$ and so on; we would keep the holomorphy property of the sequences but not their algebraicity. So, we prefer to define the generating function $B(z, u) = A(z/u, u)$. A similar proof that for the case $A(z, u)$ can be done to prove the algebraicity of $B(z, u)$. In particular, it exhibits the following algebraic equation satisfied by $B(z, u)$

$$(9u^2 + 12u - 4)(z^3 + 3z^2 + 6u - 3z - 1)B(z, u)^3 + 6z^2(9u^2 + 12u - 4)B(z, u)^2 + 6(18u^2z - 18u^2 + 6uz + 9u - 3z - 1)B(z, u) + 2(6u - 1)^2 = 0.$$

In particular, $B(0, u)$ is associated to the sequence $(t_{k,k})_k$, $[z^1]B(z, u)$ corresponds to the sequence $(t_{k-1,k})_k$ and so on. By specializing $z = 0$ in the latter algebraic equation then by resolving it through the Viète-Descartes approach for the resolution of cubic equation –detailed in the paper [Nic06]–, we obtain the following closed form formula corresponding to the branch that is analytic in 0:

$$B(0, u) = \sqrt{2} \sqrt{\frac{1 - 3u}{1 - 3u - \frac{9}{4}u^2}} \cos \left(\frac{1}{3} \arccos \left(\frac{6u - 1}{\sqrt{2}(1 - 3u)} \sqrt{\frac{1 - 3u - \frac{9}{4}u^2}{1 - 3u}} \right) \right).$$

Although the way we have represented $B(0, u)$ could suggest a singularity when the argument of the arccos function is equal to 1, the function admits an analytic continuation up to its dominant singularity ρ , solution of $1 - 3u - \frac{9}{4}u^2 = 0$, thus corresponding to $\rho = \frac{2}{3}(\sqrt{2} - 1)$. Furthermore, by studying the global generating function $B(z, u)$, we obtain its singular expansion.

Lemma 9. *Near the singularity when u tends to ρ , the function $B(z, u)$ satisfies*

$$B(z, u) \underset{u \rightarrow \rho}{=} a(z) + \frac{b(z)}{\sqrt{\rho - u}} + o\left((\rho - u)^{-1/2}\right),$$

with $a(z)$ and $b(z)$ two functions independent from u .

By using this result we deduce the asymptotic behaviors of the diagonal coefficients of $A(z, u)$.

Theorem 10. *Let i be a given integer, and k tend to infinity:*

$$t_{k+i,k} \underset{k \rightarrow \infty}{\sim} \gamma_i \frac{\rho^{-k}}{\sqrt{k}} k! \quad \text{with } \gamma_0 = \frac{1}{2} \sqrt{\frac{3}{\sqrt{2}\pi}} (\sqrt{2} - 1) \quad \text{and } \forall i \geq -1, \gamma_i = \left(\frac{1}{\sqrt{2} - 1} \right)^i \gamma_0.$$

This theorem is a direct consequence of Lemma 9. The $(\gamma_i)_i$ can be deduced by asymptotic matching.

Finally, by computing $[z^1]B(z, u)$ with the algebraic function it satisfies, we prove that its second derivative is solution of the algebraic function exhibited in OEIS A220433.

4 Uniform random generation of runs

We now introduce an algorithm to uniformly sample runs of a given arch process $A_{n,k}$. Our approach is based on the recursive equations (1) and (2) for the sequence $(t_{n,k})$. Here we deal with the cases $k \leq n$ and avoid the limit case $k = n + 1$. Although the latter limit case satisfies

this equation too, its proof is based on an other combinatorial approach, and so the construction of a run cannot be directly deduced from the combinatorial approach proposed for the cases $k \leq n$. Of course, a simple adaptation of the algorithm presented below would allow to sample in $A_{k-1,k}$, but the lack of space avoid us to present it here.

Our algorithm is a *recursive generation algorithm*. But since the objects are not specified in a classical Analytic Combinatorics's way, we can not use the results of [FZVC94]. As usual for recursive generation, the first step consists in the computation and the memorization of the value $t_{n,k}$ and all the intermediate values $(t_{i,j})$ needed for the calculation of $t_{n,k}$.

Proposition 11. *In order to compute the value $t_{n,k}$, we need to calculate the values in the bi-dimensional set $\{t_{i,j} \mid n \leq i \leq n+k \text{ and } 0 \leq j \leq k - (i - n)\}$. This computation is done with $\mathcal{O}(k^2)$ arithmetic operations.*

Recall that the coefficient computations are done only once for a given pair (n, k) , and then many runs can be drawn uniformly for $A_{n,k}$ by using the recursive generation algorithm.

