On k-11-representable graphs

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Abstract

Distinct letters x and y alternate in a word w if after deleting in w all letters but the copies of x and y we either obtain a word of the form $xyxy\cdots$ (of even or odd length) or a word of the form $yxyx\cdots$ (of even or odd length). A graph $G = (V, E)$ is word-representable if there exists a word w over the alphabet V such that letters x and y alternate in w if and only if xy is an edge in E. Thus, edges of G are defined by avoiding the consecutive pattern 11 in a word representing G, that is, by avoiding xx and yy.

In 2017, Jeff Remmel has introduced the notion of a k-11-representable graph for a non-negative integer k, which generalizes the notion of a word-representable graph. Under this representation, edges of G are defined by containing at most k occurrences of the consecutive pattern 11 in a word representing G. Thus, word-representable graphs are precisely 0-11-representable graphs. In this paper, we study properties of k-11-representable graphs for $k \geq 1$, in particular, showing that the class of word-representable graphs, studied intensively in the literature, is contained strictly in the class of 1-11-representable graphs. Another particular result that we prove is the fact that the class of interval graphs is precisely the class of 1-11-representable graphs that can be represented by uniform words containing two copies of each letter. This result can be compared with the known fact that the class of circle graphs is precisely the class of 0-11-representable graphs that can be represented by uniform words containing two copies of each letter. Also, one of our key results in this paper is the fact that any graph is k-11-representable for some $k \geq 0$.

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1 Introduction

The theory of word-representable graphs is a young but very promising research area. It was introduced by the forth author in 2004 based on the joint research with Steven Seif [12] on the celebrated *Perkins semigroup*, which has played a central role in semigroup theory since 1960, particularly as a source of examples and counterexamples. However, the first systematic study of word-representable graphs was not undertaken until the appearance in 2008 of [11], which started the development of the theory.

Up to date, about 20 papers have been written on the subject, and the core of the book [10] is devoted to the theory of word-representable graphs. It should also be mentioned that the software packages [5, 17] are often of great help in dealing with word-representation of graphs. Moreover, a recent paper [8] offers a comprehensive introduction to the theory. Some motivation points to study these graphs are given in Section 1.1.

A graph $G = (V, E)$ is *word-representable* if and only if there exists a word w over the alphabet V such that letters x and y, $x \neq y$, alternate in w if and only if $xy \in E$. In other words, $xy \in E$ if and only if the subword of w induced by x and y avoids the *consecutive pattern* 11 (which is an occurrence of xx or yy). Not all graphs are word-representable, and the minimum non-word-representable graph is the wheel graph W_5 in Figure 1, which is the only non-word-representable graph on six vertices [10, 11].

In 2017, Jeff Remmel [15] has introduced the notion of a k-11-representable graph for a non-negative integer k, which generalizes the notion of a word-representable graph. Under this representation, edges of G are defined by containing at most k occurrences of the consecutive pattern 11 in a word representing G. Thus, word-representable graphs are precisely 0-11-representable graphs. The new definition not only allows to represent at least some of non-word-representable graphs including W_5 (see Section 4), and to give a new characterization of interval graphs (see Theorem 3.1, which should be compared with Theorem 1.3 characterising circle graphs), but also it provides a way to represent *any* graph in terms of alternation of letters in words (see Theorem 2.12). The latter fact could be compared with the possibility to *u-represent any* graph, where $u \in \{1, 2\}^*$ of length at least 3 [9]. We refer the Reader to [9] for the relevant definitions just mentioning that the case of $u = 11$ corresponds to word-representable graphs.

The paper is organized as follows. In the rest of the section, we give more detail about word-representable graphs and *semi-transitive orientations* characterizing these graphs. In Section 2, we introduce rigorously the notion of a k-11-representable graph and provide a number of general results on these graphs. In particular, we show that a $(k-1)$-11-representable graph is necessarily k-11-representable (see Theorem 2.2). In Section 3, we study the class of 1-11-representable graphs. These studies are extended in Section 4, where we 1-11-represent all non-word-representable graphs on at most 7 vertices. Any 3-colorable graph is necessarily 0-11-representable, while there are non-0-11-representable 4-colorable graphs [7, 10]. In Section 5, we find a place for 4-colorable and 5-colorable graphs in the hierarchy of k-11-representable graphs. Finally, in Section 6, we state a number of open problems on k-11-representable graphs.

1.1 Word-representable graphs

Suppose that w is a word over some alphabet and x and y are two distinct letters in w. We say that x and y *alternate* in w if after deleting in w all letters but the copies of x and y we either obtain a word of the form $xyxy\cdots$ (of even or odd length) or a word of the form $yxyx\cdots$ (of even or odd length).

A graph $G = (V, E)$ is *word-representable* if there exists a word w over the alphabet V such that letters x and y alternate in w if and only if xy is an edge in E. Such a word w is called G's *word-representant*. In this paper we assume V to be $[n] = \{1, 2, \ldots, n\}$ for some $n \geq 1$. For example, the cycle graph on 4 vertices labeled by 1, 2, 3 and 4 in clockwise direction can be represented by the word 14213243. Note that a complete graph K_n can be represented by any permutation of $[n]$, while an edgeless graph (i.e. empty graph) on n vertices can be represented by $1122\cdots nn$.

The most interesting aspect of word-representable graphs from an algebraic point of view seems to be the notion of a *semi-transitive orientation* [7], which generalizes partial orders. It was shown in [7] that a graph is word-representable if and only if it admits a semi-transitive orientation.

More motivation points to study word-representable graphs include the fact exposed in [10] that these graphs generalize several important classes of graphs such as *circle graphs* [3], 3-*colourable graphs* and *comparability graphs* [14]. Relevance of word-representable graphs to scheduling problems was explained in [7] and it was based on [6]. Furthermore, the study of word-representable graphs is interesting from an algorithmic point of view as explained in [10]. For example, the *Maximum Clique problem* is polynomially solvable on word-representable graphs [10] while this problem is generally NP-complete [2]. Finally, word-representable graphs are an important class among other graph classes considered in the literature that are defined using words. Examples of other such classes of graphs are *polygon-circle graphs* [13] and *word-digraphs* [1].

The following two theorems are useful tools to study word-representable graphs. For the second theorem, we need the notion of a *cyclic shift* of a word. Let a word w be the concatenation uv of two non-empty words u and v. Then, the word vu is a cyclic shift of w.

Theorem 1.1 ([11]). *A graph is word-representable if and only if it can be represented uniformly, i.e. using the same number of copies of each letter.*

Theorem 1.2 ([11]). *Any cyclic shift of a word having the same number of copies of each letter represents the same graph.*

A *circle graph* is the intersection graph of a set of chords of a circle, i.e. it is an undirected graph whose vertices can be associated with chords of a circle such that two vertices are adjacent if and only if the corresponding chords cross each other. In this paper, we get used of the following theorem.

Theorem 1.3 ([7]). *The class of circle graphs is precisely the class of word-representable graphs that can be represented by uniform words containing two copies of each letter.*

An orientation of a graph is *transitive*, if the presence of the edges $u \to v$ and $v \to z$ implies the presence of the edge $u \to z$. An oriented graph G is a *comparability graph* if G admits a transitive orientation. A graph is *permutationally representable* if it can be represented by concatenation of permutations of (all) vertices. Thus, permutationally representable graphs are a subclass of word-representable graphs. The following theorem classifies these graphs.

Theorem 1.4 ([12]). *A graph is permutationally representable if and only if it is a comparability graph.*

1.2 Semi-transitive orientations

A *shortcut* is an *acyclic non-transitively oriented* graph obtained from a directed cycle graph forming a directed cycle on at least four vertices by changing the orientation of one of the edges, and possibly by adding more directed edges connecting some of the vertices (while keeping the graph be acyclic and non-transitive). Thus, any shortcut

- is *acyclic* (that it, there are *no directed cycles*);
- has *at least* 4 vertices;
- has *exactly one* source (the vertex with no edges coming in), *exactly one* sink (the vertex with no edges coming out), and a *directed path* from the source to the sink that goes through *every* vertex in the graph;
- has an edge connecting the source to the sink that we refer to as the *shortcutting edge*;
- is *not* transitive (that it, there exist vertices u, v and z such that $u \to v$ and $v \to z$ are edges, but there is *no* edge $u \to z$).

