GARSIDE COMBINATORICS FOR THOMPSON'S MONOID F^+ AND A HYBRID WITH THE BRAID MONOID B_∞^+

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Abstract. On the model of simple braids, defined to be the left divisors of Garside's elements Δ_n in the monoid B_∞^+, we investigate simple elements in Thompson's monoid F^+ and in a larger monoid H^+ that is a hybrid of B_∞^+ and F^+: in both cases, we count how many simple elements left divide the right lcm of the first $n-1$ atoms, and characterize their normal forms in terms of forbidden factors. In the case of H^+, a generalized Pascal triangle appears.

1. Introduction

Since the seminal work of F.A. Garside [18], as extended in [16] and [4], it is known that Artin's braid group B_n is a group of fractions for the monoid B_n^+ of positive n-strand braids and that the now called Garside element Δ_n plays a prominent role in the study of B_n^+. In particular, the divisors of Δ_n in B_n^+, called simple braids, form a family of $n!$ elements in one-to-one correspondence with the permutations of $\{1, ..., n\}$, leading to a remarkable combinatorics now at the heart of the algebraic study of B_n [1, 17], see [15, Chapter IX]. Subsequently, it was realised that such a situation can be found in many different contexts of groups and categories, always around a family of so-called simple elements resembling simple braids, and leading to various combinatorics, like, for instance, the dual Garside structure on B_n [3], whose combinatorics is that of noncrossing partitions.

Our aim in this paper is to investigate a Garside structure arising on Thompson's group F [28, 8] in connection with its submonoid F^+ generated by the standard (infinite) sequence of generators, corresponding to the presentation

$$F^+ := \langle \tau_1, \tau_2, ... \mid \tau_j\tau_i = \tau_i\tau_{j+1} \quad \text{for} \quad j \geqslant i+1\rangle^+. \tag{1.1}$$

To explain the similarity with braids and the natural questions in this non-finitely generated case, one should start from the infinite braid monoid

$$B_\infty^+ = \left\langle \sigma_1, \sigma_2, ... \;\middle|\; \begin{matrix} \sigma_j\sigma_i = \sigma_i\sigma_j & \text{for} & j \geqslant i+2 \\ \sigma_j\sigma_i\sigma_j = \sigma_i\sigma_j\sigma_i & \text{for} & j = i+1 \end{matrix} \right\rangle^+ : \tag{1.2}$$

in this case, Garside's braid Δ_n is the right lcm of the $n-1$ first atoms $\sigma_1, ..., \sigma_{n-1}$ of B_∞^+ (see Section 2.1 for a reminder about the terminology), and simple braids are those braids that left divide at least one element Δ_n in B_∞^+.

In the case of the monoid F^+, the atoms are the elements τ_i, and we shall see that there exists for each n a well defined element Δ_n that is, in F^+, the right lcm of the first $n-1$ atoms. Then we shall investigate the derived simple elements, namely the elements of F^+ that left divide at least one element Δ_n. The main results proved

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here are that, for every n, there exist 2^{n-1} simple elements left dividing Δ_n in F^+, in explicit one-to-one correspondence with the subsets of $\{1, ..., n-1\}$, and that simple elements form a Garside family in F^+[15, Def. III.1.31], thus guaranteeing the existence and properties of an associated greedy normal form in F^+. These results are established by combining the existence of a convergent rewrite system on F^+ and the reversing technique [10, 12] for analyzing the divisibility relations of a presented monoid.

The above results are technically easy, and we then switch to a combinatorially more involved situation related to another monoid H^+, which is a hybrid of the braid monoid B_∞^+ and the Thompson monoid F^+. Various hybrids of the groups B_∞ and F have already been considered, in particular the group $\widehat{BV}$ of [5, 6, 11], which is a group of fractions for a monoid, that is a Zappa-Szép product of F^+ and B_∞^+ and, therefore, inherits their Garside structures. Here we shall introduce and investigate a new hydrid, which is not a product but rather a mixture of the initial monoids F^+ and B_∞^+. Indeed, we consider

$$H^+ := \left\langle \theta_1, \theta_2, ... \;\middle|\; \begin{array}{ll} \theta_j\theta_i = \theta_i\theta_{j+1} & \text{for} \quad j \geqslant i+2 \\ \theta_j\theta_i\theta_j = \theta_i\theta_j\theta_{i+3} & \text{for} \quad j = i+1 \end{array} \right\rangle^+, \tag{1.3}$$

in which the length 2 relations are Thompson's relations as in (1.1), whereas the length 3 relations are directly reminiscent of braid relations of (1.2), but with a shift of one index. Here, we investigate the basic properties of the monoid H^+ and, specifically, the associated Garside combinatorics, if this makes sense. Actually, it does: we shall see that, for every n, the atoms $\theta_1, ..., \theta_{n-1}$ admit a right lcm, again denoted by Δ_n, so that it is natural to investigate simple elements, defined to be those that left divide some element Δ_n. The main results proved here are that, for every n, there exist $2 \cdot 3^{n-2}$ simple elements left dividing Δ_n in H^+, with an explicit description of a distinguished expression for each of them. As in the case of F^+, these results are established using a convergent rewrite system on H^+ and the reversing technique; the proofs are more difficult than for F^+ and some of them require delicate inductive arguments. We hope that the existence of this nontrivial combinatorics will draw some attention to the monoid H^+, and to the group H presented by (1.3), which remains essentially mysterious.

The paper is divided into four sections after this introduction. In Section 2, we investigate the monoid F^+ and the derived simple elements, providing a good warm-up for the sequel. In Section 3, we establish various general properties of the monoid H^+, in particular the fact that it admits cancellation on both sides. Next, in Section 4, we study the elements Δ_n of H^+ and count their left divisors by partitioning them into several families. Finally, in Section 5, we explicitly characterize the normal form (in the sense of some convergent rewrite system) of simple elements of H^+.

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2. Thompson's monoid F^+

Here we study the case of Thompson's monoid F^+, an easy first step. It is standard that (1.1) is a presentation of Thompson's group F, and, as the relations involve no inverse of the generators, it makes sense to introduce the associated monoid F^+ and to consider the associated Garside combinatorics, if it exists.

The section is divided into four parts. In Section 2.1, we recall the standard terminology for the divisibility relations in a monoid, extensively used throughout the text. Next, in Section 2.2, we define a convergent rewrite system that selects a distinguished expression for every element of F^+. In Section 2.3, we recall basic notions about word reversing, here in the new version of [14], and use them to show that F^+ is cancellative and admits right lcms (least common right multiples). Finally, in Section 2.4, we investigate the elements Δ_n and describe their left divisors explicitly.

2.1. **The divisibility relations of a monoid.** Let M be a monoid (possibly, in particular, a free one, *i.e.*, a monoid of words). For a, b in M, we say that *a left divides* b in M, or, equivalently, that b is a *right multiple* of a, written $a \preccurlyeq b$, if $ax = b$ holds for some x (of M). If M is left cancellative (meaning that $xa = xb$ implies $a = b$) and 1 is the only invertible element in M, the relation $\preccurlyeq$ is a partial ordering on M.

For a, b in M, we say that c is a *right lcm* (least common right multiple) of a and b if $a \preccurlyeq c$ and $b \preccurlyeq c$ hold, and the conjunction of $a \preccurlyeq x$ and $b \preccurlyeq x$ implies $c \preccurlyeq x$: in other words, c is a lowest upper bound of a and b with respect to $\preccurlyeq$.

The symmetric notions of a right divisor and a left multiple are defined similarly, replacing $ax = b$ with $xa = b$. Finally, we say that a is a *factor* of b if $xay = b$ holds for some x, y.

An element a of M is said to be an *atom* if it admits no decomposition $a = bc$ with $b \neq 1$ and $c \neq 1$.

2.2. **A normal form on F^+.** We begin our investigation of the monoid F^+. We recall that F^+ is defined by the presentation

$$F^+ := \langle \tau_1, \tau_2, ... \,|\, \tau_j \tau_i = \tau_i \tau_{j+1} \quad \text{for} \quad j \geqslant i+1 \rangle^+,$$

hereafter denoted by $\mathcal{P}_F$. We put $T := \{\tau_i \mid i \geqslant 1\}$, write T^* for the free monoid of all words in the alphabet T, and $\equiv$ for the congruence on T^* generated by the relations of $\mathcal{P}_F$. We use ε for the empty word. Our first tool for studying F^+ consists in defining a unique normal form using a rewrite system on T^*.

Lemma 2.1. *Let $\mathcal{E}_F$ be the rewrite system on T^* defined by the rules*

$$\tau_i \tau_{j+1} \to \tau_j \tau_i \quad \textit{for } i \geqslant 1 \textit{ and } j \geqslant i+1. \tag{2.1}$$

Then $\mathcal{E}_F$ is convergent.

Proof. As is standard, see for instance [26], we shall check that $\mathcal{E}_F$ is noetherian and locally confluent. We write $\Rightarrow$ for the one-step rewrite relation associated with the rules of (2.1), that is, for the family of all pairs

$$(w_1 \tau_i \tau_{j+1} w_2 \ , \ w_1 \tau_j \tau_i w_2) \quad \text{with } j \geqslant i+1,$$

and $\Rightarrow^*$ for the reflexive–transitive closure of $\Rightarrow$. For w in T^*, let $\rho(w)$ be the sum of the indices of the generators τ_i occurring in w. Then $w \Rightarrow w'$ implies $\rho(w) > \rho(w')$, and, therefore, there is no proper infinite sequence for $\Rightarrow$. So $\mathcal{E}_F$ is noetherian.

Next, assume $w \Rightarrow w'$ and $w \Rightarrow w''$. By definition, w' and w'' are obtained from w by replacing some length 2 factor $\tau_i \tau_{j+1}$ with the corresponding word $\tau_j \tau_i$. For local confluence, the case of disjoint factors is trivial, and the critical case of overlapping factors corresponds to $w = \tau_i\tau_{j+1}\tau_{k+2}$ with $j \geqslant i+1$ and $k \geqslant j+1$, leading to $w' = \tau_j\tau_i\tau_{k+2}$ and $w'' = \tau_i\tau_{k+1}\tau_{j+1}$. One then obtains

$$\begin{array}{ccccc} & \nearrow & \tau_j\tau_i\tau_{k+2} & \searrow^{2} & \\ \tau_i\tau_{j+1}\tau_{k+2} & & & & \tau_k\tau_j\tau_i\ , \\ & \searrow & \tau_i\tau_{k+1}\tau_{j+1} & \nearrow^{2} & \end{array} \tag{2.2}$$

It follows that $\mathcal{E}_F$ is locally confluent, hence convergent by Newman's diamond lemma [22]. $\square$

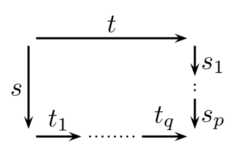
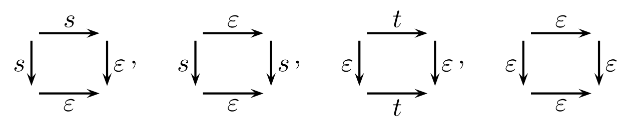
For every word w of T^*, we shall denote by $\mathsf{red}(w)$ the unique $\mathcal{E}_F$-reduced word w' satisfying $w \Rightarrow^* w'$. By definition, the words w and $\mathsf{red}(w)$ represent the same element of F^+, and $\mathsf{red}(w)$ is the unique $\mathcal{E}_F$-reduced word in the equivalence class of w in F^+. Thus, Lemma 2.1 implies

Proposition 2.2. *$\mathcal{E}_F$-reduced words provide a unique normal form for the elements of the monoid F^+.*

It directly follows from the definition that a word of T^* is $\mathcal{E}_F$-reduced if, and only if, it has no length 2 factor $\tau_i\tau_{j+1}$ with $j \geqslant i+1$, which implies that, for every n, the set of $\mathcal{E}_F$-reduced words lying in $\{\tau_1, ..., \tau_n\}^*$ is a regular language [17, 20].

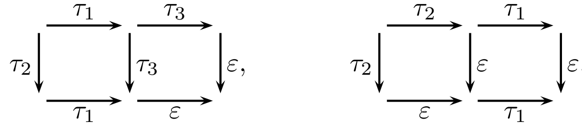
2.3. **Using word reversing.** The second method for investigating the monoid F^+ is word reversing [12], a distillation of an argument that ultimately stems from Garside's approach to braid monoids [18]. Here we shall describe reversing using the new formalism of [14], which is specially convenient in the current case (and in that of H^+ in Section 3.3). So we introduce reversing as a binary relation on pairs of words connected with a particular type of van Kampen diagram.

Definition 2.3. [14] A *reversing grid* for a monoid presentation $(\mathcal{S}, \mathcal{R})$, or $(\mathcal{S}, \mathcal{R})$-*grid*, is a rectangular diagram consisting of finitely many matching $\mathcal{S} \cup \{\varepsilon\}$-labeled pieces of the types

-  with $s, t, s_1, ..., s_p, t_1, ..., t_q$ in $\mathcal{S}$ and $st_1 \cdots t_q = ts_1 \cdots s_p$ a relation of $\mathcal{R}$,
-  with s, t in $\mathcal{S}$.

For u, v, u_1, v_1 in $\mathcal{S}^*$, we say that an $(\mathcal{S}, \mathcal{R})$-grid Γ goes *from* (u, v) *to* (u_1, v_1) if the labels of the left and top edges of Γ form the words u and v, respectively, whereas the labels of the right and bottom edges form the words u_1 and v_1. We write $(u, v) \curvearrowright_{\mathcal{R}} (u_1, v_1)$ if there exists a $(\mathcal{S}, \mathcal{R})$-grid from (u, v) to (u_1, v_1).

Example 2.4. Two typical $\mathcal{P}_F$-grids are

(2.3) 

witnessing for the relations $(\tau_2, \tau_1\tau_3) \curvearrowright (\varepsilon, \tau_1)$ and $(\tau_2, \tau_2\tau_1) \curvearrowright (\varepsilon, \tau_1)$, respectively—we omit the index in $\curvearrowright$ when there is no ambiguity. Note that, because all relations of $\mathcal{P}_F$ involve words of length 2, the pieces of the first type in Definition 2.3 are squares: the right and bottom edges each consist of one single $\mathcal{S}$-labeled arrow.

The following result is (a special case of a result) established in [14]. Below we say that a monoid presentation $(\mathcal{S}, \mathcal{R})$ is *homogeneous* if every relation in $\mathcal{R}$ has the form $w = w'$ with w, w' of the same length, and *right complemented* if it contains no relation $s... = s...$ and at most one relation $s... = t...$ for all $s \neq t$ in $\mathcal{S}$. On the other hand, two $(\mathcal{S}, \mathcal{R})$-grids Γ from (u, v) to (u_1, v_1) and Γ' from (u', v') to (u'_1, v'_1) are *equivalent* if we have $u' \equiv_\mathcal{R} u$, $v' \equiv_\mathcal{R} v$, $u'_1 \equiv_\mathcal{R} u_1$, and $v'_1 \equiv_\mathcal{R} v_1$, where $\equiv_\mathcal{R}$ is the congruence on $\mathcal{S}^*$ generated by $\mathcal{R}$—so that the monoid $\langle \mathcal{S} \,|\, \mathcal{R} \rangle^+$ is $\mathcal{S}^*/{\equiv_\mathcal{R}}$.

