Combinatorial proofs of two Euler type identities due to Andrews

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Abstract

We prove combinatorially some identities related to Euler's partition identity (the number of partitions of n into distinct parts equals the number of partitions of n into odd parts). They were conjectured by Beck and proved by Andrews via generating functions. Let $a(n)$ be the number of partitions of n such that the set of even parts has exactly one element, $b(n)$ be the difference between the number of parts in all odd partitions of n and the number of parts in all distinct partitions of n, and $c(n)$ be the number of partitions of n in which exactly one part is repeated. Then, $a(n) = b(n) = c(n)$. The identity $a(n) = c(n)$ was proved combinatorially (in greater generality) by Fu and Tang. We prove combinatorially that $a(n) = b(n)$ and $b(n) = c(n)$. Our proof relies on bijections between a set and a multiset, where the partitions in the multiset are decorated with bit strings. Let $c_1(n)$ be the number of partitions of n such that there is exactly one part occurring three times while all other parts occur only once and let $b_1(n)$ to be the difference between the total number of parts in the partitions of n into distinct parts and the total number of different parts in the partitions of n into odd parts. We prove combinatorially that $c_1(n) = b_1(n)$. In addition to these results by Andrews, we prove combinatorially that $b_1(n) = a_1(n)$, where $a_1(n)$ counts partitions of n such that the set of even parts has exactly one element and satisfying some additional conditions. Moreover, we offer an analog of these results for the number of partitions of n with exactly one part occurring two times while all other parts occur only once.

Keywords: partitions, Euler's identity, bit strings, overpartitions.

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1 Introduction

In the theory of partitions, Euler's identity among the most famous and beautiful identities. It states that the number of partitions of n into distinct parts equals the number of partitions of n into odd parts. There are many combinatorial proofs of this identity. Among them, Glaisher's bijection is the most natural. Starting with a partition into odd parts, by merging equal parts repeatedly one obtains a partition into distinct parts. Conversely, starting with a partition into distinct parts, by splitting even parts repeatedly one obtains a partition into odd parts. It is natural to ask what happens if one relaxes the conditions in Euler's identity.

Let $\emptyset(n)$ be the set of partitions of n with all parts odd and let $\mathcal{D}(n)$ be the set of partitions of n with distinct parts.

Let $\mathcal{A}(n)$ be the set of partitions of n such that the set of even parts has exactly one element. Let $\mathcal{C}(n)$ be the set of partitions of n in which exactly one part is repeated.

Let $a(n) = |\mathcal{A}(n)|$ and $c(n) = |\mathcal{C}(n)|$. Let $b(n)$ be the difference between the number of parts in all odd partitions of n and the number of parts in all distinct partitions of n. Thus, $b(n)$ is the difference between the number of parts in all partitions in $\emptyset(n)$ and the number of parts in all partitions in $\mathcal{D}(n)$. In [2], Beck conjectured that $a(n) = b(n)$. In [1], Andrews proved that $a(n) = b(n) = c(n)$ using generating functions. In [4], Fu and Tang gave two generalizations of Andrew's result. For one of the generalizations, Fu and Tang give a combinatorial proof and, as a particular case, they obtain a combinatorial proof for $a(n) = c(n)$. We give a combinatorial proof for the identities involving $b(n)$.

Theorem 1.1. *Let $n \geq 1$. Then,*

(i) $a(n) = b(n)$

(ii) $c(n) = b(n)$.

The novelty of our approach to proving this theorem is the use of partitions decorated by bit strings. This allows us to create a bijection between a set of partitions and a multiset of partitions. We distinguish the partitions in the multiset via decorations with bit strings.

Let $\mathcal{T}(n)$ be the subset of $\mathcal{C}(n)$ consisting of partitions of n in which one part is repeated exactly three times and all other parts occur only once. Let $c_1(n) = |\mathcal{T}(n)|$. Let $b_1(n)$ to be the difference between the total number of parts in the partitions of n into distinct parts and the total number of *different* parts in the partitions of n into odd parts. Thus, $b_1(n)$ is the difference between the number of parts in all partitions in $\emptyset(n)$ and the number of different parts in all partitions in $\mathcal{D}(n)$ (i.e., parts counted without multiplicity).

Let $\mathcal{A}'(n)$ be the subset of $\mathcal{A}(n)$ consisting of partitions λ of n such that the set of even parts has exactly one element and satisfying the following two conditions:

1) the even part $2^k m$, $k \geq 1, m$ odd, has odd multiplicity and

2) the largest odd factor m of the even part is a part of λ with multiplicity between 1 and $2^k - 1$.

Let $a_1(n) = |\mathcal{A}'(n)|$.

We will prove combinatorially the following Theorem. Part (i) of the theorem was conjectured by Beck [3] and proved by Andrews [1] via generating functions.

Theorem 1.2. *Let $n \geq 1$. Then,*

(i) $c_1(n) = b_1(n)$.

(ii) $a_1(n) = b_1(n)$.

We also consider the case of partitions of n with exactly one part occurring two times while all other parts occur only once.

Let $\mathcal{S}(n)$ be the be the subset of $\mathcal{C}(n)$ consisting of partitions of n in which one part is repeated exactly two times and all other parts occur only once. Let $c_2(n) = |\mathcal{S}(n)|$.