Let us present the way we exploit the recurrence equation (2) to design the sampling method. The main problem that we encounter is the presence of a minus sign in the recurrence equation. Let us rewrite it in a slightly different way: $\sigma(D_{n,k}) + \sigma(\bar{D}_{n,k}^1) = \sigma(\bar{D}_{n,k}) + \sigma(\bar{D}_{n,k}^2)$.

Recall that the structures under consideration are depicted in Fig. 3. We introduce the classes of increasingly labeled structures from $D_{n,k}$, $\bar{D}_{n,k}^1$, $\bar{D}_{n,k}$ and $\bar{D}_{n,k}^2$, respectively denoted by $I_{n,k}$, $\bar{I}_{n,k}^1$, $\bar{I}_{n,k}$ and $\bar{I}_{n,k}^2$. Remark that the number of runs of $A_{n,k}$ is equal to $|I_{n,k}|$, where the function $|\cdot|$ corresponds to the cardinality of the considered class. Obviously the equation on the cardinalities can be written directly on the classes $I_{n,k} \cup \bar{I}_{n,k}^1 = \bar{I}_{n,k} \cup \bar{I}_{n,k}^2$ (since their intersections are empty: $I_{n,k}$ and $\bar{I}_{n,k}^1$ are distinct even if they are isomorphic). Thus, we consider the problem of sampling the class $I_{n,k} \cup \bar{I}_{n,k}^1$ where we bijectively replace the runs belonging to $\bar{I}_{n,k}^1$ by ones of $I_{n,k}$ (which can be performed recursively during the sampling procedure). The Algorithm SAMPLING(n, k) is based on the correspondence depicted in the Fig. 3 and its adaptation presented above on the classes $I_{n,k} \cup \bar{I}_{n,k}^1$. In each case the algorithm completes a recursively drawn run and apply some renaming on the actions of that run. Then, it inserts the action b_1 according to the cases $\bar{I}_{n,k} \setminus \bar{I}_{n,k}^1$, $\bar{I}_{n,k}^1$ or $\bar{I}_{n,k}^2$. In the specific case $\bar{I}_{n,k}^1$, instead of b_1 , it is the action b_k that is inserted and the renaming occurs in a similar fashion to obtain a run of $I_{n,k}$ from the one of $\bar{I}_{n,k}^1$.

Theorem 12. *The Algorithm SAMPLING(n, k) builds uniformly at random a run of $A_{n,k}$ in k recursive calls, once the coefficients computations and memorizations have been done.*

Since each object of $I_{n,k}$ is sampled in two distinct ways, the uniform sampling in $I_{n,k} \cup \bar{I}_{n,k}^1$ induces the uniform sampling of $I_{n,k}$.

Focus on the run of $A_{5,4}$ depicted in Fig. 2: $\langle a_1, b_1, a_2, a_3, b_3, a_4, x_1, b_4, c_1, b_2, c_2, c_3, c_4 \rangle$. It is either obtained from a (renamed) run of $\bar{I}_{5,4}^1$: $\langle a_1, b_1, a_2, a_3, b_3, x_1, x_2, c_1, b_2, c_2, c_3 \rangle$ with $p_b = 8$ (Line 8 of the algorithm). Or it is built from $\langle a_1, a_2, b_2, a_3, x_1, b_3, x_2, b_1, c_1, c_2, c_3 \rangle$ of $\bar{I}_{5,4} \setminus \bar{I}_{5,4}^1$, with $p_b = 1$ (Line 11). But it cannot be built from a run of $\bar{I}_{5,4}^2$.

In Fig. 4, we have uniformly sampled 1000 runs for $A_{1000,1000}$ and we have represented in blue points every pair (k, n) corresponding to an increasing sub-structure from $A_{n,k}$ that has been built during the algorithm (k for abscissa and n for ordinate). Only around $4.78 \cdot 10^4$ sub-structures have been built among the $50 \cdot 10^4$ inside the red lines which are calculated for the value $t_{1000,1000}$. At the beginning $n \approx k$ and the **if** branch on Line 5 is preferred (instead of the **else** one on Line 15) because the number of actions x_i 's is too small. After some recursive calls, the number of x_i 's actions has increased and then both branches of the algorithm are taken with probabilities of the same order.

