Exponential Riordan arrays and generalized Narayana polynomials

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Abstract

Generalized Euler polynomials $\alpha_n(x) = (1-x)^{n+1} \sum_{m=0}^{\infty} p_n(m)\, x^m$, where $p_n(x)$ is the polynomial of degree n, are the numerator polynomials of the generating functions of diagonals of the ordinary Riordan arrays. Generalized Narayana polynomials $\varphi_n(x) = (1-x)^{2n+1} \sum_{m=0}^{\infty} (m+1) \ldots (m+n)\, p_n(m)\, x^m$ are the numerator polynomials of the generating functions of diagonals of the exponential Riordan arrays. In present paper we consider the constructive relationship between these two types of numerator polynomials.

1 Introduction

This paper is a continuation of the paper "Riordan arrays and generalized Euler polynomials" [1]. In Section 2 we will briefly retell its content. For the integrity of presentation, we will change some notation adopted in [1].

Subject of our study is the transformations in space of formal power series and the corresponding matrices. We associate rows and columns of matrices with the generating functions of their elements. nth coefficient of the series $a(x)$, nth row, nth descending diagonal and nth column of the matrix A will be denoted respectively by

$$[x^n]\, a(x), \qquad [n, \rightarrow]\, A, \qquad [n, \searrow]\, A, \qquad Ax^n.$$

Matrix $(f(x), g(x))$, $g_0 = 0$, nth column of which, $n = 0,\ 1,\ 2,\ \ldots$, has the generating function $f(x)\, g^n(x)$, is called Riordan array [2] – [6]. It is the product of two matrices that correspond to multiplication and composition of series:

$$(f(x), g(x)) = (f(x), x)\, (1, g(x)),$$

$$(f(x), x)\, a(x) = f(x)\, a(x), \qquad (1, g(x))\, a(x) = a(g(x)),$$

$$(f(x), g(x))\, (b(x), a(x)) = (f(x)\, b(g(x)), a(g(x))).$$

Matrices

$$|e^x|^{-1} (f(x), g(x))\, |e^x| = (f(x), g(x))_{e^x},$$

where $|e^x|$ is the diagonal matrix, $|e^x|\, x^n = x^n/n!$, are called exponential Riordan arrays. Denote

$$[n, \rightarrow]\, (f(x), g(x))_{e^x} = s_n(x), \qquad f_0 \neq 0, \qquad g_1 \neq 0.$$

Then

$$(f(x), g(x))_{e^x} (1 - \varphi x)^{-1} = |e^x|^{-1} (f(x), g(x))\, e^{\varphi x} = |e^x|^{-1} f(x) \exp(\varphi g(x)),$$

or

$$\sum_{n=0}^{\infty} \frac{s_n(\varphi)}{n!} x^n = f(x) \exp(\varphi g(x)).$$

Sequence of polynomials $s_n(x)$ is called Sheffer sequence, and in the case $f(x) = 1$, binomial sequence. The properties of the Sheffer sequences are the subject of study of the umbral calculus [7]. Matrix

$$P = \left(\frac{1}{1-x}, \frac{x}{1-x}\right) = (e^x, x)_{e^x} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & \cdots \\ 1 & 3 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is called Pascal matrix. Power of the Pascal matrix is defined by the identity

$$P^{\varphi} = \left(\frac{1}{1-\varphi x}, \frac{x}{1-\varphi x}\right) = (e^{\varphi x}, x)_{e^x}.$$

Along with the lower triangular Riordan matrices, we will consider "square" matrices $(b(x), a(x))$, $b_0 \neq 0$, $a_0 = 1$. For example,

$$\left(1, \frac{1}{1+x}\right) = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & -1 & -2 & -3 & \cdots \\ 0 & 1 & 3 & 6 & \cdots \\ 0 & -1 & -4 & -10 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This includes the upper triangular matrix $(1, 1+x)$, whose transpose is the Pascal matrix and which coincides with the matrix of shift operator:

$$(1, 1+x) = P^T = E = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 2 & 3 & \cdots \\ 0 & 0 & 1 & 3 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Matrix $(b(x), a(x))$ can be multiplied from the right by the matrix with the finite columns and from the left by the matrix with the finite rows. At first (before Section 4), we restrict ourselves to the set of matrices of the form $(1, a(x))$. Since

$$[n, \rightarrow](1, a(x)) = [n, \searrow](1, xa(x)),$$

then the matrix $(1, a(x))$ is a tool for study of the matrix $(1, xa(x))$. Denote

$$[n, \rightarrow](1, a(x) - 1) = v_n(x) = \sum_{m=1}^{n} v_m x^m, \qquad n > 0.$$

Since

$$(1, a(x) - 1)(1, 1+x) = (1, a(x)), \qquad [n, \rightarrow](1, 1+x) = \frac{x^n}{(1-x)^{n+1}},$$

then

$$[n, \rightarrow](1, a(x)) = \sum_{m=1}^{n} \frac{v_m x^m}{(1-x)^{m+1}} = \sum_{m=1}^{n} \frac{v_m x^m (1-x)^{n-m}}{(1-x)^{n+1}} = \frac{\alpha_n(x)}{(1-x)^{n+1}}.$$

If $a(x) = e^x$, then $\alpha_n(x) = A_n(x)/n!$, where $A_n(x)$ are the Euler polynomials:

$$\frac{A_n(x)}{(1-x)^{n+1}} = \sum_{m=0}^{\infty} m^n x^m, \qquad A_n(1) = n!.$$

For example,

$$A_1(x) = x, \qquad A_2(x) = x + x^2, \qquad A_3(x) = x + 4x^2 + x^3,$$

$$A_4(x) = x + 11x^2 + 11x^3 + x^4.$$

In this connection we will called these polynomials the generalized Euler polynomials (GEP).

"Square" Riordan arrays (called convolution arrays) and numerator polynomials of the generating functions of their rows were considered in the series of papers [8] – [12]. In [13] such matrices are called generalized Riordan arrays. Concept of generalized Euler polynomials (called p_n-associated Eulerian polynomials) in general form is represented in [14].

Denote

$$[n, \searrow](1, xa(x)) = \frac{\alpha_n(x)}{(1-x)^{n+1}}, \qquad [n, \to](1, \log a(x))_{e^x} = u_n(x),$$

$$[n, \to](1, a(x) - 1) = v_n(x).$$

If the sequence of polynomials has the form $c_0(x) = 1$, $[x^0]\, c_n(x) = 0$, we will bear in mind that the expression $(1/x\ )\, c_n(x)$ corresponds to the case $n > 0$. Denote

$$\frac{1}{x}\alpha_n(x) = \tilde{\alpha}_n(x), \quad \frac{1}{x}A_n(x) = \tilde{A}_n(x), \quad \frac{1}{x}u_n(x) = \tilde{u}_n(x), \quad \frac{1}{x}v_n(x) = \tilde{v}_n(x).$$

Let the symbols $(\varphi)_n$, $[\varphi]_n$ denote respectively the falling and the rising factorial:

$$(\varphi)_n = \varphi(\varphi - 1) \dots (\varphi - n + 1), \qquad [\varphi]_n = \varphi(\varphi + 1) \dots (\varphi + n - 1).$$

In Section 2 we consider the generalized Euler polynomials and associated transformations. We introduce the matrices $\tilde{U}_n$, $\tilde{V}_n$:

$$\tilde{U}_n x^p = \frac{1}{n!}(1-x)^{n-1-p}\tilde{A}_{p+1}(x), \qquad \tilde{U}_n^{-1} x^p = (x-1)_p [x+1]_{n-p-1},$$

$$\tilde{V}_n x^p = (1+x)^{n-p-1} x^p, \qquad \tilde{V}_n^{-1} x^p = (1-x)^{n-p-1} x^p, \qquad p = 0,\ 1,\ \dots,\ n-1.$$

Then

$$\tilde{U}_n \tilde{u}_n(x) = \tilde{\alpha}_n(x), \qquad \tilde{V}_n \tilde{\alpha}_n(x) = \tilde{v}_n(x).$$

We consider the series ${}_{(\beta)}a(x)$, ${}_{(0)}a(x) = a(x)$, that are defined as follows:

$${}_{(\beta)}a^{\varphi}(x) = \sum_{n=0}^{\infty} \frac{\varphi}{\varphi + n\beta} \frac{u_n(\varphi + n\beta)}{n!} x^n.$$

Denote

$$[n, \searrow]\left(1, x_{(\beta)}a(x)\right) = \frac{{}_{(\beta)}\alpha_n(x)}{(1-x)^{n+1}}, \qquad \frac{1}{x}{}_{(\beta)}\alpha_n(x) = {}_{(\beta)}\tilde{\alpha}_n(x).$$

We introduce the matrices

$$A_n^{\beta} = \tilde{U}_n E^{n\beta} \tilde{U}_n^{-1} = \tilde{V}_n^{-1} \tilde{D}\left((1+x)^{n\beta}, x\right)^T \tilde{D}^{-1} \tilde{V}_n, \qquad \tilde{D}x^n = (n+1)\, x^n.$$

Then

$$A_n^\beta \tilde{\alpha}_n(x) = {}_{(\beta)}\tilde{\alpha}_n(x).$$

We give a general formula for the GEP associated with the generalized binomial series. Namely, let

$${}_{(\beta)}a^\varphi(x) = \sum_{n=0}^{\infty} \frac{\varphi}{\varphi + n\beta} \binom{\varphi + n\beta}{n} x^n, \qquad \frac{{}_{(\beta)}\alpha_n(x)}{(1-x)^{n+1}} = \sum_{m=0}^{\infty} \frac{m}{m + n\beta} \binom{m + n\beta}{n} x^m.$$

Then

$${}_{(\beta)}\alpha_n(x) = \frac{1}{n} \sum_{m=1}^{n} \binom{n(1-\beta)}{m-1} \binom{n\beta}{n-m} x^m.$$

In Section 3 we consider the generalized Narayana polynomials $\varphi_n(x)$, which are the numerator polynomials of the matrix $(1, xa(x))_{e^x}$:

$$[n, \searrow](1, xa(x))_{e^x} = \frac{\varphi_n(x)}{(1-x)^{2n+1}} = \sum_{m=0}^{\infty} \frac{[m+1]_n u_n(m)}{n!} x^m, \qquad \frac{1}{x}\varphi_n(x) = \tilde{\varphi}_n(x).$$

We introduce the matrices $\tilde{F}_n$:

$$\tilde{F}_n x^p = (1-x)^{2n+1} \sum_{m=1}^{\infty} m^{p+1} \binom{m+n}{n} x^{m-1},$$

$$\tilde{F}_n^{-1} x^p = \frac{n!}{(2n)!} (x-1)_p [x+n+1]_{n-p-1}, \qquad p = 0,\ 1,\ \ldots,\ n-1.$$

Then

$$\tilde{F}_n \tilde{u}_n(x) = \tilde{\varphi}_n(x).$$

We introduce the matrices $\tilde{S}_n = \tilde{F}_n \tilde{U}_n^{-1}$. Then

$$\tilde{S}_n \tilde{\alpha}_n(x) = \tilde{\varphi}_n(x).$$

It turns out that

$$\tilde{S}_n = \tilde{V}_n^{-1} \tilde{C}_n \tilde{V}_n, \qquad \tilde{C}_n x^p = \frac{(n+p+1)!}{(p+1)!} x^p.$$

We give a general formula for the GNP associated with the generalized binomial series. Namely, let

$$\frac{{}_{(\beta)}\varphi_n(x)}{(1-x)^{2n+1}} = \sum_{m=0}^{\infty} \frac{m}{m + \beta n} \binom{m + \beta n}{n} [m+1]_n x^m.$$

Then

$${}_{(\beta)}\varphi_n(x) = \frac{(n+1)!}{n} \sum_{m=1}^{n} \binom{n(2-\beta)}{m-1} \binom{n\beta}{n-m} x^m.$$

In Section 4 we consider transformations of the general form. Let $g_n(x)$, $h_n(x)$ are the numerator polynomialsof of the matrices $(b(x), xa(x))$, $(b(x), xa(x))_{e^x}$, $b_0 \neq 0$, respectively. Denote

$$[n, \rightarrow](b(x), \log a(x))_{e^x} = s_n(x).$$

We introduce the matrices U_n, F_n:

$$U_n x^p = (1-x)^{n+1} \frac{1}{n!} \sum_{m=0}^{\infty} m^p x^m, \qquad U_n^{-1} x^p = (x)_p [x+1]_{n-p};$$

$$F_n x^p = (1-x)^{2n+1} \sum_{m=0}^{\infty} m^p \binom{m+n}{n} x^m, \quad F_n^{-1} x^p = \frac{n!}{(2n)!} (x)_p [x+n+1]_{n-p},$$

$p = 0, 1, \dots, n$. Then

$$U_n s_n(x) = g_n(x), \qquad F_n s_n(x) = h_n(x).$$

We introduce the matrices $S_n = F_n U_n^{-1}$. Then

$$S_n g_n(x) = h_n(x).$$

It turns out that

$$S_n = V_n^{-1} C_n V_n,$$

$$V_n x^p = (1+x)^{n-p} x^p, \quad V_n^{-1} x^p = (1-x)^{n-p} x^p, \quad C_n x^p = \frac{(n+p)!}{p!} x^p;$$

$$S_n x^p = \frac{(n+p)!\,(n-p)!}{n!} \sum_{m=p}^{n} \binom{n}{m-p} \binom{n}{n-m} x^m,$$

$$S_n^{-1} x^p = \frac{p!\,(n-p)!}{(2n)!} \sum_{m=p}^{n} \binom{-n}{m-p} \binom{2n}{n-m} x^m.$$

In Section 5 we consider the generalized Narayana polynomials of type B, which are the numerator polynomials of the matrix $(a(x), xa(x))_{e^x}$, and similar polynomials, which are the numerator polynomials of the matrix $\big((xa(x))', xa(x)\big)$.