Let $\mathcal{A}''(n)$ be the subset of $\mathcal{A}(n)$ consisting of partitions λ of n such that the set of even parts has exactly one element and satisfying the following two conditions:

1) the even part $2^k m$, $k \geq 1, m$ odd, has odd multiplicity and

2) the largest odd factor m of the even part is a part of λ with multiplicity between 0 and $2^k - 1$.

Let $a_2(n) = |\mathcal{A}''(n)|$.

Then, we have the following theorem.

Theorem 1.3. *Let $n \geq 1$. Then,* $a_2(n) = b_2(n) = c_2(n)$.

2 Preliminaries

Definition 1. A *bit string* w is a sequence of letters from the alphabet $\{0, 1\}$. The *length* of a bit string w, denoted $\ell(w)$, is the number of letters in w. We refer to *position* i in w as the ith entry from the right, where the most right entry is counted as position 0.

Note that leading zeros are allowed and are recorded. Thus 010 and 10 are different bit strings even though they are the binary representation of the same number. We have $\ell(010) = 3$ and $\ell(10) = 2$. The empty bit string has length 0 and is denoted by $\emptyset$.

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n is a sequence of non-increasing positive integers that add to n. We refer to n as the size of the partition λ and to $\lambda_1, \lambda_2, \ldots, \lambda_k$ as the parts of λ. Given a partition λ, the *length* of λ, denoted $\ell(\lambda)$, is the number of parts in λ. When using ℓ for the length, it will be clear if we mean the length of a partition or the length of a bit string as we denote partitions by lower greek letters and bit strings by w. When writing particular partitions, we separate the parts by commas. The bits of a bit string are not separated by commas.

We employ two conventions for writing partitions:

(1) in non-increasing order of parts; for example $\lambda = (7,7,4,3,2,2,2,1,1,1,1)$.

(2) using exponential notation $\lambda = (1^{j_1}, 2^{j_2}, \ldots)$, where j_i is the multiplicity of i in λ.

We often have to refer to the largest odd divisor of a part. Thus, we will write parts as $2^k m$, where $k \geq 0$ and m is odd. This is the only instance where exponents are used to mean powers. All other uses of exponents when writing partitions represent multiplicities of parts.

An overpartition of n is a partition of n where the first occurrence of each part may be overlined or not. For example, the overpartitions of 3 are:

$$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1.$$

3 Combinatorial proofs of Theorem 1.1

3.1 $b(n)$ as the cardinality of a multiset of partitions

In order to prove Theorem 1.1 combinatorially, we need to interpret the number $b(n)$ as the cardinality of a set.

First, we recall Glaisher's bijection φ used to prove Euler's identity. It is the map from the set of partitions with odd parts to the set of partitions with distinct maps which merges equal parts repeatedly.

Example 1. For $n = 7$, Glaisher's bijection is given by

$$\begin{array}{ccc}
(7) & \xrightarrow{\varphi} & (7) \\[1em]
(5, \underbrace{1,1}) & \longrightarrow & (5,2) \\[1em]
(\underbrace{3,3},1) & \longrightarrow & (6,1) \\[1em]
(3, \underbrace{\underbrace{1,1}, \underbrace{1,1}}) & \longrightarrow & (4,3) \\[1em]
(\underbrace{\underbrace{1,1}, \underbrace{1,1}}, \underbrace{1,1}, 1) & \longrightarrow & (4,2,1)
\end{array}$$

Thus, each partition $\lambda \in \mathcal{O}(n)$, has at least as many parts as its image $\varphi(\lambda) \in \mathcal{D}(n)$.

When calculating $b(n)$, the difference between the number of parts in all odd partitions of n and the number of parts in all distinct partitions of n, we sum up the differences in the number of parts in each pair $(\lambda, \varphi(\lambda))$. Write each part

μ_j of $\mu = \varphi(\lambda)$ as $\mu_j = 2^{k_j} m_j$ with m_j odd. Then, μ_j was obtained by merging 2^{k_j} parts in λ and thus contributes an excess of $2^{k_j} - 1$ parts to the difference. Therefore, the difference between the number of parts of λ and the number of parts of $\varphi(\lambda)$ is $\displaystyle\sum_{j=1}^{\ell(\varphi(\lambda))} (2^{k_j} - 1)$.

Definition 2. Given a partition μ with parts $\mu_j = 2^{k_j} m_j$, where m_j odd, the *weight* of the partition is $wt(\mu) = \displaystyle\sum_{j=1}^{\ell(\mu)} (2^{k_j} - 1)$.

Thus, $wt(\mu) > 0$ if and only if μ contains at least one even part.

We denote by $\mathcal{MD}(n)$ the multiset of partitions of n with distinct parts in which every partition $\mu \in \mathcal{D}(n)$ appears exactly $wt(\mu)$ times. For example, $wt(4, 2, 1) = 4$ and $(4, 2, 1)$ appears four times in $\mathcal{MD}(7)$. Since $wt(7) = 0$, the partition (7) does not appear in $\mathcal{MD}(7)$.

The discussion above proves the following interpretation of $b(n)$.