Lemma 2.5. [14, Propositions 1.12, 1.14, 1.16] *Assume that $(\mathcal{S}, \mathcal{R})$ is a homogeneous right complemented monoid presentation and, for every s in $\mathcal{S}$ and every relation $w = w'$ in $\mathcal{R}$,*

($\diamondsuit$) *for every grid from (s, w), there is an equivalent grid from (s, w'), and vice versa.*

(i) *Two words u, v of $\mathcal{S}^*$ represent the same element of the monoid $\langle \mathcal{S} \,|\, \mathcal{R} \rangle^+$ if, and only if, $(u, v) \curvearrowright (\varepsilon, \varepsilon)$ holds.*

(ii) *The monoid $\langle \mathcal{S} \,|\, \mathcal{R} \rangle^+$ is left cancellative.*

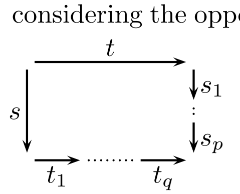
(iii) *Two elements a, b of $\langle \mathcal{S} \,|\, \mathcal{R} \rangle^+$ represented by u and v in $\mathcal{S}^*$ admit a common right multiple if, and only if, $(u, v) \curvearrowright_\mathcal{R} (u_1, v_1)$ holds for some u_1, v_1; in this case, the element represented by uv_1 is a right lcm of a and b. In the special case when, for all $s \neq t$ in $\mathcal{S}$, there exist s', t' in $\mathcal{S}$ such that $st' = ts'$ is a relation of $\mathcal{R}$, there always exist u_1, v_1 as above, and any two elements of $\langle \mathcal{S} \,|\, \mathcal{R} \rangle^+$ admit a right lcm.*

Applying Lemma 2.5, we deduce:

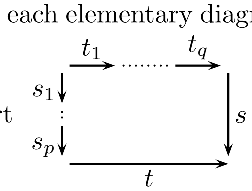
Proposition 2.6. *The monoid F^+ is left and right cancellative. Any two elements of F^+ admit a right lcm. Any two elements of F^+ that admit a common left multiple admit a left lcm.*

Proof. In view of applying Lemma 2.5, we observe that the presentation $\mathcal{P}_F$ is homogeneous (all relations are of the form $w = w'$ with w and w' of length two), right complemented with one relation $\tau_i... = \tau_j...$ for all i, j, and that Condition ($\diamondsuit$) holds for every τ_i and every relation of $\mathcal{P}_F$. To this end, we consider all pairs $(\tau_i, \tau_j\tau_{k+1})$ with $k \geqslant j+1$, and compare the reversing grids from $(\tau_i, \tau_j\tau_{k+1})$ and from $(\tau_i, \tau_k\tau_j)$: the two grids of Example 2.4 are typical, corresponding to $i = 2$, $j = 1$, and $k = 2$, and they are indeed equivalent, since both admit as output (ε, τ_1). The number of triples (i, j, k) to consider is infinite but only finitely many patterns may occur, according to the position of i with respect to j and k. We skip the details, which are fairly obvious. Having established ($\diamondsuit$), we deduce from Lemma 2.5(ii) that the monoid F^+ is left cancellative and from Lemma 2.5(iii) that any two elements of F^+ admit a right lcm.

To study left multiples, we observe that the presentation $\mathcal{P}_F$ is also left complemented (in the obvious sense), and consider the notion of a left reversing grid, which is symmetric to the above notion of a right reversing grid (which amounts to considering the opposed monoid). To this end, we replace each elementary diagram



of Definition 2.3 with its counterpart



for

$s_1 \cdots s_p t = t_1 \cdots t_q s$ in $\mathcal{R}$ and, similarly, replace $\begin{array}{ccc} & \xrightarrow{s} & \\ s\big\downarrow & & \big\downarrow\varepsilon \\ & \xrightarrow[\varepsilon]{} & \end{array}$ with $\begin{array}{ccc} & \xrightarrow{\varepsilon} & \\ \varepsilon\big\downarrow & & \big\downarrow s \\ & \xrightarrow[s]{} & \end{array}$. Then one easily checks that the counterpart of $(\diamondsuit)$ is satisfied and one deduces, by the counterpart of Lemma 2.5(ii), that F^+ is right cancellative. Finally, the counterpart of Lemma 2.5(iii) implies that any two elements of F^+ that admit a common left multiple admit a left lcm. However, two elements of F^+ need not always admit a common left multiple: there is no relation $...\tau_1 = ...\tau_2$ in $\mathcal{P}_F$, and, therefore, the counterpart of Lemma 2.5(iii) implies that τ_1 and τ_2 admit no common left-multiple in F^+. $\square$

It follows from Proposition 2.6 and Ore's classical theorem [23] that the monoid F^+ embeds in its enveloping group, which is the group presented by $\mathcal{P}_F$, namely Thompson's group F, and that the latter is a group of right fractions for F^+, that is, every element of F can be expressed as ab^{-1} with a, b in F^+. The expression is unique if, in addition, we require that the fraction be irreducible, meaning that a and b admit no common right divisor.

Remark 2.7. As explained in [13], there exists a (more redundant) positive presentation $\mathcal{P}_F^*$ of the group F in terms of a family of generators τ_s^* with s a finite sequence of 0s and 1s such that τ_i coincides with $\tau_{1^{i-1}}^*$ and that F is a group both of left and right fractions for the monoid F^{+*} defined by $\mathcal{P}_F^*$. The latter admits left and right lcms and is a sort of counterpart for the dual braid monoid of [3]. The main relations in $\mathcal{P}_F^*$ correspond to the MacLane–Stasheff pentagon.

2.4. **Garside combinatorics for F^+.** The monoid F^+ resembles the braid monoid B_∞^+ in that it is cancellative and admits right lcms and, therefore, it makes sense to consider the counterpart of the Garside elements Δ_n and their divisors.

As the presentation $\mathcal{P}_F$ is homogeneous, the atoms of F^+ are the elements τ_i with $i \geqslant 1$. So, exactly as in the case of B_∞^+, we shall consider the element Δ_n that is the right lcm of $\tau_1, ..., \tau_{n-1}$—we might use a different notation, for instance Δ_n^F, but there will be no risk of ambiguity here. We start from an explicit expression.

Definition 2.8. We put $\underline{\Delta}_1 := \varepsilon$, and, for $n \geqslant 2$, we put $\underline{\Delta}_n := \tau_1 \tau_3 \tau_5 \cdots \tau_{2n-3}$. We denote by Δ_n the class of $\underline{\Delta}_n$ in F^+.

It is clear that Δ_n left divides Δ_{n+1} for each n, and one inductively checks that the $\mathcal{E}_F$-normal form of Δ_n is $\tau_{n-1}\tau_{n-2} \cdots \tau_2 \tau_1$.

Lemma 2.9. *For every $n \geqslant 2$, the element Δ_n is the right lcm of $\tau_1, ..., \tau_{n-1}$. No element τ_i with $i \geqslant n$ left divides Δ_n.*

Proof. We prove using induction on $n \geqslant 2$ that Δ_n is the right lcm of $\tau_1, ..., \tau_{n-1}$. The result is trivial for $n = 2$. Assume $n \geqslant 3$. A direct computation gives

$$(\tau_{n-1}, \underline{\Delta}_{n-1}) \curvearrowright (\tau_{2n-3}, \underline{\Delta}_{n-1}).$$

By Lemma 2.5(iii), this implies that $\underline{\Delta}_n$ represents the right lcm of τ_{n-1} and Δ_{n-1}. By induction hypothesis, Δ_{n-1} is the right lcm of $\tau_1, ..., \tau_{n-2}$, so Δ_n is the right lcm of $\tau_1, ..., \tau_{n-1}$. On the other hand, for $i \geqslant n$, we find $(\tau_i, \underline{\Delta}_n) \curvearrowright (\tau_{i+n-1}, \underline{\Delta}_n)$, which shows that the right lcm of τ_i and Δ_n is not Δ_n, so τ_i does not left divide Δ_n. $\square$

The main notion in Garside theory [15] is the notion of a simple element, defined as the (left) divisors of the distinguished element(s) Δ.

Definition 2.10. An element a of F^+ is called *simple* if $a \preccurlyeq \Delta_n$ holds for some n.

Our aim is to understand the structure of simple elements of F^+, typically to characterize their normal forms. To this end, the key point will be the following exhaustive description of the expressions of Δ_n. Below, we write $\mathfrak{S}_n$ for the group of all permutations of $\{1, ..., n\}$, and s_i for the transposition $(i, i+1)$.

Lemma 2.11. *The expressions of Δ_n are the words w_f with f in $\mathfrak{S}_{n-1}$, where, for $p \leqslant n-1$, we put*

$$\widehat{f}(p) := \#\{i < f^{-1}(p) \mid f(i) > p\} \quad \textit{and} \quad \widetilde{f}(p) := 2f^{-1}(p) - 1 - \widehat{f}(p),$$

and let w_f be the word $\tau_{\widetilde{f}(1)}\tau_{\widetilde{f}(2)} \cdots \tau_{\widetilde{f}(n-1)}$.

Proof. We first establish the following technical result:

(2.4) If $f^{-1}(p) < f^{-1}(p+1)$ (*resp.*, $>$) holds, then so does $\widetilde{f}(p)+1 < \widetilde{f}(p+1)$ (*resp.*, $\widetilde{f}(p) > \widetilde{f}(p+1)$); applying a relation of $\mathcal{P}_F$ to w_f in position p yields the word $w_{s_p f}$.

So assume $f^{-1}(p+1) = f^{-1}(p) + m$ with $m \geqslant 1$. The definition gives

$$\widehat{f}(p+1) = \widehat{f}(p) + \#\{i \mid f^{-1}(p) < i < f^{-1}(p+1) \text{ and } f(i) > p+1\},$$

whence $\widehat{f}(p+1) \leqslant \widehat{f}(p) + m - 1$ and, for there, $\widetilde{f}(p+1) \geqslant \widetilde{f}(p) + 2$. Let $g := s_p f$. We find $g^{-1}(p) = f^{-1}(p+1)$, $g^{-1}(p+1) = f^{-1}(p)$, then $\widehat{g}(p) = \widehat{f}(p+1) + 1$ and $\widehat{g}(p+1) = \widehat{f}(p)$, because $f^{-1}(p)$ contributes to $\widehat{g}(p)$ but not to $\widehat{f}(p+1)$, and, finally, $\widetilde{g}(p) = \widetilde{f}(p+1) - 1$ and $\widetilde{g}(p+1) = \widetilde{f}(p)$, with $\widetilde{g}(q) = \widetilde{f}(q)$ for $q \neq p, p+1$. So w_g is the result of applying the rule $\tau_{\widetilde{f}(p)}\tau_{\widetilde{f}(p+1)} \to \tau_{\widetilde{f}(p+1)-1}\tau_{\widetilde{f}(p)}$ to w_f in position p.

On the other hand, for $f^{-1}(p) = f^{-1}(p+1)+m$ with $m \geqslant 1$, we find $\widehat{f}(p) \leqslant \widehat{f}(p+1)+m$, leading to $\widetilde{f}(p) \geqslant \widetilde{f}(p+1)+1$. For $g := s_p f$, we find now, $\widehat{g}(p) = \widehat{f}(p+1)$ and $\widehat{g}(p+1) = \widehat{f}(p)-1$, whence $\widetilde{g}(p) = \widetilde{f}(p+1)$ and $\widetilde{g}(p+1) = \widetilde{f}(p)+1$, with $\widetilde{g}(q) = \widetilde{f}(q)$ for $q \neq p, p+1$. So w_g is the result of applying the rule $\tau_{\widetilde{f}(p)}\tau_{\widetilde{f}(p+1)} \to \tau_{\widetilde{f}(p+1)}\tau_{\widetilde{f}(p)+1}$ to w_f in position p.

Now, (2.4) implies that the family $W := \{w_f \mid f \in \mathfrak{S}_{n-1}\}$ is closed under $\equiv$. As the transpositions s_i generate $\mathfrak{S}_{n-1}$, this family W is the $\equiv$-equivalence class of the word w_{id}, which, by definition, is $\underline{\Delta}_n$. □

From there, a complete description of simple elements of F^+ follows:

Proposition 2.12. *For every a in F^+, the following are equivalent:*

(i) *The element a is simple, i.e., a left divides some element Δ_n;*

(ii) *The element a is a factor of some element Δ_n;*

(iii) *The normal form of a has the form $\tau_{i_1} \cdots \tau_{i_\ell}$ with $i_1 > \cdots > i_\ell$.*

Moreover, a left divides Δ_n if, and only if, $\mathrm{NF}(a)$ is $\tau_{i_1} \cdots \tau_{i_\ell}$ with $n > i_1 > \cdots > i_\ell$.

Proof. By definition, (i) implies (ii). Next, assume that a is a factor of Δ_n, say $\Delta_n = a_1 a a_2$. Let w_1, w, w_2 be the normal forms of a_1, a, and a_2, respectively. Then $w_1 w w_2$ is an expression of Δ_n, so, by Lemma 2.11(ii), it is a word w_f for some permutation f. Moreover, because w is $\mathcal{E}_F$-reduced, no rule of $\mathcal{E}_F$ may apply to it: by (2.4), this implies that the indices of the generators τ_i in w make a decreasing sequence. So (ii) implies (iii).

Assume now $w = \tau_{i_1} \cdots \tau_{i_\ell}$ with $i_1 > \cdots > i_\ell$. By inserting intermediate letters when $i_p \geqslant i_{p+1} + 2$, we obtain a word w' that is the normal form of Δ_{i_1+1}. Then, repeatedly applying to w' some relations $\tau_i\tau_j \to \tau_j\tau_{i+1}$, we push the new letters to the right starting with the last one and finishing with the first one. In this way, one obtains a new expression of Δ_{i_1+1} that begins with w. So w is the normal form of a prefix of Δ_{i_1+1}, hence of a simple element. Si (iii) implies (i).

For the last sentence, if a left divides Δ_n, then so does the first generator of $\mathrm{NF}(a)$: by Lemma 2.9, the latter cannot be τ_i with $i \geqslant n$. Conversely, the above proof of (iii) $\Rightarrow$ (i) shows that $\tau_{i_1} \cdots \tau_{i_\ell}$ left divides Δ_{i_1+1}, hence Δ_n for $n > i_1$. □

Corollary 2.13. (i) *For every n, the number of left divisors of Δ_n in F^+ is 2^{n-1}.*
(ii) *Simple elements of F^+ make a Garside family in F^+.*

Proof. (i) By the last statement in Proposition 2.12, mapping a subset of $\{1, ..., n-1\}$ to the decreasing enumeration of the corresponding elements τ_i establishes a one-to-one correspondence between $\mathfrak{P}(\{1, ..., n-1\})$ and the left divisors of Δ_n.

(ii) By definition, the family of simple elements in F^+ is closed under right lcm: the conjunction of $a \preccurlyeq \Delta_n$ and $b \preccurlyeq \Delta_p$ implies that the right lcm of a and b left divides $\Delta_{\max(n,p)}$. On the other hand, a right divisor of a simple element must be a factor of some Δ_n, hence, by Proposition 2.12, it is simple. By [15, Coro. IV.2.29], this implies that simple elements form a Garside family. □

As simple elements form a Garside family, every element of F^+ admits a unique *greedy decomposition* in terms of simple elements, namely a decomposition $a_1, ..., a_p$ with $a_1, ..., a_p$ simple, $a_p \neq 1$, and, for each i, the entry a_i is the maximal simple left divisor of $a_i \cdots a_p$, [15, Prop. IV.1.20]. In the current case, the greedy decomposition is directly connected with the $\mathcal{E}_F$-normal form: $\mathrm{NF}(a_1), ..., \mathrm{NF}(a_p)$ are the maximal decreasing factors of $\mathrm{NF}(a)$. For instance, for $\mathrm{NF}(a) = \tau_4\tau_3\tau_2\tau_3\tau_1\tau_1\tau_2$, the greedy decomposition has four entries, namely $\tau_4\tau_3\tau_2$, $\tau_3\tau_1$, τ_1, and τ_2.