In Section 6 we return to the series ${}_{(\beta)}a(x)$ from Section 2 and consider the transformations

$$G_n^\beta = U_n E^{n\beta} U_n^{-1} = V_n^{-1} \Big((1+x)^{n\beta}, x\Big)^T V_n,$$

$$H_n^\beta = F_n E^{n\beta} F_n^{-1} = S_n G_n^\beta S_n^{-1} = V_n^{-1} C_n \Big((1+x)^{n\beta}, x\Big)^T C_n^{-1} V_n,$$

$$T_n^\beta = \tilde{F}_n E^{n\beta} \tilde{F}_n^{-1} = \tilde{S}_n A_n^\beta \tilde{S}_n^{-1} = \tilde{V}_n^{-1} \tilde{C}_n \tilde{D} \Big((1+x)^{n\beta}, x\Big)^T \tilde{D}^{-1} \tilde{C}_n^{-1} \tilde{V}_n.$$

Let ${}_{(\beta)}g_n(x)$, ${}_{(\beta)}h_n(x)$ are the numerator polynomials of the matrices

$$\Big(b\left(x{}_{(\beta)}a^\beta(x)\right)\Big(1 + x\beta\big(\log {}_{(\beta)}a(x)\big)'\Big), x{}_{(\beta)}a(x)\Big),$$

$$\Big(b\left(x{}_{(\beta)}a^\beta(x)\right)\Big(1 + x\beta\big(\log {}_{(\beta)}a(x)\big)'\Big), x{}_{(\beta)}a(x)\Big)_{e^x},$$

respectively, ${}_{(\beta)}\varphi_n(x)$ are the numerator polynomials of the matrix $\big(1, x{}_{(\beta)}a(x)\big)_{e^x}$. Then

$$G_n^\beta g_n(x) = {}_{(\beta)}g_n(x), \quad H_n^\beta h_n(x) = {}_{(\beta)}h_n(x), \quad T_n^\beta \tilde{\varphi}_n(x) = {}_{(\beta)}\tilde{\varphi}_n(x).$$

Matrices G_n^β, H_n^β, T_n^β are characterized by the fact that, in comparison with them, the columns and rows of the matirices $G_n^{-\beta}$, $H_n^{-\beta}$, $T_n^{-\beta}$ are rearranged in the reverse order. Columns of the matrix G_n^β are expressed by the general formula:

$$G_n^\beta x^p = \sum_{m=0}^{n} \binom{-n\beta+p}{m} \binom{n\beta+n-p}{n-m} x^m.$$

2 Generalized Euler polynomials

Let

$$u_n(x) = \sum_{p=1}^{n} u_p x^p, \qquad n > 0.$$

Since

$$a^m(x) = \sum_{n=0}^{\infty} \frac{u_n(m)}{n!} x^n, \qquad u_0(x) = 1,$$

then

$$\frac{\alpha_n(x)}{(1-x)^{n+1}} = \sum_{m=0}^{\infty} \frac{u_n(m)}{n!} x^m = \frac{1}{n!} \sum_{m=0}^{\infty} x^m \sum_{p=1}^{n} u_p m^p = \frac{1}{n!} \sum_{p=1}^{n} \sum_{m=0}^{\infty} u_p m^p x^m =$$

$$= \frac{1}{n!} \sum_{p=1}^{n} \frac{u_p A_p(x)}{(1-x)^{p+1}} = \frac{\frac{1}{n!} \sum_{p=1}^{n} u_p (1-x)^{n-p} A_p(x)}{(1-x)^{n+1}}.$$

We introduce the matrices $\tilde{U}_n$:

$$\tilde{U}_n x^p = \frac{1}{n!} (1-x)^{n-1-p} \tilde{A}_{p+1}(x), \qquad p = 0,\ 1,\ \ldots,\ n-1.$$

For example,

$$\tilde{U}_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad \tilde{U}_3 = \frac{1}{3!} \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 4 \\ 1 & -1 & 1 \end{pmatrix}, \qquad \tilde{U}_4 = \frac{1}{4!} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 3 & 11 \\ 3 & -1 & -3 & 11 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

Then

$$\tilde{U}_n \tilde{u}_n(x) = \tilde{\alpha}_n(x).$$

Since

$$\frac{x^{p+1}}{(1-x)^{n+1}} = \sum_{m=0}^{\infty} \binom{m+n-p-1}{n} x^m = \sum_{m=0}^{\infty} \frac{[m-p]_n}{n!} x^m, \qquad 0 \le p < n,$$

then

$$\tilde{U}_n^{-1} x^p = \frac{1}{x} \prod_{i=0}^{n-1} (x-p+i) = (x-1)_p [x+1]_{n-p-1}.$$

For example,

$$\tilde{U}_2^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad \tilde{U}_3^{-1} = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \tilde{U}_4^{-1} = \begin{pmatrix} 6 & -2 & 2 & -6 \\ 11 & -1 & -1 & 11 \\ 6 & 2 & -2 & -6 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

We introduce the matrices J_n corresponding to the operator rearranging the coefficients of the polynomial of degree n in the reverse order. For example,

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Denote $\tilde{J}_n = J_{n-1}$.
Theorem 1.
$$\tilde{U}_n\,(1,-x)\,\tilde{U}_n^{-1} = (-1)^{n-1}\tilde{J}_n.$$

Proof.
$$(1,-x)\,(x-1)_p[x+1]_{n-p-1} = (-x-1)_p[-x+1]_{n-p-1} = (-1)^{n-1}(x-1)_{n-p-1}[x+1]_p,$$
or
$$(1,-x)\,\tilde{U}_n^{-1}x^p = (-1)^{n+1}\tilde{U}_n^{-1}x^{n-p-1}, \qquad (1,-x)\,\tilde{U}_n^{-1} = (-1)^{n+1}\tilde{U}_n^{-1}\tilde{J}_n.$$
Thus,
$$(-1)^{n-1}\tilde{J}_n\tilde{\alpha}_n\,(x) = \tilde{U}_n\tilde{u}_n\,(-x)\,.$$
Denote
$$[n,\searrow]\left(1, xa^{-1}\,(x)\right) = \frac{\alpha_n^{(-1)}\,(x)}{(1-x)^{n+1}}.$$
Since
$$[n,\rightarrow]\left(1, \log a^{-1}\,(x)\right)_{e^x} = u_n\,(-x)\,,$$
then
$$\alpha_n^{(-1)}\,(x) = (-1)^n xJ_n\alpha_n\,(x)\,.$$
Theorem 2.
$$\alpha_n\,(1) = (a_1)^n.$$
Proof. Denote $\tilde{U}_n x^p = \tilde{U}_p\,(x)$. Since
$$a_1 = [x]\log a\,(x)\,, \quad (a_1)^n = [x^n]\,u_n\,(x)\,; \quad \tilde{U}_p\,(1) = 0, \quad p < n-1; \quad \tilde{U}_{n-1}\,(1) = 1.$$
then
$$\alpha_n\,(1) = \sum_{p=0}^{n-1} u_{p+1}\tilde{U}_p\,(1) = u_n = (a_1)^n.$$

The case when $a_1 = 0$, the degree of polynomial $u_n\,(x)$ is less than n and matrix $(1, \log a\,(x))_{e^x}$ has no inverse, is possible. This possibility is reflected in the following theorem.
Theorem 3. *If $s_{n-m}\,(x)$ is a polynomial of degree $n-m-1$, then*
$$\tilde{U}_n s_{n-m}\,(x) = (1-x)^m\frac{(n-m)!}{n!}\tilde{U}_{n-m}s_{n-m}\,(x)\,.$$
Respectively, if $c_{n-m}\,(x)$ is a polynomial of degree $< n-m$, then
$$\tilde{U}_n^{-1}(1-x)^m c_{n-m}\,(x) = \frac{n!}{(n-m)!}\tilde{U}_{n-m}^{-1}c_{n-m}\,(x)\,.$$

Proof. Let I_n is the identity square matrix of order $n+1$, corresponding to the operator annihilating excess columns or rows of matrices. Denote $\tilde{I}_n = I_{n-1}$. Then it is obvious that
$$\left((1-x)^{-m}, x\right)\tilde{U}_n\tilde{I}_{n-m} = \frac{(n-m)!}{n!}\tilde{U}_{n-m},$$
or
$$\tilde{U}_n\tilde{I}_{n-m} = ((1-x)^m, x)\,\frac{(n-m)!}{n!}\tilde{U}_{n-m}.$$
Respectively,
$$\tilde{U}_n^{-1}\,((1-x)^m, x)\,\tilde{I}_{n-m} = \frac{n!}{(n-m)!}\tilde{U}_{n-m}^{-1}.$$

We introduce the matrices $\tilde{V}_n = \tilde{J}_n E \tilde{J}_n = ((1+x)^n, x)\, P^{-1} \tilde{I}_n$. For example,

$$\tilde{V}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \qquad \tilde{V}_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

Then, as we found in the Introduction,

$$\tilde{V}_n^{-1} \tilde{v}_n(x) = \tilde{\alpha}_n(x),$$

and, hence

$$\tilde{U}_n^{-1} \tilde{V}_n^{-1} \tilde{v}_n(x) = \tilde{u}_n(x).$$

By Theorem 3 we find:

$$\tilde{U}_n^{-1} \tilde{V}_n^{-1} x^p = \tilde{U}_n^{-1} (1-x)^{n-p-1} x^p = \frac{n!}{(p+1)!} (x-1)_p =$$

$$= \frac{n!}{(p+1)!} \sum_{m=1}^{p+1} s(p+1,\ m)\, x^{m-1},$$

where $s(p+1,\ m)$ are the Stirling numbers of the first kind. Hence

$$\tilde{V}_n \tilde{U}_n x^p = \frac{1}{n!} \sum_{m=1}^{p+1} m! S(p+1,\ m)\ x^{m-1},$$

where $S(p+1,\ m)$ are the Stirling numbers of the second kind. For example,

$$\tilde{U}_4^{-1} \tilde{V}_4^{-1} = 4! \begin{pmatrix} 1 & -1 & 2 & -6 \\ 0 & 1 & -3 & 11 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3!} & 0 \\ 0 & 0 & 0 & \frac{1}{4!} \end{pmatrix},$$

$$\tilde{V}_4 \tilde{U}_4 = \frac{1}{4!} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3! & 0 \\ 0 & 0 & 0 & 4! \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Generalized binomial series,

$$_{(\beta)}a^{\varphi}(x) = \sum_{n=0}^{\infty} \frac{\varphi}{\varphi + n\beta} \binom{\varphi + n\beta}{n} x^n;$$

$$_{(0)}a(x) = 1 + x, \qquad _{(1)}a(x) = (1-x)^{-1}, \qquad _{(2)}a(x) = \frac{1 - (1-4x)^{1/2}}{2x},$$

$$_{(-1)}a(x) = \frac{1 + (1+4x)^{1/2}}{2}, \qquad _{(1/2\,)}a(x) = \left(\frac{x}{2} + \left(1 + \frac{x^2}{4}\right)^{1/2} \right)^2,$$

takes important place in our studies. Generalization that underlies it can be extended to each formal power series $a(x)$, $a_0 = 1$. Each such series is associated by means of the Lagrange transform

$$a^{\varphi}(x) = \sum_{n=0}^{\infty} \frac{x^n}{a^{\beta n}(x)} [x^n] \left(1 - x\beta (\log a(x))'\right) a^{\varphi + \beta n}(x)$$

with the set of series ${}_{(\beta)}a\left(x\right)$, ${}_{(0)}a\left(x\right)=a\left(x\right)$, such that

$$ {}_{(\beta)}a\left(xa^{-\beta}\left(x\right)\right)=a\left(x\right), \qquad a\left(x{}_{(\beta)}a^{\beta}\left(x\right)\right)={}_{(\beta)}a\left(x\right), $$

$$ [x^n]\,{}_{(\beta)}a^{\varphi}\left(x\right)=[x^n]\left(1-x\beta\frac{a'\left(x\right)}{a\left(x\right)}\right)a^{\varphi+\beta n}\left(x\right)=\frac{\varphi}{\varphi+\beta n}[x^n]\,a^{\varphi+\beta n}\left(x\right), $$

$$ [x^n]\left(1+x\beta\frac{{}_{(\beta)}a'\left(x\right)}{{}_{(\beta)}a\left(x\right)}\right){}_{(\beta)}a^{\varphi}\left(x\right)=\frac{\varphi+\beta n}{\varphi}[x^n]\,{}_{(\beta)}a^{\varphi}\left(x\right)=[x^n]\,a^{\varphi+\beta n}\left(x\right), $$

$$ \left(1,x{}_{(\beta)}a^{\varphi}\left(x\right)\right)^{-1}=\left(1,x{}_{(\beta-\varphi)}a^{-\varphi}\left(x\right)\right), $$

$$ \left(1+x\varphi\left(\log{}_{(\beta)}a\left(x\right)\right)',x{}_{(\beta)}a^{\varphi}\left(x\right)\right)^{-1}=\left(1-x\varphi\left(\log{}_{(\beta-\varphi)}a\left(x\right)\right)',x{}_{(\beta-\varphi)}a^{-\varphi}\left(x\right)\right). $$

Denote

$$ [n,\rightarrow]\left(1,\log{}_{(\beta)}a\left(x\right)\right)_{e^x}={}_{(\beta)}u_n\left(x\right). $$

Then

$$ {}_{(\beta)}a^{\varphi}\left(x\right)=\sum_{n=0}^{\infty}\frac{\varphi}{\varphi+n\beta}\frac{u_n\left(\varphi+n\beta\right)}{n!}x^n, $$

$$ {}_{(\beta)}u_n\left(x\right)=x(x+n\beta)^{-1}u_n\left(x+n\beta\right). $$

Let

$$ \left(1,\log a\left(x\right)\right)^{-1}=\left(1,q\left(x\right)\right). $$

Then

$$ \left(1,\log{}_{(\beta)}a\left(x\right)\right)^{-1}=\left(1,q\left(x\right)e^{-\beta x}\right). $$

Denote

$$ \left(1,q\left(x\right)e^{-\beta x}\right)_{e^x}x^n={}_{(\beta)}q_n\left(x\right). $$

Then

$$ {}_{(\beta)}q_n\left(x\right)=(1+n\beta x)^{-1}q_n\left(\frac{x}{1+n\beta x}\right), $$

$$ \sum_{n=0}^{\infty}{}_{(\beta)}u_n\left(\varphi\right){}_{(\beta)}q_n\left(x\right)=(1-\varphi x)^{-1}. $$

Series ${}_{(\beta)}a\left(x\right)$ for integer β, denoted by $S_\beta\left(x\right)$, were introduced in [15]. In [16] these series, called generalized Lagrange series, are considered in connection with the Riordan arrays. Properties of these series intersect with the properties of Sheffer sequences, therefore the identities associated with them can be found in the umbral calculus.