An orientation of a graph is *semi-transitive* if it is *acyclic* and *shortcut-free*. An equivalent definition of a semi-transitive orientation is as follows. An acyclic orientation is semi-transitive if and only if for any directed path $u_0 \to u_1 \to \cdots \to u_t$, $t \geq 3$, either there is *no* edge $u_0 \to u_t$, or there is the edge $u_i \to u_j$ for any $0 \leq i < j \leq t$. It is easy to see from definitions that *any* transitive orientation is necessary semi-transitive. The converse is *not* true, as is evident, for example, from the path graph on vertices $\{u, v, z\}$ oriented as $u \to v \to z$. Thus, semi-transitive orientations generalize transitive orientations.

As is mentioned above, a key result in the theory of word-representable graphs is the fact proved in [7] that a graph is word-representable if and only if it is semi-transitively orientable. Next, we follow [7] to sketch the idea of the proof of this result, to let the Reader compare the proof with the orientations approach we use in proving Theorem 5.1 below.

Given a word w representing a graph G, and a pair of alternating in w letters x and y, we direct the edge of G from the vertex x to the vertex y if the first occurrence of x is before that of y in w. It is not difficult to see that such an orientation is semi-transitive [7].

For the opposite direction, the basic idea is to represent the *non-edges* incident with each vertex v in a semi-transitively oriented graph with n vertices by a word w_v in which each of the n letters occurs exactly twice. Then concatenating all such w_v's gives a word w representing the given graph with the orientations removed. We conclude this section with giving the explicit construction of w_v. In what follows, for an acyclic directed graph $D = (V, E)$, we let $u \rightsquigarrow v$ denote the fact that there exists a directed path from u to v in D. Also, we say that a permutation P of the set V is a *topological sort* of D if for every distinct $u, v \in V$ such that $u \rightsquigarrow v$, the letter u precedes v in P. By definition, $u \rightsquigarrow u$.

Let $I(v) = \{u : u \to v\}$ be the set of all in-neighbors of v, and $O(v) = \{u : v \to u\}$ be the set of all out-neighbors of v. Also, let $A(v) = \{u \in V : u \rightsquigarrow v\} \setminus I$ be the set of v's non-neighboring vertices that can reach v, and $B(v) = \{u \in V : v \rightsquigarrow u\} \setminus O$ be the set of v's non-neighboring vertices that can be reached from v. Finally, let $T(v) = V \setminus (\{v\} \cup I(v) \cup O(v) \cup A(v) \cup B(v))$ be the set of remaining vertices. Note that the sets $I(v)$, $O(v)$, $A(v)$, $B(v)$ and $T(v)$ are pairwise disjoint and some of them can be empty. Denote by A, B, I, O and T topological sorts of the corresponding digraphs induced by the sets $A(v)$, $B(v)$, $I(v)$, $O(v)$ and $T(v)$, respectively. Then

$$w_v = A\ I\ T\ A\ v\ O\ I\ v\ B\ T\ O\ B.$$

2 Definitions and general results

A *factor* in a word $w_1 w_2 \dots w_n$ is a word $w_i w_{i+1} \dots w_j$ for $1 \leq i \leq j \leq n$. For a letter or a word x, we let x^k denote $\underbrace{x \dots x}_{k \text{ times}}$. For any word w, we let $\pi(w)$ denote the *initial permutation* of w obtained by reading w from left to right and recording the leftmost occurrences of the letters in w. For example, if $w = 2535214421$ then $\pi(w) = 25314$. Similarly, the *final permutation* $\sigma(w)$ of w is obtained by reading w from right to left and recording the rightmost occurrences of w. For the w above, $\sigma(w) = 35421$. Also, for a word w, we let $r(w)$ denote the *reverse* of w, that is, w written in the reverse order. For example, if $w = 22431$ then $r(w) = 13422$. Finally, for a pair of letters x and y in a word w, we let $w|_{\{x,y\}}$ denote the word induced by the letters x and y. For example, for the word $w = 2535214421$, $w|_{\{2,5\}} = 25522$. The last definition can be extended in a straightforward way to defining $w|_S$ for a set of letters S. For example, for the same w, $w|_{\{1,2,3\}} = 232121$.

Throughout this paper, we denote by $G \backslash v$ the graph obtained from a graph G by deleting a vertex $v \in V(G)$ and all edges adjacent to it.

Let $k \geq 0$. A graph $G = (V, E)$ is k-11-*representable* if there exists a word w over the alphabet V such that the word $w|_{\{x,y\}}$ contains in total at most k occurrences of the factors in $\{xx, yy\}$ if and only if xy is an edge in E. Such a word w is called G's k-*representant*. A *uniform* (resp., k-*uniform*) representation of a graph G is a word, satisfying the required properties, in which each letter occurs the same (resp., k) number of times. As is stated above, in this paper we assume V to be $[n] = \{1, 2, \dots, n\}$ for some $n \geq 1$. Note that 0-11-representable graphs are precisely word-representable graphs, and that 0-representants are precisely word-representants. We also note that the "11" in "k-11-representable" refers to counting occurrences of the *consecutive pattern* 11 in the word induced by a pair of letters $\{x, y\}$, which is exactly the total number of occurrences of the factors in $\{xx, yy\}$. Throughout the paper, we normally omit the word "consecutive" in "consecutive pattern" for brevity. Finally, we let $\mathcal{G}^{(k)}$ denote the class of k-11-representable graphs.

Lemma 2.1. *Let $k \geq 0$ and a word w k-11-represent a graph G. Then the word $r(\pi(w))w$ $(k+1)$-11-represents G. Also, the word $wr(\sigma(w))$ $(k+1)$-11-represents G. Moreover, if $k = 0$ then the word ww 1-11-represents G.*

Proof. Suppose x and y are two vertices in G. If xy is an edge in G then $w|_{\{x,y\}}$ contains at most k occurrences of the pattern 11, so $(r(\pi(w))w)|_{\{x,y\}}$ (resp., $(wr(\sigma(w)))|_{\{x,y\}}$) contains at most $k+1$ occurrences of the pattern 11, and xy will be an edge in the new representation. On the other hand, if xy is not an edge in G, then $w|_{\{x,y\}}$ contains at least $k+1$ occurrences of the pattern 11, so $(r(\pi(w))w)|_{\{x,y\}}$ (resp., $(wr(\sigma(w)))|_{\{x,y\}}$) contains at least $k+2$ occurrences of the pattern 11, and xy will not be an edge in the new representation.

Finally, if x and y alternate in w, then ww contains at most one occurrence of xx or yy, while non-alternation of x and y in w leads to at least two occurrence of the pattern 11 in ww, which involves x or/and y. These observations prove the last claim. □

Theorem 2.2. *We have $\mathcal{G}^{(k)} \subseteq \mathcal{G}^{(k+1)}$ for any $k \geq 0$.*

Proof. This is an immediate corollary of Lemma 2.1. □

Lemma 2.3. *Let $k \geq 0$, G be a k-11-representable graph, and i and j be vertices in G, possibly $i = j$. Then there are infinitely many words w k-representing G such that $w = iw'j$ for some words w'.*

Proof. Let u k-represent G. Then note that any word v of the form $\pi(u) \cdots \pi(u)u\sigma(u) \cdots \sigma(u)$ k-represents G. Deleting all letters to the left of the leftmost i in v, and all letters to the right of the rightmost j in v, we clearly do not change the number of occurrence of the pattern 11 for any pair of letters $\{x, y\}$. The obtained word w satisfies the required properties. □

There is a number of properties that is shared between word-representable graphs and k-11-representable graphs for any $k \geq 1$. These properties can be summarized as follows:

- The class $\mathcal{G}^{(k)}$ is hereditary. Indeed, if a word w k-11-represents a graph G, and v is a vertex in G, then clearly the word obtained from w by removing v k-11-represents the graph $G \backslash \{v\}$.
- In the study of k-11-representable graphs, we can assume that graphs in question are connected (see Theorem 2.4).
- In the study of k-11-representable graphs, we can assume that graphs in question have no vertices of degree 1 (see Theorem 2.5).
- In the study of k-11-representable graphs, we can assume that graphs in question have no two vertices having the same neighbourhoods up to removing these vertices, if they are connected (see Theorem 2.6).