From there, all results involving greedy decompositions are valid in F^+. However, this Garside structure of F^+ is mostly trivial, exactly parallel to the case of the free commutative monoid $\mathbb{Z}_{\geqslant 0}^{(\infty)}$, where simple elements also correspond to finite subsets of generators. In fact, the relations of $\mathcal{P}_F$ are in essence a shifted version of the commutation rules of a free commutative monoid.

3. The monoid H^+

The previous results are elementary and easy, and we now switch to a combinatorially more intricate and interesting situation, connected with the new hybrid H^+ between Thompson's monoid F^+ and Artin's braid monoid B_∞^+ mentioned in the introduction. Our aim will be to develop the same analysis as in the case of F^+, namely understanding the structure of simple elements, defined as the left divisors of the right lcms of atoms. To this end, we shall follow the same scheme as in Section 2 and use both a normal form associated with a rewrite system (Section 3.2) and the reversing transformation associated with the presentation (Section 3.3).

3.1. **Presentation and first properties.**

We recall that H^+ is the monoid defined by the explicit presentation called (1.3) in the introduction

$$H^+ := \left\langle \theta_1, \theta_2, ... \;\middle|\; \begin{array}{ll} \theta_j\theta_i = \theta_i\theta_{j+1} & \text{for} \quad j \geqslant i+2 \\ \theta_j\theta_i\theta_j = \theta_i\theta_j\theta_{i+3} & \text{for} \quad j = i+1 \end{array} \right\rangle^+,$$

hereafter denoted by $\mathcal{P}_H$. We put $\Theta := \{\theta_i \mid i \geqslant 1\}$, and write $\equiv$ for the congruence on Θ^* generated by the relations of $\mathcal{P}_H$. For w a word of Θ^*, we write $[w]$ for the $\equiv$-class of w. The relations of $\mathcal{P}_H$ should appear as a mixture of the Thompson relations (as for length 2 relations), and of braid relations (as for length 3 relations). We immediately see that $\mathcal{P}_H$ is a homogeneous presentation, and we can refer without ambiguity to the length $|a|$ of an element a of H^+, defined to be the common length of all words of Θ^* that represent a. We also observe that the relations are invariant under shifting the indices of the θ_is by $+1$, implying that mapping θ_i to θ_{i+1} for each i induces a well defined endomorphism of H^+.

Unlike the case of B_∞^+, the family of generators occurring in a word is not invariant under $\equiv$: for instance, $\theta_3\theta_1$ is equal to $\theta_1\theta_4$. However, we can easily construct an upper bound on the indices of the generators possibly occurring in the expressions of an element.

Lemma 3.1. *Define the* ceiling $\lceil w\rceil$ *of a nonempty word* $w = \theta_{i_1}\cdots\theta_{i_\ell}$ *of* Θ^* *by*

$$\lceil w\rceil := \max\{i_p + \ell - p \mid p = 1, ..., \ell\}. \tag{3.1}$$

Then $\lceil w\rceil$ *is invariant under* $\equiv$.

Proof. It suffices to consider the case of two words w, w' deduced from one another by applying one relation of $\mathcal{P}_H$. For $w = u\theta_j\theta_i v$ and $w' = u\theta_i\theta_{j+1}v$, with $j \geqslant i+2$, one finds $\lceil w\rceil = \max(\lceil u\rceil + |v| + 2, j+1+|v|, \lceil v\rceil) = \lceil w'\rceil$. Similarly, for $w = u\theta_i\theta_{i+1}\theta_{i+3}v$ and $w' = u\theta_{i+1}\theta_i\theta_{i+1}v$, one obtains $\lceil w\rceil = \max(\lceil u\rceil + |v| + 3, i + 3 + |v|, \lceil v\rceil) = \lceil w'\rceil$. □

For a in H^+, we write $\lceil a\rceil$ for the common value of $\lceil w\rceil$ for w representing a. A direct application is the following a priori nontrivial result:

Proposition 3.2. *The word problem for* $\mathcal{P}_H$ *is decidable.*

Proof. For every word w in Θ^*, the $\equiv$-class of w is finite: indeed, $w' \equiv w$ implies both $|w'| = |w|$ and $\lceil w'\rceil = \lceil w\rceil$, and the number of words w' satisfying these conditions is bounded above by $\lceil w\rceil^{|w|}$. Therefore, starting from two words w, w', one can decide whether $w' \equiv w$ holds by saturating $\{w\}$ with respect to the relations of $\mathcal{P}_H$, eventually obtaining in finitely many steps an exhaustive enumeration of the $\equiv$-class of w. Then one compares w' with the elements of the list so constructed. □

Another property that directly follows from the presentation is the fact that the monoid F^+ is a quotient of H^+:

Proposition 3.3. *The map* $\pi : \theta_i \mapsto \tau_i$ *induces a surjective homomorphism from the monoid* H^+ *onto the Thompson monoid* F^+.

Proof. Let π^* be the extension of π into a homomorphism from the free monoid Θ^* to the monoid F^+. We claim that $w \equiv w'$ implies $\pi^*(w) = \pi^*(w')$. It is enough to check this when $w{=}w'$ is a relation of $\mathcal{P}_H$. The case of length 2 relations is trivial, as the latter are relations of $\mathcal{P}_F$. For length 3 relations, we find in F^+

$$\pi^*(\theta_{i+1}\theta_i\theta_{i+1}) = \tau_{i+1}\tau_i\tau_{i+1} = \tau_i\tau_{i+2}\tau_{i+1} = \tau_i\tau_{i+1}\tau_{i+3} = \pi^*(\theta_i\theta_{i+1}\theta_{i+3}).$$

So π^* induces a homomorphism from H^+ to F^+. The latter is surjective since each generator τ_i lies in the image. □

The projection π from H^+ to F^+ provided by Proposition 3.3 is not injective: $\theta_2\theta_1$ and $\theta_1\theta_3$ are distinct in H^+ since no relation of $\mathcal{P}_H$ applies to the corresponding words, but they both project to $\tau_2\tau_1$ in F^+.

3.2. **A normal form on H^+.** Like in the case of F^+, our first method for investigating the monoid H^+ is to construct a normal form using a rewrite system.

Lemma 3.4. *Let $\mathcal{E}_H$ be the rewrite system on Θ^* defined by the rules*

$$\theta_i\theta_{j+1} \to \theta_j\theta_i \quad \text{for } i \geqslant 1 \text{ and } j \geqslant i+2, \tag{3.2}$$

$$\theta_i\theta_{i+1}\theta_{i+3} \to \theta_{i+1}\theta_i\theta_{i+1} \quad \text{for } i \geqslant 1. \tag{3.3}$$

Then $\mathcal{E}_H$ is convergent.

Proof. As in the case of $\mathcal{E}_F$, we show that $\mathcal{E}_H$ is noetherian and locally confluent, and appeal to Newman's diamond lemma. Let π denote the homomorphism from Θ^* on T^* that maps θ_i to τ_i for every i. Then, for every w in Θ^* and every integer m,

$$w \Rightarrow_H^m w' \quad \text{implies} \quad \pi(w) \Rightarrow^p \pi(w') \quad \text{for some } p \text{ satisfying } m \leqslant p \leqslant 2m. \tag{3.4}$$

Indeed, up to applying π, (3.2) is a rule of $\mathcal{E}_F$, whereas, for (3.3), we find

$$\pi(\theta_i\theta_{i+1}\theta_{i+3}) = \tau_i\tau_{i+1}\tau_{i+3} \Rightarrow \tau_i\tau_{i+2}\tau_{i+1} \Rightarrow \tau_{i+1}\tau_i\tau_{i+1} = \pi(\theta_{i+1}\theta_i\theta_{i+1}).$$

Then an infinite nontrivial sequence of $\mathcal{E}_H$-reductions would project to an infinite nontrivial sequence of $\mathcal{E}_F$-reductions, so the noetherianity of $\mathcal{E}_F$ implies that of $\mathcal{E}_H$.

We now check local confluence. As in Lemma 2.1, it is sufficient to consider the critical cases where two rules overlap. As there are two types of rules, four patterns are possible. Twice using (3.2) has already been seen (up to a change of letters) in (2.2). The remaining three cases then correspond to the confluence diagrams

$$\begin{array}{ccccc} & \nearrow_H & \theta_{i+1}\theta_i\theta_{i+1}\theta_{i+4}\theta_{i+6} & \searrow_H^5 & \\ \theta_i\theta_{i+1}\theta_{i+3}\theta_{i+4}\theta_{i+6} & & & & \theta_{i+2}\theta_{i+1}\theta_{i+2}\theta_i\theta_{i+1}\ , \\ & \searrow_H & \theta_i\theta_{i+1}\theta_{i+4}\theta_{i+3}\theta_{i+4} & \nearrow_H^5 & \end{array} \tag{3.5}$$

$$\begin{array}{ccccc} & \nearrow_H & \theta_j\theta_i\theta_{j+2}\theta_{j+4} & \searrow_H^3 & \\ \theta_i\theta_{j+1}\theta_{j+2}\theta_{j+4} & & & & \theta_{j+1}\theta_j\theta_{j+1}\theta_i \text{ with } j \geqslant i+2, \\ & \searrow_H & \theta_i\theta_{j+2}\theta_{j+1}\theta_{j+2} & \nearrow_H^3 & \end{array} \tag{3.6}$$

$$\begin{array}{ccccc} & \nearrow_H & \theta_{i+1}\theta_i\theta_{i+1}\theta_{j+1} & \searrow_H^3 & \\ \theta_i\theta_{i+1}\theta_{i+3}\theta_{j+1} & & & & \theta_{j-2}\theta_{i+1}\theta_i\theta_{i+1} \text{ with } j \geqslant i+5, \\ & \searrow_H & \theta_i\theta_{i+1}\theta_j\theta_{i+3} & \nearrow_H^3 & \end{array} \tag{3.7}$$

which complete the verification. □

We deduce:

Proposition 3.5. *$\mathcal{E}_H$-reduced words provide a unique normal form for the elements of the monoid H^+.*

For a in H^+, we shall denote by $\mathrm{NF}(a)$ the unique $\mathcal{E}_H$-reduced word that represents a. For w in Θ^*, we denote by $\mathsf{red}(w)$ the unique $\mathcal{E}_H$-reduced word to which w is $\mathcal{E}_H$-reducible. As in Section 2.2, we note that a Θ-word is $\mathcal{E}_H$-reduced if, and only if, it contains no factor in a list of obstructions, here

$$\mathcal{O} := \{\theta_i\theta_j \mid j \geqslant i+3\} \cup \{\theta_i\theta_{i+1}\theta_{i+3} \mid i \geqslant 1\}. \tag{3.8}$$

This implies that, for every n, the family of all $\mathcal{E}_F$-reduced words lying in $\{\theta_1, ..., \theta_n\}^*$ is a regular language. The above characterization of $\mathcal{E}_H$-reduced words implies the following useful properties:

Corollary 3.6. (i) *Every factor of an $\mathcal{E}_H$-reduced word is $\mathcal{E}_H$-reduced.*
(ii) *A word w is $\mathcal{E}_H$-reduced if, and only if, all length 3 factors of w are.*
(iii) *If uv and vw are $\mathcal{E}_H$-reduced, then uvw is $\mathcal{E}_H$-reduced, except for:*
- $|v| = 0$, $u = u'\theta_i$, *and* $w = \theta_j w'$ *with* $j \geqslant i + 3$,
- $|v| = 0$, $u = u'\theta_i$, *and* $w = \theta_{i+1}\theta_{i+3}w'$,
- $|v| = 0$, $u = u'\theta_i\theta_{i+1}$, *and* $w = \theta_{i+3}w'$,
- $|v| = 1$, $u = u'\theta_i$, $v = \theta_{i+1}$, *and* $w = \theta_{i+3}w'$.

Proof. Points (i) and (ii) directly follow from the characterization of $\mathcal{E}_H$-reduced words, and so does the fact that uvw is not $\mathcal{E}_H$-reduced if one is in one of the four listed cases. The point is, assuming that uvw is not $\mathcal{E}_H$-reduced, to prove that one is necessarily in one of the listed cases. Now the assumption that uvw is not $\mathcal{E}_H$-reduced means that at least one rule of $\mathcal{E}_H$ can be applied, and, owing to (ii), the assumption about uv and vw requires that v has length at most one. Considering the various possibilities yields the four identified cases. □

The next result shows that, if w is an $\mathcal{E}_H$-reduced word, then the $\mathcal{E}_H$-reduced form of $w\theta_i$ is obtained by pushing θ_i to the left as much as possible:.

Lemma 3.7. *If w is $\mathcal{E}_H$-reduced, then, for every i, we have $\mathsf{red}(w\theta_i) = w_1\theta_{i-|w_2|}w_2$ for some decomposition (w_1, w_2) of w.*

Proof. We use induction on the length of w. For w empty, the result is obvious. So assume $|w| \geqslant 1$. Then we have $w = w'\theta_k$ for some k. As $w'\theta_k$ is $\mathcal{E}_H$-reduced, the possible rewritings of $w'\theta_k\theta_i$ necessarily involve the final letter θ_i.

For $i \leqslant k + 1$, no rule applies to $w'\theta_k\theta_i$, so $w\theta_i$ is $\mathcal{E}_H$-reduced, and the result is true for $(w_1, w_2) := (w, \varepsilon)$.

For $i \geqslant k + 3$, we have $w'\theta_k\theta_i \Rightarrow_H w'\theta_{i-1}\theta_k$. Now w' is $\mathcal{E}_H$-reduced and shorter than w. Hence, by induction hypothesis, there exists a decomposition $w' = w'_1w'_2$ satisfying $\mathsf{red}(w'\theta_{i-1}) = w'_1\theta_jw'_2$ with $j = i - 1 - |w'_2|$. Now $w'_1\theta_jw'_2$ is $\mathcal{E}_H$-reduced, and so is $w'_2\theta_k$ as a factor of the $\mathcal{E}_H$-reduced word $w'_1w'_2\theta_k$. By Corollary 3.6, $w'_1\theta_jw'_2\theta_k$ is $\mathcal{E}_H$-reduced, as we have $j \geqslant k + 2 - |w'_2|$. Hence, $\mathsf{red}(w\theta_i)$ is $w'_1\theta_jw'_2\theta_k$, and the result is true with $(w_1, w_2) := (w'_1, w'_2\theta_k)$.

There remains the case $i = k + 2$. For $w' = \varepsilon$, we have $w\theta_i = \theta_{i-2}\theta_i$, which is $\mathcal{E}_H$-reduced, and the result is true for $(w_1, w_2) := (w, \varepsilon)$. Otherwise, we write $w' = w''\theta_\ell$. For $\ell \neq i - 3$, we find $w\theta_i = w''\theta_\ell\theta_{i-2}\theta_i$, which is $\mathcal{E}_H$-reduced as, by assumption, $w''\theta_\ell\theta_{i-2}$ is $\mathcal{E}_H$-reduced. So the result is true for $(w_1, w_2) := (w, \varepsilon)$.