Denote

$$ \frac{1}{x}{}_{(\beta)}u_n\left(x\right)={}_{(\beta)}\tilde{u}_n\left(x\right), \quad [n,\searrow]\left(1,x{}_{(\beta)}a\left(x\right)\right)=\frac{{}_{(\beta)}\alpha_n\left(x\right)}{(1-x)^{n+1}}, \quad \frac{1}{x}{}_{(\beta)}\alpha_n\left(x\right)={}_{(\beta)}\tilde{\alpha}_n\left(x\right). $$

Then

$$ E^{n\beta}\tilde{u}_n\left(x\right)=\tilde{u}_n\left(x+n\beta\right)={}_{(\beta)}\tilde{u}_n\left(x\right), \qquad U_nE^{n\beta}U_n^{-1}\tilde{\alpha}_n\left(x\right)={}_{(\beta)}\tilde{\alpha}_n\left(x\right). $$

Denote

$$ \tilde{U}_nE^{n\beta}\tilde{U}_n^{-1}=A_n^{\beta}. $$

Since

$$ (1,-x)\,E^{n\beta}\,(1,-x)=E^{-n\beta}, \qquad \tilde{U}_n\,(1,-x)\,\tilde{U}_n^{-1}=(-1)^{n-1}\tilde{J}_n, $$

then

$$ \tilde{J}_nA_n^{\beta}\tilde{J}_n=A_n^{-\beta}. $$

For example,

$$A_2=\begin{pmatrix}2&1\\-1&0\end{pmatrix},\quad A_3=\begin{pmatrix}5&5/2&1\\-6&-2&0\\2&1/2&0\end{pmatrix},\quad A_4=\begin{pmatrix}14&7&3&1\\-28&-35/3&-10/3&0\\20&22/3&5/3&0\\-5&-5/3&-1/3&0\end{pmatrix};$$

$$A_2^{-1}=\begin{pmatrix}0&-1\\1&2\end{pmatrix},\quad A_3^{-1}=\begin{pmatrix}0&1/2&2\\0&-2&-6\\1&5/2&5\end{pmatrix},\quad A_4^{-1}=\begin{pmatrix}0&-1/3&-5/3&-5\\0&5/3&22/3&20\\0&-10/3&-35/3&-28\\1&3&7&14\end{pmatrix}.$$

Since $[x]\,{}_{(\beta)}a\left(x\right)=[x]\,a\left(x\right)=a_1$, then ${}_{(\beta)}\alpha_n\left(1\right)=\alpha_n\left(1\right)$ and sum of the elements of each column of the matrix A_n^β is 1. From Theorem 3 it follows that

$$\left(\left(1-x\right)^{-m},x\right)A_n^\beta\left(\left(1-x\right)^m,x\right)\tilde{I}_{n-m}=A_{n-m}^{\frac{n\beta}{n-m}}.$$

We introduce the diagonal matrix $\tilde{D}$, $\tilde{D}x^n=\left(n+1\right)x^n$:

$$\tilde{D}=\begin{pmatrix}1&0&0&\cdots\\0&2&0&\cdots\\0&0&3&\cdots\\\vdots&\vdots&\vdots&\ddots\end{pmatrix},\qquad \tilde{D}^{-1}=\begin{pmatrix}1&0&0&\cdots\\0&\frac{1}{2}&0&\cdots\\0&0&\frac{1}{3}&\cdots\\\vdots&\vdots&\vdots&\ddots\end{pmatrix}.$$

Theorem 4.

$$A_n^\beta=\tilde{V}_n^{-1}\tilde{D}\Big(\left(1+x\right)^{n\beta},x\Big)^T\tilde{D}^{-1}\tilde{V}_n.$$

Proof. Columns of the matrices $\tilde{V}_n\tilde{U}_n$, $\tilde{U}_n^{-1}\tilde{V}_n^{-1}$ are connected a certain way with the rows of the matrices $\left(e^x,e^x-1\right)_{e^x}$, $\left(\left(1+x\right)^{-1},\log\left(1+x\right)\right)_{e^x}$:

$$n!\left|e^x\right|\tilde{D}^{-1}\tilde{V}_n\tilde{U}_nx^p=[p,\to]\left(e^x,e^x-1\right)_{e^x},$$

$$\left(1/n!\,\right)\tilde{U}_n^{-1}\tilde{V}_n^{-1}\left|e^x\right|^{-1}\tilde{D}x^p=[p,\to]\left(\left(1+x\right)^{-1},\log\left(1+x\right)\right)_{e^x}.$$

Since

$$\left(\left(1+x\right)^{-1},\log\left(1+x\right)\right)_{e^x}\left(e^{n\beta},x\right)_{e^x}\left(e^x,e^x-1\right)_{e^x}=\Big(\left(1+x\right)^{n\beta},x\Big)_{e^x},$$

then

$$\tilde{V}_n\tilde{U}_nE^{n\beta}\tilde{U}_n^{-1}\tilde{V}_n^{-1}=\tilde{D}\Big(\left(1+x\right)^{n\beta},x\Big)^T\tilde{D}^{-1}\tilde{I}_n.$$

Thus,

$$A_2=\begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}1&0\\0&2\end{pmatrix}\begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&\frac{1}{2}\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix},$$

$$A_3=\begin{pmatrix}1&0&0\\-2&1&0\\1&-1&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}\begin{pmatrix}1&3&3\\0&1&3\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&\frac{1}{2}&0\\0&0&\frac{1}{3}\end{pmatrix}\begin{pmatrix}1&0&0\\2&1&0\\1&1&1\end{pmatrix},$$

$$A_4=\begin{pmatrix}1&0&0&0\\-3&1&0&0\\3&-2&1&0\\-1&1&-1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&2&0&0\\0&0&3&0\\0&0&0&4\end{pmatrix}\begin{pmatrix}1&4&6&4\\0&1&4&6\\0&0&1&4\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\frac{1}{2}&0&0\\0&0&\frac{1}{3}&0\\0&0&0&\frac{1}{4}\end{pmatrix}\begin{pmatrix}1&0&0&0\\3&1&0&0\\3&2&1&0\\1&1&1&1\end{pmatrix}.$$

Denote

$$[n,\to]\left(1,{}_{(\beta)}a\left(x\right)-1\right)={}_{(\beta)}v_n\left(x\right),\qquad \frac{1}{x}{}_{(\beta)}v_n\left(x\right)={}_{(\beta)}\tilde{v}_n\left(x\right).$$

Then
$$\tilde{D}\left((1+x)^{n\beta},x\right)^T\tilde{D}^{-1}\tilde{v}_n(x)={}_{(\beta)}\tilde{v}_n(x).$$

Let ${}_{(\beta)}a(x)$ is the generalized binomial series. Then $a(x)=1+x$,
$$\frac{{}_{(\beta)}\alpha_n(x)}{(1-x)^{n+1}}=\sum_{m=0}^{\infty}\frac{m}{m+n\beta}\binom{m+n\beta}{n}x^m.$$

Theorem 5.
$${}_{(\beta)}\alpha_n(x)=\frac{1}{n}\sum_{m=1}^{n}\binom{n(1-\beta)}{m-1}\binom{n\beta}{n-m}x^m.$$

Proof. We use the factorial representation of binomial coefficients, i.e. we prove the theorem for positive integers β. By polynomial argument (binomial coefficients under consideration are polynomials in β) this is equivalent to the general proof . Since $\tilde{v}_n(x)=x^{n-1}$, then
$${}_{(\beta)}\tilde{v}_n(x)=\tilde{D}\left((1+x)^{n\beta},x\right)^T\tilde{D}^{-1}x^{n-1}=\sum_{m=0}^{n-1}\frac{m+1}{n}\binom{n\beta}{n-m-1}x^m.$$

Since
$$[m,\to]\tilde{V}_n^{-1}=\sum_{i=0}^{m}\binom{m-n}{m-i}x^i=\sum_{i=0}^{m}(-1)^{m-i}\binom{n-i-1}{m-i}x^i,$$

then
$$[x^m]\,{}_{(\beta)}\tilde{\alpha}_n(x)=[x^m]\,\tilde{V}_n^{-1}\,{}_{(\beta)}\tilde{v}_n(x)=$$
$$=\sum_{i=0}^{m}(-1)^{m-i}\binom{n-i-1}{m-i}\frac{(i+1)}{n}\binom{n\beta}{n-i-1}\frac{(n\beta-n+m+1)!}{(n\beta-n+m+1)!}=$$
$$=\frac{1}{n}\binom{n\beta}{n-m-1}\sum_{i=0}^{m}(-1)^{m-i}(i+1)\binom{n\beta-n+m+1}{m-i}=$$
$$=\frac{1}{n}\binom{n\beta}{n-m-1}(-1)^m\binom{n\beta-n+m-1}{m}=\frac{1}{n}\binom{n\beta}{n-m-1}\binom{n(1-\beta)}{m}.$$

Note that
$${}_{(0)}\alpha_n(x)=x^n,\qquad {}_{(1)}\alpha_n(x)=x,\qquad {}_{(1/2\,)}\alpha_{2n}(x)=\frac{1}{2}(1+x)x^n;$$

since
$${}_{(1-\beta)}a(x)={}_{(\beta)}a^{-1}(-x),$$

then
$${}_{(1-\beta)}\alpha_n(x)=xJ_{n\,(\beta)}\alpha_n(x).$$

3 Generalized Narayana polynomials

Constructive relationships between the ordinary and the exponential Riordan arrays exist. Particular manifestations of these relationships resemble the details of construction, the general plan of which is a secret for us. Following [17] – [19], we will consider some of such manifestations associated with the numerator polynomials. Since
$$[n,\searrow](1,xa(x))=\sum_{m=0}^{\infty}\frac{u_n(m)}{n!}x^m,$$

then

$$[n, \searrow](1, xa(x))_{e^x} = \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} \frac{u_n(m)}{n!} x^m = \sum_{m=0}^{\infty} \frac{[m+1]_n u_n(m)}{n!} x^m.$$

If $a_1 \neq 0$, then $[x+1]_n u_n(x)$ is the polynomial of degree $2n$, so that

$$\sum_{m=0}^{\infty} \frac{[m+1]_n u_n(m)}{n!} x^m = \frac{\varphi_n(x)}{(1-x)^{2n+1}},$$

where

$$\varphi_n(x) = x \frac{(2n)!}{n!} \tilde{U}_{2n} [x+1]_n \tilde{u}_n(x).$$

Since $[x+1]_n u_n(x) = 0$ when $x = 0, -1, \ldots, -n$, then, in accordance with the Theorem1,

$$\sum_{m=0}^{\infty} \frac{[-m+1]_n u_n(-m)}{n!} x^m = \frac{(-1)^{2n} x J_{2n} \varphi_n(x)}{(1-x)^{2n+1}} = \frac{(-1)^{2n} x^{n+1} J_n \varphi_n(x)}{(1-x)^{2n+1}},$$

i.e. $\varphi_n(x)$ is the polynomial of degree $\leq n$. Since

$$\left[x^{2n}\right] [x+1]_n u_n(x) = (a_1)^n,$$

then, in accordance with the Theorem 2,

$$\varphi_n(1) = (a_1)^n \frac{(2n)!}{n!}.$$

If $a(x) = (1-x)^{-1}$, then

$$\varphi_n(x) = (n+1)! N_n(x) = (1-x)^{2n+1} \sum_{m=0}^{\infty} [m+1]_n \binom{m+n-1}{n} x^m,$$

where

$$N_n(x) = \frac{1}{n} \sum_{m=1}^{n} \binom{n}{m-1} \binom{n}{n-m} x^m$$

is the Narayana polynomials. In this connection we will called polynomials $\varphi_n(x)$ the generalized Narayana polynomials (GNP).