- Glueing two k-11-representable graphs in a vertex gives a k-11-representable graph (see Theorem 2.7).

- Connecting two k-11-representable graphs by an edge gives a k-11-representable graph (see Theorem 2.8).

Theorem 2.4. *Let $k \geq 0$. A graph G is k-11-representable if and only if each connected component of G is k-11-representable.*

Proof. If G is k-11-representable then each of G's connected components is k-11-representable by the hereditary property of k-11-representable graphs.

Conversely, suppose that C_i's are the connected components of G for $1 \leq i \leq \ell$, and w_i k-11-represents C_i. Adjoining several copies of $\pi(w_i)$ to the left of w_i, if necessary, we can assume that each letter in any w_i occurs at least $k+2$ times. But then, the word $w = w_1 w_2 \cdots w_\ell$ k-11-represents G, since

- edges/non-edges in each C_i are represented by the w_i, and

- for $x \in C_i$ and $y \in C_j$, $i \neq j$, the word $w|_{\{x,y\}}$ contains at least $2k+2$ occurrences of the pattern 11 making x and y be disconnected in G,

we are done. □

Theorem 2.5. *Let $k \geq 0$, G be a graph with a vertex x, and G_{xy} be the graph obtained from G by adding to it a vertex y connected only to x. Then, G is k-11-representable if and only if G_{xy} is k-11-representable.*

Proof. The backward direction follows directly from the hereditary nature of k-11-representability. For the forward direction, suppose that w k-11-represents G. Adjoining several copies of $\pi(w)$ to the left of w, if necessary, we can assume that x occurs at least $2k+2$ times in w. Replacing every other occurrence of x in w, starting from the leftmost one, with yxy, we obtain a word w' that k-11-represents G_{xy}. Indeed, clearly, the letters x and y alternate in w' so xy is an edge in G_{xy} no matter what k is. On the other hand, if $z \neq x$ is a vertex in G, then $w'|_{\{z,y\}}$ has at least $k+1$ occurrences of the pattern 11 (formed by y's) ensuring that zy is not an edge in G_{xy}. Any other alternation of letters in w is the same as that in w'. □

Theorem 2.6. *Let $k \geq 0$ and G be a graph having two, possibly connected vertices, x and y, with the same neighbourhoods up to removing x and y. Then, G is k-11-representable if and only if $G \setminus x$ is k-11-representable.*

Proof. The forward direction follows directly from the hereditary nature of k-11-representability. For the backward direction, let w k-11-represent $G \setminus x$. If x and y are connected in G, then replacing each y by xy in w clearly gives a k-11-representation of G because x and y will have the same properties and they will be strictly alternating. On the other hand, if x and y are not connected in G, then adjoining several copies of $\pi(w)$ to the left of w, if necessary, we can assume that y occurs at least $k+2$ times in w. We then replace every even occurrence of y in w (from left to right) by yx, and every odd occurrence by xy. This will ensure that in the subword induced by x and y, the number of occurrences of the pattern 11 is at least $k+1$ making x and y be not connected in G. On the other hand, still x and y have the same alternating properties with respect to other letters. Thus, the obtained word k-11-represents G, as desired. $\square$

Theorem 2.7. *Let $k \geq 0$, G_1 and G_2 be k-11-representable graphs, and the graph G is obtained from G_1 and G_2 by identifying a vertex x in G_1 with a vertex y in G_2. Then, G is k-11-representable.*

Proof. Let w_1 and w_2 be k-11-representations of the graphs G_1 and G_2, respectively. Recall that if a word w k-11-represents a graph H, then the word $w' = \pi(w)w$ obtained from w by adding the initial permutation $\pi(w)$ of w in front of w also k-11-represents H. Applying this observation, we may assume that the number of occurrences of x in the word w_1 equals to that of the letter y in the word w_2. In addition, by Lemma 2.3, we may further assume that w_1 starts with the letter x, and w_2 starts with the letter y. That is, $w_1 = xg_1xg_2\dots xg_m$, where g_i's are words over $V(G_1) \setminus \{x\}$, and $w_2 = yh_1yh_2\dots yh_m$, where h_i's are words over $V(G_2) \setminus \{y\}$. Let π_1 (resp., π_2) be the initial permutation of the word $g_1g_2\dots g_m$ (resp., $h_1h_2\dots h_m$). In other words, $\pi(w_1) = x\pi_1$ and $\pi(w_2) = y\pi_2$.

Let z be the vertex in G which corresponds to the vertices x and y, i.e. $z = x = y$ in G. We claim that the word $w(G) := (z\pi_1\pi_2z\pi_2\pi_1)^{k+1}zg_1h_1zg_2h_2\dots zg_mh_m$ k-11-represents the graph G. The induced subword of $w(G)$ on $V(G_1)$ is precisely $\pi(w_1)^{2k+2}w_1$ which k-11-represents the graph G_1. Similarly, the induced subword of $w(G)$ on $V(G_2)$ k-11-represents the graph G_2. Now, consider $v_1 \neq x$ in $V(G_1)$ and $v_2 \neq y$ in $V(G_2)$. By the definition of G, the vertices v_1 and v_2 are not adjacent in G. Thus, it remains to show that the induced subword $w(G)|_{\{v_1,v_2\}}$ has at least $k+1$ occurrences of the pattern 11, which is easy to see from $(v_1v_2v_2v_1)^{k+1}$ being a factor of $w(G)|_{\{v_1,v_2\}}$. Therefore, the word $w(G)$ indeed k-11-represents the graph G. $\square$

Theorem 2.8. *Let $k \geq 0$, G_1 and G_2 be k-11-representable graphs, and the graph G is obtained from G_1 and G_2 by connecting a vertex x in G_1 with a vertex y in G_2 by an edge. Then G is k-11-representable.*

Proof. Let w_1 and w_2 be k-11-representations of G_1 and G_2, respectively. By the same argument as in Theorem 2.7, we can assume that the number of occurrences of the letter x in the word w_1 equals that of the letter y in the word w_2. By Lemma 2.3, we can assume that w_1 begins with x, and w_2 ends with y. In addition, we can assume that the initial permutation of w_2 ends with y. Suppose the initial permutation of w_2 does not end with y, and let AyB be the initial permutation. It is clear that the word $w_2' = BAyBw_2$ also k-11-represents G_2, so that we can consider w_2' instead of w_2, and the initial permutation of w_2' ends with y.

Now we can write $w_1 = xg_1xg_2\dots xg_m$, where g_i's are words over $V(G_1) \setminus \{x\}$, and $w_2 = h_1yh_2y\dots h_my$, where h_i's are words over $V(G_2) \setminus \{y\}$. Let π_1 (resp., π_2) be the initial permutation of the word $g_1g_2\dots g_m$ (resp., $h_1h_2\dots h_m$). Observe that $\pi(w_1) = x\pi_1$ and $\pi(w_2) = \pi_2 y$. We claim that the word $w(G) := (x\pi_1\pi_2 y\pi_2 xy\pi_1)^{k+1}xg_1h_1yxg_2h_2y\dots xg_mh_my$ is a k-11-representation of G. As in Theorem 2.7, it is clear that the word $w(G)$ k-11-represents the graphs G_1 and G_2, when restricted to $V(G_1)$ and $V(G_2)$, respectively. Also, $w(G)$ makes the vertices x and y be adjacent, because $w(G)|_{\{x,y\}} = (xy)^{2k+m+2}$. Hence, it remains to show that for every $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ such that $v_1 \neq x$ or $v_2 \neq y$, which must be non-adjacent in G, the induced subword $w(G)|_{\{v_1,v_2\}}$ has at least $k+1$ occurrences of the pattern 11. This is obviously the case, because $w(G)|_{\{v_1,v_2\}}$ contains $(v_1v_2v_2v_1)^{k+1}$ having at least $2k+1$ occurrences of the pattern 11. Therefore, the word $w(G)$ k-11-represents the graph G. □

Theorem 2.9. *Let G be a graph with a vertex v. If $G \setminus v$ is k-uniform word-representable for $k \geq 1$, then G is $(k-1)$-11-representable.*

Proof. Let w be a k-uniform word that represents the graph $G \setminus v$. Let $N(v) \subset V(G)$ be the set of all neighbors of v in G, and let $N^c(v)$ be the complement of $N(v)$ in $V(G) \setminus \{v\}$, i.e. $N^c(v) = V(G) \setminus (N(v) \cup \{v\})$. We will describe how to construct a $(k-1)$-11-representation $w(G)$ of G from the word w. Recall that $r(\pi(w))$ is the reverse of the initial permutation $\pi(w)$ of the word w.