Finally, for $\ell = i - 3$, we have $w''\theta_{i-3}\theta_{i-2}\theta_i \equiv w''\theta_{i-2}\theta_{i-3}\theta_{i-2}$. As w'' is $\mathcal{E}_H$-reduced and shorter than w, the induction hypothesis gives a decomposition $w'' = w''_1w''_2$ satisfying $\mathsf{red}(w''\theta_{i-2}) = w''_1\theta_jw''_2$ with $j = i - 2 - |w''_2|$. It remains to show that $w''_1\theta_jw''_2\theta_{i-3}\theta_{i-2}$ is $\mathcal{E}_H$-reduced. Now $w''_1\theta_jw''_2$ is $\mathcal{E}_H$-reduced, and so is $w''_2\theta_{i-3}\theta_{i-2}$, as a factor of the $\mathcal{E}_H$-reduced word $w''_1w''_2\theta_{i-3}\theta_{i-2}$. By Corollary 3.6, $w''_1\theta_jw''_2\theta_{i-3}\theta_{i-2}$ is $\mathcal{E}_H$-reduced, and $j = i - 2 - |w''_2|$ holds. Therefore, $\mathsf{red}(w\theta_i)$ is $w''_1\theta_jw''_2\theta_{i-3}\theta_{i-2}$, and the result is true for $(w_1, w_2) := (w''_1, w''_2\theta_{i-3}\theta_{i-2})$. □

We shall now apply the normal form provided by $\mathcal{E}_H$ to studying right cancellativity in H^+. At this point, we shall not obtain a complete answer, but only a (surprising) connection between left and right cancellativity.

Proposition 3.8. *If H^+ is left cancellative, then it is right cancellative as well.*

Proof. We assume that H^+ is left cancellative, and aim at proving that any equality $a\theta_i = b\theta_i$ implies $a = b$. So assume $a\theta_i = b\theta_i$. Let $u := \mathrm{NF}(a)$ and $v := \mathrm{NF}(b)$. By Lemma 3.7, there exist u_1, u_2, v_1, v_2 satisfying

$$u = u_1u_2 \quad \text{and} \quad \mathsf{red}(u\theta_i) = u_1\theta_j u_2 \quad \text{with } j = i - |u_2|, \tag{3.9}$$

$$v = v_1v_2 \quad \text{and} \quad \mathsf{red}(v\theta_i) = v_1\theta_k v_2 \quad \text{with } k = i - |v_2|. \tag{3.10}$$

By assumption, we have $a\theta_i = b\theta_i$, hence $\mathsf{red}(u\theta_i) = \mathsf{red}(v\theta_i)$, and, from there, $u_1\theta_j u_2 = v_1\theta_k v_2$. We consider the various possible cases.

Assume first $j \neq k$, say $j < k$. By (3.9) and (3.10), we obtain

$$|u_2| = i - j > i - k = |v_2|, \quad \text{whence} \quad |u_1| < |v_1|.$$

So u_1 is a proper prefix of v_1, and v_2 is a proper suffix of u_2. As u_1 is a proper prefix of v_1, the word $u_1\theta_j$ is a prefix of v_1, and there exists w satisfying $v_1 = u_1\theta_j w$. We find $u_1\theta_j u_2 = u_1\theta_j w\theta_k v_2$, hence $u_2 = w\theta_k v_2$. Therefore, we have

$$u = u_1 w\theta_k v_2, \quad v = u_1\theta_j w v_2, \quad \text{and} \quad \mathsf{red}(u\theta_i) = \mathsf{red}(v\theta_i) = u_1\theta_j w\theta_k v_2.$$

The equality $\mathsf{red}(v\theta_i) = u_1\theta_j w\theta_k v_2$ implies $v\theta_i \equiv u_1\theta_j w\theta_k v_2$, that is, $u_1\theta_j w v_2\theta_i \equiv u_1\theta_j w\theta_k v_2$. As H^+ is left cancellative, left cancelling $u_1\theta_j w$ yields $v_2\theta_i \equiv \theta_k v_2$. By assumption, u_1 is $\mathcal{E}_H$-reduced, hence, by Corollary 3.6(i), so is its suffix $\theta_k v_2$. The equivalence $v_2\theta_i \equiv \theta_k v_2$ implies $\mathsf{red}(v_2\theta_i) = \theta_k v_2$, hence $v_2\theta_i \Rightarrow_H^* \theta_k v_2$, whence $u\theta_i = u_1 w\theta_k v_2\theta_i \Rightarrow_H^* u_1 w\theta_k\theta_k v_2$. Now, as a prefix of u, the word $u_1 w\theta_k$ is $\mathcal{E}_H$-reduced, whereas $\theta_k\theta_k$ is $\mathcal{E}_H$-reduced by definition. By Corollary 3.6(iii), $u_1 w\theta_k\theta_k$ is reduced. On the other hand, as a suffix of u_1, the word $\theta_k v_2$ is $\mathcal{E}_H$-reduced, so, by Corollary 3.6(iii) again, $\theta_k\theta_k v_2$ is $\mathcal{E}_H$-reduced. Finally, $u_1 w\theta_k\theta_k$ and $\theta_k\theta_k w$ are $\mathcal{E}_H$-reduced, hence, by Corollary 3.6(ii), $u_1 w\theta_k\theta_k v_2$ is $\mathcal{E}_H$-reduced. So the two words $u_1\theta_j w\theta_k v_2$ and $u_1 w\theta_k\theta_k v_2$ are $\mathcal{E}_H$-reduced, both equivalent to $u\theta_i$. Hence they must coincide: $u_1 w\theta_k\theta_k v_2 = u_1\theta_j w\theta_k v_2$ holds. Deleting the prefix u_1 and the suffix $\theta_k v_2$ on both sides, we deduce

$$w\theta_k = \theta_j w. \tag{3.11}$$

An induction on $|w|$ shows that the word equality (not equivalence) (3.11) is possible only for $j = k$: for $|w| \geqslant 2$, a word w satisfying (3.11) must begin with θ_j and finish with θ_k, leading to $w = \theta_j w'\theta_k$ with w' satisfying $w'\theta_k = \theta_j w'$. But this contradicts the assumption $j \neq k$.

So, the only possibility is $j = k$. Then (3.9) and (3.10) imply $|u_2| = |v_2|$, whence $u_2 = v_2$, and, from there, $u_1 = v_1$ and $u = v$, implying $a = b$. □

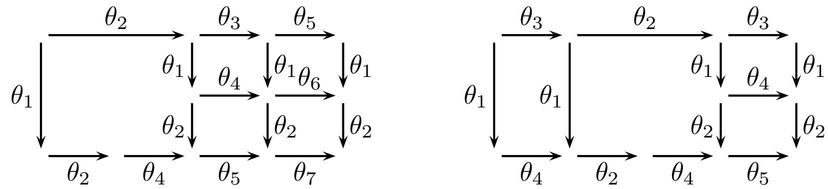
Another application of the normal form in H^+ is a solution for the word problem of the presentation $\mathcal{P}_H$ that is much more efficient than the “stupid” solution of Proposition 3.2: two words w, w' represent the same element in H^+ if, and only if, $\mathsf{red}(w)$ and $\mathsf{red}(w')$ coincide. It is easy to see that, from a word of length ℓ, at most $\binom{\ell}{2}$ rules can be applied, leading to a solution for the word problem whose overall complexity is quadratic in ℓ. We do not go into details here.

3.3. **Reversing for H^+.** Continuing as in Section 2.3 for F^+, we now investigate the (right) reversing relation associated with the presentation $\mathcal{P}_H$ of H^+, in view of possibly establishing that it is left cancellative and admits right lcms.

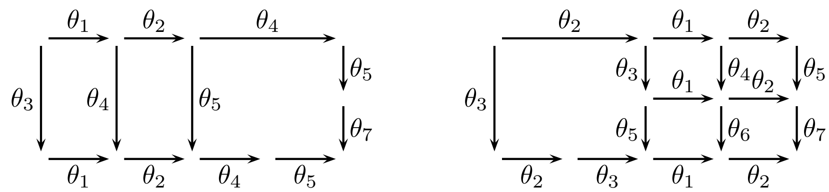
For applying Lemma 2.5, the first step is to check Condition ($\diamondsuit$).

Lemma 3.9. *For every generator θ_i and for every relation $w = w'$ of $\mathcal{P}_H$, Condition ($\diamondsuit$) is satisfied: for every $\mathcal{P}_H$-grid from (θ_i, w), there is an equivalent grid from (θ_i, w'), and vice versa.*

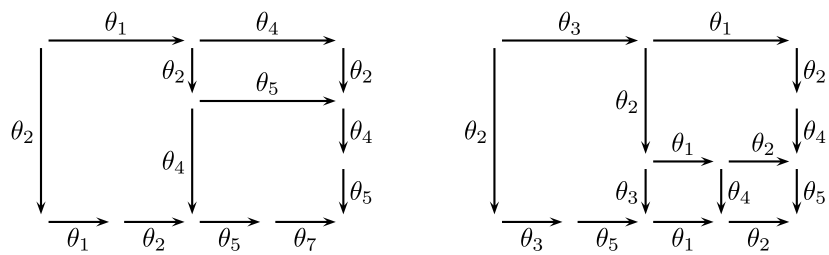
Proof. Because there exists exactly one relation of the form $\theta_i ... = \theta_j ...$ in $\mathcal{P}_H$ for all i, j, a $\mathcal{P}_H$-grid is unique when it exists, so we only have to check that the involved grids either exist and are equivalent, or they do not exist. As in the case of F^+, there are infinitely many generators and relations, but only finitely many patterns occur, according to the relative positions of the indices of the involved generators. In the case of θ_i and the relation $\theta_j \theta_{j+1} \theta_{j+3} = \theta_{j+1} \theta_j \theta_{j+1}$, the only cases that do not just result in shifting the indices are $j = i+1$ and $j = i-2$, for which we find (for readability, we draw the diagrams for $i = 1, j = 2$, and for $i = 3, j = 1$)

(3.12)


and we check $\theta_2\theta_4\theta_5\theta_7 \equiv \theta_2\theta_5\theta_4\theta_5 \equiv \theta_4\theta_2\theta_4\theta_5$, so the grids are equivalent, and

(3.13)


and we check $\theta_1\theta_2\theta_4\theta_5 \equiv \theta_2\theta_1\theta_2\theta_5 \equiv \theta_2\theta_1\theta_4\theta_2 \equiv \theta_2\theta_3\theta_1\theta_2$, so the grids are equivalent. Similarly, in the case of θ_i and a relation $\theta_j \theta_{k+1} = \theta_k \theta_j$ with $k \geqslant j+2$, the only nontrivial case is for $i = j+1$ and $k = j+2$, where we find (here for $i = 2$, $j = 1$, and $k = 3$)

(3.14)


and we check $\theta_1\theta_2\theta_5\theta_7 \equiv \theta_1\theta_4\theta_2\theta_7 \equiv \theta_3\theta_1\theta_2\theta_7 \equiv \theta_3\theta_1\theta_6\theta_2 \equiv \theta_3\theta_5\theta_1\theta_2$, so the grids are equivalent. □

Lemma 3.9 shows that the presentation $\mathcal{P}_H$, which is homogeneous, is eligible for Lemma 2.5. So, in particular, for all u, v in Θ^*, we have the equivalence

$$u \equiv v \quad \Longleftrightarrow \quad (u, v) \curvearrowright (\varepsilon, \varepsilon), \tag{3.15}$$

that is, u and v represent the same element of H^+ if, and only if there exists a $\mathcal{P}_H$-grid from (u, v) to $(\varepsilon, \varepsilon)$. As an application, we deduce

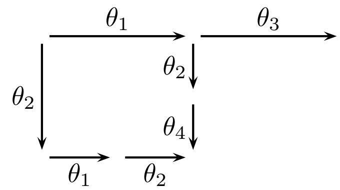
Proposition 3.10. *The monoid H^+ is left and right cancellative.*

Proof. Lemma 2.5(ii) implies that H^+ is left cancellative. By Proposition 3.8, this implies that H^+ is right cancellative as well. □

As for common right multiples, owing to the fact that a $\mathcal{P}_H$-grid with a given source is unique when it exists, Lemma 2.5(iii) directly implies:

Proposition 3.11. *Two elements* $[u], [v]$ *of* H^+ *admit a common right multiple in* H^+ *if, and only if, there exists a* $\mathcal{P}_H$*-grid from* (u, v)*; in this case,* $[u]$ *and* $[v]$ *admit a right lcm, represented by* uv_1 *with* (u_1, v_1) *the output of the grid from* (u, v)*.*

Proposition 3.11 is optimal: there exist elements of H^+ without a common right multiple, typically θ_2 and $\theta_1\theta_3$. Indeed, if we try to construct a $\mathcal{P}_H$-grid from $(\theta_2, \theta_1\theta_3)$, we must start with



and the process cannot terminate, since the pending pattern $(\theta_2\theta_4, \theta_3)$ is, up to a symmetry, the image of $(\theta_2, \theta_1\theta_3)$ under shifting the indices. By Proposition 3.11, this is enough to conclude that θ_2 and $\theta_1\theta_3$ admit no common right multiple in H^+.

Remark 3.12. In the above example, the non-existence of a common right multiple is easily established, as constructing a $\mathcal{P}_H$-grid enters an explicit non-terminating loop. The general question of the existence of a $\mathcal{P}_H$-grid from a given pair of words is a priori difficult, and it is not clear whether it is algorithmically decidable. In fact, it is, but this is nontrivial. The method consists in identifying an explicit family of words Θ' that is closed under reversing, in the sense that, if u and v belong to Θ' and $(u, v) \curvearrowright (u_1, v_1)$ holds, then u_1 and v_1 lie in Θ'. Then, one easily shows that, if the existence of common multiples can be decided for pairs of words of Θ', it can decided for arbitrary pairs of words. So the question is to find a convenient family Θ' and analyze the existence of $\mathcal{P}_H$-grids for words of Θ'. In the case of $\mathcal{P}_H$, this program is successfully completed in [27] for $\Theta' := \Theta_1 \cup \Theta_2 \cup \{\varepsilon\}$ with

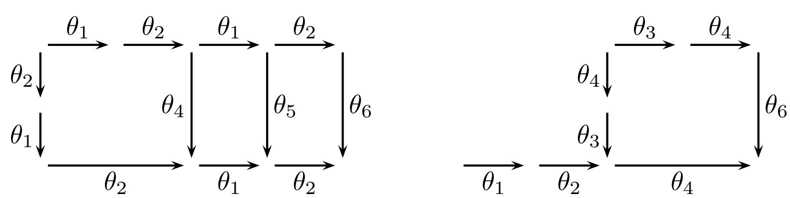
$$\Theta_1 := \{\theta_i\theta_{i+2}\theta_{i+4}\cdots\theta_{i+2k} \mid i \geqslant 1, k \geqslant 0\}, \tag{3.16}$$

$$\Theta_2 := \{\theta_i\theta_{i+2}\theta_{i+4}\cdots\theta_{i+2k}\theta_{i+2k+1} \mid i \geqslant 1, k \geqslant 0\}, \tag{3.17}$$

Therefore the existence of common right multiples in H^+ is decidable.

Trying to apply the same approach to studying right cancellativity and left multiples in H^+ fails. Indeed, one easily checks that the left diagram below is a legitimate "left $\mathcal{P}_H$-grid" from $(\theta_6, \theta_2\theta_1\theta_2)$, whereas the right diagram shows that constructing an equivalent left $\mathcal{P}_H$-grid from $(\theta_6, \theta_1\theta_2\theta_4)$ fails, since there is no relation $...\theta_2 = ...\theta_3$ in $\mathcal{P}_H$:

(3.18)



So the counterpart of Condition ($\diamondsuit$) fails for θ_6 and the relation $\theta_1\theta_2\theta_4 = \theta_2\theta_1\theta_2$, and the counterpart of Lemma 2.5 cannot be appealed to. This says nothing about H^+, but just leaves the questions open. As for right cancellativity, the question was solved using the normal form of Section 3.2. As for left multiples, nothing is clear, in particular about the possible existence of left lcms. However, what is clear is that, for instance, θ_i and θ_{i+1} admit no common left multiple, since their projections in F^+ do not.