Proposition 1. *Let $n \geq 1$. Then, $b(n) = |\mathcal{MD}(n)|$.*

To create bijections from $\mathcal{A}(n)$ to $\mathcal{MD}(n)$ and from $\mathcal{C}(n)$ to $\mathcal{MD}(n)$, we need to distinguish identical elements of $\mathcal{MD}(n)$ and thus view it as a set. Recall that all partitions in $\mathcal{MD}(n)$ have distinct parts and at least one even part.

Definition 3. A *decorated partition* is a partition μ with at least one even part in which one single *even* part, called the decorated part, has a bit string w as an index. If the decorated part is $\mu_i = 2^k m$, where $k \geq 1$ and m is odd, the index w has length $0 \leq \ell(w) \leq k - 1$.

Since there are 2^t distinct bit strings of length t, there are $2^k - 1$ distinct bit strings w of length $0 \leq \ell(w) \leq k - 1$. Thus, for each even part $\mu_i = 2^k m$ of μ there are $2^k - 1$ possible indices and for each partition μ there are precisely $wt(\mu)$ possible decorated partitions with the same parts as μ.

We denote by $\mathcal{DD}(n)$ the set of decorated partitions of n with distinct parts. Note that, by definition, a decorated partition has at least one even part. Then,

$$|\mathcal{MD}(n)| = |\mathcal{DD}(n)|$$

and therefore

$$b(n) = |\mathcal{DD}(n)|.$$

3.2 A combinatorial proof for $a(n) = b(n)$

We prove that $a(n) = b(n)$ by establishing a one-to-one correspondence between $\mathcal{A}(n)$ and $\mathcal{DD}(n)$.

From $\mathcal{DD}(n)$ to $\mathcal{A}(n)$:

Start with a decorated partition $\mu \in \mathcal{DD}(n)$. Suppose part $\mu_i = 2^k m$, with $k \geq 1$ and m odd, is decorated with bit string w of length $\ell(w)$. Then, $0 \leq \ell(w) \leq k-1$. Let d_w be the decimal value of w. We set $d_\emptyset = 0$.

The decorated part will split into two kinds of parts: $2^j m$ and m. The length of w, $\ell(w)$, determines the size of $2^j m$ and d_w determines the number of parts of size $2^j m$ that split from the decorated part. All even parts with the same odd part that are larger than the decorated part split into parts equal to $2^j m$. All even parts with the same odd part that are less than the decorated part split into parts equal to m. Each of the other even parts splits completely into odd parts. We describe the precise algorithm below.

1. Split part μ_i into $d_w + 1$ parts of size $2^{k-\ell(w)} m$ and parts of size m. Thus, there will be $2^k - (d_w + 1)2^{k-\ell(w)}$ parts of size m. Since $d_w + 1 \leq 2^{\ell(w)}$, the resulting number of parts equal to m is non-negative. Moreover, after the split there is at least one even part.

2. Every part of size $2^t m$, with $t > k$ (if it exists), splits completely into parts of size $2^{k-\ell(w)} m$, i.e., into $2^{t-k+\ell(w)}$ parts of size $2^{k-\ell(w)} m$.

3. Every other even part splits into odd parts of equal size. I.e., every part $2^u v$ with v odd, such that $2^u v \neq 2^s m$ for some $s \geq k$, splits into 2^u parts of size v.

The resulting partition, λ is in $\mathcal{A}(n)$. Its set of even parts is $\{2^{k-\ell(w)} m\}$.

Example 2. Consider the decorated partition

$$\mu = (96, 35, 34, 24_{01}, 6, 2) = (2^5 \cdot 3, 35, 2 \cdot 17, (2^3 \cdot 3)_{01}, 2 \cdot 3, 2 \cdot 1) \in \mathcal{DD}(197).$$

We have $k = 3, m = 3, \ell(w) = 2, d_w = 1$.

1. Part $24 = 2^3 \cdot 3$ splits into two parts of size 6 and four parts of size 3.

2. Part $96 = 2^5 \cdot 3$ splits into 16 parts of size 6.

3. All other even parts split into odd parts. Thus, part 34 splits into two parts of size 17, part 6 splits into two parts of size 3, and part 2 splits into two parts of size 1.

The resulting partition is $\lambda = (35, 17^2, 6^{18}, 3^6, 1^2) \in \mathcal{A}(197)$.

Similarly, the transformation maps the decorated partition $(96, 35, 34, 24_{10}, 6, 2) \in \mathcal{DD}(197)$ to $(35, 17^2, 6^{19}, 3^4, 1^2) \in \mathcal{A}(197)$.

From $\mathcal{A}(n)$ to $\mathcal{DD}(n)$:

Start with partition $l \in \mathcal{A}(n)$. Then there is one and only one even number $2^k m$, $k \geq 1, m$ odd, among the parts of l. Let s be the multiplicity of the even part in l. As in Glaisher's bijection, we merge equal parts repeatedly until we obtain a partition μ with distinct parts. Since l has an even part, μ will also have an even part.

Next, we determine the decoration of μ. Consider the parts μ_{j_i} of the form $2^{r_i}m$, with m odd and $r_i \geq k$. We have $j_1 < j_2 < \cdots$. For notational convenience, set $\mu_{j_0} = 0$. Let h be the positive integer such that

$$\sum_{i=0}^{h-1} \mu_{j_i} < s \cdot 2^k m \leq \sum_{i=0}^{h} \mu_{j_i}. \tag{1}$$

Then, we will decorate part $\mu_{j_h} = 2^{r_h}m$.