We start with the word $\pi(w)|_{N(v)}v\pi(w)|_{N^c(v)}\ w$, where $\pi(w)|_{N(v)}$ and $\pi(w)|_{N^c(v)}$ are the induced subwords of $\pi(w)$ on $N(v)$ and $N^c(v)$, respectively. In each step, we adjoin the words $r(\pi(w))v$ and $\pi(w)v$, in turn, from the left side of the word constructed in the previous step. We stop when the current word, denoted by $w(G)$, has exactly k v's. For example, the word $w(G)$, when $k = 6$, is given by

$$w(G) = r(\pi(w))v\ \pi(w)v\ r(\pi(w))v\ \pi(w)v\ r(\pi(w))v\ \pi(w)|_{N(v)}v\pi(w)|_{N^c(v)}\ w.$$

Next, we will show that the word $w(G)$ $(k-1)$-11-represents G. First, take a vertex $x \neq v$ in G. If $x \in N(v)$, then $w(G)|_{\{x,v\}} = xv\dots xv\ w|_{\{x\}}$ has $k-1$ occurrences of the pattern 11 since $w|_{\{x\}} = x^k$. If $x \in N^c(v)$, then $w(G)|_{\{x,v\}} = xv\dots xv\ vx\ w|_{\{x\}}$ has $k+1$ occurrences of the pattern 11. Thus $w(G)$ preserves all the (non-)adjacencies of v. Now, take two distinct vertices, y, z in $V(G) \setminus \{v\}$. Without loss of generality, we can assume that $\pi(v)|_{\{y,z\}} = yz$. If y and z are adjacent in $G \setminus v$, then $w|_{\{y,z\}} = yzyz\dots yz$. Hence, the induced subword

$$w(G)|_{\{y,z\}} = \dots zy\ yz\ zy\ (\pi(w)|_{N(v)}v\pi(w)|_{N^c(v)})|_{\{y,z\}}\ yzyz\dots yz$$

has $k-1$ occurrences of the pattern 11 since the part $\dots zy\ yz\ zy$ is of length $2(k-1)$, and $(\pi(w)|_{N(v)}v\pi(w)|_{N^c(v)})|_{\{y,z\}}$ is either yz or zy. If y and z are not adjacent in $G \setminus v$, then $w|_{\{y,z\}}$ has at least one occurrence of the pattern 11 and it starts with y. Hence, $w(G)|_{\{y,z\}} = \dots zy\ yz\ zy\ (\pi(w)|_{N(v)}v\pi(w)|_{N^c(v)})|_{\{y,z\}}\ w|_{\{y,z\}}$ has at least k occurrences of the pattern 11 since the only difference from the previous case is $w|_{\{y,z\}}$, which now has at least one occurrence of the pattern 11. This proves that $w(G)$ is a $(k-1)$-11-representation of G. □

Theorem 2.10. *For any non-negative integers m and k satisfying $2m-k-1 > 0$, the following holds. Let G be a graph with a vertex v. If $G \setminus v$ is m-uniform k-11-representable, then G is $(3m-k-1)$-uniform $(2m-2)$-11-representable.*

Proof. Let w be an m-uniform k-11-representation of $G \setminus v$, $N(v) \subset V(G)$ be the set of all neighbors of v in G, and let $N^c(v) = V(G) \setminus (N(v) \cup \{v\})$. We will describe how to construct a $(3m-k-1)$-uniform $(2m-2)$-11-representation $w(G)$ of G from the word w. Similarly to the proof of Theorem 2.9, we start with the word $\pi(w)|_{N(v)} v \pi(w)|_{N^c(v)}\ w$, and in each step, we adjoin $r(\pi(v))v$ and $\pi(w)v$, in turn, from the left side until $w(G)$ has exactly $2m-k-1$ occurrences of v. Then, we adjoin v^m from the left side. For example, when $k = 3$ and $m = 4$, the word $w(G)$ is given by

$$w(G) = vvvv\ r(\pi(w))v\ \pi(w)v\ r(\pi(v))v\ \pi(w)|_{N(v)} v \pi(w)|_{N^c(v)}\ w.$$

It is easy to see that $w(G)$ is $(3m-k-1)$-uniform. Indeed, if $x \in V(G) \setminus \{v\}$, then $w(G)$ contains $(2m-k-1) + m = 3m-k-1$ x's since w is m-uniform; also, $w(G)$ contains $m + (2m-k-1) = 3m-k-1$ v's. Next, we will show that $w(G)$ $(2m-2)$-11-represents G.

Let $x \in V(G) \setminus \{v\}$. If $x \in N(v)$, then $w(G)|_{\{x,v\}} = v^m\ xv \dots xv\ x^m$. Thus $w(G)|_{\{x,v\}}$ has $2m-2$ occurrences of the pattern 11. If $x \in N^c(v)$, then the only difference from the previous case in $w(G)|_{\{x,v\}}$ is that $\pi(w)|_{N(v)} v \pi(w)|_{N^c(v)}$ is vx, not xv. Thus, $w(G)|_{\{x,v\}}$ has $2m$ occurrences of the pattern 11. Now take two distinct vertices $x, y \in V(G) \setminus \{v\}$. Without loss of generality, we can assume that $\pi(w)|_{\{x,y\}} = xy$. If x, y are adjacent in $G \setminus v$, then $w|_{\{x,y\}}$ has at most k occurrences of the pattern 11. Hence,

$$w(G)|_{\{x,y\}} = \dots yx\ xy\ yx\ (\pi(w)|_{N(v)} v \pi(w)|_{N^c(v)})|_{\{x,y\}}\ w|_{\{x,y\}}.$$

Since the length of $\dots yx\ xy\ yx$ is $4m-2k-4$ and $(\pi(w)|_{N(v)} v \pi(w)|_{N^c(v)})|_{\{x,y\}}$ is xy or yx, $w(G)|_{\{x,y\}}$ has at most $(2m-k-3)+1+k = 2m-2$ occurrences of the pattern 11. If x, y are not adjacent in $G \setminus v$, then $w|_{\{x,y\}}$ has at least $k+1$ occurrences of the pattern 11. In this case, the only difference from the previous case in $w(G)$ is $w|_{\{x,y\}}$ and so $w(G)|_{\{x,y\}}$ has at least $(2m-k-3)+1+k+1 = 2m-1$ occurrences of the pattern 11. This proves that $w(G)$ is a $(2m-2)$-11-representation of G. □

The following corollary to Theorem 2.10 is of fundamental importance since it plays an important role in Theorem 2.12 below to show that any graph belongs to $\mathcal{G}^{(k)}$ for some $k \geq 0$.

Corollary 2.11. *For any non-negative integers n and k satisfying $2n+k-7 > 0$, if each graph on n vertices is $(k+n-3)$-uniformly k-11-representable, then every graph on $n+1$ vertices is $(2k+3n-10)$-uniformly $(2k+2n-8)$-11-representable.*

Proof. This is a direct consequence of Theorem 2.10. Suppose every graph on n vertices is $(k+n-3)$-uniformly k-11-representable, and G is a graph on $n+1$ vertices. Clearly, $k+n-3$ is a positive integer since we have $2n+k-7 > 0$. Then for any vertex v in G, the graph $G \setminus v$ obtained from G by removing a vertex v is $(k+n-3)$-uniformly k-11-representable. Since $2(k+n-3)-k-1 = 2n+k-7 > 0$, we can apply Theorem 2.10, concluding that the graph G is $(2k+3n-10)$-uniform $(2k+2n-8)$-11-representable. □

In particular, Corollary 2.11 holds for any integers $n \geq 5$ and $k \geq 0$.