Since

$$\frac{\varphi_n(t)}{(n+1)!(1-t)^{2n+1}} = \sum_{m=0}^{\infty} \frac{1}{n+1} \binom{n+m}{m} [x^n] a^m(x) t^m =$$

$$= \frac{1}{n+1} [x^n] (1 - ta(x))^{-n-1} = [x^n] b(x),$$

where

$$b(x) = \frac{1}{1 - ta(xb(x))}, \qquad (1, xb(x)) = (1, x(1 - ta(x)))^{-1},$$

then

$$\sum_{n=0}^{\infty} \varphi_n(t) \frac{x^n}{(n+1)!} = (1-t) b\left(x(1-t)^2\right).$$

For example,

$$a(x) = \frac{1}{1-x}, \qquad b(x) = \frac{1 + x - t - \sqrt{1 - 2x - 2t - 2xt + x^2 + t^2}}{2x},$$

$$\sum_{n=0}^{\infty} \varphi_n(t) \frac{x^n}{(n+1)!} = \sum_{n=0}^{\infty} N_n(t) x^n = \frac{1 + x(1-t) - \sqrt{1 - 2x(1+t) + x^2(1-t)^2}}{2x}.$$

Generating functions of the numerator polynomials of the matrices $(1, e^x - 1)_{e^x}$, $(e^x, e^x - 1)_{e^x}$ are considered in [20].

We introduce the matrices $\tilde{F}_n$:

$$\tilde{F}_n = \frac{(2n)!}{n!} \tilde{U}_{2n}([x+1]_n, x) \tilde{I}_n, \qquad \tilde{F}_n^{-1} = \frac{n!}{(2n)!} ([x+1]_n, x)^{-1} \tilde{U}_{2n}^{-1} \tilde{I}_n;$$

$$\tilde{F}_n x^p = \frac{(2n)!}{n!} \tilde{U}_{2n} x^p [x+1]_n = (1-x)^{2n+1} \sum_{m=1}^{\infty} m^{p+1} \binom{m+n}{n} x^{m-1},$$

$$\tilde{F}_n^{-1} x^p = \frac{n!}{(2n)!} (x-1)_p [x+n+1]_{n-p-1}, \qquad p = 0,\ 1,\ \ldots,\ n-1.$$

For example,

$$\tilde{F}_2 = 3 \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}, \qquad \tilde{F}_3 = 4 \begin{pmatrix} 1 & 1 & 1 \\ -2 & 3 & 13 \\ 1 & -4 & 16 \end{pmatrix}, \qquad \tilde{F}_4 = 5 \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & 3 & 15 & 39 \\ 3 & -9 & 9 & 171 \\ -1 & 5 & -25 & 125 \end{pmatrix};$$

$$\tilde{F}_2^{-1} = \frac{2!}{4!} \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}, \quad \tilde{F}_3^{-1} = \frac{3!}{6!} \begin{pmatrix} 20 & -4 & 2 \\ 9 & 3 & -3 \\ 1 & 1 & 1 \end{pmatrix}, \quad \tilde{F}_4^{-1} = \frac{4!}{8!} \begin{pmatrix} 210 & -30 & 10 & -6 \\ 107 & 19 & -13 & 11 \\ 18 & 10 & 2 & -6 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Denote $(1/x\ )\, \varphi_n(x) = \tilde{\varphi}_n(x)$. Then

$$\tilde{F}_n \tilde{u}_n(x) = \tilde{\varphi}_n(x).$$

Let $\alpha_n(x)\, |a(x)$, $\varphi_n(x)\, |a(x)$ denotes respectively GEP, GNP associated with the matrices $(1, xa(x))$, $(1, xa(x))_{e^x}$. Then

$$x\tilde{F}_n x^{n-1} = \varphi_n(x)\, |e^x.$$

Theorem 6.

$$\tilde{F}_n E^n(1, -x) \tilde{F}_n^{-1} = (-1)^{n-1} \tilde{J}_n.$$

Proof.

$$E^n(1, -x)(x-1)_p [x+n+1]_{n-p-1} = (-x-n-1)_p [-x+1]_{n-p-1} =$$

$$= (-1)^{n-1} (x-1)_{n-p-1} [x+n+1]_p,$$

or

$$E^n(1, -x) \tilde{F}_n^{-1} x^p = (-1)^{n-1} \tilde{F}_n^{-1} x^{n-1-p}, \qquad E^n(1, -x) \tilde{F}_n^{-1} = (-1)^{n-1} \tilde{F}_n^{-1} \tilde{J}_n.$$

Thus,

$$(-1)^{n-1} \tilde{J}_n \tilde{\varphi}_n(x) = \tilde{F}_n \tilde{u}_n(-x-n).$$

We denote by using the notation for the series $_{(\beta)}a(x)$:

$$(1, xa(x))^{-1} = \left(1, x_{(-1)} a^{-1}(x)\right).$$

Then

$$[n, \rightarrow] \left(1, \log_{(-1)} a^{-1}(x)\right)_{e^x} = -x\tilde{u}_n(-x-n).$$

Denote

$$[n,\searrow]\left(1,xa\left(x\right)\right)_{e^x}^{-1}=\frac{\varphi_n^{[-1]}\left(x\right)}{\left(1-x\right)^{2n+1}}.$$

Then

$$\varphi_n^{[-1]}\left(x\right)=\left(-1\right)^n xJ_n\varphi_n\left(x\right).$$

Thus, if the matrix $\left(1,xa\left(x\right)\right)$ is a pseudo-involution, i.e. $\left(1,xa\left(x\right)\right)^{-1}=\left(1,xa\left(-x\right)\right)$, then $\varphi_n\left(x\right)=xJ_n\varphi_n\left(x\right)$.

We introduce the matrices $\tilde{S}_n$:

$$\tilde{S}_n=\tilde{F}_n\tilde{U}_n^{-1},\qquad \tilde{S}_n^{-1}=\tilde{U}_n\tilde{F}_n^{-1}.$$

For example,

$$\tilde{S}_2=3!\begin{pmatrix}1&0\\1&2\end{pmatrix},\qquad \tilde{S}_3=4!\begin{pmatrix}1&0&0\\3&5/2&0\\1&5/2&5\end{pmatrix},\qquad \tilde{S}_4=5!\begin{pmatrix}1&0&0&0\\6&3&0&0\\6&8&7&0\\1&3&7&14\end{pmatrix};$$

$$\tilde{S}_2^{-1}=\frac{2!}{4!}\begin{pmatrix}2&0\\-1&1\end{pmatrix},\quad \tilde{S}_3^{-1}=\frac{3!}{6!}\begin{pmatrix}5&0&0\\-6&2&0\\2&-1&1\end{pmatrix},\quad \tilde{S}_4^{-1}=\frac{4!}{8!}\begin{pmatrix}14&0&0&0\\-28&14/3&0&0\\20&-16/3&2&0\\-5&5/3&-1&1\end{pmatrix}.$$

Then

$$\tilde{S}_n\tilde{\alpha}_n\left(x\right)=\tilde{\varphi}_n\left(x\right).$$

Theorem 7.

$$\tilde{S}_n=\tilde{V}_n^{-1}\tilde{C}_n\tilde{V}_n,\qquad \tilde{C}_nx^p=\frac{(n+p+1)!}{(p+1)!}x^p.$$

Proof. We use Theorem 3 and the identities

$$\tilde{U}_n^{-1}\tilde{V}_n^{-1}x^p=\frac{n!}{(p+1)!}(x-1)_p,\qquad \tilde{U}_{n+p+1}^{-1}x^p=(x-1)_p[x+1]_n.$$

Then

$$\tilde{F}_n\tilde{U}_n^{-1}\tilde{V}_n^{-1}x^p=\frac{(2n)!}{n!}\tilde{U}_{2n}\frac{n!}{(p+1)!}(x-1)_p[x+1]_n=$$

$$=\frac{(2n)!}{(p+1)!}\left(1-x\right)^{n-p-1}\frac{(n+p+1)!}{(2n)!}\tilde{U}_{n+p+1}(x-1)_p[x+1]_n=$$

$$=\frac{(n+p+1)!}{(p+1)!}\left(1-x\right)^{n-p-1}x^p,$$

or

$$\tilde{S}_n\tilde{V}_n^{-1}=\tilde{V}_n^{-1}\tilde{C}_n.$$

Thus,

$$\tilde{S}_2=\begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}3!&0\\0&4!/2!\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix},$$

$$\tilde{S}_3=\begin{pmatrix}1&0&0\\-2&1&0\\1&-1&1\end{pmatrix}\begin{pmatrix}4!&0&0\\0&5!/2!&0\\0&0&6!/3!\end{pmatrix}\begin{pmatrix}1&0&0\\2&1&0\\1&1&1\end{pmatrix},$$

$$\tilde{S}_4=\begin{pmatrix}1&0&0&0\\-3&1&0&0\\3&-2&1&0\\-1&1&-1&1\end{pmatrix}\begin{pmatrix}5!&0&0&0\\0&6!/2!&0&0\\0&0&7!/3!&0\\0&0&0&8!/4!\end{pmatrix}\begin{pmatrix}1&0&0&0\\3&1&0&0\\3&2&1&0\\1&1&1&1\end{pmatrix}.$$

Note that since $\alpha_n(x)|(1-x)^{-1}=x$, then

$$x\tilde{S}_n x^0=\varphi_n(x)|(1-x)^{-1}=(n+1)!N_n(x);$$

since

$$x\frac{(2n)!}{n!}\tilde{F}_n^{-1}x^0=x(x+n+1)...(x+n+n-1)=[n,\rightarrow](1,\log C(x))_{e^x}$$

where $C(x)$ is the Catalan series, then

$$x\frac{(2n)!}{n!}\tilde{S}_n^{-1}x^0=\alpha_n(x)|C(x)=\frac{1}{n}\sum_{m=1}^{n}\binom{-n}{m-1}\binom{2n}{n-m}x^m.$$

Let ${}_{(\beta)}a(x)$ is the generalized binomial series. Denote

$$[n,\searrow]\left(1,x_{(\beta)}a(x)\right)_{e^x}=\frac{{}_{(\beta)}\varphi_n(x)}{(1-x)^{2n+1}}=\sum_{m=0}^{\infty}\frac{m}{m+\beta n}\binom{m+\beta n}{n}[m+1]_n x^m.$$

Theorem 8.

$$_{(\beta)}\varphi_n(x)=\frac{(n+1)!}{n}\sum_{m=1}^{n}\binom{n(2-\beta)}{m-1}\binom{n\beta}{n-m}x^m.$$

Proof. Taking into account the polynomial argument, we prove the theorem for positive integers β. Since

$$\tilde{V}_{n(\beta)}\tilde{\alpha}_n(x)={}_{(\beta)}\tilde{v}_n(x)=\sum_{m=0}^{n-1}\frac{m+1}{n}\binom{n\beta}{n-m-1}x^m,$$

$$[m,\rightarrow]\tilde{V}_n^{-1}=\sum_{i=0}^{m}\binom{m-n}{m-i}x^i=\sum_{i=0}^{m}(-1)^{m-i}\binom{n-i-1}{m-i}x^i,$$

then

$$[x^m]\,{}_{(\beta)}\tilde{\varphi}_n(x)=[x^m]\,\tilde{V}_n^{-1}\tilde{C}_{n(\beta)}\tilde{v}_n(x)=$$

$$=\sum_{i=0}^{m}(-1)^{m-i}\binom{n-i-1}{m-i}\frac{(n+i+1)!}{(i+1)!}\frac{(i+1)}{n}\binom{n\beta}{n-i-1}\frac{(n\beta-n+m+1)!}{(n\beta-n+m+1)!}=$$

$$=\frac{(n+1)!}{n}\binom{n\beta}{n-m-1}\sum_{i=0}^{m}(-1)^{m-i}\binom{n+i+1}{i}\binom{n\beta-n+m+1}{m-i}=$$

$$=\frac{(n+1)!}{n}\binom{n\beta}{n-m-1}(-1)^m\binom{n\beta-2n+m-1}{m}=$$

$$=\frac{(n+1)!}{n}\binom{n\beta}{n-m-1}\binom{n(2-\beta)}{m}.$$

Note that

$$_{(0)}\varphi_n(x)=\frac{(2n)!}{n!}x^n,\qquad {}_{(1)}\varphi_n(x)=(n+1)!N_n(x),\qquad {}_{(2)}\varphi_n(x)=\frac{(2n)!}{n!}x;$$

since

$$\left(1,x_{(\beta)}a(x)\right)^{-1}=\left(1,x_{(\beta-1)}a^{-1}(x)\right),\qquad {}_{(\beta-1)}a^{-1}(-x)={}_{(2-\beta)}a(x),$$

then

$$_{(2-\beta)}\varphi_n(x)=xJ_{n(\beta)}\varphi_n(x).$$

4 Transformations of general form

We introduce the matrices U_n:

$$U_n x^p = \frac{1}{n!}(1-x)^{n-p} A_p(x), \qquad A_0(x) = 1, \qquad p = 0,\ 1,\ \ldots,\ n.$$

Or

$$U_n x^p = (1-x)^{n+1} \frac{1}{n!} \sum_{m=0}^{\infty} m^p x^m.$$

For example,

$$U_0 = (1), \quad U_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad U_2 = \frac{1}{2!}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \quad U_3 = \frac{1}{3!}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 1 & 1 \\ 3 & -2 & 0 & 4 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