To determine the decoration, let N_h be the number of parts $2^k m$ in l that merged to form all parts of the form $2^r m > \mu_{j_h}$. Thus,

$$N_h = \frac{\sum_{i=0}^{h-1} \mu_{j_i}}{2^k m}.$$

Then, (1) becomes

$$2^k m N_h < s \cdot 2^k m \leq 2^k m N_h + 2^{r_h} m,$$

which in turn implies $N_h < s \leq N_h + 2^{r_h - k}$. Therefore, $0 < s - N_h \leq 2^{r_h - k}$.

Let $d = s - N_h - 1$ and $\ell = r_h - k$. We have $0 \leq \ell \leq r_h - 1$. Consider the binary representation of d and insert leading zeros to form a bit string w of length ℓ. Decorate μ_{j_h} with w. The resulting decorated partition is in $\mathcal{DD}(N)$.

Example 3. Consider the partition $\lambda = (35, 17^2, 6^{18}, 3^6, 1^2) \in \mathcal{A}(197)$. We have $k = 1, m = 3, s = 18$. Glaisher's bijection produces the partition $\mu = (96, 35, 34, 24, 6, 2) \in \mathcal{MD}(197)$. The parts of the form $2^{r_i} \cdot 3$ with $r_i \geq 1$ are $96, 24, 6$. Since $96 < 18 \cdot 6 \leq 96 + 24$ the decorated part will be $24 = 2^3 \cdot 3$. We have $N_h = 96/6 = 16$. To determine the decoration, let $d = 18 - 16 - 1 = 1$ and $\ell = 3 - 1 = 2$. The binary representation of d is 1. To form a bit string of length 2 we introduce one leading 0. Thus, the decoration is $w = 01$ and the resulting decorated partition is $(96, 35, 34, 24_{01}, 6, 2) \in \mathcal{DD}(197)$. Similarly, starting with $(35, 17^2, 6^{19}, 3^4, 1^2) \in \mathcal{A}(197)$, after applying Glaisher's bijection we obtain $\mu = (96, 35, 34, 24, 6, 2) \in \mathcal{MD}(197)$. All parameters are the same as in the previous example with the exception of $s = 19$. As before, the decorated part is 24 and $\ell = 2$. We have $d = 19 - 16 - 1 = 2$ whose binary representation is 10 and already has length 2. Thus $w = 10$ and the resulting decorated partition is $(96, 35, 34, 24_{10}, 6, 2) \in \mathcal{DD}(197)$.

3.3 A combinatorial proof for $c(n) = b(n)$

We could compose the bijection of section 3.2 with the bijection of [4] (for $k = 2$) to obtain a combinatorial proof of part (ii) of Theorem 1.1. We give an alternative proof that $c(n) = b(n)$ by establishing a one-to-one correspondence between $\mathcal{C}(n)$ and $\mathcal{DD}(n)$ that does not involve the bijection of [4].

From $\mathcal{DD}(n)$ to $\mathcal{C}(n)$:

Start with a decorated partition $\mu \in \mathcal{DD}(n)$. Suppose part $\mu_i = 2^k m$, with $k \geq 1$ and m odd, is decorated with bit string w of length $\ell(w)$ and decimal value d_w. Then, $0 \leq \ell(w) \leq k-1$.

If $w = \emptyset$, split part μ_i into two equal parts of size $2^{k-\ell(w)-1}m$. The partition λ obtained in this way from μ is in $\mathcal{C}(n)$. If $w \neq \emptyset$ we obtain λ from μ by performing the following steps.

1. Split μ_i into $2(d_w+1)$ parts of size $2^{k-\ell(w)-1}m$ and a part of size $2^{i+k-\ell(w)}m$ for each i such that there is a 0 in position i in w.

2. Each part $2^t m$ with $k-\ell(w)-1 \leq t < k$ splits completely into parts of size $2^{k-\ell(w)-1}m$, i.e., into $2^{t-k+\ell(w)+1}$ parts of size $2^{k-\ell(w)-1}m$.

Since $2(d_w+1) \geq 2$, the obtained partition λ is in $\mathcal{C}(n)$. The repeated part is $2^{k-\ell(w)-1}m$.

Note: In step 1. above, after splitting off $2(d_w+1)$ parts of size $2^{k-\ell(w)-1}m$ from μ_i, we are left with $r = 2^{k-\ell(w)}(2^{\ell(w)} - d_w - 1)m$ to split into distinct parts. We do this by splitting repeatedly as in Glaisher's bijection. Thus, after the splitting, we will have a part equal to $2^j m$ if and only if the binary representation of r has a 1 in position j. However, $2^{\ell(w)} - 1$ is a bit string of length $\ell(w)$ with every entry equal to 1. Then, the binary representation of $2^{\ell(w)} - d_w - 1$ (filled with leading zeros if necessary to create a bit string of length $\ell(w)$) is precisely the complement of w, i.e., the bit string obtained from w by replacing every 0 by 1 and every 1 by 0.