4. Garside combinatorics for H^+

We now show that the monoid H^+ admits interesting combinatorial properties similar to those of F^+, in connection with distinguished elements Δ_n defined as right lcms of atoms, and with the left divisors of the latter, called simple elements. So our goal is to establish for H^+ results similar to those of Section 2.4. We shall see that the results are indeed similar, but with more delicate and interesting proofs.

4.1. **The elements Δ_n and their divisors.** The atoms of the monoid H^+ are the elements θ_i with $i \geqslant 1$. On the shape of the braid and Thompson cases, we shall introduce for every n a distinguished element of H^+, again denoted Δ_n, which is the right lcm of the first $n-1$ atoms. As in Section 2, it will be convenient to start from explicit word representatives. Moreover, in view of subsequent computations, we shall simultaneously introduce, for every n, an element $\Delta_{n+0.5}$ that is intermediate between Δ_n and Δ_{n+1}.

Definition 4.1. We put $\underline{\Delta}_1 = \underline{\Delta}_{1.5} := \varepsilon$, $\underline{\Delta}_2 := \theta_1$, and, for $n \geqslant 2$,

$$\underline{\Delta}_n := \underline{\Delta}_{n-1}\theta_{3n-7}\theta_{3n-5} \quad \text{and} \quad \underline{\Delta}_{n+0.5} := \underline{\Delta}_n\theta_{3n-4}. \tag{4.1}$$

We denote by Δ_n (*resp.,* $\Delta_{n+0.5}$) the class of $\underline{\Delta}_n$ (*resp.,* $\underline{\Delta}_{n+0.5}$) in H^+.

So, by definition, the word $\underline{\Delta}_n$ (*resp.,* $\underline{\Delta}_{n+0.5}$) is the increasing enumeration from 1 to $3n-5$ (*resp.,* $3n-4$) of all generators θ_i with $i \neq 0 \mod 3$. We immediately obtain for every $n \geqslant 2$

$$\Delta_n \preccurlyeq \Delta_{n+0.5} \preccurlyeq \Delta_{n+1}, \tag{4.2}$$

where we recall $\preccurlyeq$ denotes the left divisibility relation. For $n \geqslant 3$, the word $\underline{\Delta}_n$ is not $\mathcal{E}_H$-reduced; an easy induction gives the values

$$\text{NF}(\Delta_n) = \theta_{n-1} \cdot \theta_{n-2}\theta_{n-1} \cdot \theta_{n-3}\theta_{n-2} \cdot \cdots \cdot \theta_2\theta_3 \cdot \theta_1\theta_2, \tag{4.3}$$

$$\text{NF}(\Delta_{n+0.5}) = \theta_{n-1}\theta_n \cdot \theta_{n-2}\theta_{n-1} \cdot \theta_{n-3}\theta_{n-2} \cdot \cdots \cdot \theta_2\theta_3 \cdot \theta_1\theta_2. \tag{4.4}$$

Note that (4.4) implies that $\Delta_{n+0.5}$, which left divides $\Delta_{n+1.5}$ by (4.2), also right divides it. It also implies, for $n \geqslant 2$, the equality

$$\Delta_n = \theta_{n-1}\Delta_{n-0.5}. \tag{4.5}$$

The first step in studying the element Δ_n is to establish that it is indeed the right lcm of the expected atoms.

Lemma 4.2. *For every $n \geqslant 2$, the element Δ_n is the right lcm of $\theta_1, \ldots, \theta_{n-1}$. No element θ_i with $i \geqslant n$ left divides either Δ_n or $\Delta_{n+0.5}$.*

Proof. We first prove using induction on $n \geqslant 2$ that Δ_n is the right lcm of $\theta_1, \ldots, \theta_{n-1}$. The result is trivial for $n \geqslant 2$, so assume $n \geqslant 3$. A direct computation gives

$$(\theta_{n-1}, \underline{\Delta}_{n-1}) \curvearrowright (\theta_{3n-7}\theta_{3n-5}, \underline{\Delta}_{n-0.5}).$$

By Lemma 2.5(iii), this implies that $\underline{\Delta}_n$ represents the right lcm of θ_{n-1} and Δ_{n-1}. By induction hypothesis, Δ_{n-1} is the right lcm of $\theta_1, ..., \theta_{n-2}$, so Δ_n is the right lcm of $\theta_1, ..., \theta_{n-1}$. On the other hand, for $i \geqslant n$, similar computations give

$$(\theta_i, \underline{\Delta}_{n+0.5}) \curvearrowright (\theta_{i+2n-2}, \underline{\Delta}_{n+0.5}),$$

showing that the right lcm of θ_n and $\Delta_{n+0.5}$ exists but is not $\Delta_{n+0.5}$, so θ_i does not left divide $\Delta_{n+0.5}$. Hence, by (4.2), θ_i does not left divide Δ_n either. ☐

We already mentioned that θ_1 and θ_2 admit no common left multiple in H^+: it follows that Δ_n cannot be a left lcm for $\theta_1, ..., \theta_{n-1}$.

Definition 4.3. An element a of H^+ is called *simple* if $a \preccurlyeq \Delta_n$ holds for some n; in this case, the least such n is called the *index* of a, denoted by $\mathsf{ind}(a)$. For $n \geqslant 1$ and $\ell \geqslant 0$, we put

$$\Sigma_{n,\ell} := \{a \in H^+ \mid a \preccurlyeq \Delta_n \text{ and } |a| = \ell\}, \quad \text{and} \quad \Sigma_n := \bigcup\nolimits_{\ell \geqslant 0} \Sigma_{n,\ell}.$$

For instance, Σ_3 is the family of all left divisors of Δ_3; one easily checks that it consists of the six elements 1, θ_1, θ_2, $\theta_1\theta_2$, $\theta_2\theta_1$, and Δ_3. On the other hand, Lemma 4.2 implies that $\Sigma_{n,1}$ is equal to $\{\theta_1, ..., \theta_{n-1}\}$ for every n.

As Δ_n left divides Δ_{n+1} for every n, if a is simple, the values of n satisfying $a \preccurlyeq \Delta_n$ make an interval $[p, \infty[$, and $\mathsf{ind}(a)$ is the number p. Thus, $a \preccurlyeq \Delta_n$ is equivalent to $\mathsf{ind}(a) \leqslant n$, and $\mathsf{ind}(a) = n$ is equivalent to the conjunction of $a \preccurlyeq \Delta_n$ and $a \not\preccurlyeq \Delta_{n-1}$.

We shall subsequently need an upper bound for the ceiling of simple elements (as introduced in Lemma 3.1). This can be deduced from Lemma 4.2.

Lemma 4.4. *For $a \preccurlyeq \Delta_{n+0.5}$ in H^+ with $n \geqslant 2$, one has*

$$\lceil a \rceil \leqslant n + |a| - 2. \tag{4.6}$$

Proof. Use induction on $|a| \geqslant 1$. For $|a| = 1$, Lemma 4.2 implies $a = \theta_i$ with $i \leqslant n-1$, whence $\lceil a \rceil = i \leqslant n - 1 = n + |a| - 2$.

Assume now $|a| \geqslant 2$, and write $a = b\theta_i$. Then $b \preccurlyeq \Delta_{n+0.5}$ holds as well, and the induction hypothesis implies $\lceil b \rceil \leqslant n + |a| - 3$. Write $b\theta_i c = \Delta_{n+0.5}$. By definition, the contribution of θ_i to the ceiling of $\Delta_{n+0.5}$ is $i + |c|$, *i.e.*, $i + 2n - |a| - 2$. The explicit definition of (4.1) gives $\lceil \Delta_{n+0.5} \rceil = 3n - 4$, and we deduce $i \leqslant n + |a| - 2$, whence, finally, $\lceil a \rceil = \lceil b\theta_i \rceil = \max(\lceil b \rceil + 1, i) \leqslant n + |a| - 2$. ☐

4.2. **Expressions of Δ_n and $\Delta_{n+0.5}$.** Our aim is to analyze simple elements of H^+ precisely. To this end, it will be crucial to control the various expressions of the elements Δ_n and $\Delta_{n+0.5}$, and we establish here technical results in this direction. Obtaining a complete description as in Lemma 2.11 seems hopeless, but it will be sufficient to connect the expressions of Δ_n with those of $\Delta_{n-0.5}$ (*i.e.*, of $\Delta_{n-1}\theta_{3n-7}$). A similar connection will be stated between the expressions of $\Delta_{n-0.5}$ and $\Delta_{n-1.5}$ in Lemma 4.6 below.

Lemma 4.5. *For $n \geqslant 2$, every expression of Δ_n has the form $w_1\theta_k w_2$ with $w_1 w_2 \equiv \underline{\Delta}_{n-0.5}$ and $k = n + |w_1| - 1$; in this case, $w_1\theta_k \equiv \theta_{n-1} w_1$ holds.*

Proof. The result is trivial for $n = 2$. We assume $n \geqslant 3$, and establish the existence of w_1 and w_2 for an expression w of Δ_n using induction on the combinatorial distance d between $\underline{\Delta}_n$ and w, *i.e.*, the minimal number of relations of $\mathcal{P}_H$ needed to transform $\underline{\Delta}_n$ to w. For $d = 0$, the result is trivial with $w_1 := \underline{\Delta}_{n-0.5}$

and $w_2 := \varepsilon$. Assume now $d \geqslant 1$, and let w' satisfying $\mathsf{dist}(\underline{\Delta}_n, w') = p-1$ and $\mathsf{dist}(w', w) = 1$. Write $w' = u_1 v' u_2$, $w = u_1 v u_2$ with $v'{=}v$ a relation of $\mathcal{P}_H$. By induction hypothesis, there exist w'_1, w'_2 satisfying $w' = w'_1 \theta_{k'} w'_2$ with $w'_1 w'_2 \equiv \underline{\Delta}_{n-0.5}$ and $k' := n + |w'_1| - 1$.

We consider the various possibilities for the position of v' inside w'. If $u_1 v'$ is a prefix of w'_1, *i.e.*, if we have $w'_1 = u_1 v' u''_1$ for some u''_1, we find $w' = u_1 v' u''_1 \theta_{k'} w'_2$ and $w = u_1 v u''_1 \theta_{k'} w'_2$. Now we have $u_1 v u''_1 w'_2 \equiv u_1 v' u''_1 w'_2 = w'_1 w'_2 \equiv \underline{\Delta}_{n-0.5}$, whence the result with $w_1 := u_1 v u''_1$, $w_2 := w'_2$, and $k := k'$. The argument is similar if $v' u_2$ is a suffix of w'_2.

There remain the cases when $\theta_{k'}$ occurs in v', *i.e.*, $\theta_{k'}$ is involved in going from w' to w. Assume first that $v'{=}v$ is a length 2 relation. Then v' is either $\theta_i\theta_j$ with $j \geqslant i+3$, or $\theta_i\theta_j$ with $j \leqslant i-2$. As $\theta_{k'}$ occurs in v' in position either 1 or 2, four cases are to be considered.

(i) θ_i is the last letter of w'_1 and $j = k' \geqslant i+3$ holds. Putting $w'_1 := w''_1\theta_i$, we obtain $w' = w''_1 \theta_i \theta_{k'} w'_2$ and $w = w''_1 \theta_{k'-1} \theta_i w'_2 = w''_1 \theta_{n+|w''_1|-1} \theta_i w'_2$. Moreover, $w''_1 \theta_i w'_2 \equiv \underline{\Delta}_{n-0.5}$ holds, whence the result for $w_1 := w''_1$, $w_2 := \theta_i w'_2$, and $k := k'{-}1$.

(ii) θ_i is the last letter of w'_1 and $j = k' \leqslant i-2$ holds. This is impossible, because a letter θ_i in this position would contribute $i+1+|w'_2|$, hence at least $k'+3+|w'_2|$, *i.e.*, $3n-2$, to the ceiling $\lceil w' \rceil$, which is that of Δ_n, namely $3n-5$.

(iii) θ_i is $\theta_{k'}$ and θ_j is the first letter of w'_2, with $j \geqslant i+3$. This is impossible for the same ceiling reason as in (ii).

(iv) θ_i is $\theta_{k'}$ and θ_j is the first letter of w'_2, with $j \leqslant i-2$. This is similar to (i).

We now similarly handle the case when $v'{=}v$ is a length 3 relation. Then v' is either $\theta_i\theta_{i+1}\theta_{i+3}$, or $\theta_{i+1}\theta_i\theta_{i+1}$. This time, $\theta_{k'}$ occurs in v' in position 1, 2, or 3, so six cases are a priori possible.

(i) $\theta_i\theta_{i+1}$ is the final factor of w'_1, and $i{+}3 = k'$ holds. Putting $w'_1 := w''_1\theta_{k'-3}\theta_{k'-2}$, we obtain $w' = w''_1 \theta_{k'-3}\theta_{k'-2}\theta_{k'} w'_2$ and $w = w''_1 \theta_{k'-2}\theta_{k'-3}\theta_{k'-2} w'_2$. Moreover, we find $w''_1 \theta_{k'-3}\theta_{k'-2} w'_2 \equiv \underline{\Delta}_{n-0.5}$, whence the result for $w_1 := w''_1$, $w_2 := \theta_{k-3}\theta_{k-2} w'_2$, and $k := k'$.

(ii), (iii) θ_i is the last letter of w'_1 with $i{+}1 = k'$, and θ_{i+3} is the first letter in w'_2, or we have $i = k'$ and $\theta_{i+1}\theta_{i+3}$ is a prefix of w'_2. These cases are impossible because $\lceil w' \rceil = 3n{-}5$ holds, whereas the letter $\theta_{k'+2}$ would contribute $n{+}|w'_1|{+}1{+}|w'_2|{-}1 = n + 2n - 3 - 1 = 3n - 4$ to the ceiling of w'.

(iv), (v) $\theta_{i+1}\theta_i$ is a suffix of w'_1 and $i+1 = k$ holds, or θ_{i+1} is the last letter of w'_1 and $i = k$ holds and θ_{i+1} is the first letter of w'_2. These cases are impossible because θ_{k+2} would contribute $3n-3$ to the ceiling of w'.

(vi) $\theta_i\theta_{i+1}$ is a prefix of w'_2, with $i+1 = k'$. Putting $w'_2 = \theta_{k'-1}\theta_{k'} w''_2$, we obtain $w' = w'_1 \theta_{k'}\theta_{k'-1}\theta_{k'} w''_2$ and $w = w'_1 \theta_{k'-1}\theta_{k'}\theta_{k'+2} w''_2$. Moreover, we find $w'_1 \theta_{k'-1}\theta_{k'} w''_2 \equiv \underline{\Delta}_{n-0.5}$, whence the result for $w_1 := w'_1\theta_{k'-1}\theta_{k'}$, $w_2 := w''_2$, and $k := k'+2$.

This completes the induction. For the final equivalence, we find, using (4.5),

$$w_1 \theta_k w_2 \equiv \underline{\Delta}_n \equiv \theta_{n-1} \underline{\Delta}_{n-0.5} \equiv \theta_{n-1} w_1 w_2.$$

By right cancelling w_2, we deduce $w_1\theta_k \equiv \theta_{n-1} w_1$. $\square$

We now state a similar result for the expressions of $\Delta_{n-0.5}$. The latter is equal to $\Delta_{n-1.5}\theta_{3n-8}\theta_{3n-7}$ and the two letters θ_{3n-8} and θ_{3n-7} can be moved left.

Lemma 4.6. *For $n \geqslant 3$, every expression of $\Delta_{n-0.5}$ has the form $w_1\theta_k w_2\theta_\ell w_3$ with $w_1w_2w_3 \equiv \underline{\Delta}_{n-1.5}$, $k = n-2+|w_1|$, and $\ell = n-1+|w_1w_2|$; in this case, $w_1\theta_k w_2 w_3$ represents Δ_{n-1}.*

We skip the proof, which is entirely similar to that of Lemma 4.5—but with more cases, as one has to take care of the positions of two letters.