Let

$$[n, \to]\left(b(x), \log a(x)\right) = c_n(x) = \sum_{m=0}^{n} c_m x^m, \qquad b_0 \neq 0,$$

$$[n, \to]\left(b(x), \log a(x)\right)_{e^x} = s_n(x) = \sum_{m=0}^{n} s_m x^m,$$

$$[n, \to]\left(b(x), a(x)\right) = \frac{g_n(x)}{(1-x)^{n+1}}.$$

Since

$$\left(b(x), a(x)\right) = \left(b(x), \log a(x)\right)(1, e^x),$$

then

$$\frac{g_n(x)}{(1-x)^{n+1}} = \sum_{p=0}^{n} \frac{c_p A_p(x)/p!}{(1-x)^{p+1}} = \frac{1}{n!}\sum_{p=0}^{n} \frac{s_p A_p(x)}{(1-x)^{p+1}} = \frac{\frac{1}{n!}\sum_{p=0}^{n} s_p (1-x)^{n-p} A_p(x)}{(1-x)^{n+1}}.$$

Thus,

$$b(x)\, a^m(x) = \sum_{n=0}^{\infty} \frac{s_n(m)}{n!} x^n, \qquad \frac{g_n(x)}{(1-x)^{n+1}} = \sum_{m=0}^{\infty} \frac{s_n(m)}{n!} x^m, \qquad g_n(x) = U_n s_n(x).$$

Since

$$\frac{1}{(1-x)^{n+1}} = \sum_{m=0}^{\infty} \frac{[m+1]_n}{n!} x^m,$$

then

$$U_n^{-1} x^0 = [x+1]_n, \qquad U_n^{-1} x^p = x \tilde{U}_n^{-1} x^{p-1},$$

or

$$U_n^{-1} x^p = (x)_p [x+1]_{n-p}.$$

For example,

$$U_3^{-1} = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 11 & 2 & -1 & 2 \\ 6 & 3 & 0 & -3 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Theorem 9.
$$U_nE\left(1,-x\right)U_n^{-1}=\left(-1\right)^nJ_n.$$

Proof.
$$E\left(1,-x\right)\left(x\right)_p[x+1]_{n-p}=\left(-x-1\right)_p[-x]_{n-p}=(-1)^n(x)_{n-p}[x+1]_p,$$
or
$$E\left(1,-x\right)U_n^{-1}=(-1)^nU_n^{-1}J_n.$$

Thus,
$$\left(-1\right)^nJ_ng_n\left(x\right)=U_ns_n\left(-x-1\right).$$
Since
$$\left(b\left(x\right),\log a\left(x\right)\right)\left(1,-x\right)\left(e^x,x\right)=\left(b\left(x\right)a^{-1}\left(x\right),\log a^{-1}\left(x\right)\right),$$
then polynomials $\left(-1\right)^nJ_ng_n\left(x\right)$ are the numerator polynomials of the matrix
$$\left(b\left(x\right)a^{-1}\left(x\right),xa^{-1}\left(x\right)\right).$$
In particular,
$$\left(-1\right)^nJ_n\alpha_n\left(x\right)=\tilde{\alpha}_n^{(-1)}\left(x\right).$$

Denote
$$V_n=J_nEJ_n=\left(\left(1+x\right)^{n+1},x\right)P^{-1}I_n=\tilde{V}_{n+1},$$
$$[n,\rightarrow]\left(b\left(x\right),a\left(x\right)-1\right)=w_n\left(x\right).$$
Since
$$\left(b\left(x\right),a\left(x\right)\right)=\left(b\left(x\right),a\left(x\right)-1\right)\left(1,1+x\right),$$
then
$$g_n\left(x\right)=V_n^{-1}w_n\left(x\right).$$

Theorem 10. *If $s_{n-m}\left(x\right)$ is a polynomial of degree $n-m$, then*
$$U_ns_{n-m}\left(x\right)=\left(1-x\right)^m\frac{(n-m)!}{n!}U_{n-m}s_{n-m}\left(x\right).$$
Respectively, if $c_{n-m}\left(x\right)$ is a polynomial of degree $\le n-m$, then
$$U_n^{-1}(1-x)^mc_{n-m}\left(x\right)=\frac{n!}{(n-m)!}U_{n-m}^{-1}c_{n-m}\left(x\right).$$

Proof.
$$\left(\left(1-x\right)^{-m},x\right)U_nI_{n-m}=\frac{(n-m)!}{n!}U_{n-m},$$
$$U_n^{-1}\left(\left(1-x\right)^m,x\right)I_{n-m}=\frac{n!}{(n-m)!}U_{n-m}^{-1}.$$

From this we find:
$$U_n^{-1}V_n^{-1}x^p=U_n^{-1}(1-x)^{n-p}x^p=\frac{n!}{p!}(x)_p=\frac{n!}{p!}\sum_{m=0}^{p}s\left(p,\ m\right)x^m,$$
$$V_nU_nx^p=\frac{1}{n!}\sum_{m=0}^{p}m!S\left(p,\ m\right)\ x^m.$$

Remark 1. Matrices U_n, U_n^{-1} are associated with the matrices $\tilde{U}_n$, $\tilde{U}_n^{-1}$ by the identities
$$\tilde{U}_n=\left(x,x\right)^TU_n\left(x,x\right),\qquad\tilde{U}_n^{-1}=\left(x,x\right)^TU_n^{-1}\left(x,x\right),$$

and, since

$$\frac{\tilde{\alpha}_n(x)}{(1-x)^{n+1}} = \sum_{m=0}^{\infty} \frac{u_n(m+1)}{n!} x^m, \qquad \tilde{\alpha}_n(x) = U_n E u_n(x),$$

then

$$\tilde{U}_n = U_n E(x,x) I_{n-1}, \qquad \tilde{U}_n^{-1} = (x,x)^T E^{-1} U_n^{-1} I_{n-1}.$$

Denote

$$[n, \searrow](b(x), xa(x))_{e^x} = \sum_{m=0}^{\infty} \frac{[m+1]_n s_n(m)}{n!} x^m = \frac{h_n(x)}{(1-x)^{2n+1}},$$

Then

$$h_n(x) = \frac{(2n)!}{n!} U_{2n}[x+1]_n s_n(x).$$

Polynomials $g_n(x)$, $h_n(x)$ will be called respectively the ordinary and the exponential numerator polynomials. Names GEP and GNP we will fix for the polynomials $\alpha_n(x)$, $\varphi_n(x)$.

We introduce the matrices F_n:

$$F_n = \frac{(2n)!}{n!} U_{2n}([x+1]_n, x) I_n, \qquad F_n^{-1} = \frac{n!}{(2n)!}([x+1]_n, x)^{-1} U_{2n}^{-1} I_n;$$

Since

$$U_{2n}[x+1]_n = (1-x)^n \frac{n!}{(2n)!} U_n [x+1]_n = \frac{n!}{(2n)!}(1-x)^n,$$

then

$$F_n x^0 = (1-x)^n, \qquad F_n x^p = x \tilde{F}_n x^{p-1},$$

or

$$F_n x^p = (1-x)^{2n+1} \sum_{m=0}^{\infty} m^p \binom{m+n}{n} x^m.$$

For example,

$$F_2 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 3 & 3 \\ 1 & -3 & 9 \end{pmatrix}, \qquad F_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 4 & 4 & 4 \\ 3 & -8 & 12 & 52 \\ -1 & 4 & -16 & 64 \end{pmatrix}.$$

Respectively,

$$F_n^{-1} x^0 = \frac{n!}{(2n)!}[x+n+1]_n, \qquad F_n^{-1} x^p = x \tilde{F}_n^{-1} x^{p-1},$$

or

$$F_n^{-1} x^p = \frac{n!}{(2n)!}(x)_p [x+n+1]_{n-p}.$$

For example,

$$F_2^{-1} = \frac{2!}{4!} \begin{pmatrix} 12 & 0 & 0 \\ 7 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad F_3^{-1} = \frac{3!}{6!} \begin{pmatrix} 120 & 0 & 0 & 0 \\ 74 & 20 & -4 & 2 \\ 15 & 9 & 3 & -3 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Thus,

$$F_n s_n(x) = h_n(x).$$

Example 1. We explain the identity

$$F_n[x+n+1]_n = \frac{(2n)!}{n!}.$$

Analog of the identity for ordinary Riordan arrays

$$[n,\searrow]\left(a\left(x\right),xa\left(x\right)\right) = \frac{\tilde{\alpha}_n\left(x\right)}{\left(1-x\right)^{n+1}}$$

is the identity for exponential Riordan arrays

$$[n,\searrow]\left((xa\left(x\right))',xa\left(x\right)\right)_{e^x} = \frac{\tilde{\varphi}_n\left(x\right)}{\left(1-x\right)^{2n+1}}.$$

Since

$$[n,\rightarrow]\left(1+x(\log a\left(x\right))',\log a\left(x\right)\right)_{e^x} = \left(x+n\right)\tilde{u}_n\left(x\right),$$

$$\left((xa\left(x\right))',\log a\left(x\right)\right) = \left(1+x(\log a\left(x\right))',\log a\left(x\right)\right)\left(e^x,x\right),$$

then

$$[n,\rightarrow]\left((xa\left(x\right))',\log a\left(x\right)\right)_{e^x} = \left(x+n+1\right)\tilde{u}_n\left(x+1\right).$$

If $a\left(x\right)=C\left(x\right)$, then

$$\tilde{\varphi}_n\left(x\right)=\frac{(2n)!}{n!}, \qquad \tilde{u}_n\left(x\right)=[x+n+1]_{n-1}, \qquad \left(x+n+1\right)\tilde{u}_n\left(x+1\right)=[x+n+1]_n.$$

Theorem 11.

$$F_nE^{n+1}\left(1,-x\right)F_n^{-1} = (-1)^nJ_n.$$

Proof.

$$E^{n+1}\left(1,-x\right)(x)_p[x+n+1]_{n-p} = (-x-n-1)_p[-x]_{n-p} = (-1)^n(x)_{n-p}[x+n+1]_p,$$

or

$$E^{n+1}\left(1,-x\right)F_n^{-1} = (-1)^nF_n^{-1}J_n.$$

Thus,

$$(-1)^nJ_nh_n\left(x\right) = F_nEs_n\left(-x-n\right).$$

Since (see Remark 2)

$$s_n\left(-x-n\right) = [n,\rightarrow]\left(b\left(x_{(-1)}a^{-1}\left(x\right)\right)\left(1+x\left(\log{}_{(-1)}a^{-1}\left(x\right)\right)'\right),\log{}_{(-1)}a^{-1}\left(x\right)\right)_{e^x},$$

where

$$\left(1,x_{(-1)}a^{-1}\left(x\right)\right) = \left(1,xa\left(x\right)\right)^{-1}, \qquad {}_{(-1)}a^{\varphi}\left(x\right) = \sum_{m=0}^{\infty}\frac{\varphi}{(\varphi-n)}\frac{u_n\left(\varphi-n\right)}{n!}x^m,$$

then polynomials $(-1)^nJ_nh_n\left(x\right)$ are the numerator polynomials of the matrix

$$\left(b\left(x_{(-1)}a^{-1}\left(x\right)\right)\left(x_{(-1)}a^{-1}\left(x\right)\right)',x_{(-1)}a^{-1}\left(x\right)\right)_{e^x}.$$

In particular,

$$(-1)^nJ_n\varphi_n\left(x\right) = \tilde{\varphi}_n^{[-1]}\left(x\right).$$

Remark 2. We represent the matrix $(b(x), a^{-1}(x))^T$ in the form

$$\left(b(x), a^{-1}(x)\right)^T = \begin{pmatrix} s_0(0) & s_1(0) & s_2(0) & s_3(0) & \cdots \\ s_0(-1) & s_1(-1) & s_2(-1) & s_3(-1) & \cdots \\ s_0(-2) & s_1(-2) & s_2(-2) & s_3(-2) & \cdots \\ s_0(-3) & s_1(-3) & s_2(-3) & s_3(-3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} |e^x|,$$

where $s_0(x) = b_0$. From the Lagrange inversion theorem it follows that

$$[n, \searrow] \left(b(x), a^{-1}(x)\right)^T = b\left(x_{(-1)}a^{-1}(x)\right)\left(1 + x\left(\log_{(-1)}a^{-1}(x)\right)'\right)_{(-1)}a^{-n}(x).$$

We introduce the matrices S_n:

$$S_n = F_n U_n^{-1}, \qquad S_n^{-1} = U_n F_n^{-1}.$$

For example,

$$S_2 = 2! \begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 0 \\ 1 & 3 & 6 \end{pmatrix}, \quad S_3 = 3! \begin{pmatrix} 1 & 0 & 0 & 0 \\ 9 & 4 & 0 & 0 \\ 9 & 12 & 10 & 0 \\ 1 & 4 & 10 & 20 \end{pmatrix}, \quad S_4 = 4! \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 16 & 5 & 0 & 0 & 0 \\ 36 & 30 & 15 & 0 & 0 \\ 16 & 30 & 40 & 35 & 0 \\ 1 & 5 & 15 & 35 & 70 \end{pmatrix};$$

$$S_2^{-1} = \frac{2!}{4!} \begin{pmatrix} 6 & 0 & 0 \\ -8 & 2 & 0 \\ 3 & -1 & 1 \end{pmatrix}, \quad S_3^{-1} = \frac{3!}{6!} \begin{pmatrix} 20 & 0 & 0 & 0 \\ -45 & 5 & 0 & 0 \\ 36 & -6 & 2 & 0 \\ -10 & 2 & -1 & 1 \end{pmatrix},$$

$$S_4^{-1} = \frac{4!}{8!} \begin{pmatrix} 70 & 0 & 0 & 0 & 0 \\ -224 & 14 & 0 & 0 & 0 \\ 280 & -28 & 14/3 & 0 & 0 \\ -160 & 20 & -16/3 & 2 & 0 \\ 35 & -5 & 5/3 & -1 & 1 \end{pmatrix}.$$