Example 4. Consider the decorated partition

$$\begin{aligned}\mu &= (768, 384_{0110}, 105, 96, 25, 12, 9, 6, 2)\\ &= (2^8\cdot 3, (2^7\cdot 3)_{0110}, 105, 2^5\cdot 3, 25, 2^2\cdot 3, 9, 2\cdot 3, 2\cdot 1) \in \mathcal{DD}(1407).\end{aligned}$$

We have $k = 7, w = 0110, \ell(w) = 4, d_w = 6$. The decorated part is μ_2.

1. Since $2(d_w+1) = 14$ and w has zeros in positions 0 and 3, μ_2 splits into 14 parts of size $2^2\cdot 3$ and one part each of sizes $2^3\cdot 3$ and $2^6\cdot 3$.

2. The parts of the form $2^t\cdot 3$ with $2 \leq t < 7$ are $\mu_4 = 2^5\cdot 3$ and $\mu_6 = 2^2\cdot 3$. Then, μ_4 splits into 2^3 parts of size $2^2\cdot 3$ and μ_6 "splits" into one part of size $2^2\cdot 3$.

We obtain the partition

$$\begin{aligned}\lambda &= (2^8\cdot 3, 2^6\cdot 3, 105, 25, 2^3\cdot 3, \underbrace{2^2\cdot 3, \ldots, 2^2\cdot 3}_{23 \text{ times}}, 9, 2\cdot 3, 2\cdot 1)\\ &= (768, 192, 105, 25, 24, 12^{23}, 9, 6, 2) \in \mathcal{C}(n).\end{aligned}$$

From $\mathcal{C}(n)$ to $\mathcal{DD}(n)$:

Start with partition $l \in \mathcal{C}(n)$. Then there is one and only one repeated part among the parts of l. Suppose the repeated part is $2^k m$, $k \geq 0, m$ odd, and denote by $s \geq 2$ its multiplicity in l. As in Glaisher's bijection, we merge equal parts repeatedly until we obtain a partition μ with distinct parts. Since l has a repeated part, μ will have at least one even part.

Next, we determine the decoration of μ. In this case, we want to work to the parts of μ from the right to the left (i.e., from smallest to largest part). Let $\tilde{\mu}_q = \mu_{\ell(\mu)-q+1}$. Consider the parts $\tilde{\mu}_{j_i}$ of the form $2^{r_i}m$, with m odd and $r_i \geq k$. We have $j_1 < j_2 < \cdots$.

As before, we set $\tilde{\mu}_{j_0} = 0$. Let h be the positive integer such that

$$\sum_{i=0}^{h-1} \tilde{\mu}_{j_i} < s \cdot 2^k m \leq \sum_{i=0}^{h} \tilde{\mu}_{j_i}. \tag{2}$$

Then, we will decorate part $\tilde{\mu}_{j_h} = 2^{r_h}m$.

To determine the decoration, let N_h be the number of parts $2^k m$ in l that merged to form all parts of the form $2^r m < \tilde{\mu}_{j_h}$. Thus,

$$N_h = \frac{\sum_{i=0}^{h-1} \tilde{\mu}_{j_i}}{2^k m}.$$

Then, (2) becomes

$$2^k m N_h < s \cdot 2^k m \leq 2^k m N_h + 2^{r_h} m,$$

which in turn implies $N_h < s \leq N_h + 2^{r_h - k}$. Therefore, $0 < s - N_h \leq 2^{r_h - k}$.

Let $d = \frac{s - N_h}{2} - 1$ and $\ell = r_h - k - 1$. We have $0 \leq \ell \leq r_h - 1$. Consider the binary representation of d and insert leading zeros to form a bit string w of length ℓ. Decorate $\tilde{\mu}_{j_h}$ with w. The resulting decorated partition (with parts written in non-increasing order) is in $\mathcal{DD}(N)$.

Note: To see that $s - N_h$ above is always even consider the three cases below.

(i) If $h = 1$, then $N_h = 0$. In this case we must have had $s = 2$. Thus, $s - N_h$ is even.

(ii) If s is odd, then after the merge we have one part equal to $2^k m$ contributing to N_h. All other parts contributing to N_h are divisible by $2 \cdot 2^k m$. Thus, N_h is odd and $s - N_h$ is even.

(iii) If s is even and at least 2, then after the merge we have no part equal to $2^k m$ contributing to N_h. All parts contributing to N_h are divisible by $2 \cdot 2^k m$. Thus, N_h is even and $s - N_h$ is even.

Example 5. Consider the partition

$$\begin{aligned}\lambda &= (2^8 \cdot 3, 2^6 \cdot 3, 105, 25, 2^3 \cdot 3, \underbrace{2^2 \cdot 3, \ldots, 2^2 \cdot 3}_{23 \text{ times}}, 9, 2 \cdot 3, 2 \cdot 1) \\ &= (768, 192, 105, 25, 24, 12^{23}, 9, 6, 2) \in \mathcal{C}(n).\end{aligned}$$

We have $k = 2$ and $s = 23$. Glaisher's bijection transforms λ as follows.