Theorem 2.12. *Let G be a graph on n vertices. Then, G is uniformly k-11-representable, where $0 \leq k \leq 2^{n-3}$, and thus every graph on n vertices is uniformly $O(2^n)$-11-representable.*

Proof. If $n \leq 4$, then G is uniformly 0-11-representable (due to Theorem 1.1 and the fact that all graphs on at most five vertices are word-representable [11, 10]), so the statement is obviously true. Hence, we can assume that $n \geq 5$. By induction on n, we will show that for $n \geq 5$, every graph on n vertices is $(2^{n-3}-n+3)$-uniformly $(2^{n-3}-2n+6)$-11-representable.

Note that every graph on five vertices is a circle graph, which follows from [4], or from comparison of the sequences A000088 for the number of all graphs and A156809 for the number of circle graphs for $n = 5$ in [16]. Thus, the statement is true in the base case of $n = 5$, since circle graphs are 2-uniformly 0-11-representatable by Theorem 1.3.

Assume that the statement is true for every graph on n vertices. By Corollary 2.11, every graph on $n+1$ vertices is $(2^{n-2}-n+2)$-uniform $(2^{n-2}-2n+4)$-11-representable, and $2^{(n+1)-3}-(n+1)+3 = 2^{n-2}-n+2$ and $2^{(n+1)-3}-2(n+1)+6 = 2^{n-2}-2n+4$, which completes the proof. □

3 1-11-representable graphs

An *interval graph* has one vertex for each interval in a family of intervals, and an edge between every pair of vertices corresponding to intervals that intersect. Not all interval graphs are word-representable [10]. However, all interval graphs are 1-11-representable using two copies of each letter, as shown in the following theorem. This shows that the notion of an interval graph admits a natural generalization in terms of 1-11-representable graphs (instead of 2-uniform 1-11-representations, one can deal with m-uniform 1-11-representations for $m \geq 3$).

Theorem 3.1. *A graph is an interval graph if and only if it is 2-uniformly 1-11-representable.*

Proof. Let G be a 1-11-representable graph on n vertices and $w = w_1w_2 \dots w_{2n}$ be a word that 2-uniformly 1-11-represents G. For any $v \in V(G) = [n]$, consider the interval $I_v = [v_1, v_2]$ on the real line such that $w_{v_1} = w_{v_2} = v$. Note that uv is an edge in G if and only if I_u and I_v overlap. But then, G is the interval graph given by the family of intervals $\{I_v : v \in [n]\}$.

To see that any interval graph G is necessarily 1-11-representable, we note a well-known easy to see fact that in the definition of an interval graph, one can assume that overlapping intervals overlap in more than one point. But then, the endpoints of an interval I_v will give the positions of the letter v in a word w constructed by recording relative positions of all the intervals. As above, one can see that such an w 1-11-represents G. □

Given a graph G with an edge xy, we let $G^{\triangle}_{xy}$ be the graph obtained from G by adding a vertex z connected only to the vertices x and y. Thus, $G^{\triangle}_{xy}$ is obtained from G by adding a triangle. If G is word-representable, that is, $G \in \mathcal{G}^{(0)}$, then $G^{\triangle}_{xy}$ is not necessarily word-representable. This can be seeing on the non-word-representable graph D_1 in Figure 2. Indeed, removing, for example, the top vertex in that graph, we obtain a word-representable graph, since the only non-word-representable graph on six vertices is the wheel W_5 [10, 11]. The following theorem establishes that adding a triangle is a safe operation in the case of 1-11-representable graphs.

Theorem 3.2. *Let $G \in \mathcal{G}^{(1)}$ and xy be an edge in G. Then $G^{\triangle}_{xy} \in \mathcal{G}^{(1)}$.*

Proof. Let w be an 1-11-representation of G. Note that, since x and y are adjacent in G, the letters x and y are either alternating in the word w, or $w|_{\{x,y\}}$ has exactly one occurrence of the pattern 11. In each case, we will construct a word $\tilde{w}$ over $V(G^{\triangle}_{xy})$, which 1-11-represents the graph $G^{\triangle}_{xy}$.

Case 1. Suppose that x and y are alternating in w. By Lemma 2.3, we can assume that w starts with x and ends with y, i.e. $w = x\ g_1\ y\ g_2 \ldots x\ g_m\ y$, where g_i is a word on $V(G) \setminus \{x, y\}$. Also, we can assume that $m \geq 3$; if not, adjoin the initial permutation $\pi(w)$ to the left of w. Now, we claim that the word

$$\tilde{w} := zxz\ g_1\ y\ g_2\ x\ g_3\ zyz\ g_4\ x\ g_5\ yz\ g_6 \ldots x\ g_m\ yz$$

1-11-represents the graph $G^{\triangle}_{xy}$, where $z \in V(G^{\triangle}_{xy}) \setminus V(G)$.

It is clear that $\tilde{w}$ respects the whole structure of G since the restriction of $\tilde{w}$ to $V(G)$ is w. Since $\tilde{w}|_{\{x,z\}} = zxzxzzxz \ldots xz$ and $\tilde{w}|_{\{y,x\}} = zzyzyzyz \ldots yz$, z is adjacent to x and y. On the other hand, for each $v \in V(G) \setminus \{x, y\}$, it is obvious that the induced subword $\tilde{w}|_{\{v,z\}}$ has at least two occurrences of the pattern 11, hence z is not adjacent to v. Therefore, $\tilde{w}$ 1-11-represents the graph $G^{\triangle}_{xy}$.

Case 2. Suppose $w|_{\{x,y\}}$ has exactly one occurrence of the pattern 11. Without loss of generality, we can assume that $w|_{\{x,y\}}$ contains the occurrence of the factor yy. By Lemma 2.3, we can also assume that w starts with x and ends with x, i.e.

$$w = x\ g_1\ y\ g_2 \ldots x\ g_{m-1}\ y\ g_m\ y\ h_1\ x\ h_2 \ldots y\ h_l\ x$$

for some positive integers m, l, and words g_i, h_j on $V(G) \setminus \{x, y\}$. We claim that the word

$$\tilde{w} := zxz\ g_1\ y\ g_2\ xz\ g_3\ y\ g_4 \ldots xz\ g_{m-3}\ y\ g_{m-2}\ x\ g_{m-1}\ zyz\ g_m\ y\ h_1\ xz\ h_2 \ldots y\ h_l\ xz$$

1-11-represents the graph $G^{\triangle}_{xy}$.

It is clear that $\tilde{w}$ respects the whole structure of G since the restriction of $\tilde{w}$ to $V(G)$ is w. Since $\tilde{w}|_{\{x,z\}} = zxzx \ldots zxzzxz \ldots xz$ and $\tilde{w}|_{\{y,z\}} = zzyzyz \ldots yz$, z is adjacent to x and y. On the other hand, for each $v \in V(G) \setminus \{x, y\}$, the induced subword $\tilde{w}|_{\{v,z\}}$ has at least two occurrences of the pattern 11, hence z is not adjacent to v. Therefore, $\tilde{w}$ 1-11-represents the graph $G^{\triangle}_{xy}$.

□

For the next theorem, Theorem 3.3, recall the definition of a permutationally representable graph in Section 1.1. Note that the proof of Theorem 3.3 is similar to that of Theorem 2.9, while Theorem 3.3 deals with a stricter assumption. However, the stricter assumption is compensated by a stronger conclusion, justifying us having Theorem 2.9.