Then

$$S_n g_n(x) = h_n(x).$$

Theorem 12.

$$S_n = V_n^{-1} C_n V_n, \qquad C_n x^p = \frac{(n+p)!}{p!} x^p.$$

Proof. We use Theorem 10 and the identities

$$U_n^{-1} V_n^{-1} x^p = \frac{n!}{p!}(x)_p, \qquad U_{n+p}^{-1} x^p = (x)_p [x+1]_n.$$

Then

$$F_n U_n^{-1} V_n^{-1} x^p = \frac{(2n)!}{n!} U_{2n} \frac{n!}{p!}(x)_p [x+1]_n =$$

$$= \frac{(2n)!}{p!}(1-x)^{n-p} \frac{(n+p)!}{(2n)!} U_{n+p}(x)_p [x+1]_n = \frac{(n+p)!}{p!}(1-x)^{n-p} x^p,$$

or

$$S_n V_n^{-1} = V_n^{-1} C_n.$$

Theorem 13.

$$S_n x^p = \frac{(n+p)!\,(n-p)!}{n!} \sum_{m=p}^{n} \binom{n}{m-p} \binom{n}{n-m} x^m.$$

Proof.

$$[m,\to]\,V_n^{-1}=\sum_{i=0}^{m}\binom{m-n-1}{m-i}x^i=\sum_{i=0}^{m}(-1)^{m-i}\binom{n-i}{m-i}x^i,$$

$$C_nV_nx^p=\sum_{i=p}^{n}\frac{(n+i)!}{i!}\binom{n-p}{i-p}x^i,$$

$$[x^m]\,V_n^{-1}C_nV_nx^p=\sum_{i=p}^{m}(-1)^{m-i}\binom{n-i}{m-i}\frac{(n+i)!}{i!}\binom{n-p}{i-p}=$$

$$=\frac{(n+p)!\,(n-p)!}{(n-m)!m!}\sum_{i=p}^{m}(-1)^{m-i}\binom{n+i}{i-p}\binom{m}{i}=$$

$$=\frac{(n+p)!\,(n-p)!}{(n-m)!m!}(-1)^{m-p}\binom{m-n-p-1}{m-p}=$$

$$=\frac{(n+p)!\,(n-p)!}{n!}\binom{n}{m-p}\binom{n}{n-m}.$$

Theorem 14.

$$S_n^{-1}x^p=\frac{p!\,(n-p)!}{(2n)!}\sum_{m=p}^{n}\binom{-n}{m-p}\binom{2n}{n-m}x^m.$$

Proof.

$$[x^m]\,V_n^{-1}C_n^{-1}V_nx^p=\sum_{i=p}^{m}(-1)^{m-i}\binom{n-i}{m-i}\frac{i!}{(n+i)!}\binom{n-p}{i-p}=$$

$$=\frac{p!\,(n-p)!}{(n-m)!\,(n+m)!}\sum_{i=p}^{m}(-1)^{m-i}\binom{i}{i-p}\binom{n+m}{m-i}=$$

$$=\frac{p!\,(n-p)!}{(n-m)!\,(n+m)!}(-1)^{m-p}\binom{n+m-p-1}{m-p}=$$

$$=\frac{p!\,(n-p)!}{(2n)!}\binom{-n}{m-p}\binom{2n}{n-m}.$$

5 Generalized Narayana polynomials of type B

Polynomials

$${}^{B}N_n(x)=\sum_{m=0}^{n}\binom{n}{m}^2x^m=(1-x)^{2n+1}\sum_{m=0}^{\infty}\binom{m+n}{n}^2x^m$$

are called Narayana polynomials of type B. Denote

$$[n,\searrow](a(x),xa(x))_{e^x}=\frac{{}^{B}\varphi_n(x)}{(1-x)^{2n+1}}.$$

Let $\tilde{\alpha}_n(x)\,|a(x)$, ${}^{B}\varphi_n(x)\,|a(x)$ denotes respectively polynomials $\tilde{\alpha}_n(x)$, ${}^{B}\varphi_n(x)$, associated with the matrices $(a(x),xa(x))$, $(a(x),xa(x))_{e^x}$. Then

$${}^{B}\varphi_n(x)\,|(1-x)^{-1}=n!\,{}^{B}N_n(x).$$

In this connection we will called polynomials ${}^{B}\varphi_n(x)$ the generalized Narayana polynomials of type B. Since

$$[n,\searrow](a(x),xa(x))=\frac{\tilde{\alpha}_n(x)}{(1-x)^{n+1}}=\sum_{m=0}^{\infty}\frac{u_n(m+1)}{n!}x^m,$$

then

$$[n,\searrow](a(x), xa(x))_{e^x} = \sum_{m=0}^{\infty} \frac{[m+1]_n u_n(m+1)}{n!} x^m,$$

$$^B\varphi_n(x) = \frac{(2n)!}{n!} U_{2n}[x+1]_n u_n(x+1) = F_n u_n(x+1).$$

We introduce the matrices $^BF_n = F_n E$:

$$^BF_n x^p = (1-x)^{2n+1} \sum_{m=0}^{\infty} (m+1)^p \binom{m+n}{n} x^m, \quad ^BF_n^{-1} x^p = \frac{n!}{(2n)!}(x-1)_p [x+n]_{n-p}.$$

For example,

$$^BF_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad ^BF_2 = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 7 \\ 1 & -2 & 4 \end{pmatrix}, \qquad ^BF_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & 1 & 9 & 25 \\ 3 & -5 & -1 & 67 \\ -1 & 3 & -9 & 27 \end{pmatrix},$$

$$^BF_1^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad ^BF_2^{-1} = \frac{2!}{4!}\begin{pmatrix} 6 & -2 & 2 \\ 5 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}, \quad ^BF_3^{-1} = \frac{3!}{6!}\begin{pmatrix} 60 & -12 & 6 & -6 \\ 47 & 5 & -7 & 11 \\ 12 & 6 & 0 & -6 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then

$$^BF_n u_n(x) = {}^B\varphi_n(x).$$

In particular,

$$^BF_n x^n = {}^B\varphi_n(x) \,|e^x.$$

For the matrices ^BF_n, Theorem 11 takes the simpler form. Since

$$E^{n-1}(1,-x)(x-1)_p[x+n]_{n-p} = (-x-n)_p[-x+1]_{n-p} = (-1)^n (x-1)_{n-p}[x+n]_p,$$

then

$$^BF_n E^{n-1}(1,-x)\, {}^BF_n^{-1} = (-1)^n J_n, \qquad (-1)^n J_n {}^B\varphi_n(x) = F_n u_n(-x-n),$$

where

$$u_n(-x-n) = [n,\rightarrow]\Big(1 + x\big(\log_{(-1)} a^{-1}(x)\big)', \log_{(-1)} a^{-1}(x)\Big)_{e^x}.$$

Denote

$$[n,\searrow]\left(1 + x(\log a(x))', xa(x)\right)_{e^x}^{-1} = \frac{^B\varphi_n^{[-1]}(x)}{(1-x)^{2n+1}}.$$

Since

$$\left(1 + x(\log a(x))', xa(x)\right)_{e^x}^{-1} = \Big(1 + x\big(\log_{(-1)} a^{-1}(x)\big)', x_{(-1)} a^{-1}(x)\Big)_{e^x},$$

then

$$^B\varphi_n^{[-1]}(x) = (-1)^n J_n {}^B\varphi_n(x).$$

Let all polynomials $^B\varphi_n(x)$ are symmetric, i.e.

$$^B\varphi_n(x) = J_n {}^B\varphi_n(x).$$

Then $(xa(x))' = a^2(x)$, or $\sum_{m=0}^{n} a_{n-m} a_m = (n+1) a_n$. This is possible only in the case $a(x) = (1-\beta x)^{-1}$:

$$^B\varphi_n |(1-\beta x)^{-1} = \beta^n n! {}^BN_n(x).$$

Example 2. Since

$$\tilde{\alpha}_n(x)\,|1+x = x^{n-1},$$

then

$${}^{B}\varphi_n(x)\,|1+x = S_n x^{n-1} = \frac{(2n)!}{n!2}(1+x)\,x^{n-1};$$

$$(1+x, x(1+x))_{e^x} = |e^x|^{-1}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 4 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 6 & 5 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 4 & 10 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}|e^x|.$$

If $a(x) = 1+x$, then ${}_{(-1)}a^{-1}(x) = C(-x)$ and, hence,

$$[n,\searrow]\left(1+x(\log C(x))', xC(x)\right)_{e^x} = \frac{(2n)!}{n!2}\frac{1+x}{(1-x)^{2n+1}};$$

$$\left(1+x(\log C(x))', xC(x)\right)_{e^x} = |e^x|^{-1}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 10 & 6 & 3 & 1 & 0 & 0 & \cdots \\ 35 & 20 & 10 & 4 & 1 & 0 & \cdots \\ 126 & 70 & 35 & 15 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}|e^x|.$$

We introduce ordinary numerator polynomials similar to the polynomials ${}^{B}\varphi_n(x)$. Denote

$$[n,\searrow]\left((xa(x))', xa(x)\right) = \frac{{}^{B}\alpha_n(x)}{(1-x)^{n+1}}, \quad [n,\searrow]\left(1+x(\log a(x))', xa^{-1}(x)\right) = \frac{{}^{B}\alpha_n^{(-1)}(x)}{(1-x)^{n+1}}.$$

Then

$${}^{B}\alpha_n^{(-1)}(x) = (-1)^n J_n {}^{B}\alpha_n(x).$$

Example 3. Let $\tilde{\varphi}_n(x)\,|a(x)$, ${}^{B}\alpha_n(x)\,|a(x)$ denotes respectively polynomials $\tilde{\varphi}_n(x)$, ${}^{B}\alpha_n(x)$, associated with the matrices $\left((xa(x))', xa(x)\right)_{e^x}$, $\left((xa(x))', xa(x)\right)$. Since

$$\tilde{\varphi}_n(x)\,|C(x) = \frac{(2n)!}{n!},$$

then

$${}^{B}\alpha_n(x)\,|C(x) = \frac{(2n)!}{n!}S_n^{-1}x^0 = \sum_{m=0}^{n}\binom{-n}{m}\binom{2n}{n-m}x^m;$$

$$\left((xC(x))', xC(x)\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 20 & 10 & 4 & 1 & 0 & 0 & \cdots \\ 70 & 35 & 15 & 5 & 1 & 0 & \cdots \\ 252 & 126 & 56 & 21 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Respectively, polynomials

$$(-1)^n \sum_{m=0}^{n} \binom{2n}{m} \binom{-n}{n-m} x^m$$

are the numerator polynomials of the matrix

$$\left(1 + x(\log \mathrm{C}(x))', xC^{-1}(x)\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 10 & 1 & -1 & 1 & 0 & 0 & \cdots \\ 35 & 4 & 0 & -2 & 1 & 0 & \cdots \\ 126 & 15 & 1 & 0 & -3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example 4. Since

$$\tilde{\varphi}_n(x)\,|1 + x = \frac{(2n)!}{n!} x^{n-1},$$

then

$$^{B}\alpha_n(x)\,|1 + x = \frac{(2n)!}{n!} S_n^{-1} x^{n-1} = (2 - x)\, x^{n-1};$$

$$\left((x(1+x))', x(1+x)\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 5 & 5 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 9 & 6 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 7 & 14 & 7 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Respectively, polynomials $(-1)^n(-1 + 2x)$ are the numerator polynomials of the matrix

$$\left(1 + x(\log(1+x))', x(1+x)^{-1}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & \cdots \\ -1 & 2 & 0 & -2 & 1 & 0 & 0 & \cdots \\ 1 & -3 & 2 & 2 & -3 & 1 & 0 & \cdots \\ -1 & 4 & -5 & 0 & 5 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example 5. Since $\left(x(1-x)^{-1}\right)' = (1-x)^{-2}$, then

$$[n, \searrow]\left(\left(\frac{x}{1-x}\right)', \frac{x}{1-x}\right) = \frac{1}{x}\left(\frac{1}{(1-x)^{n+1}} - 1\right), \qquad {}^{B}\alpha_n(x) = \frac{1 - (1-x)^{n+1}}{x}.$$

Respectively, numerator polynomials of the matrix $\left((1-x)^{-1}, x(1-x)\right)$ are the polynomials $(1-x)^{n+1} + (-x)^n$.