$$(2^8 \cdot 3, 2^6 \cdot 3, 105, 25, 2^3 \cdot 3, \underbrace{2^2 \cdot 3, \dots, 2^2 \cdot 3}_{23 \text{ times}}, 9, 2 \cdot 3, 2 \cdot 1)$$
$$\downarrow$$
$$(2^8 \cdot 3, 2^6 \cdot 3, 105, 25, \underbrace{2^3 \cdot 3, \dots, 2^3 \cdot 3}_{12 \text{ times}}, 2^2 \cdot 3, 9, 2 \cdot 3, 2 \cdot 1)$$
$$\downarrow$$
$$(2^8 \cdot 3, 2^6 \cdot 3, 105, \underbrace{2^4 \cdot 3, \dots, 2^4 \cdot 3}_{6 \text{ times}}, 25, 2^2 \cdot 3, 9, 2 \cdot 3, 2 \cdot 1)$$
$$\downarrow$$
$$(2^8 \cdot 3, 2^6 \cdot 3, 105, 2^5 \cdot 3, 2^5 \cdot 3, 2^5 \cdot 3, 25, 2^2 \cdot 3, 9, 2 \cdot 3, 2 \cdot 1)$$
$$\downarrow$$
$$(2^8 \cdot 3, 2^6 \cdot 3, 2^6 \cdot 3, 105, 2^5 \cdot 3, 25, 2^2 \cdot 3, 9, 2 \cdot 3, 2 \cdot 1)$$
$$\downarrow$$
$$\mu = (2^8 \cdot 3, 2^7 \cdot 3, 105, 2^5 \cdot 3, 25, 2^2 \cdot 3, 9, 2 \cdot 3, 2 \cdot 1)$$

The parts of the form $2^r \cdot 3$ with $r \geq 2$ are $\tilde{\mu}_4 = 2^2 \cdot 3 = 12, \tilde{\mu}_6 = 2^5 \cdot 3 = 96, \tilde{\mu}_8 = 2^7 \cdot 3 = 384$, and $\tilde{\mu}_9 = 2^8 \cdot 3 = 768$. Since $12 + 96 < 23 \cdot 2^2 \cdot 3 \leq 12 + 96 + 384$, the decorated part will be $2^7 \cdot 3 = 384$. We have $h = 3$ and $N_3 = \frac{2^2 \cdot 3 + 2^5 \cdot 3}{2^2 \cdot 3} = 1 + 2^3 = 9$. Thus $d = \frac{23-9}{2} - 1 = 6$ and $\ell = 7 - 2 - 1 = 4$. Thus $w = 0110$ and the resulting decorate partition is

$$\begin{aligned} \mu &= (2^8 \cdot 3, (2^7 \cdot 3)_{0110}, 105, 2^5 \cdot 3, 25, 2^2 \cdot 3, 9, 2 \cdot 3, 2 \cdot 1) \\ &= (768, 384_{0110}, 105, 96, 25, 12, 9, 6, 2) \in \mathcal{DD}(1407). \end{aligned}$$

4 Combinatorial proof of Theorem 1.2

4.1 $b_1(n)$ as the cardinality of a set of overpartitions

As in section 3.1, we use Glaisher's bijection and calculate $b_1(n)$ by summing up the difference between the number of parts of $\varphi(\lambda)$ and the number of different parts of λ of each partition $\lambda \in \mathcal{O}(n)$. In $\varphi(\lambda)$ there is be a part of size $2^i m$, with m odd if and only if there is an 1 is position i of the binary representation of the multiplicity of m in λ. After the merge, each odd part in λ creates as many parts in $\varphi(\lambda)$ as the number of 1s in the binary representation of its multiplicity. Moreover, if we write the parts of $\varphi(\lambda)$ as $2^{k_i} m_i$ with m_i odd and $k_i \geq 0$, all parts $2^s m$ with the same largest odd factor m are obtained by merging parts equal to m in λ.

For each positive odd integer $2j-1$, denote by $oddm(2j-1)$ the number of parts of $\varphi(\lambda)$ of the form $2^s(2j-1)$ for some $s \geq 0$. Then, given $\lambda \in \emptyset(n)$, the difference between the number of parts of $\varphi(\lambda)$ and the number of different parts of λ equals

$$\sum_{\substack{j \\ oddm(2j-1)\neq 0}} (oddm(2j-1)-1).$$

Let $\overline{\mathcal{D}}(n)$ be the set of overpartitions of n with distinct parts in which exactly one part is overlined. Part 2^sm with $s \geq 0$ and m odd may be overlined only if there is a part 2^tm with $t < s$. In particular, no odd part can be overlined. By an overpartition with distinct parts we mean that all parts have multiplicity one, In particular, p and $\bar{p}$ cannot not both appear as parts of the overpartition. The discussion above proves the following interpretation of $b_1(n)$.

Proposition 2. *Let $n \geq 1$. Then, $b_1(n) = |\overline{\mathcal{D}}(n)|$.*

4.2 A combinatorial proof for $c_1(n) = b_1(n)$

From $\overline{\mathcal{D}}(n)$ to $\mathcal{T}(n)$:

Start with an overpartition $\mu \in \overline{\mathcal{D}}(n)$. Suppose the overlined part is $\mu_i = 2^sm$ for some $s \geq 1$ and m odd. Then there is a part $\mu_j = 2^tm$ of μ with $t < s$. Let k be the largest positive integer such that s^km is a part of μ and $k < s$. To obtain $\lambda \in \mathcal{T}(n)$ from μ, split μ_i into two parts equal to 2^km and one part equal to 2^jm whenever there is a 1 in position j of the binary representation of $(2^s - 2^{k+1})$, i.e, one part equal to 2^jm for each $j = k+1, k+2, \ldots, s-1$.