Theorem 3.3. *Let G be a graph with a vertex v. If $G \setminus v$ is permutationally representable (equivalently, by Theorem 1.4, if $G \setminus v$ is a comparability graph) then G is 1-11-representable.*

Proof. Let w be a 0-11-representation of $G \setminus v$. Since $G \setminus v$ is permutationally representable, we can assume that w is of the form $w = \pi_1\pi_2 \ldots \pi_k$ for some positive integer k and permutations π_i of $V(G \setminus v)$. Let $N(v)$ be the set of neighbours of v in G and let $N^c(v) := V(G) \setminus (N(v) \cup \{v\})$. We claim that the word

$$w(G) := r(\pi(w))\ v\ \pi(w)|_{N(v)}\ v\ \pi(w)|_{N^c(v)}\ \pi_1 v \pi_2 v \ldots v \pi_k.$$

1-11-represents the graph G.

For each $x \in V(G) \setminus \{v\}$, if $x \in N(v)$ then the induced subword $w(G)|_{\{x,v\}} = xvxv \ldots xvx$ is alternating, which should be the case. If $x \in N^c(v)$, then the induced subword $w(G)|_{\{x,v\}} = xvvxxvxv \ldots xvx$ has two occurrences of the pattern 11, which, again, should be the case. Thus, $w(G)$ respects all adjacencies of the vertex v. Now, take $y, z \in V(G) \setminus \{v\}$. If y and z are adjacent in $G \setminus v$, then $w|_{\{y,z\}}$ has alternating y and z. Without loss of generality, assume that $w|_{\{y,z\}} = yzyz \ldots yz$. Then, the induced subword $w(G)|_{\{y,z\}} = zy\ (\pi(w)|_{N(v)}\ \pi(w)|_{N^c(v)})|_{\{y,z\}}\ yzyz \ldots yz$ has at most one occurrence of the pattern 11 as $(\pi(w)|_{N(v)}\ \pi(w)|_{N^c(v)})|_{\{y,z\}}$ is either yz or zy. If y and z are not adjacent in $G \setminus v$, then $w|_{\{y,z\}}$ is not alternating, i.e. it contains either yy or zz. Without loss of generality, assume that $w|_{\{y,z\}}$ contains yy. If $\pi(w)|_{\{y,z\}} = yz$, then with the assumption on an occurrence of yy, at least one occurrence of the factor zz is not avoidable in w, so at least two occurrences of the pattern 11 in $w(G)|_{\{y,z\}}$ are guaranteed. Otherwise, $w|_{\{y,z\}} = zy \ldots zy\ yz \ldots$. Then, $w(G)|_{\{y,z\}} = yz\ (\pi(w)|_{N(v)}\ \pi(w)|_{N^c(v)})|_{\{y,z\}}\ zy \ldots zy\ yz \ldots$ has two occurrences of the pattern 11, as $(\pi(w)|_{N(v)}\ \pi(w)|_{N^c(v)})|_{\{y,z\}}$ is either yz or zy. In any case, $w(G)$ preserves the (non-)adjacency of y and z. Therefore the word $w(G)$ 1-11-represents the graph G. ∎

Theorem 3.4. *Let G be a word-representable graph and e be an edge in G. Let $G \setminus e$ be the graph obtained from G by removing e. Then, $G \setminus e$ is 1-11-representable.*

Proof. Let $e = xy$ and w be G's uniform word-representant that exists by Theorem 1.1. Without loss of generality, we can assume that $w|_{\{x,y\}} = xyxy \ldots xy$. We claim that the graph G' on $V(G)$, which is 1-11-represented by the word $w' := yxwwyx$, is precisely the graph $G \setminus e$.

It is clear that x and y are not adjacent in G' since $w'|_{\{x,y\}} = yxxy \ldots xyyx$. Since the word ww is a 1-11-representation of G, it remains to show that for every vertex $z \in V(G) \setminus \{x, y\}$, and a vertex $i \in \{x, y\}$, G' contains the edge iz whenever iz is an edge in G. Suppose iz is an edge in G. Then, $ww|_{\{i,z\}}$ is either $iz \ldots iz$, or $zi \ldots zi$. It follows that $w'|_{\{i,z\}}$ is either $iiz \ldots izi$, or $izi \ldots zii$. Thus, iz is an edge in G'. If iz is not an edge in G, then $ww|_{\{i,z\}}$ will contain at least two occurrences of the pattern 11, so iz is not an edge in G'. This shows that $G' = G \setminus e$. ∎

The following two theorems generalize Theorem 3.4. The reason that we keep Theorem 3.4 as a separate result is that it is very useful in 1-11-representing 25 non-word-representable graphs (see Section 4).

Theorem 3.5. *Let G be a word-representable graph and K be a vertex subset in G. Let G_K be the graph obtained from G by removing the edges $\{xy \in E(G) : x, y \in K\}$. Then, G_K is* 1-11-*representable.*

Proof. Let w be a uniform word-representant of G that exists by Theorem 1.1. Let p be the reverse of the initial permutation of $w|_K$, and let q be the reverse of the final permutation of $w|_K$. Note that if K is a clique in G, then $p = q$. It is straightforward to check that the word $w' := pwwq$ 1-11-represents the graph G_K. □

Theorem 3.6. *Let G be a word-representable graph, v be a vertex in G, and N be a set of some (not necessarily all) neighbors of v in G. Let G_N be the graph obtained from G by removing the edges $\{uv : u \in N\}$. Then, G_N is* 1-11-*representable.*

Proof. Let $N = \{v_1, \ldots, v_k\}$ and w be a uniform word-representant of G. Since w is uniform, by Theorem 1.2, we can assume that v is the first letter in w. Without loss of generality, assume that $v_1 \ldots v_k$ is the initial permutation of $w|_N$. Then, it is easy to check that the word $w' := v_k \ldots v_1 v w w v_k \ldots v_1 v$ 1-11-represents the graph G_N. □

4 1-11-representing non-word-representable graphs

All graphs on at most five vertices are word-representable, and there is only one non-word-representable graph, the wheel W_5, on six vertices (see Figure 1). Also, there are 25 non-word-representable graphs on seven vertices, which are shown in Figure 2.

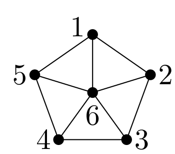

Figure 1: The wheel graph W_5

The following theorem shows that the notion of k-11-representability allows us to enlarge the class of word-representable graphs ($\mathcal{G}^{(0)}$), still by using alternating properties of letters in words.

Theorem 4.1. *We have $\mathcal{G}^{(0)} \subsetneq \mathcal{G}^{(1)}$.*

Proof. By Theorem 2.2, we have $\mathcal{G}^{(0)} \subseteq \mathcal{G}^{(1)}$. To show that the inclusion is strict, we give a word 1-11-representing the non-word-representable wheel graph W_5 in Figure 1. We start with 0-11-representing the cycle graph induced by all vertices but the vertex 6 by the 2-uniform word $w = 1521324354$. This word, and a generic approach to find it, is found on page 36 in [10]. Note that the initial permutation $\pi(w)$ is 15234, and thus, by Lemma 2.1, the word $r(\pi(w))w = 432511521324354$ 1-11-represents the cycle graph. Inserting a 6 in w to obtain $u = 4325161521324354$ gives a word 1-11-representing W_5 (which is easy to see). Note that the word $6u6$ gives a 3-uniform 1-11-representation of W_5. ☐