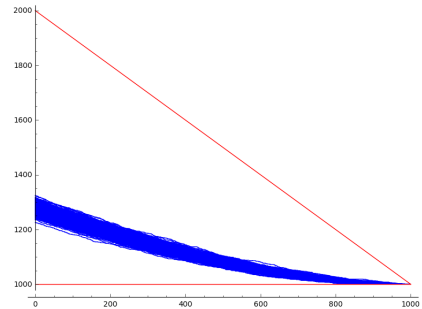


Figure 4: The terms $t_{i,j}$ needed for the sampling of 1000 runs of $A_{1000,1000}$

Algorithm 1 Uniform random sample for $I_{n,k}$

```

1: function SAMPLING( $n, k$ )
2:   if  $k = 0$  then
3:     return  $\langle x_1, x_2, \dots, x_n \rangle$ 
4:    $r := \text{RAND\_INT}(0, 2 \cdot t_{n,k} - 1)$  ▷ an uniform integer between 0 and  $2 \cdot t_{n,k} - 1$  in  $r$ 
5:   if  $r < \bar{I}_{n,k}$  then ▷ generation in  $\bar{I}_{n,k}$ 
6:      $U := \text{SAMPLING}(n, k - 1)$ 
7:      $p_b := 1 + r // t_{n,k-1}$  ▷ The position of the new  $b$  to insert
8:     if  $p_b > p_{x_1}$  then ▷ generation in  $\bar{I}_{n,k}^1$ 
9:       Rename  $x_1$  by  $a_k$  ; and each  $x_i$  with  $i > 1$  by  $x_{i-1}$ 
10:      Insert  $b_k$  at position  $p_b$  ; and  $c_k$  at the end of  $U$ 
11:     else ▷ generation in  $\bar{I}_{n,k} \setminus \bar{I}_{n,k}^1$ 
12:       In  $U$ , rename each  $a_i$  (resp.  $c_i$  and  $b_i$ ) by  $a_{i+1}$  (resp.  $c_{i+1}$  and  $b_{i+1}$ )
13:       Rename  $x_{n-k+1}$  by  $c_1$ 
14:       Insert  $b_1$  at position  $p_b$  ; and  $a_1$  at the head of  $U$ 
15:     else ▷ generation in  $\bar{I}_{n,k}^2$ 
16:        $U := \text{SAMPLING}(n + 1, k - 1)$ 
17:        $p_b := 2 + (r - (n + 2k - 1) \cdot t_{n,k-1}) // t_{n+1,k-1}$ 
18:       Rename  $x_{p_b}$  by  $b_1$  and  $x_{n-k+2}$  by  $c_1$  ; and each  $x_i$  with  $i > p_b$  by  $x_{i-1}$ 
19:       Insert  $a_1$  at the head of  $U$ 
20:   return  $U$ 

```

Line 4 and 17 : the binary operator $//$ denotes the Euclidean division.

The position of an action in a run is its arrival number (from 1 to the number of actions).

Recall that the constants γ_i 's (cf. Theorem 10) are evolving with an exponential growth. Finally, we observe that only a small number of diagonals are necessary for the samplings. Since the diagonals (t_{n_i, k_i}) for increasing sequences $(n_i)_i$ and $(k_i)_i$ follow P-recurrences (cf. [Lip88]), a lazy calculations of the terms of the necessary diagonals that envelop the blue points (but that are much narrow to the blue points than both red lines) would allow to minimize the pre-computations of Proposition 13.

We close this section with the presentation of an *unranking* algorithm for the construction of the runs of a given arch process $A_{n,k}$. This type of algorithm has been developed during the 70's by Nijenhuis and Wilf [NW75] and introduced in the context of Analytic Combinatorics by Martínez and Molinero [MM03]. Our algorithm is based on a bijection between the set of integers $\{0, \dots, t_{n,k} - 1\}$ and the set of runs of $A_{n,k}$. Here again we restrict ourselves to the values $k \leq n$. As usual for unranking algorithms, the first step consists in the computation and the memorization of the values of a sequence. But compared to the uniform random sampling, here we need more information than the one given by the sequence $(t_{n,k})$.