Example 6. Since $(xe^x)' = (1+x)e^x$, then

$$[n, \searrow]\left((xe^x)', xe^x\right) = \sum_{m=0}^{\infty} \frac{(m+1)^n + n(m+1)^{n-1}}{n!} x^m,$$

$$ {}^{B}\alpha_n(x)\,|e^x = \frac{1}{n!}\left(\tilde{A}_n(x) + n(1-x)\tilde{A}_{n-1}(x)\right), \qquad \tilde{A}_0(x) = A_0(x). $$

Respectively, numerator polynomials of the matrix $(1+x, xe^{-x})$ are the polynomials

$$ \frac{(-1)^n}{n!}\left(A_n(x) - n(1-x)A_{n-1}(x)\right). $$

Note that

$$ \tilde{\varphi}_n(x)\,|e^x = F_n(x+n+1)(x+1)^{n-1} = {}^{B}F_n(x+n)\,x^{n-1}. $$

In general case

$$ \tilde{\varphi}_n(x) = F_n(x+n+1)\,\tilde{u}_n(x+1) = {}^{B}F_n(x+n)\,\tilde{u}_n(x), $$

or

$$ F_n(x+n+1,x)\,EI_{n-1} = {}^{B}F_n(x+n,x)\,I_{n-1} = \tilde{F}_n. $$

Hence,

$$ [x^n]\,F_n x^p(x+n+1) = [x^n]\,{}^{B}F_n x^p(x+n) = 0, \qquad p<n. $$

Here the identities for the nth elements of the columns of the matrices F_n, ${}^{B}F_n$ are manifested:

$$ \sum_{m=0}^{n}(-1)^{n-m}\binom{2n+1}{n-m}m^p\binom{m+n}{n} = (-1)^{n+p}(n+1)^p, $$

$$ \sum_{m=0}^{n}(-1)^{n-m}\binom{2n+1}{n-m}(m+1)^p\binom{m+n}{n} = (-1)^{n+p}n^p, \qquad p\le n. $$

6 Numerator polynomials and generalized Lagrange series

We return to the series ${}_{(\beta)}a(x)$, ${}_{(0)}a(x) = a(x)$, from Section 2. Parameter β is defined by the identity

$$ a\left(x\,{}_{(\beta)}a^\beta(x)\right) = {}_{(\beta)}a(x), $$

so that

$$ a^\beta\left(x\,{}_{(\beta)}a^\beta(x)\right) = {}_{(\beta)}a^\beta(x). $$

Denote $a^\beta(x) = c(x)$, ${}_{(\beta)}a^\beta(x) = d(x)$. By the Lagrange inversion theorem, if

$$ b(x)\,c^\varphi(x) = \sum_{n=0}^{\infty} f_n(\varphi)\,x^n, $$

where $f_n(x)$ are the polynomials, then

$$ b(xd(x))\left(1 + x(\log d(x))'\right)d^\varphi(x) = \sum_{n=0}^{\infty} f_n(\varphi+n)x^n. $$

Here

$$ f_n(x) = \frac{s_n(\beta x)}{n!}, \qquad s_n(x) = [n,\to]\,(b(x), \log a(x))_{e^x}. $$

Thus,

$$ s_n(\beta x + \beta n) = [n,\to]\left(b\left(x\,{}_{(\beta)}a^\beta(x)\right)\left(1 + x\big(\log{}_{(\beta)}a^\beta(x)\big)'\right), \log{}_{(\beta)}a^\beta(x)\right)_{e^x}, $$

$$ s_n(x + \beta n) = [n,\to]\left(b\left(x\,{}_{(\beta)}a^\beta(x)\right)\left(1 + x\big(\log{}_{(\beta)}a^\beta(x)\big)'\right), \log{}_{(\beta)}a(x)\right)_{e^x}. $$

Denote

$$[n,\searrow]\left(b\left(x_{(\beta)}a^{\beta}\left(x\right)\right)\left(1+x\beta\left(\log_{(\beta)}a\left(x\right)\right)'\right),x_{(\beta)}a\left(x\right)\right)=\frac{{}_{(\beta)}g_n\left(x\right)}{\left(1-x\right)^{n+1}}.$$

We introduce the matrices $G_n^{\beta}=U_nE^{n\beta}U_n^{-1}$. For example,

$$G_2=\begin{pmatrix}6&3&1\\-8&-3&0\\3&1&0\end{pmatrix},\ G_3=\begin{pmatrix}20&10&4&1\\-45&-20&-6&0\\36&15&4&0\\-10&-4&-1&0\end{pmatrix},\ G_4=\begin{pmatrix}70&35&15&5&1\\-224&-105&-40&-10&0\\280&126&45&10&0\\-160&-70&-24&-5&0\\35&15&5&1&0\end{pmatrix}.$$

Then

$$G_n^{\beta}g_n\left(x\right)={}_{(\beta)}g_n\left(x\right).$$

Theorem 15.

$$G_n^{-\beta}=J_nG_n^{\beta}J_n.$$

Proof. Since $E\left(1,-x\right)=\left(1,-x\right)E^{-1}$, by Theorem 6

$$J_nU_nE^{n\beta}U_n^{-1}J_n=U_nE\left(1,-x\right)E^{n\beta}E\left(1,-x\right)U_n^{-1}=$$

$$=U_n\left(1,-x\right)E^{n\beta}\left(1,-x\right)U_n^{-1}=U_nE^{-n\beta}U_n^{-1}.$$

Thus,

$$G_2^{-1}=\begin{pmatrix}0&1&3\\0&-3&-8\\1&3&6\end{pmatrix},\ G_3^{-1}=\begin{pmatrix}0&-1&-4&-10\\0&4&15&36\\0&-6&-20&-45\\1&4&10&20\end{pmatrix},\ G_4^{-1}=\begin{pmatrix}0&1&5&15&35\\0&-5&-24&-70&-160\\0&10&45&126&280\\0&-10&-40&-105&-224\\1&5&15&35&70\end{pmatrix}.$$

Theorem 16.

$$G_n^{\beta}=V_n^{-1}\left(\left(1+x\right)^{n\beta},x\right)^TV_n.$$

Proof. Since

$$n!\left|e^x\right|V_nU_nx^p=[p,\to]\left(1,e^x-1\right)_{e^x},$$

$$\left(1/n!\ \right)U_n^{-1}V_n^{-1}\left|e^x\right|^{-1}x^p=[p,\to]\left(1,\log\left(1+x\right)\right)_{e^x},$$

$$\left(1,\log\left(1+x\right)\right)_{e^x}\left(e^{n\beta},x\right)_{e^x}\left(1,e^x-1\right)_{e^x}=\left(\left(1+x\right)^{n\beta},x\right)_{e^x},$$

then

$$V_nU_nE^{n\beta}U_n^{-1}V_n^{-1}=\left(\left(1+x\right)^{n\beta},x\right)^TI_n.$$

For example,

$$G_3=\begin{pmatrix}1&0&0&0\\-3&1&0&0\\3&-2&1&0\\-1&1&-1&1\end{pmatrix}\begin{pmatrix}1&3&3&1\\0&1&3&3\\0&0&1&3\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\3&1&0&0\\3&2&1&0\\1&1&1&1\end{pmatrix}.$$

Theorem 17.

$$G_n^{\beta}x^p=\sum_{m=0}^{n}\binom{-n\beta+p}{m}\binom{n\beta+n-p}{n-m}x^m.$$

Proof. Taking into account the polynomial argument, we prove the theorem for positive integers β.

$$\left((1+x)^{n\beta},x\right)^T V_n x^p=\sum_{i=0}^{n}\binom{n\beta+n-p}{n-i}x^i,$$

$$[x^m]\,V_n^{-1}\left((1+x)^{n\beta},x\right)^T V_n x^p=$$

$$=\sum_{i=0}^{m}(-1)^{m-i}\binom{n-i}{m-i}\binom{n\beta+n-p}{n-i}\frac{(n\beta+m-p)!}{(n\beta+m-p)!}=$$

$$=\binom{n\beta+n-p}{n-m}\sum_{i=0}^{m}(-1)^{m-i}\binom{n\beta+m-p}{m-i}=$$

$$=\binom{n\beta+n-p}{n-m}(-1)^m\binom{n\beta+m-p-1}{m}=\binom{n\beta+n-p}{n-m}\binom{-n\beta+p}{m}.$$

We introduce the matrices $X_n=V_n^{-1}(x,x)^T V_n$. Since $V_n^{-1}=\left((1-x)^{n+1},x\right)PI_n$, we find:

$$X_n x^0=\frac{1-x-(1-x)^{n+1}}{x},\qquad X_n x^p=x^{p-1}(1-x).$$

Then

$$G_n^\beta=(I_n+X_n)^{n\beta}=\sum_{m=0}^{n}\binom{n\beta}{m}X_n^m.$$

For example,

$$G_3=I_3+3\begin{pmatrix}3&1&0&0\\-6&-1&1&0\\4&0&-1&1\\-1&0&0&-1\end{pmatrix}+3\begin{pmatrix}3&2&1&0\\-8&-5&-2&1\\7&4&1&-2\\-2&-1&0&1\end{pmatrix}+\begin{pmatrix}1&1&1&1\\-3&-3&-3&-3\\3&3&3&3\\-1&-1&-1&-1\end{pmatrix},$$

$$G_3^{-1}=I_3-3\begin{pmatrix}3&1&0&0\\-6&-1&1&0\\4&0&-1&1\\-1&0&0&-1\end{pmatrix}+6\begin{pmatrix}3&2&1&0\\-8&-5&-2&1\\7&4&1&-2\\-2&-1&0&1\end{pmatrix}-10\begin{pmatrix}1&1&1&1\\-3&-3&-3&-3\\3&3&3&3\\-1&-1&-1&-1\end{pmatrix}.$$

Thus,

$$I_n+X_n=G_n^{1/n}=U_nEU_n^{-1}.$$

For example,

$$G_2^{1/2}=\begin{pmatrix}3&1&0\\-3&0&1\\1&0&0\end{pmatrix},\qquad G_3^{1/3}\begin{pmatrix}4&1&0&0\\-6&0&1&0\\4&0&0&1\\-1&0&0&0\end{pmatrix},\qquad G_4^{1/4}=\begin{pmatrix}5&1&0&0&0\\-10&0&1&0&0\\10&0&0&1&0\\-5&0&0&0&1\\1&0&0&0&0\end{pmatrix},$$

$$G_2^{-1/2}=\begin{pmatrix}0&0&1\\1&0&-3\\0&1&3\end{pmatrix},\quad G_3^{-1/3}\begin{pmatrix}0&0&0&-1\\1&0&0&4\\0&1&0&-6\\0&0&1&4\end{pmatrix},\quad G_4^{-1/4}=\begin{pmatrix}0&0&0&0&1\\1&0&0&0&-5\\0&1&0&0&10\\0&0&1&0&-10\\0&0&0&1&5\end{pmatrix}.$$

From Theorem 10 it follows that

$$\left((1-x)^{-m},x\right)G_n^\beta\left((1-x)^m,x\right)I_{n-m}=G_{n-m}^{\frac{n\beta}{n-m}}.$$

In particular,
$$\left((1-x)^{-m},x\right)G_n^{1/n}\left((1-x)^{m},x\right)I_{n-m}=G_{n-m}^{1/(n-m)}.$$

Example 7. Let $b(x)=1$, $a(x)=1+x$, ${}_{(\beta)}a(x)$ is the generalized binomial series. Then polynomials ${}_{(\beta)}g_n(x)=G_n^{\beta}x^n$ are the numerator polynomials of the matrix
$$\left(1+x\big(\log{}_{(\beta)}a^{\beta}(x)\big)',x_{(\beta)}a(x)\right).$$

But since ${}_{(1)}g_n(x)=1$, $G_n^{\beta}x^0=G_n^{\beta+1}x^n$, then this matrix can also be represented in the form
$$\left({}_{(\beta)}a(x)\left(1+x\big(\log{}_{(\beta)}a^{\beta-1}(x)\big)'\right),x_{(\beta)}a(x)\right).$$
Hence, here the property of the generalized binomial series is manifested:
$${}_{(\beta)}a(x)\left(1+x\big(\log{}_{(\beta)}a^{\beta-1}(x)\big)'\right)=1+x\big(\log{}_{(\beta)}a^{\beta}(x)\big)'.$$

Example 8. Let ${}_{(\beta)}a(x)$ is the generalized binomial series. Then polynomials $G_n^{\beta}x$ are the numerator polynomials of the matrix
$$\left(1+x\big(\log{}_{(\beta+1)}a^{\beta}(x)\big)',x_{(\beta+1)}a(x)\right),$$

polynomials $G_n^{\beta}x^{n-1}$ are the numerator polynomials of the matrix
$$\left({}_{(\beta)}a(x)\left(1+x\big(\log{}_{(\beta)}a^{\beta}(x)\big)'\right),x_{(\beta)}a(x)\right).$$

Since $G_n^{-\beta}x=J_nG_n^{\beta}x^{n-1}$, then matrix
$$\left(1+x\big(\log{}_{(1-\beta)}a^{-\beta}(x)\big)',x_{(1-\beta)}a(x)\right)$$

coincides with the matrix
$$(1,-x)\left(1+x\big(\log{}_{(\beta)}a^{\beta}(x)\big)',x_{(\beta)}a^{-1}(x)\right)(1,-x),$$

matrix
$$\left({}_{(-\beta)}a(x)\left(1+x\big(\log{}_{(-\beta)}a^{-\beta}(x)\big)'\right),x_{(-\beta)}a(x)\right)$$
coincides with the matrix
$$(1,-x)\left({}_{(\beta+1)}a^{-1}(x)\left(1+x\big(\log{}_{(\beta+1)}a^{\beta}(x)\big)'\right),x_{(\beta+1)}a^{-1}(x)\right)(1,-x).$$

Denote
$$[n,\searrow]\left(b\left(x_{(\beta)}a^{\beta}(x)\right)\left(1+x\beta\big(\log{}_{(\beta)}a(x)\big)'\right),x_{(\beta)}a(x)\right)_{e^x}=\frac{{}_{(\beta)}h_n(x)}{(1-x)^{2n+1}}.$$

We introduce the matrices $H_n^{\beta}=F_nE^{n\beta}F_n^{-1}$. For example,
$$H_2=\frac{1}{6}\begin{pmatrix}15&5&1\\-12&2&4\\3&-1&1\end{pmatrix},\qquad H_3=\frac{1}{20}\begin{pmatrix}84&28&7&1\\-108&-4&15&9\\54&-6&-1&9\\-10&2&-1&1\end{pmatrix},$$
$$H_4=\frac{1}{70}\begin{pmatrix}495&165&135/3&9&1\\-880&-110&160/3&44&16\\660&0&-90/3&24&36\\-240&20&0&-6&16\\35&-5&5/3&-1&1\end{pmatrix}.$$