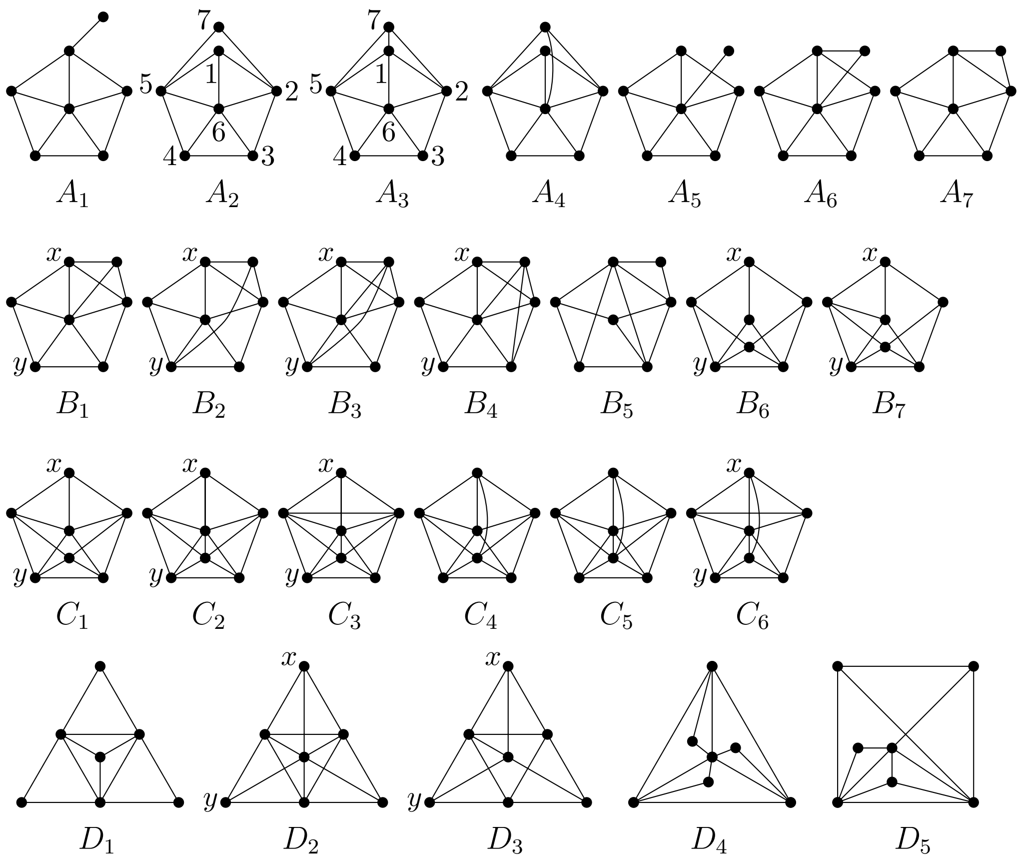

Figure 2: The 25 non-isomorphic non-word-representable graphs on 7 vertices

We do not know whether $\mathcal{G}^{(1)}$ coincides with the class of all graphs, but at least we can show that all 25 graphs in Figure 2 are 1-11-representable, which we do next. We will use the fact that all graphs on at most six vertices are 1-11-representable, which follows from the proof of Theorem 4.1, where we 1-11-represent the only non-word-representable graph on six vertices.

The graphs A_1 and A_5 are 1-11-representable by Theorem 2.5, since they have a vertex of degree 1. Theorem 2.6 can be applied to the graphs A_4, C_4 and C_5 since each of these graphs have a pair of vertices whose neighbourhoods are the same up to removing these vertices. Further, Theorem 3.2 gives 1-11-representability of the graphs A_6, A_7, B_5, D_1, D_4 and D_5

since each of these graphs has a triangle with a vertex of degree 2. Explicit easy-to-check 1-11-representations of the graphs A_2 and A_3 are, respectively, 437257161521324354 and 437251761521324354. For each graph G of the remaining 12 graphs in Figure 2, we provide vertices x and y connecting which by an edge results in a word-representable graph G_{xy}, so that Theorem 3.4 can be applied (removing the edge xy from G_{xy}) to see that G is 1-11-representable. The fact that G_{xy} is word-representable follows from it not being isomorphic to any of the graphs in Figure 2, where all non-word-representable graphs on seven vertices are presented. Alternatively, one can use the software packages [5, 17] to see that G_{xy} is word-representable (the software can produce an easy to check word representing G_{xy}).

5 5-colorable graphs

It is known [10] that any 3-colorable graph is word-representable, but there are examples of 4-colorable non-word-representable graphs (e.g. D_1 in Figure 2). We still do not know whether 4-colorable graphs are i-11-representable for $i \in \{1, 2, 3, 4, 5\}$, but we can prove that 5-colorable graphs, and thus 4-colorable graphs, are 6-11-representable.

Theorem 5.1. *Let G be a 5-colorable graph. Then $G \in \mathcal{G}^{(6)}$.*

Proof. Our proof is based on the approach sketched in Section 1.2, with an appropriate to our case modifications. Since G is 5-colorable, we can consider a partition $V_1 \cup V_2 \cup \cdots \cup V_5 = V(G)$, where each V_i is an independent set. Let $V_i = \{v_{i_1}, \ldots, v_{i_{m_i}}\}$. Also, let $V_{-1} = V_4, V_0 = V_5, V_6 = V_1, V_7 = V_2$. We will consider $|V(G)|$ acyclic orientations (not necessarily distinct) of the graph G. Namely, for every $1 \leq i \leq 5$, and for every vertex $v \in V_i$, we assign the acyclic orientation $V_{i-2} \to V_{i-1} \to V_i \to V_{i+1} \to V_{i+2}$ to G such that for every pair of indices $i-2 \leq j < j' \leq i+2$ and a pair of adjacent vertices $v_j \in V_j, v_{j'} \in V_{j'}$, the edge $v_j v_{j'}$ is oriented as $v_j \to v_{j'}$. For example, for a vertex $v \in V_1$, we consider the acyclic orientation $V_{-1} = V_4 \to V_0 = V_5 \to V_1 \to V_2 \to V_3$, while for a vertex $v \in V_4$, such an orientation is $V_2 \to V_3 \to V_4 \to V_5 \to V_6 = V_1$.

Recall the notation $u \rightsquigarrow v$ in Section 1.2. Let $I(v) = \{u : u \to v\}$, $O(v) = \{u : v \to u\}$, $A(v) = \{u \in V : u \rightsquigarrow v\} \setminus I$, $B(v) = \{u \in V : v \rightsquigarrow u\} \setminus O$, and $T(v) = V \setminus (\{v\} \cup I(v) \cup O(v) \cup A(v) \cup B(v))$. Note that every directed path that either starts with v, or ends with v, has length at most 2. Thus, we observe that every edge between $u_1 \in A(v)$ and $u_2 \in I(v)$ is oriented by $u_1 \to u_2$, and that every edge between $u_1 \in O(v)$ and $u_2 \in B(v)$ is oriented by $u_1 \to u_2$. In addition, by the definition of $A(v)$ and $B(v)$, the orientation of vertices between $T(v)$ and the others must be given by $A(v) \to T(v)$, $I(v) \to T(v)$, $T(v) \to O(v)$, and $T(v) \to B(v)$. Observe that v is not adjacent to any vertex in $A(v) \cup B(v) \cup T(v)$. See Figure 3 for an illustration of this situation.

We say that a word w *preserves the orientation*, if the induced subword of w on $\{u_1, u_2\}$ starts with u_1 for each directed edge $u_1 \to u_2$. For each vertex $v \in V(G)$, we will define a word w_v preserving the orientation depicted in Figure 3, so that w_v contains a "sufficiently large" number of occurrences of the pattern 11 between v and the vertices not adjacent to v, while w_v contains "only a few" occurrences of the pattern 11 for any other pair of vertices. Then,

$$
\begin{tikzcd}
 & & T(v) \arrow[dr] \arrow[drr] & & \\
A(v) \arrow[r] \arrow[urr] \arrow[rrr, bend right] \arrow[rrrr, bend right=50] & I(v) \arrow[r] \arrow[ur] \arrow[rrr, bend right] & v \arrow[r] & O(v) \arrow[r] & B(v)
\end{tikzcd}
$$

Figure 3: The orientation of G corresponding to a vertex v

we will concatenate the words w_v's to obtain a 6-11-representation $w(G) := W_1W_2W_3W_4W_5$ of G, where $W_i := w_{v_{i_1}} \dots w_{v_{i_m}}$. Let

$$
\begin{aligned}
w_v := \ & AvITOB\ vAITOB\ AvITOB\ vAITOB \\
& AITvOB\ AIvTOB\ AITvOB\ AIvTOB \\
& AITOBv\ AITOvB\ AITOBv\ AITOvB,
\end{aligned}
$$

where A, I, T, O, B are fixed permutations of the corresponding sets that preserve the orientations. Note that w_v contains four occurrences of the pattern 11 for each pair (u, v) such that $u \in A \cup B \cup T$. In addition, for each directed edge $u_1 \to u_2$ with $u_1, u_2 \neq v$, the induced subword $w_v|_{\{u_1,u_2\}}$ is alternating and starts with u_1, thus preserving the orientation of G. If $u \in I$, then $w_v|_{\{u,v\}} = vu\ vu\ vu\ vu\ uv \dots uv$, which begins with v and has one occurrence of the pattern 11. If $u \in O$, then $w_v|_{\{u,v\}} = vu \dots vu\ uv\ uv\ uv\ uv$, which begins with v and has one occurrence of the pattern 11. In either case, the word w_v preserves the orientation.