Applying Lemmas 4.5 and 4.6, we can now easily establish various properties of simple elements, paving the way for a partition of these elements in several families.

Lemma 4.7. (i) *For $n \geqslant 2$, every left divisor of Δ_n either left divides $\Delta_{n-0.5}$, or has the form $a\theta_k b$ with $ab \preccurlyeq \Delta_{n-0.5}$, $k = n+|a|-1$, and $a\theta_k = \theta_{n-1}a$.*

(ii) *For $n \geqslant 2$ and $a \preccurlyeq \Delta_n$, the conditions $a \preccurlyeq \Delta_{n-0.5}$ and $\theta_{n-1} \not\preccurlyeq a$ are equivalent.*

(iii) *For $n \geqslant 3$, every left divisor of $\Delta_{n-0.5}$ either left divides Δ_{n-1}, or has the form $\theta_{n-2}\theta_{n-1}a$ with $a \preccurlyeq \underline{\Delta}_{n-1.5}$.*

Proof. (i) If $[w] \preccurlyeq \Delta_n$ holds, then Δ_n has an expression of form ww' for some w'. By Lemma 4.5, we can write $ww' = w_1\theta_k w_2$ with $w_1w_2 \equiv \underline{\Delta}_{n-0.5}$. Then either w is a prefix of w_1, and then we have $[w] \preccurlyeq \Delta_{n-0.5}$, or w has the form $w_1\theta_k w_2'$ with w_2' a prefix of w_2, and then we have $[w] = [w_1]\theta_k[w_2']$ with $[w_1][w_2'] = [w_1w_2'] \preccurlyeq \Delta_{n-0.5}$. Moreover, $w_1\theta_k \equiv \theta_{n-1}w_1$ implies $[w_1]\theta_k = \theta_{n-1}[w_1]$.

(ii) Assume $a \preccurlyeq \Delta_{n-0.5}$. By Lemma 4.2, θ_{n-1} does not left divide $\Delta_{n-0.5}$, so, a fortiori, θ_{n-1} does not left divide a.

Conversely, assume $a \preccurlyeq \Delta_n$ and $\theta_{n-1} \not\preccurlyeq a$. By (i), there are two possibilities: either we have $a \preccurlyeq \Delta_{n-0.5}$, as expected, or a can be decomposed as $b\theta_k c$ with $bc \preccurlyeq \Delta_{n-0.5}$ and $b\theta_k = \theta_{n-1}b$, implying $\theta_{n-1} \preccurlyeq a$ and contradicting the assumption. So $a \preccurlyeq \Delta_{n-0.5}$ is the only possibility.

(iii) If $[w] \preccurlyeq \Delta_{n-0.5}$ holds, then $\Delta_{n-0.5}$ has an expression of form ww' for some w'. By Lemma 4.6, we can write $ww' = w_1\theta_k w_2\theta_\ell w_3$ with $w_1w_2w_3 \equiv \underline{\Delta}_{n-1.5}$, $k = n-2+|w_1|$, and $\ell = n-1+|w_1w_2|$. Then three cases may arise. Either w is a prefix of w_1, and then we have $[w] \preccurlyeq \Delta_{n-1.5}$, whence a fortiori $[w] \preccurlyeq \Delta_{n-1}$. Or w is $w_1\theta_k w_2'$ for some prefix w_2' of w_2. By Lemma 4.6, we have $[w] \preccurlyeq \Delta_{n-1}$ again. Or w is $w_1\theta_k w_2\theta_\ell w_3'$ for some prefix w_3' of w_3, say $w_3 = w_3'w_3''$. Applying (4.4), we find $ww_3'' \equiv \underline{\Delta}_{n-0.5} \equiv \theta_{n-2}\theta_{n-1}\underline{\Delta}_{n-1.5} \equiv \theta_{n-2}\theta_{n-1}w_1w_2w_3'w_3''$. Right cancelling w_3'', we deduce $w \equiv \theta_{n-2}\theta_{n-1}w_1w_2w_3'$, with w_1, w_2, w_3' satisfying $[w_1w_2w_3'] \preccurlyeq \Delta_{n-1.5}$. □

4.3. **Partitioning the sets $\Sigma_{n,\ell}$.** With the preparatory results of Section 4.2, it is now easy to describe the simple elements of the monoid H^+ more precisely. To this end, we introduce subfamilies of H^+. We shall eventually see that these subfamilies form a partition of the set $\Sigma_{n,\ell}$ of all length ℓ left divisors of Δ_n.

Definition 4.8. For $n \geqslant 2$ and $0 \leqslant \ell \leqslant 2n-3$, we put

(type 0) $\Sigma^0_{n,\ell} := \{a \mid a \preccurlyeq \Delta_{n-1} \text{ and } |a| = \ell\}$ for $n \geqslant 2, \ell \geqslant 0$,

(type I) $\Sigma^{\mathrm{I}}_{n,\ell} := \{\theta_{n-1}a \mid a \preccurlyeq \Delta_{n-1} \text{ and } |a| = \ell-1\}$ for $n \geqslant 2, \ell \geqslant 1$,

(type II$_1$) $\Sigma^{\mathrm{II}_1}_{n,\ell} := \{\theta_{n-2}\theta_{n-1}a \mid a \preccurlyeq \Delta_{n-1.5} \text{ and } |a| = \ell-2\}$ for $n \geqslant 3, \ell \geqslant 2$,

(type II$_2$) $\Sigma^{\mathrm{II}_2}_{n,\ell} := \{\theta_{n-1}\theta_{n-2}\theta_{n-1}a \mid \theta_{n-2}a \preccurlyeq \Delta_{n-1} \text{ and } |a| = \ell-3\}$ for $n, \ell \geqslant 3$,

completed with $\Sigma^0_{n,\ell} = \Sigma^{\mathrm{I}}_{n,\ell} = \Sigma^{\mathrm{II}_1}_{n,\ell} = \Sigma^{\mathrm{II}_2}_{n,\ell} := \emptyset$ for other values of n and ℓ.

The first step is to check that the above sets consist of left divisors of Δ_n.

Lemma 4.9. *For all n, ℓ, the sets $\Sigma^0_{n,\ell}, \Sigma^{\mathrm{I}}_{n,\ell}, \Sigma^{\mathrm{II}_1}_{n,\ell}$ et $\Sigma^{\mathrm{II}_2}_{n,\ell}$ are included in $\Sigma_{n,\ell}$.*

Proof. By definition, all elements of $\Sigma^0_{n,\ell}, \Sigma^{\mathrm{I}}_{n,\ell}, \Sigma^{\mathrm{II}_1}_{n,\ell}$, and $\Sigma^{\mathrm{II}_2}_{n,\ell}$ have length ℓ, so the point is to check that they left divide Δ_n. As Δ_{n-1} left divides Δ_n, the result is obvious for $\Sigma^0_{n,\ell}$. Next, $a \preccurlyeq \Delta_{n-1}$ implies $\theta_{n-1}a \preccurlyeq \theta_{n-1}\Delta_{n-1}$, whence $\theta_{n-1}a \preccurlyeq \theta_{n-1}\Delta_{n-1}\theta_{3n-7} = \Delta_n$. So $\Sigma^{\mathrm{I}}_{n,\ell}$ is included in Σ_n. Then, by (4.4), we have $\Delta_n = \theta_{n-2}\theta_{n-1}\Delta_{n-1.5}\theta_{3n-5}$, so $a \preccurlyeq \Delta_{n-1.5}$ implies $\theta_{n-2}\theta_{n-1}a \preccurlyeq \theta_{n-2}\theta_{n-1}\Delta_{n-1.5}$, whence $\theta_{n-2}\theta_{n-1}a \preccurlyeq \Delta_n$. So $\Sigma^{\mathrm{II}_1}_{n,\ell}$ is included in Σ_n. Finally, for $b = \theta_{n-1}\theta_{n-2}\theta_{n-1}a$ with $\theta_{n-2}a \preccurlyeq \Delta_{n-1}$, we have $\Delta_{n-1} = \theta_{n-2}\Delta_{n-1.5}$ by (4.5), whence $a \preccurlyeq \Delta_{n-1.5}$ by left cancelling θ_{n-2}. A direct computation gives $\Delta_n = \theta_{n-1}\theta_{n-2}\theta_{n-1}\Delta_{n-1.5}$, so $a \preccurlyeq \Delta_{n-1.5}$ implies $b \preccurlyeq \theta_{n-1}\theta_{n-2}\theta_{n-1}\Delta_{n-1.5} = \Delta_n$. So $\Sigma^{\mathrm{II}_2}_{n,\ell}$ is included in Σ_n. □

The second step consists in showing that the various sets $\Sigma^0_{n,\ell}, \ldots, \Sigma^{\mathrm{II}_2}_{n,\ell}$ are pairwise disjoint. This is more delicate, in that it involves proving that certain words are not equivalent. According to Lemma 2.5(i), this can be seen using $\mathcal{P}_H$-grids.

Lemma 4.10. *For all n, ℓ, the sets $\Sigma^0_{n,\ell}, \Sigma^{\mathrm{I}}_{n,\ell}, \Sigma^{\mathrm{II}_1}_{n,\ell}$, and $\Sigma^{\mathrm{II}_2}_{n,\ell}$ are pairwise disjoint.*

Proof. To prove that $\Sigma^0_{n,\ell}$ is disjoint from $\Sigma^{\mathrm{I}}_{n,\ell}, \Sigma^{\mathrm{II}_1}_{n,\ell}$, and $\Sigma^{\mathrm{II}_2}_{n,\ell}$, it suffices to prove that no element of the latter three sets left divides Δ_{n-1}. Now, by definition, θ_{n-1} left divides every element of $\Sigma^{\mathrm{I}}_{n,\ell}$ and $\Sigma^{\mathrm{II}_2}_{n,\ell}$, whereas, by Lemma 4.2, θ_{n-1} does not left divide Δ_{n-1}. So $\Sigma^{\mathrm{I}}_{n,\ell}$ and $\Sigma^{\mathrm{II}_2}_{n,\ell}$ are disjoint from $\Sigma^0_{n,\ell}$.

Next, a direct computation gives $(\theta_{n-2}\theta_{n-1}, \underline{\Delta}_{n-1}) \curvearrowright (\theta_{3n-7}, \underline{\Delta}_{n-1.5})$, which, by Lemma 2.5(i), proves $\theta_{n-2}\theta_{n-1} \not\preccurlyeq \Delta_{n-1}$. As $\theta_{n-2}\theta_{n-1}$ left divides every element of $\Sigma^{\mathrm{II}_1}_{n,\ell}$, it follows that $\Sigma^{\mathrm{II}_1}_{n,\ell}$ is disjoint from $\Sigma^0_{n,\ell}$.

Assume now $a \in \Sigma^{\mathrm{I}}_{n,\ell} \cap \Sigma^{\mathrm{II}_1}_{n,\ell}$. Then, by definition, we have both $\theta_{n-1} \preccurlyeq a$ and $a \preccurlyeq \theta_{n-2}\theta_{n-1}\Delta_{n-1.5}$, whence $\theta_{n-1} \preccurlyeq \theta_{n-2}\theta_{n-1}\Delta_{n-1.5}$. This is impossible: a direct computation gives $(\theta_{n-1}, \theta_{n-2}\theta_{n-1}\underline{\Delta}_{n-1.5}) \curvearrowright (\theta_{3n-10}, \theta_{n-1}\theta_{n-2}\underline{\Delta}_{n-1.5})$, which, by Lemma 2.5(ii), proves $\theta_{n-1} \not\preccurlyeq \theta_{n-2}\theta_{n-1}\Delta_{n-1.5}$. Hence $\Sigma^{\mathrm{I}}_{n,\ell}$ and $\Sigma^{\mathrm{II}_1}_{n,\ell}$ are disjoint.

Assume next $a \in \Sigma^{\mathrm{I}}_{n,\ell} \cap \Sigma^{\mathrm{II}_2}_{n,\ell}$. We have both $a = \theta_{n-1}b$ with $b \in \Sigma_{n-1,\ell-1}$, and $a = \theta_{n-1}\theta_{n-2}\theta_{n-1}c$ with $\theta_{n-2}c \preccurlyeq \Delta_{n-1}$. By left cancelling θ_{n-1}, we deduce $b = \theta_{n-2}\theta_{n-1}c$, whence $\theta_{n-2}\theta_{n-1}c \preccurlyeq \Delta_{n-1}$ and, a fortiori, $\theta_{n-2}\theta_{n-1} \preccurlyeq \Delta_{n-1}$, what we saw above is false. Hence $\Sigma^{\mathrm{I}}_{n,\ell}$ and $\Sigma^{\mathrm{II}_2}_{n,\ell}$ are disjoint.

Finally, assume $a \in \Sigma^{\mathrm{II}_1}_{n,\ell} \cap \Sigma^{\mathrm{II}_2}_{n,\ell}$. By definition, we have $a = \theta_{n-2}\theta_{n-1}b = \theta_{n-1}\theta_{n-2}\theta_{n-1}c$ with $b \preccurlyeq \Delta_{n-1.5}$ and $\theta_{n-2}c \preccurlyeq \Delta_{n-1}$. As $\theta_{n-1}\theta_{n-2}\theta_{n-1}$ is also $\theta_{n-2}\theta_{n-1}\theta_{n+1}$, we deduce $\theta_{n-2}\theta_{n-1}b = \theta_{n-2}\theta_{n-1}\theta_{n+1}c$, whence $b = \theta_{n+1}c$ by left cancelling $\theta_{n-2}\theta_{n-1}$, and, from there, $\theta_{n+1} \preccurlyeq b \preccurlyeq \Delta_{n-1.5}$. Now, by Lemma 4.2, θ_{n+1} does not left divide $\Delta_{n-1.5}$. Hence $\Sigma^{\mathrm{II}_1}_{n,\ell}$ and $\Sigma^{\mathrm{II}_2}_{n,\ell}$ are disjoint. □

We are now ready to establish the expected partition result:

Proposition 4.11. *For all n, ℓ, the sets $\Sigma^0_{n,\ell}$, $\Sigma^{\mathrm{I}}_{n,\ell}$, $\Sigma^{\mathrm{II}_1}_{n,\ell}$, and $\Sigma^{\mathrm{II}_2}_{n,\ell}$ form a partition of $\Sigma_{n,\ell}$.*

Proof. Owing to Lemmas 4.9 and 4.10, the only point remaining to be proved is that every element of $\Sigma_{n,\ell}$ belongs to one of the sets $\Sigma^0_{n,\ell}$, $\Sigma^{\mathrm{I}}_{n,\ell}$, $\Sigma^{\mathrm{II}_1}_{n,\ell}$, $\Sigma^{\mathrm{II}_2}_{n,\ell}$. So let a belong to $\Sigma_{n,\ell}$. By Lemma 4.7(i), we have either $a \preccurlyeq \Delta_{n-0.5}$, or $a = b\theta_k c$ with $bc \preccurlyeq \Delta_{n-0.5}$ and $b\theta_k = \theta_{n-1}b$. Assume first $a \preccurlyeq \Delta_{n-0.5}$. By Lemma 4.7(iii), we have either $a \preccurlyeq \Delta_{n-1}$, whence $a \in \Sigma^0_{n,\ell}$, or $a = \theta_{n-2}\theta_{n-1}d$ with $d \preccurlyeq \Delta_{n-1.5}$, whence $a \in \Sigma^{\mathrm{II}_1}_{n,\ell}$.