Example 6. Let

$$\mu = (\overline{768}, 48, 46, 9, 6, 5, 2) = (\overline{2^8 \cdot 3}, 2^4 \cdot 3, 2 \cdot 23, 9, 2 \cdot 3, 5, 2 \cdot 1) \in \overline{\mathcal{D}}(884).$$

Then $2^8 \cdot 3$ splits into two parts equal to $2^4 \cdot 3$ and one part each of size $2^5 \cdot 3, 2^6 \cdot 3, 2^7 \cdot 3$. Thus, we obtain the partition

$$\lambda = (2^7 \cdot 3, 2^6 \cdot 3, 2^5 \cdot 3, 2^4 \cdot 3, 2^4 \cdot 3, 2^4 \cdot 3, 2 \cdot 23, 9, 2 \cdot 3, 5, 2 \cdot 1)$$
$$= (384, 192, 96, 48^3, 46, 9, 6, 5, 2) \in \mathcal{T}(884).$$

Similarly, $(768, \overline{48}, 46, 9, 6, 5, 2) = (2^8 \cdot 3, \overline{2^4 \cdot 3}, 2 \cdot 23, 9, 2 \cdot 3, 5, 2 \cdot 1) \in \overline{\mathcal{D}}(884)$ transforms into $(2^8 \cdot 3, 2 \cdot 23, 2^3 \cdot 3, 2^2 \cdot 3, 9, 2 \cdot 3, 2 \cdot 3, 2 \cdot 3, 5, 2 \cdot 1) =$ $(768, 46, 24, 12, 9, 6^3, 5, 2) \in \mathcal{T}(884)$.

From $\mathcal{T}(n)$ to $\overline{\mathcal{D}}(n)$:

Start with a partition $\lambda \in \mathcal{T}(n)$. Merge the parts of λ repeatedly using Glaisher's bijection φ to obtain a partition μ with distinct parts. Overline the smallest part of μ that is not a part of λ. Note that if the thrice repeated part of λ is 2^km for some $k \geq 0$ and m odd, then in μ there is a part equal to 2^km and the overlined part is of the form 2^tm for some $t > k$. Thus, we obtain an overpartition in $\overline{\mathcal{D}}(n)$.

Example 7. Let

$$\lambda = (2^7 \cdot 3, 2^6 \cdot 3, 2^5 \cdot 3, 2^4 \cdot 3, 2^4 \cdot 3, 2^4 \cdot 3, 2 \cdot 23, 9, 2 \cdot 3, 5, 2 \cdot 1)$$
$$= (384, 192, 96, 48^3, 46, 9, 6, 5, 2) \in \mathcal{T}(884).$$

Merging equal parts as in Glaisher's bijection, we obtain the partition $\mu = (768, 48, 46, 9, 6, 5, 2) = (2^8 \cdot 3, 2^4 \cdot 3, 2 \cdot 23, 9, 2 \cdot 3, 5, 2 \cdot 1) \in \mathcal{D}(884)$. The smallest part of μ that is not a part of λ is 768. Thus, we obtain the overpartition $(\overline{768}, 48, 46, 9, 6, 5, 2) \in \overline{\mathcal{D}}(884)$.

Similarly, after applying Glaisher's bijection, the partition

$$\lambda = (2^8 \cdot 3, 2 \cdot 23, 2^3 \cdot 3, 2^2 \cdot 3, 9, 2 \cdot 3, 2 \cdot 3, 2 \cdot 3, 5, 2 \cdot 1)$$
$$= (768, 46, 24, 12, 9, 6^3, 5, 2) \in \mathcal{T}(884)$$

maps to $\mu = (768, 48, 46, 9, 6, 5, 2) = (2^8 \cdot 3, 2^4 \cdot 3, 2 \cdot 23, 9, 2 \cdot 3, 5, 2 \cdot 1) \in \mathcal{D}(884)$. The smallest part of μ not appearing in λ is 48. Thus, we obtain the overpartition $(768, \overline{48}, 46, 9, 6, 5, 2) \in \overline{\mathcal{D}}(884)$.

Remark 1. We could have obtained the transformation above from the combinatorial proof of part (ii) of Theorem 1.1. In the transformation from $\mathcal{C}(n)$ to $\mathcal{DD}(n)$, we have $s = 3$, $h = 1$, and $N_h = 1$. Thus $d = 0$ and the decorated part is the smallest part in the transformed partition μ that did not occur in the original partition λ. Then

$$r_h = 1 + \max\{j \mid 2^k m, 2^{k+1} m, \ldots, 2^{k+i} m \text{ are all parts of } \lambda\}.$$

Thus, in μ, the decorated part $2^{r_h} m$ is decorated with a bit string consisting of all zeros and of length $r_h - k - 1$, one less than the difference in exponents of 2 of the decorated part and the next smallest part with the same largest odd factor m. Since the decoration of a partition in $\mathcal{DD}(n)$ is completely determined by the part being decorated, we could simply just overline the part.