We claim that $w(G)$ 6-11-represents G. Let x and y be two distinct vertices in G. If x, y are not adjacent in G, then $w_x|_{\{x,y\}}$ and $w_y|_{\{x,y\}}$ each have four occurrences of the pattern 11. Thus, $w(G)|_{\{x,y\}}$ has at least eight occurrences of the pattern 11. It remains to consider the case of adjacent x and y with the edge oriented as $x \to y$. There are ten cases to consider: $x \in V_i$ and $y \in V_j$, where $1 \leq i < j \leq 5$. Here, we only consider the case of $x \in V_1, y \in V_2$, but all the other cases can be easily considered in a very similar way.

(i) For $v \in V_1$, the word w_v deals with the orientation $V_4 \to V_5 \to V_1 \to V_2 \to V_3$ on G. Then, $w_v|_{\{x,y\}} = xy \dots xy$ if $v \neq x$ and $w_x|_{\{x,y\}} = xy \dots xyyx \dots yx$. Thus, $W_1|_{\{x,y\}}$ is either $xy \dots xyyx \dots yxxy \dots xy$ with two occurrences of the pattern 11, or $xy \dots xyyx \dots yx$ with one occurrence of the pattern 11.

(ii) For $v \in V_2$, the word w_v deals with the orientation $V_5 \to V_1 \to V_2 \to V_3 \to V_4$ on G. Then, $w_v|_{\{x,y\}} = xy \dots xy$ if $v \neq y$ and $w_y|_{\{x,y\}} = yx \dots yxxy \dots xy$. Thus $W_2|_{\{x,y\}}$ is $xy \dots xyyx \dots yxxy \dots xy$ with two occurrences of the pattern 11, or $yx \dots yxxy \dots xy$ with one occurrence of the pattern 11.

(iii) For $v \in V_3$, the word w_v deals with the orientation $V_1 \to V_2 \to V_3 \to V_4 \to V_5$ on G. Then, $w_v|_{\{x,y\}} = xy \dots xy$, so $W_3|_{\{x,y\}} = xy \dots xy$, which is alternating.

(iv) For $v \in V_4$, the word w_v deals with the orientation $V_2 \to V_3 \to V_4 \to V_5 \to V_1$ on G. Then, $w_v|_{\{x,y\}} = yx \dots yx$, so $W_4|_{\{x,y\}} = yx \dots yx$, which is alternating.

(v) For $v \in V_5$, the word w_v deals with the orientation $V_3 \to V_4 \to V_5 \to V_1 \to V_2$ on G. Then, $w_v|_{\{x,y\}} = xy \dots xy$, so $W_5|_{\{x,y\}} = xy \dots xy$, which is alternating.

Some extra occurrences of the pattern 11 can occur when we concatenate the words $W_k|_{\{x,y\}}$ to obtain $w(G)|_{\{x,y\}} = W_1|_{\{x,y\}}W_2|_{\{x,y\}}W_3|_{\{x,y\}}W_4|_{\{x,y\}}W_5|_{\{x,y\}}$. However, in any case, it is clear that $w(G)|_{\{x,y\}}$ has at most six occurrences of the pattern 11. $\square$

We note that the methods to prove Theorem 5.1 do not work for k-colorable graphs for $k \geq 6$, so in general, we do not know how to estimate the value of ℓ such that all k-colorable graphs belong to $\mathcal{G}^{(\ell)}$.

6 Open problems on k-11-representable graphs

Probably the most intriguing open question in the theory of k-11-representable graphs is the following.

Problem 1. Is it true that any graph is 1-11-representable?

If the answer to the question in Problem 1 is yes, then some of the problems below will be automatically solved. However, the remaining problems would still be interesting and challenging.

Problem 2. Is it true that $\mathcal{G}^{(k)} \subsetneq \mathcal{G}^{(k+1)}$ for any $k \geq 1$. Recall that this is true for $k = 0$ by Theorem 4.1.

By Theorem 1.1, any word-representable graph can be represented by a uniform word. Thus, the following question is natural.

Problem 3. Is it true that any k-11-representable graph can be represented by a uniform word?

Problem 4. Given a graph G, we know by Theorem 2.12 that G is k-11-representable for some $k \geq 0$. Theorem 2.12 also gives an upper bound on such a k. Can this bound be (significantly) improved if not for a generic graph then for, say, a t-coloarble graph, or a planar graph, or a non-word-representable graph from another (well known) class of graphs? In particular, what is the true value of k for 4-colorable graphs? Recall that currently, we only know that $k \leq 6$ for these graphs (see Theorem 5.1).

It is known [7] that if a graph G with n vertices is word-representable, then it can be represented by a uniform word of length at most $2n(n-\kappa)$ where κ is the size of a maximum clique in G. However, we have no upper bounds for words k-11-representing graphs.

Problem 5. Provide an upper bound for words k-11-representing graphs. In particular, is there such a bound in the case of $k = 1$?

Problem 6. Theorem 3.1 shows that the class of interval graphs is precisely the class of 1-11-representable graphs that can be represented 2-uniformly. Does the class of m-uniformly 1-11-representable graphs, for $m \geq 3$, have any interesting/useful properties? In particular, is there a description of such graphs in terms of forbidden subgraphs? A good starting point to answer the last question should be the case of $m = 3$.

Subdividing an edge uv in a graph is replacing the edge by a path $ux_1x_2\cdots x_tv$, $t \geq 1$. It was shown in [11] that the graph obtained by subdividing *each* edge in *any* graph into at least three edges ($t \geq 2$) can be 0-11-represented by a 3-uniform word. On the other hand, it is easy to use the notion of a semi-transitive orientation to show that if each edge in a graph is subdivided arbitrarily then the resulting graph is always 0-11-representable (i.e. word-representable). Indeed, orienting subdivisions of the form ux_1v as $u \rightarrow x_1 \leftarrow v$, and any other subdivisions as $u \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_{t-1} \rightarrow x_t \leftarrow v$, we clearly obtain no cycles or shortcuts. Thus, by Theorem 4.1, subdividing each edge in a graph we obtain a 1-11-representable graph, so the following problem is well defined.

Problem 7. What is the minimum number of letters we need to (uniformly) 1-11-represent the graph obtained by a subdivision of each edge in a given graph.

Not all planar graphs are word-representable (e.g. W_5, as well as several planar graphs in Figure 2, are non-word-representable). It would be interesting to answer the following question.

Problem 8. Are all planar graphs 1-11-representable?

In Section 1.2, we introduced the notion of a semi-transitive orientation characterizing word-representable graphs. k-11-representable graphs generalize the notion of a word-representable graph, and thus it is natural to try to find orientations, generalizing semi-transitive orientations, that characterize k-11-representable graphs. Two possible approaches in the case of $k = 1$ might be

- Changing the direction of a shortcutting edge that will result in a cycle. Such a cycle might correspond to defining an edge using exactly one occurrence of the pattern 11.
- Allowing shortcutting edges, which might correspond to defining an edge using exactly one occurrence of the pattern 11.

However, we were not able to find such a generalization of a semi-transitive orientation, so we state the following problem.

Problem 9. Find orientations of graphs characterizing k-11-representable graphs for $k \geq 1$. Even the case $k = 1$ is of substantial interest here.

We conclude by recalling that certain orientations were used by us in the proof of Theorem 5.1, and they might be useful in solving Problem 9.

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