Assume now $a = b\theta_k c$ with $bc \preccurlyeq \Delta_{n-0.5}$ and $b\theta_k = \theta_{n-1}b$, whence $a = \theta_{n-1}bc$. By Lemma 4.7(iii), we have either $bc \preccurlyeq \Delta_{n-1}$, whence $a \in \Sigma^{\mathrm{I}}_{n,\ell}$, or $bc = \theta_{n-2}\theta_{n-1}d$ with $d \preccurlyeq \Delta_{n-1.5}$. In the latter case, we find $a = \theta_{n-1}\theta_{n-2}\theta_{n-1}d$. Moreover, $d \preccurlyeq \Delta_{n-1.5}$ implies $\theta_{n-2}d \preccurlyeq \theta_{n-2}\Delta_{n-1.5} = \Delta_{n-1}$, whence $a \in \Sigma^{\mathrm{II}_2}_{n,\ell}$. □

With the partition of Proposition 4.11, we can now count the left divisors of Δ_n.

Lemma 4.12. *For $n \geqslant 3$, let $F^0_{n,\ell}$ be the identity map on $\Sigma_{n-1,\ell}$, let $F^{\mathrm{I}}_{n,\ell}$ be the map $a \mapsto \theta_{n-1}a$ on $\Sigma_{n-1,\ell-1}$, and let $F^{\mathrm{II}}_{n,\ell}$ be the map on $\Sigma_{n-1,\ell-2}$ defined by $F(a) := \theta_{n-2}\theta_{n-1}a$ if $a \preccurlyeq \Delta_{n-1.5}$ holds, and $F(a) := \theta_{n-1}\theta_{n-2}\theta_{n-1}b$ with $a = \theta_{n-2}b$ otherwise. Then $F^0_{n,\ell}$, $F^{\mathrm{I}}_{n,\ell}$, and $F^{\mathrm{II}}_{n,\ell}$ respectively establish bijections*

$$\Sigma_{n-1,\ell} \leftrightarrow \Sigma^0_{n,\ell}, \quad \Sigma_{n-1,\ell-1} \leftrightarrow \Sigma^{\mathrm{I}}_{n,\ell}, \quad \textit{and} \quad \Sigma_{n-1,\ell-2} \leftrightarrow \Sigma^{\mathrm{II}_1}_{n,\ell} \cup \Sigma^{\mathrm{II}_2}_{n,\ell}.$$

Proof. The result for $F^0_{n,\ell}$ directly follows from the definition of $\Sigma^0_{n,\ell}$. For $F^{\mathrm{I}}_{n,\ell}$, it follows from the definition of $\Sigma^{\mathrm{I}}_{n,\ell}$ and the left cancellativity of H^+, which ensures that $F^{\mathrm{I}}_{n,\ell}$ is injective. Finally, for $F^{\mathrm{II}}_{n,\ell}$, put

$$X_1 := \{a \in \Sigma_{n-1,\ell-2} \mid a \preccurlyeq \Delta_{n-1.5}\} \quad \text{and} \quad X_2 := \{a \in \Sigma_{n-1,\ell-2} \mid a \not\preccurlyeq \Delta_{n-1.5}\}.$$

It follows from the definition of $\Sigma^{\mathrm{II}_1}_{n,\ell}$ and the left cancellativity of H^+ that $F^{\mathrm{II}}_{n,\ell}$ establishes a bijection from X_1 to $\Sigma^{\mathrm{II}_1}_{n,\ell}$. On the other hand, for a in X_1, Lemma 4.7(ii) implies $\theta_{n-2} \preccurlyeq a$, say $a = \theta_{n-2}b$, and then the left cancellativity of H^+ implies that $F^{\mathrm{II}}_{n,\ell}$ establishes a bijection from X_2 to $\Sigma^{\mathrm{II}_2}_{n,\ell}$. As $\Sigma^{\mathrm{II}_1}_{n,\ell}$ and $\Sigma^{\mathrm{II}_2}_{n,\ell}$ are disjoint, this completes the proof. □

Lemma 4.12 immediately implies that, if we denote by $N_{n,\ell}$ the cardinal of $\Sigma_{n,\ell}$, then the numbers $N_{n,\ell}$ are determined by the inductive rule

$$N_{n,\ell} = N_{n-1,\ell} + N_{n-1,\ell-1} + N_{n-1,\ell-2}, \tag{4.7}$$

starting from the initial values $N_{2,0} = N_{2,1} = 1$. It follows that the numbers $N_{n,\ell}$ appear in the generalized Pascal triangle in which each entry is the sum of the three entries above it, starting from the row $(1,1)$, see Figure 1. An obvious induction from (4.7) shows that $N_{n,\ell}$ is the coefficient of $x^{\ell-1}$ in $(1+x)(1+x+x^2)^{n-2}$, that $N_{n,\ell} = N_{n,2n-3-\ell}$ holds for $n-1 \leqslant \ell \leqslant 2n-3$, and that, for $0 \leqslant \ell \leqslant n-2$, the number $N_{n,\ell}$ is the number of (compact-rooted) directed animals of size $n-1$ with $n-1-\ell$ source points, see [19, Table 1] and [24, sequence 005773]. In particular, the highest value occurring in the $n-1$st row of Figure 1 (the one that corresponds to the divisors of Δ_n), namely $N_{n,n-2}$ and $N_{n,n-1}$,—that is, the sequence 1, 2, 5, 13, 35, ...—is the number of directed animals of size $n-1$ with one source point. Finding an explicit direct bijection between the divisors of Δ_n in H^+ and size $n-1$ directed animals [19, 29]—or, equivalently, “arbres guingois” or bicolored Motzkin paths [2]—is a natural open question.

From (4.7) again, it is clear that the total number of left divisors of Δ_n triples when one goes from a row of the triangle to the next one, and, as Δ_2 admits two left divisors, we obtain

Proposition 4.13. *For $n \geqslant 2$, the number of left divisors of Δ_n in H^+ is $2 \cdot 3^{n-2}$.*

The number of simple elements of index n is $\sum_\ell N_{n,\ell} - \sum_\ell N_{n-1,\ell}$, hence it is $4 \cdot 3^{n-3}$: so 2/3 of the left divisors of Δ_n have index n, whereas 1/3 has index $< n$.

				1	1				
			1	2	2	1			
		1	3	5	5	3	1		
	1	4	9	13	13	9	4	1	
1	5	14	26	35	35	26	14	5	1

Figure 1. Generalized Pascal triangle generating the numbers $N_{n,\ell}$: each entry is the sum of the three entries above it: for instance, we find $N_{5,2} = 9 = 1+3+5 = N_{4,0}+N_{4,1}+N_{4,2}$; missing values are 0.

5. Normal form of simple elements

We complete the investigation of simple elements in H^+ by determining their normal form. In Section 2.4, we saw that normal elements of F^+ are those, whose normal form is a word in which the indices of the generators decrease, which amounts to saying that a word is the normal form of a simple element if, and only if, it has no factor $\theta_i\theta_j$ with $j \geqslant i$. We shall establish below a similar result characterizing the normal form of simple elements in terms of forbidden factors of length 2 and 3.

5.1. **The key lemma.** A direct inspection shows that the normal forms of the six simple elements of index $\leqslant 3$, *i.e.*, of the six left divisors of Δ_3, are ε, θ_1, θ_2, $\theta_1\theta_2$, $\theta_2\theta_1$, and $\theta_2\theta_1\theta_2$. The following result will then enable one to inductively determine the normal form of a simple element according to its position in the partition of Proposition 4.11.

Lemma 5.1. *For every simple element a of index $n \geqslant 4$ in H^+, there exists b of index $< n$ such that exactly one of the following holds:*

$$\text{(5.1)}\qquad (\textit{type}\ \mathrm{I})\qquad b \preccurlyeq \Delta_{n-1} \quad \textit{and} \quad \mathrm{NF}(a) = \theta_{n-1}\mathrm{NF}(b),$$

$$\text{(5.2)}\qquad (\textit{type}\ \mathrm{II}_1)\qquad b \preccurlyeq \Delta_{n-1.5} \quad \textit{and} \quad \mathrm{NF}(a) = \theta_{n-2}\theta_{n-1}\mathrm{NF}(b),$$

$$\text{(5.3)}\qquad (\textit{type}\ \mathrm{II}_2)\qquad \theta_{n-2}b \preccurlyeq \Delta_{n-1} \quad \textit{and} \quad \mathrm{NF}(a) = \theta_{n-1}\theta_{n-2}\theta_{n-1}\mathrm{NF}(b).$$

Proof. Let $\ell := |a|$. By assumption, a belongs to $\Sigma_{n,\ell} \setminus \Sigma_{n-1,\ell}$. Then, by Proposition 4.11, a belongs to exactly one of $\Sigma^{\mathrm{I}}_{n,\ell}$, $\Sigma^{\mathrm{II}_1}_{n,\ell}$, or $\Sigma^{\mathrm{II}_2}_{n,\ell}$. So there exists b such that exactly one of the following holds:

$$\text{(5.4)}\qquad (\text{type I})\qquad b \preccurlyeq \Delta_{n-1} \quad \text{and} \quad a = \theta_{n-1}\, b,$$

$$\text{(5.5)}\qquad (\text{type II}_1)\qquad b \preccurlyeq \Delta_{n-1.5} \quad \text{and} \quad a = \theta_{n-2}\theta_{n-1}\, b,$$

$$\text{(5.6)}\qquad (\text{type II}_2)\qquad \theta_{n-2}b \preccurlyeq \Delta_{n-1} \quad \text{and} \quad a = \theta_{n-1}\theta_{n-2}\theta_{n-1}\, b.$$

In the case of (5.4), we have $b \preccurlyeq \Delta_{n-1}$, so b is simple with $\mathsf{ind}(b) \leqslant n-1$. In the case of (5.5), we have $b \preccurlyeq \Delta_{n-1.5} \preccurlyeq \Delta_{n-1.5}\theta_{3n-8} = \Delta_{n-1}$, so, again, b is simple with $\mathsf{ind}(b) \leqslant n-1$. Finally, in the case of (5.6), $\theta_{n-2}b \preccurlyeq \Delta_{n-1}$ implies $b \preccurlyeq \Delta_{n-1.5}$ by (4.5), whence $b \preccurlyeq \Delta_{n-1}$, so b is simple with $\mathsf{ind}(b) \leqslant n-1$. So, in every case, b is simple with $\mathsf{ind}(b) < n = \mathsf{ind}(a)$. Then, by definition of NF and by Proposition 4.2,

$$\text{(5.7)}\qquad \mathrm{NF}(b) \text{ is an } \mathcal{E}_H\text{-reduced word and its first letter is among } \theta_1, \ldots, \theta_{n-2}.$$

In the case of (5.4), (5.7) implies that $\theta_{n-1}\mathrm{NF}(b)$ is $\mathcal{E}_H$-reduced, hence it must the normal form of $\theta_{n-1}b$, *i.e.*, of a, and (5.1) is true. In the case of (5.5), (5.7) implies that $\theta_{n-2}\theta_{n-1}\mathrm{NF}(b)$ is $\mathcal{E}_H$-reduced, hence it must the normal form of $\theta_{n-2}\theta_{n-1}b$, *i.e.*, of a, and (5.1) is true. Finally, in the case of (5.6), (5.7) implies that $\theta_{n-1}\theta_{n-2}\theta_{n-1}\mathrm{NF}(b)$ is $\mathcal{E}_H$-reduced, hence it must the normal form of $\theta_{n-1}\theta_{n-2}\theta_{n-1}b$, *i.e.*, of a, and (5.1) is true. □

An easy application of Lemma 5.1 is that, in addition to the obstructions of $\mathcal{O}$, certain factors cannot appear in the normal form of a simple element.

Lemma 5.2. *Put*

(5.8) $\mathcal{O}_\Sigma := \{\theta_i^2 \mid i \geqslant 1\} \cup \{\theta_i\theta_{i+2} \mid i \geqslant 1\} \cup \{\theta_i\theta_{i+1}\theta_i \mid i \geqslant 1\} \cup \{\theta_i\theta_{i+1}\theta_{i+2} \mid i \geqslant 1\}.$

Then the normal form of a simple element of H^+ contains no factor in $\mathcal{O}_\Sigma$.

Proof. We prove the result for a simple element a using induction on the index n of a. For $n \leqslant 3$, a direct inspection of the six possible words gives the result. Assume $n \geqslant 4$. By Lemma 5.1, there exists b simple of index $<n$ such that exactly one of (5.1), (5.2), or (5.3) holds. By induction hypothesis, the word $\mathrm{NF}(b)$ contains no factor of $\mathcal{O}_\Sigma$, and we only have to check that the letters added to transform $\mathrm{NF}(b)$ into $\mathrm{NF}(a)$ create no factor in $\mathcal{O}_\Sigma$. As the index of b is $< n$, Lemma 4.2 guarantees that the first letter of $\mathrm{NF}(b)$ must be among $\theta_1, ..., \theta_{n-2}$.

In the case of (5.1), $\mathrm{NF}(a)$ begins with $\theta_{n-1}\theta_j$ with $1 \leqslant j \leqslant n-2$: this length 2 word is not in $\mathcal{O}_\Sigma$, and it is not the prefix of a word of $\mathcal{O}_\Sigma$ either. Similarly, in the case of (5.2), $\mathrm{NF}(a)$ begins with $\theta_{n-2}\theta_{n-1}\theta_j$ with $1 \leqslant j \leqslant n-3$, and this length 3 word includes no factor in $\mathcal{O}_\Sigma$, nor can it contribute to a factor in $\mathcal{O}_\Sigma$. Finally, in the case of (5.3), $\mathrm{NF}(a)$ begins with $\theta_{n-1}\theta_{n-2}\theta_{n-1}\theta_j$ with $1 \leqslant j \leqslant n-3$, and, again, this length 4 word includes no factor in $\mathcal{O}_\Sigma$, nor can it either contribute to a factor in $\mathcal{O}_\Sigma$. So, in every case, the word $\mathrm{NF}(a)$ has no factor in $\mathcal{O}_\Sigma$. □

We use Lemma 5.1 once more to establish a constraint about the first letter of a normal word.

Lemma 5.3. *If $a \preccurlyeq \Delta_n$ and $\theta_{n-1} \preccurlyeq a$ hold, the first letter of $\mathrm{NF}(a)$ is θ_{n-1}.*

Proof. Assume $a \preccurlyeq \Delta_n$ and $\theta_{n-1} \preccurlyeq a$. So a is simple with $\mathsf{ind}(a) \leqslant n$. If we had $\mathsf{ind}(a) \leqslant n-1$, hence $a \preccurlyeq \Delta_{n-1}$, then $\theta_{n-1} \preccurlyeq a$ would be impossible. So we must have $\mathsf{ind}(a) = n$. For $n \leqslant 3$, a direct inspection of the six possible normal words shows that the result is true. Otherwise, we apply Lemma 5.1. In the cases (5.1) and (5.3), $\mathrm{NF}(a)$ explicitly begins with θ_{n-1}. There remains the case of (5.2). Assume $a = \theta_{n-2}\theta_{n-1}b$ with $b \preccurlyeq \Delta_{n-1.5}$, let w represent b. By constructing a $\mathcal{P}_H$-grid from $(\theta_{n-1}, \theta_{n-2}\theta_{n-1}w)$, we see that $\theta_{n-1} \preccurlyeq a$ is equivalent to $\theta_{n+1} \preccurlyeq b$, hence it implies $\theta_{n+1} \preccurlyeq b \preccurlyeq \Delta_{n-1.5} \preccurlyeq \Delta_{n-1.5}\theta_{3n-8} = \Delta_{n-1}$, which contradicts Lemma 4.2. So $\theta_{n-1} \preccurlyeq a$ excludes (5.2). □

5.2. **The normal form of simple elements.** Our goal is now to establish that the necessary condition of Lemma 5.2 is also sufficient, thus obtaining a combinatorial characterization of the normal form of simple elements. We begin with a preliminary observation about the indices of the generators θ_i that may appear in words with no factor in $\mathcal{O}_\Sigma$.