Then
$$H_n^\beta h_n(x) = {}_{(\beta)}h_n(x).$$

Theorem 18.
$$H_n^{-\beta} = J_n H_n^\beta J_n.$$

Proof. By Theorem 11
$$J_n F_n E^{n\beta} F_n^{-1} J_n = F_n E^{n+1}(1,-x) E^{n\beta} E^{n+1}(1,-x) F_n^{-1} =$$
$$= F_n(1,-x) E^{n\beta}(1,-x) F_n^{-1} = F_n E^{-n\beta} F_n^{-1}.$$

Matrix H_n^β can be represented in the form
$$H_n^\beta = S_n G_n^\beta S_n^{-1} = V_n^{-1} C_n\Big((1+x)^{n\beta}, x\Big)^T C_n^{-1} V_n.$$

For example,
$$H_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 20 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{10} & 0 \\ 0 & 0 & 0 & \frac{1}{20} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Denote
$$t_n(\varphi|\beta, x) = \sum_{m=0}^{n} \binom{\varphi}{m}\binom{\beta}{n-m} x^m.$$

Theorem 19.
$$H_n^\beta x^p = \sum_{m=p}^{n} \binom{n-p}{n-m}\binom{n+m}{m}^{-1} (1-x)^{n-m} t_m(-n\beta+n+m|n\beta, x).$$

Proof. Since
$$\frac{1}{n!}[x^m] V_p^{-1} C_n\Big((1+x)^{n\beta}, x\Big)^T x^p =$$
$$= \sum_{i=0}^{m} (-1)^{m-i}\binom{p-i}{m-i}\binom{n\beta}{p-i}\binom{n+i}{i}\frac{(n\beta+m-p)!}{(n\beta+m-p)!} =,$$
$$= \binom{n\beta}{p-m}\sum_{i=0}^{m}(-1)^{m-i}\binom{n+i}{i}\binom{n\beta+m-p}{m-i} =$$
$$= \binom{n\beta}{p-m}(-1)^m\binom{n\beta+m-p-n-1}{m} = \binom{n\beta}{p-m}\binom{-n\beta+n+p}{m},$$
then
$$V_n^{-1} C_n\Big((1+x)^{n\beta}, x\Big)^T x^p = \left((1-x)^{n-p}, x\right) V_p^{-1} C_n\Big((1+x)^{n\beta}, x\Big)^T x^p =$$
$$= n!(1-x)^{n-p} t_p(-n\beta+n+p|n\beta, x).$$
It remains to add that
$$C_n^{-1} V_n x^p = \frac{1}{n!}\sum_{m=p}^{n}\binom{n-p}{n-m}\binom{n+m}{m}^{-1} x^m.$$

In particular,
$$H_n^\beta x^n = \binom{2n}{n}^{-1}\sum_{m=0}^{n}\binom{-n\beta+2n}{m}\binom{n\beta}{n-m} x^m.$$

Respectively, by Theorem 18

$$H_n^{\beta}x^0=\binom{2n}{n}^{-1}\sum_{m=0}^{n}\binom{-n\beta}{m}\binom{n\beta+2n}{n-m}x^m.$$

Example 9. Let ${}_{(\beta)}a\left(x\right)$ is the generalized binomial series. Then polynomials ${}_{(\beta)}h_n\left(x\right)=\frac{(2n)!}{n!}H_n^{\beta}x^n$ are the numerator polynomials of the matrix

$$\left(1+x\left(\log{}_{(\beta)}a^{\beta}\left(x\right)\right)',x{}_{(\beta)}a\left(x\right)\right)_{e^x}.$$

Since ${}_{(2)}h_n\left(x\right)=\frac{(2n)!}{n!}$, then polynomials $\frac{(2n)!}{n!}H_n^{\beta}x^0=\frac{(2n)!}{n!}H_n^{\beta+2}x^n$ are the numerator polynomials of the matrix

$$\left(1+x\left(\log{}_{(\beta+2)}a^{\beta+2}\left(x\right)\right)',x{}_{(\beta+2)}a\left(x\right)\right)_{e^x}.$$

Since

$$H_n^{-\beta}x^0=J_nH_n^{\beta}x^n,\qquad\left(1,x{}_{(\beta)}a\left(x\right)\right)^{-1}=\left(1,x{}_{(\beta-1)}a^{-1}\left(x\right)\right),$$
$$\left(x{}_{(\beta-1)}a^{-1}\left(x\right)\right)'\left(1,x{}_{(\beta-1)}a^{-1}\left(x\right)\right)\left(1+x\left(\log{}_{(\beta)}a^{\beta}\left(x\right)\right)'\right)=$$
$$=\left(1+x\left(\log{}_{(\beta-1)}a^{\beta-1}\left(x\right)\right)'\right){}_{(\beta-1)}a^{-1}\left(x\right),$$

then matrix

$$\left(1+x\left(\log{}_{(2-\beta)}a^{2-\beta}\left(x\right)\right)',x{}_{(2-\beta)}a\left(x\right)\right)$$

coincides with the matrix

$$\left(1,-x\right)\left(\left(1+x\left(\log{}_{(\beta-1)}a^{\beta-1}\left(x\right)\right)'\right){}_{(\beta-1)}a^{-1}\left(x\right),x{}_{(\beta-1)}a^{-1}\left(x\right)\right)\left(1,-x\right).$$

Denote

$$[n,\searrow]\left(1,x{}_{(\beta)}a\left(x\right)\right)_{e^x}=\frac{{}_{(\beta)}\varphi_n\left(x\right)}{\left(1-x\right)^{2n+1}},\qquad\frac{1}{x}{}_{(\beta)}\varphi_n\left(x\right)={}_{(\beta)}\tilde{\varphi}_n\left(x\right).$$

We introduce the matrices $T_n^{\beta}=\tilde{F}_nE^{n\beta}\tilde{F}_n^{-1}$. For example,

$$T_2=\frac{1}{2}\begin{pmatrix}3&1\\-1&1\end{pmatrix},\qquad T_3=\frac{1}{5}\begin{pmatrix}12&4&1\\-9&2&3\\2&-1&1\end{pmatrix},\qquad T_4=\frac{1}{14}\begin{pmatrix}55&55/3&5&1\\-66&0&10&6\\30&-6&0&6\\-5&5/3&-1&1\end{pmatrix}.$$

Then

$$T_n^{\beta}\tilde{\varphi}_n\left(x\right)={}_{(\beta)}\tilde{\varphi}_n\left(x\right).$$

Theorem 20.

$$T_n^{-\beta}=\tilde{J}_nT_n^{\beta}\tilde{J}_n.$$

Proof. By Theorem 6

$$\tilde{J}_n\tilde{F}_nE^{n\beta}\tilde{F}_n^{-1}\tilde{J}_n=\tilde{F}_nE^n\left(1,-x\right)E^{n\beta}E^n\left(1,-x\right)\tilde{F}_n^{-1}=$$
$$=\tilde{F}_n\left(1,-x\right)E^{n\beta}\left(1,-x\right)\tilde{F}_n^{-1}=\tilde{F}_nE^{-n\beta}\tilde{F}_n^{-1}.$$

Matrix T_n^{β} can be represented in the form

$$T_n^{\beta}=\tilde{S}_nA_n^{\beta}\tilde{S}_n^{-1}=\tilde{V}_n^{-1}\tilde{C}_n\tilde{D}\left(\left(1+x\right)^{n\beta},x\right)^T\tilde{D}^{-1}\tilde{C}_n^{-1}\tilde{V}_n,$$

where

$$\tilde{C}_n\tilde{D}x^p=(n+1)!\binom{n+1+p}{p}x^p.$$

For example,

$$T_4=\begin{pmatrix}1&0&0&0\\-3&1&0&0\\3&-2&1&0\\-1&1&-1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&6&0&0\\0&0&21&0\\0&0&0&56\end{pmatrix}\begin{pmatrix}1&4&6&4\\0&1&4&6\\0&0&1&4\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\frac{1}{6}&0&0\\0&0&\frac{1}{21}&0\\0&0&0&\frac{1}{56}\end{pmatrix}\begin{pmatrix}1&0&0&0\\3&1&0&0\\3&2&1&0\\1&1&1&1\end{pmatrix}.$$

Theorem 21.

$$T_n^{\beta}x^p=\sum_{m=p}^{n-1}\binom{n-1-p}{n-1-m}\binom{n+1+m}{m}^{-1}(1-x)^{n-m-1}t_m\left(-n\beta+n+m+1|n\beta,x\right).$$

Proof. In this case $p=0,\ 1,\ \ldots\ ,n-1$. Since

$$\frac{1}{(n+1)!}\left[x^m\right]\tilde{V}_{p+1}^{-1}\tilde{C}_n\tilde{D}\Big((1+x)^{n\beta},x\Big)^Tx^p=$$

$$=\sum_{i=0}^{m}(-1)^{m-i}\binom{p-i}{m-i}\binom{n\beta}{p-i}\binom{n+1+i}{i}\frac{(n\beta+m-p)!}{(n\beta+m-p)!}=$$

$$=\binom{n\beta}{p-m}\sum_{i=0}^{m}(-1)^{m-i}\binom{n+1+i}{i}\binom{n\beta+m-p}{m-i}=$$

$$=\binom{n\beta}{p-m}(-1)^m\binom{n\beta+m-p-n-2}{m}=\binom{n\beta}{p-m}\binom{-n\beta+n+p+1}{m},$$

then

$$\tilde{V}_n^{-1}\tilde{C}_n\tilde{D}\Big((1+x)^{n\beta},x\Big)^Tx^p=\left((1-x)^{n-p-1},x\right)\tilde{V}_{p+1}^{-1}\tilde{C}_n\tilde{D}\Big((1+x)^{n\beta},x\Big)^Tx^p=$$

$$=(n+1)!(1-x)^{n-p-1}t_p\left(-n\beta+n+p+1|n\beta,x\right).$$

It remains to add that

$$\tilde{D}^{-1}\tilde{C}_n^{-1}\tilde{V}_nx^p=\frac{1}{(n+1)!}\sum_{m=p}^{n-1}\binom{n-1-p}{n-1-m}\binom{n+1+m}{m}^{-1}x^m.$$

In particular,

$$T_n^{\beta}x^{n-1}=\binom{2n}{n-1}^{-1}\sum_{m=0}^{n-1}\binom{n\,(2-\beta)}{m}\binom{n\beta}{n-1-m}x^m.$$

Respectively, by Theorem 20

$$T_n^{\beta}x^0=\binom{2n}{n-1}^{-1}\sum_{m=0}^{n-1}\binom{-n\beta}{m}\binom{n\,(2+\beta)}{n-1-m}x^m.$$

References

[1] E. Burlachenko, Riordan arrays and generalized Euler polynomials, arXiv:1709.02229.

[2] L. Shapiro, S. Getu, W. Woan, L. Woodson, The Riordan group, Discrete Appl. Math. 34 (1991) 229-339.

[3] R. Sprugnoli, Riordan arrays and combinatorial sums, Discrete Math.132 (1994) 267-290.

[4] D. Merlini, D. G. Rogers, R. Sprugnoli and M. C. Verri, On some alternative characterizations of Riordan arrays, Can. J. Math. 49 (1997) 301-320.

[5] W, Wang, T. Wang, Generalized Riordan arrays, Discrete Math. 308 (2008) 6466-6500.

[6] P. Barry, A study of integer sequences, Riordan arrays, Pascal-like arrays and Hankel transforms, University College Cork, 2009.

[7] S. M. Roman, The Umbral Calculus, Academic Press, 1984.

[8] V. E. Hoggatt and Marjorie Bicknell, Convolution triangles, Fibonacci Quart. 10 (1972) 599-609.

[9] V. E. Hoggatt, and G. E. Bergum, Generalized convolution arrays, Fibonacci Quart. 13 (1975) 193-198.

[10] V. E. Hoggatt and Marjorie Bicknell-Johnson, Numerator polynomial coefficient arrays for Catalan and related sequence convolution triangles, Fibonacci Quart. 15 (1977) 30-34.

[11] G. E. Bergum and V. E. Hoggatt, An application of the characteristic of the generalized Fibonacci sequence, Fibonacci Quart. 15 (1977) 215-220.

[12] V. E. Hoggatt and Marjorie Bicknell-Johnson, Convolution arrays for Jacobsthal and Fibonacci polynomials, 16 (1978) 385-402.

[13] Peter Bala, Notes on generalized Riordan arrays, https://oeis.org/A260492/a260492.pdf

[14] M. Koutras, Eulerian nambers associated with sequences of polynomials, Fibonacci Quart. 32 (1994) 44-57.

[15] V. E. Hoggatt and Paul S. Bruckman, H-convolution transform, Fibonacci Quart. 13 (1975) 357-368.

[16] E. V. Burlachenko, Riordan arrays and generalized Lagrange series, Mathematical Notes, 100 (2016) 531-539.

[17] B. Drake, An inversion theorem for labeled trees and some limits of areas under lattice paths, A dissertation presented to the Faculty of the Graduate School of Arts and Sciences of Brandeis University, 2008.

[18] Peter Bala, Diagonals of triangles with generating function exp(t*F(x)), https://oeis.org/A112007/a112007.txt

[19] Wolfdieter Lang, On Generating functions of Diagonals Sequences of Sheffer and Riordan Number Triangles, arXiv:1708.01421.

[20] Dmitry V. Kruchinin, Vladimir V. Kruchinin, A generating function for the Euler numbers of the second kind and its application, arXiv:1802.09003.

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