4.3 A combinatorial proof for $a_1(n) = b_1(n)$

Once we proved the identity $c_1(n) = b_1(n)$ which was conjectured by Beck and proved by Andrews, is was natural to look for the analogue of Theorem 1.1 (i) involving $b_1(n)$. If in Euler's partition identity we relax the condition in $\mathcal{D}(n)$ to allow one part to be repeated exactly three times, how should we relax the condition on $\mathcal{O}(n)$ to obtain an identity? We can search for the condition by following the proof of Theorem 1.1 part (i) but only for decorated partitions from $\overline{\mathcal{D}}(n)$, were an even part is decorated with a bit string consisting entirely of zeros as in Remark 1. Following the algorithm, we see that the set $\mathcal{T}(n)$ has the same cardinality as the set of partitions with exactly one even part $2^k \cdot m$, $k \geq 1, m$ odd, which appears with odd multiplicity. Moreover, part m appears with multiplicity at least 2^{k-1} and the multiplicity of m must belong to an interval $[2^s - 2^{k-1}, 2^s - 1]$ for some $s \geq k$. Given the elegant description of the partitions in $\mathcal{T}(n)$ it would be desirable to find a nicer set of the same cardinality consisting of partitions with only one even part under some constraints. Below we suggest such a set.

Let $\mathcal{A}'(n)$ be the subset of $\mathcal{A}(n)$ consisting of partitions λ of n such that the set of even parts has exactly one element and satisfying the following two conditions:

1) the even part $2^k m$, $k \geq 1, m$ odd, has odd multiplicity and

2) the largest odd factor m of the even part is a part of λ with multiplicity between 1 and $2^k - 1$.

Let $a_1(n) = |\mathcal{A}'(n)|$. We show that $a_1(n) = b_1(n)$.

Proof. From $\overline{\mathcal{D}}(n)$ to $\mathcal{A}'(n)$:

Start with an overpartition $\mu \in \overline{\mathcal{D}}(n)$. Suppose the overlined part is $\mu_i = 2^s m, s \geq 1, m$ odd. Then there is a part $\mu_j = 2^t m$ of μ with $0 \leq t < s$. Keep part $2^s m$ and remove its overline. Split each part of the form $2^u m$ with $u > s$ (if it exists) into 2^{u-s} parts equal to $2^s m$. Split each part of the form $2^v m$ with $0 \leq v < s$ into 2^v parts equal to m. Split every other even part into odd parts. Call the obtained partition λ. Then the multiplicity of $2^s m$ in λ is odd. Since there is a part $\mu_j = 2^t m$ of μ with $0 \leq t < s$, there will be at least one part equal to m in λ. The largest possible multiplicity of m in λ is $2^{s-1} + 2^{s-2} + \cdots + 2 + 1 = 2^s - 1$. Thus $\lambda \in \mathcal{A}'(n)$.

From $\mathcal{A}'(n)$ to $\overline{\mathcal{D}}(n)$:

Let $\lambda \in \mathcal{A}'(n)$. Merge equal terms repeatedly (as in Glaisher's bijection) to obtain a partition with distinct parts. Overline the part equal to the even part in λ. Call the obtained overpartition μ. Since the even part $2^k m$ in λ has odd multiplicity, there will be a part in μ equal to $2^k m$. Since m has multiplicity between 1 and $2^k - 1$ in λ, there will be a part of size $2^i m$ in μ whenever there is a 1 in position i in the binary representation of the multiplicity of m in λ. The binary representation of $2^k - 1$ is a bit string of length $k - 1$ consisting entirely of ones. Thus, in μ there is at least one part of size $2^t m$ with $0 \leq t < k$ and $\mu \in \overline{\mathcal{D}}(n)$. $\square$

5 Combinatorial proof of Theorem 1.3

Recall that $\mathcal{S}(n)$ is the be the subset of $\mathcal{C}(n)$ consisting of partitions of n in which one part is repeated exactly two times and all other parts occur only once and $c_2(n) = |\mathcal{S}(n)|$.

Suppose that in the proof of Theorem 1.1 part (ii) we let $s = 2$ in the transformation from $\mathcal{C}(n)$ to $\mathcal{DD}(n)$. Then $h = 1$ and $N_h = 0$. As in Remark 1, the decoration of the partition in $\mathcal{DD}(n)$ is completely determined by the part being decorated and we can just overline the part. Moreover, after merging equal parts in of a partition in $\mathcal{S}(n)$, in the obtained partition there might be no part with the same largest odd factor as the merged parts. Thus, we obtain a bijection from $\mathcal{S}(n)$ to the set $\overline{\mathcal{D}}'(n)$ of overpartitions of n with distinct parts in which exactly one part is overlined and the overlined part is even. By a similar argument to that in Section 4.1, $|\overline{\mathcal{D}}'(n)|$ equals the number $b_2(n)$ of parts in all partitions of n into distinct parts.

Moreover, we defined $\mathcal{A}''(n)$ to be the subset of $\mathcal{A}(n)$ consisting of partitions λ of n such that the set of even parts has exactly one element and satisfying the following two conditions:

1) the even part $2^k m$, $k \geq 1, m$ odd, has odd multiplicity and

2) the largest odd factor m of the even part is a part of λ with multiplicity between 0 and $2^k - 1$.

Let $a_2(n) = |\mathcal{A}''(n)|$.

The combinatorial proof for $a_2(n) = b_2(n)$ follows closely the proof of Theorem 1.2 part (ii) in section 4.3.

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