To be able to reconstruct the run associated to a given rank, we need to know the position of the action x_1 in the recursively drawn run in order to decide if the action b_1 appears before or after it. First suppose $k < n$ and let $t_{n,k,\ell}$ be the number of runs in $\mathcal{A}_{n,k}$ whose action x_1 appears at position ℓ . Let us denote by $I_{n,k,\ell}$ the associated combinatorial class. We obtain directly a constructive recurrence for the sequence.

$$t_{n,k,\ell} = (\ell - 2) t_{n,k-1,\ell-2} + (n - k) t_{n+1,k-1,\ell-1} \quad \text{and} \quad t_{n,0,1} = 1; \quad t_{n,0,\ell>1} = 0.$$

Proposition 13. *The computation of $t_{n,k,\ell}$ is done with $\mathcal{O}(k^2)$ arithmetic operations.*

The UNRANKING algorithm computes a run given its rank in the following total order:

$$\alpha \preceq_{n,k} \beta \quad \text{iff.} \quad \begin{cases} \alpha \in I_{n,k,i_0} \text{ and } \beta \in I_{n,k,i_1} & \wedge & i_0 < i_1, & \text{or} \\ \alpha, \beta \in I_{n,k,i} & \wedge & \begin{array}{l} \alpha \text{ is built recursively from } I_{n,k-1,i-2} \text{ and} \\ \beta \text{ is built recursively from } I_{n+1,k-1,i-1} \end{array} & \text{or} \\ \alpha, \beta \in I_{n,k,i} & \wedge & \begin{array}{l} \alpha, \beta \in I_{n,k-1,i-2} \text{ (resp. } I_{n+1,k-1,i-1}) \text{ and} \\ \alpha_0, \beta_0 \text{ inducing } \alpha, \beta \text{ satisfy } \alpha_0 \preceq_{n,k-1} \beta_0. \end{array} \end{cases}$$

The run example of Fig. 2 has rank 479 among the 1270 runs of $A_{5,4}$. Note that in the case $k = n$ (at the end there is no x_1) the algorithm is easily extended by considering the position of b_1 as the one of x_1 .

Algorithm 2 Unranking for $I_{n,k}$

```
1: function UNRANKING( $n, k, r$ )
2:    $\ell := k + 1$ 
3:   while  $r \geq 0$  do
4:      $r := r - t_{n,k,\ell}$ 
5:      $\ell := \ell + 1$ 
6:   return CONS( $n, k, \ell, r$ )
7: function CONS( $n, k, \ell, r$ )
8:   if  $k = 0$  then
9:     return  $(x_1, x_2, \dots, x_n)$ 
10:  if  $r < (\ell - 2) \cdot t_{n,k-1,\ell-2}$  then ▷ generation in  $I_{n,k-1,\ell-2}$ 
11:     $rr := r \% t_{n,k-1,\ell-2}$ 
12:     $U := \text{CONS}(n, k - 1, \ell - 2, rr)$ 
13:     $p_b := 1 + r // t_{n,k-1,\ell-2}$  ▷ The position of the new  $b$  to insert
14:    In  $U$ , rename each  $a_i$  (resp.  $c_i$  and  $b_i$ ) by  $a_{i+1}$  (resp.  $c_{i+1}$  and  $b_{i+1}$ )
15:    Rename  $x_{n-k+1}$  by  $c_1$ 
16:    Insert  $b_1$  at position  $p_b$  ; and  $a_1$  at the head of  $U$ 
17:  else ▷ generation in  $I_{n+1,k-1,\ell-1}$ 
18:     $r' := r - (\ell - 2) \cdot t_{n,k-1,\ell-2}$ 
19:     $rr := r' \% t_{n+1,k-1,\ell-1}$ 
20:     $U := \text{CONS}(n + 1, k - 1, \ell - 1, rr)$ 
21:     $p_b := 2 + r' // t_{n+1,k-1,\ell-1}$ 
22:    Rename  $x_{p_b}$  by  $b_1$  and  $x_{n-k+2}$  by  $c_1$  ; and each  $x_i$  with  $i > p_b$  by  $x_{i-1}$ 
23:    Insert  $a_1$  at the head of  $U$ 
24:  return  $U$ 
```

Line 11 and 19 : the binary operator $\%$ denotes the Euclidean division remainder.

Theorem 14. *The Algorithm UNRANKING(n, k, r) builds the r -th run of $A_{n,k}$ in k recursive calls, once the coefficient memorizations $t_{n,k,\ell}$, for all ℓ such that $k + 1 \leq \ell \leq 2k + 1$ (and the necessary n and k), have been done.*

Note that the implementation of both algorithms can be much more efficient than the pseudocode exhibited above. Actually, only the absolute positions of the b_i 's are important in a run, because all other actions have their positions determined by b_i 's positions. However, such implementations are much more cryptic to read, and so we preferred to present here easy-to-read algorithms.

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