Definition 5.4. We put $\mathsf{ht}(\varepsilon) := 0$, and, for w nonempty in Θ^*, we write $\mathsf{ht}(w)$ for the largest i such that θ_i occurs in w.

Lemma 5.5. *If $\theta_i v$ is $\mathcal{E}_H$-reduced with no factor in $\mathcal{O}_\Sigma$, then $\mathsf{ht}(v) \leqslant i+1$ holds.*

Proof. We use induction on $|v|$. For $|v| = 0$, the result is vacuously true. Assume $|v| \geqslant 1$, and write $v = \theta_j w$. As $\theta_i v$, *i.e.*, $\theta_i\theta_j w$, is $\mathcal{E}_H$-reduced, it contains no factor in $\mathcal{O}$, hence $j \geqslant i+3$ is excluded. On the other hand, as $\theta_i v$ has no factor in $\mathcal{O}_\Sigma$, the values $j = i$ and $j = i+2$ are impossible. So the only possible values for j are $1, ..., i-1$, and $i+1$.

Assume first $j \leqslant i-1$. As a factor of $\theta_i v$, the word $\theta_j w$ is reduced with no factor in $\mathcal{O}_\Sigma$. Then the induction hypothesis implies $\mathsf{ht}(w) \leqslant j+1$, whence $\mathsf{ht}(v) = \max(j, \mathsf{ht}(w)) \leqslant j+1 \leqslant i+1$, as expected.

Assume now $j = i+1$. The result is true for $|v| = 1$: the word $\theta_i\theta_{i+1}$ has no factor in $\mathcal{O}_\Sigma$ and its height is $i+1$. Assume now $|v| \geqslant 2$, and write $v = \theta_{i+1}\theta_k w$. As v has no factor in $\mathcal{O}$, the values $k \geqslant j+3 = i+4$ are forbidden, and, as $\theta_i v$ has no factor in $\mathcal{O}_\Sigma$, the values $k = i$, $k = i+1$, and $k = i+2$ are also excluded. So we must have $k \leqslant i-1$. As $\theta_k w$ is reduced with no factor in $\mathcal{O}_\Sigma$, the induction hypothesis implies $\mathsf{ht}(w) \leqslant k+1$, whence $\mathsf{ht}(v) = \max(i+1, k, \mathsf{ht}(w)) \leqslant \max(i+1, k+1) = i+1$, as expected. □

Completing the characterization of the normal forms of simple elements then relies on a long inductive argument.

Lemma 5.6. *If u is a reduced word of Θ^* with no factor in $\mathcal{O}_\Sigma$, then u is the normal form of a simple element with index at most* $\mathsf{ht}(u)+1$.

Proof. We will show using induction on $m \geqslant 0$ that, if u is an $\mathcal{E}_H$-reduced word with no factor in $\mathcal{O}_\Sigma$ and satisfying $\mathsf{ht}(u) = m$, then $[u] \preccurlyeq \Delta_{m+1}$ holds. This will imply that $[u]$ is simple with index $\leqslant \mathsf{ht}(u)+1$, giving the expected result when m varies. So, herafter, we assume that u is $\mathcal{E}_H$-reduced, has no factor in $\mathcal{O}_\Sigma$, and satisfies $\mathsf{ht}(u) = m$; our aim is to establish $[u] \preccurlyeq \Delta_{m+1}$. As can be expected, the various types of Proposition 4.11 will appear when we consider the possible cases.

For $m = 0$, the word u must be empty. We then find $[u] = 1 \preccurlyeq \Delta_1 = \Delta_{m+1}$, as expected. For $m = 1$, the only letter occurring in u is θ_1, so u is θ_1^ℓ for some $\ell \geqslant 1$. The assumption that u has no factor in $\mathcal{O}_\Sigma$ requires $\ell = 1$, whence $u = \theta_1$. We then find $[u] = \theta_1 \preccurlyeq \theta_1 = \Delta_2 = \Delta_{m+1}$, as expected.

From now on, we assume $m \geqslant 2$. The word u cannot be empty, so it has a first letter, say θ_i. By assumption, we have $m = \mathsf{ht}(u)$, hence $i \leqslant m$. On the other hand, Lemma 5.5 implies $\mathsf{ht}(u) \leqslant i+1$, hence $m \leqslant i+1$. Therefore, u must begin either by θ_m, or by θ_{m-1}.

Case 1. The first letter of u is θ_{m-1}, say $u = \theta_{m-1}v$. The word v cannot be empty, for otherwise we would have $u = \theta_{m-1}$ and $\mathsf{ht}(u) = m-1$, contradicting the assumption. Let θ_j be the first letter of v. By definition, we have $j \leqslant \mathsf{ht}(u) = m$. Moreover, u has no factor in $\mathcal{O}_\Sigma$, so $j = m-1$ is impossible. On the other hand, v, as a factor of u, is $\mathcal{E}_H$-reduced and has no factor in $\mathcal{O}_\Sigma$, so Lemma 5.5 implies $\mathsf{ht}(v) \leqslant j+1$, and $j \leqslant m-2$ would imply $\mathsf{ht}(u) \leqslant \max(m-1, \mathsf{ht}(v)) \leqslant m-1$, contradicting the assumption $m = \mathsf{ht}(u)$. So the only possibility is $j = m$, *i.e.*, u begins with $\theta_{m-1}\theta_m$, say $u = \theta_{m-1}\theta_m w$.

If w is the empty word, we have $u = \theta_{m-1}\theta_m$. Applying (4.5) twice gives $\Delta_{m+1} = \theta_{m-1}\theta_m\Delta_{m-0.5}\theta_{3m-2}$, which implies $[u] \preccurlyeq \Delta_{m+1}$, as expected.

Assume now that w is nonempty, and let θ_k be its first letter. As w is a factor of u, we must have $k \leqslant m$. As u, and its factor v, have no factor in $\mathcal{O}_\Sigma$, the values $k = m-1$ and $k = m$ are impossible as they would respectively create some factor $\theta_{m-1}\theta_m\theta_{m-1}$ and θ_m^2. So, we necessarily have $k < m-1$. Since w, as a factor of u, is $\mathcal{E}_H$-reduced and has no factor in $\mathcal{O}_\Sigma$, Lemma 5.5 implies $\mathsf{ht}(w) \leqslant k+1$, whence $\mathsf{ht}(w) \leqslant m-1$. The word w is $\mathcal{E}_H$-reduced with no factor in $\mathcal{O}_\Sigma$, so the induction hypothesis implies $[w] \preccurlyeq \Delta_{\mathsf{ht}(w)+1}$, hence a fortiori $[w] \preccurlyeq \Delta_m$. Moreover, we know that the first letter of w is not θ_{m-1}. By Lemma 5.3, we deduce $\theta_{m-1} \not\preccurlyeq [w]$, and then, by Lemma 4.7(ii), $[w] \preccurlyeq \Delta_{m-0.5}$. By definition, this means that $[u]$ belongs to $\Sigma^{\mathrm{II}_1}_{m+1,|u|}$ and, therefore, implies $[u] \preccurlyeq \Delta_{m+1}$, as expected.

Case 2. The first letter of u is θ_m, say $u = \theta_m v$. If v is empty, we have $u = \theta_m$, which has height m, and $[u] = \theta_m \preccurlyeq \Delta_{m+1}$, as expected.

We now suppose v nonempty. Let θ_j be its first letter. The assumption $\mathsf{ht}(u) = m$ implies $j \leqslant m$. As u has no factor in $\mathcal{O}_\Sigma$, the value $j = m$ is impossible, since it would create an initial factor θ_m^2. So we have $j \leqslant m-1$.

Subcase 2.1. We have $j \leqslant m-2$. Then Lemma 5.5 implies $\mathsf{ht}(v) \leqslant m-1$. Moreover, as a factor of u, the word v is $\mathcal{E}_H$-reduced and has no factor in $\mathcal{O}_\Sigma$. The induction hypothesis then implies $[v] \preccurlyeq \Delta_{\mathsf{ht}(v)+1}$, hence a fortiori $[v] \preccurlyeq \Delta_m$. Therefore, we have $[u] = \theta_m[v]$ with $[v] \preccurlyeq \Delta_m$. This means that $[u]$ lies in $\Sigma^{\mathrm{I}}_{m+1,|u|}$, implying $[u] \preccurlyeq \Delta_{m+1}$, as expected.

Subcase 2.2. We have $j = m-1$. Write $v = \theta_{m-1}w$, yielding $u = \theta_m\theta_{m-1}w$.

If w is empty, we have $u = \theta_m\theta_{m-1}$. Applying (4.5) twice gives the equality $\Delta_{m+1} = \theta_m\theta_{m-1}\Delta_{m-1}\theta_{3n-7}\theta_{3m-4}$, whence $[u] \preccurlyeq \Delta_{m+1}$, as expected.

We assume now that w is nonempty, with first letter θ_k. The assumption $\mathsf{ht}(u) = m$ implies $k \leqslant m$. Moreover, as u has no factor in $\mathcal{O}_\Sigma$, the value $k = m-1$ is impossible, since it would create in position 2 a factor θ_{m-1}^2.

Subsubcase 2.2.1. We have $k \leqslant m-2$. As a factor of u, the word w is $\mathcal{E}_H$-reduced and has no factor in $\mathcal{O}_\Sigma$, so Lemma 5.5 implies $\mathsf{ht}(w) \leqslant m-1$, whence $\mathsf{ht}(v) = m-1$. As a factor of u, the word v is $\mathcal{E}_H$-reduced and has no factor in $\mathcal{O}_\Sigma$, so the induction hypothesis implies $[v] \preccurlyeq \Delta_{\mathsf{ht}(v)+1}$, hence a fortiori $[v] \preccurlyeq \Delta_m$. This means that $[u]$ lies in $\Sigma^{\mathrm{I}}_{m+1,|u|}$ and implies $[u] \preccurlyeq \Delta_{m+1}$, as expected.

Subsubcase 2.2.2. We have $k = m$. Write $w = \theta_m u'$, yielding $u = \theta_m\theta_{m-1}\theta_m u'$. If u' is empty, we have $u = \theta_m\theta_{m-1}\theta_m$. A direct computation from (4.5) gives $\Delta_{m+1} = \theta_m\theta_{m-1}\theta_m\Delta_{m-1}\theta_{3n-7}$, whence $[u] \preccurlyeq \Delta_{m+1}$, as expected.

We assume now that u' is nonempty, with first letter θ_ℓ. The assumption $\mathsf{ht}(u) = m$ implies $\ell \leqslant m$. The assumption that u has no factor in $\mathcal{O}_\Sigma$ excludes $\ell = m-1$ and $\ell = m$, as these values would create factors $\theta_{m-1}\theta_m\theta_{m-1}$ or θ_m^2 in u. Next, Lemma 5.5 implies $\mathsf{ht}(u') \leqslant m-1$. Moreover, as a factor of u, the word u' is $\mathcal{E}_H$-reduced and has no factor in $\mathcal{O}_\Sigma$, so the induction hypothesis implies $[u'] \preccurlyeq \Delta_m$. By Lemma 5.3, if we had $\theta_{m-1} \preccurlyeq [u']$, the first letter of u' should be θ_{m-1}, contradicting $\ell \leqslant m-2$. Hence we have $\theta_{m-1} \not\preccurlyeq [u']$, whence $[u'] \preccurlyeq \Delta_{m-0.5}$ by Lemma 4.7. We then find $\theta_{m-1}[u'] \preccurlyeq \theta_{m-1}\Delta_{m-0.5} = \Delta_m$. Therefore, $[u]$ has the form $\theta_m\theta_{m-1}\theta_m[u']$ with $\theta_{m-1}[u'] \preccurlyeq \Delta_m$. This means that $[u]$ lies in $\Sigma^{\mathrm{II}_2}_{m+1,|u|}$ and implies $[u] \preccurlyeq \Delta_{m+1}$, as expected.

Thus, $[u] \preccurlyeq \Delta_{m+1}$ holds in every possible case, and this completes the proof. $\square$

Merging Lemmas 5.2 and 5.6, we finally obtain:

Proposition 5.7. *A word of Θ^* is the normal form of a simple element of H^+ if, and only if, it contains no factor in $\mathcal{O}$ or $\mathcal{O}_\Sigma$.*

Thus, the monoid H^+ gives rise to a Garside combinatorics that is quite similar to that of the Thompson monoid F^+. In both cases, we have a family of simple elements that is filtered by the sequence $(\Delta_n)_{n\geqslant 1}$, with finitely many elements below Δ_n, namely 2^{n-1} in the case of F^+ and $2\cdot 3^{n-2}$ in the case of H^+, and the normal forms of simple elements are characterized in terms of finitely many types of forbidden factors of length 2 or 3, namely the factors $\tau_i\tau_j$ with $j\geqslant i$ in the case of F^+, and the factors $\theta_i\theta_j$ with $j\geqslant i+2$ or $j=i$ and the factors $\theta_i\theta_{i+1}\theta_j$ with $j=i$ or $j=i+2$ in the case of H^+.

However, the parallel is not complete, as, in the case of H^+, simple elements do not form a Garside family. Indeed, the element $\theta_2\theta_4$ is not simple, although it right divides the simple element $\theta_1\theta_2\theta_4$, *i.e.*, Δ_3. It is easy to check that every element of H^+ admits a greatest simple left divisor, namely its greatest common left divisor with Δ_n for n sufficiently large, and, from there, to show for every element the existence of a greedy decomposition in terms of simple pieces, but the decompositions so obtained fail to obey the good properties that make Garside families interesting. In particular, the "domino rule" of [15, Prop. III.1.45], implying that the elements of H^+ have no well defined degree in terms of simple elements.

The enveloping group of the monoid H^+ is the group H defined by the presentation $\mathcal{P}_H$. At this point, the most puzzling open problem about H^+ is

Question 5.8. *Does the monoid H^+ embed in the group H?*

The monoid H^+ is cancellative, but some pairs of elements of H^+ fail to admit a common left multiple, or a common right multiple, and, therefore, contrary to F^+ and F, the group H is not a group of (left or right) fractions for H^+. As checking the Malcev conditions [9] for H^+ seems problematic, a more realistic way for proving that H^+ embeds in H could be to construct a faithful representation of H^+ in a group of matrices. No such representation is known so far, but mapping θ_i to the surjection F_i from $\mathbb{Z}_{>0}$ to itself defined by

$$F_i(k) := k \text{ for } k\leqslant i+1,\ F_i(i+2) := i,\ \text{and } F_i(k) := k-1 \text{ for } k\geqslant i+3$$

provides a representation ρ of H^+ that does not factor through F^+. The images of $\theta_1^2\theta_2$ and $\theta_1\theta_2\theta_3$ under ρ coincide, so ρ is not faithful, but experiments reported in [27] suggest that the polynomial deformation $\widetilde{\rho}$ of ρ that maps θ_i to the linear transformation $\widetilde{F}_i$ defined by $\widetilde{F}_i(\vec{x})_k := x_k$ for $k\leqslant i$, $\widetilde{F}_i(\vec{x})_k := x_{k-1}$ for $k\geqslant i+3$, plus

$$\widetilde{F}_i(\vec{x})_{i+1} := tx_i + (1-t)x_{i+1} \quad\text{and}\quad \widetilde{F}_i(\vec{x})_{i+2} := (1+t)x_i - tx_{i+1}$$

could be faithful. The involved matrices are not invertible, so proving that $\widetilde{\rho}$ is faithful would not solve Question 5.8 directly, but it could be a promising first step.

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