ON ENUMERATION OF DYCK PATHS WITH COLORED HILLS

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ABSTRACT. We continue to investigate the properties of the earlier defined functions f_m and g_m, which depend on an initial arithmetic function f_0. In this papers values of f_0 are the Fine numbers. We investigate functions $f_i, g_i, (i = 1, 2, 3, 4)$. For each function, we derive an explicit formula and give a combinatorial interpretation. It appears that g_2 and g_3 are well-known combinatorial object called the Catalan triangles.

We finish with an identity consisting of ten items.

1. INTRODUCTION

This paper is a continuation of the investigations of restricted words from the author's previous papers, where two quantities $f_m(n)$ and $g_m(n,k)$ are defined as follows. For an initial arithmetic function f_0, $f_m, (m > 1)$ is the mth invert transform of f_0. The function $g_m(n,k)$ is defined in the following way:

$$g_m(n,k) = \sum_{i_1+i_2+\cdots+i_k=n} f_{m-1}(i_1)\cdot f_{m-1}(i_2)\cdots f_{m-1}(i_k). \tag{1}$$

Also, the following equation holds:

$$f_m(n) = \sum_{k=1}^{n} g_m(n,k). \tag{2}$$

We restate [2, Propositions 10] which will be used throughout the paper.

Proposition 1. Let f_0 the arithmetic function which values are nonnegative integers, and $f_0(1) = 1$. Assume next that, for $n \geq 1$, we have $f_{m-1}(n)$ words of length $n-1$ over a finite alphabet α. Let x be a letter which is not in α. Then, the value of $g_m(n,k)$ is the number of words of length $n-1$ over the alphabet $\alpha \cup \{x\}$ in which x appears exactly $k-1$ times.

We denote by $G_m(n)$ the array $g_m(n,k)$ viewed as a lower triangular matrix of order n. It is proved in [2, Proposition 6] that

$$G_m(n) = G_1(n)\cdot L_n^{m-1}. \tag{3}$$

In particular, we have

$$g_m(n,k) = \sum_{i=k}^{n} \binom{i-1}{k-1} c_{m-1}(n,i), \tag{4}$$

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and

$$\sum_{n=k}^{\infty} g_m(n,k)x^n = \left(\sum_{i=1}^{\infty} f_{m-1}(i)x^i\right)^k. \tag{5}$$

2. A combinatorial result

We start with a combinatorial proof that the sequence $C_0, C_1, \ldots$ of the Catalan numbers is the invert transform of the sequence $\mathbb{F}_1, \mathbb{F}_2, \ldots$ of the Fine numbers with $\mathbb{F}_1 = 1$. We define $f_0(n) = \mathbb{F}_n, (n \geq 1)$. Thus, $f_0(1) = 1, f_0(2) = 0, f_0(3) = 1$, and so on. It is a well-known fact that $\mathbb{F}_n$ is the number of Dyck paths of semi-length $n-1$ with no hills.

All investigation in the paper are based on the following result.

Theorem 1. For $m \geq 1$, we have

(1) The value of $g_m(n,k)$ is the number of Dyck paths of semilength $n-1$ having hills in m colors, of which $k-1$ are in color m.
(2) The value of $f_m(n)$ is the number of Dyck paths of semilength $n-1$ having hills in m colors.

Proof. We use induction on m. We have $f_0(n) = \mathbb{F}_n$. It is well-known that $f_0(n)$ is the number of Dyck paths of semilength $n-1$ with no hills. If we consider the symbol x in Proposition 1 as a hill (of color 1), then the first assertion holds for $m = 1$. The second assertion holds by (2).

Assume that the assertion is true for $m > 1$, that is, Assume that $f_{m-1}(n)$ equals the number of Dyck paths of semilength $n-1$ having hills in $m-1$ colors. Since $f_{m-1}(1) = 1$ and since the empty Dyck path has no hills, we may apply (2) to obtain the assertion. □

We state two particular cases. Firstly, for $m = 1$, the value of $f_1(n)$ is the number of Dyck paths of semilength $n-1$, which equals the Catalan number C_{n-1}. Hence,

$$f_1(n) = C_{n-1}.$$

Since $f_1(1), f_1(2), \ldots$ is the invert transform of $f_0(1), f_0(2), \ldots$, we obtain the following relation between Fine and Catalan numbers.

Corollary 1. The sequence $C_0, C_1, \ldots$ of the Catalan numbers is the invert transform of the sequence $\mathbb{F}_1, \mathbb{F}_2, \ldots$ of the Fine numbers.

From [2, Identity 12], by the use of the identity $i \cdot \binom{i-1}{k-1} = k\binom{i}{k}$, we obtain the following identity relating the Fine and the Catalan numbers via the partial Bell polynomials.

Identity 1.

$$(k-1)!B_{n,k}(C_0, 2! \cdot C_1, 3! \cdot C_2, \ldots) = \sum_{i=k}^{n} \binom{i}{k}(i-1)!B_{n,k}(\mathbb{F}_1, 2! \cdot \mathbb{F}_2, 3! \cdot \mathbb{F}_3, \ldots).$$

3. Catalan triangle $g_2(n,k)$

According to Theorem 1, we have

Corollary 2. (1) The value of $g_1(n,k)$ is the number of Dyck paths of semilength $n-1$ having $k-1$ hills.

(2) The number of Dyck paths of semilngth $n-1$ equals $f_1(n)$.

We conclude that
$$f_1(n) = C_{n-1},$$
where C_{n-1} is the Catalan number.

An explicit formula for $g_1(n,k)$ will be derived later.

The Segner's formula means that the sequence $C_1, C_2, \ldots$ of the Catalan numbers is the invert transform of the sequence $C_0, C_1, \ldots$. This yields that $f_2(n) = C_n$, for all n. We thus obtain the following combinatorial interpretation of the Catalan numbers.

Corollary 3. The Catalan number C_n is the number of Dyck paths of semilength $n-1$ having hills in two colors.

Remark 1. Note that this property of Catalan number is equivalent to Stanley [6, BE.52].

We also have the following identity relating the Catalan numbers and the partial Bell polynomials.

Identity 2.
$$(k-1)!B_{n,k}(C_1, 2!\cdot C_1, 3!\cdot C_3, \ldots) = \sum_{i=k}^{n}\binom{i}{k}(i-1)!B_{n,k}(C_0, 2!\cdot C_1, 3!\cdot C_2, \ldots).$$

We firstly derive an explicit formula for $g_2(n,k)$. It is known that
$$g(x) = \frac{1-\sqrt{1-4x}}{2x}$$
is the ordinary generating function for the sequence $C_0, C_1, \ldots$. It follows from (5) that $\sum_{n=k}^{\infty} g_2(n,k)x^n$ is the expansion of $[xg(x)]^k$ into powers of x. Using the binomial theorem and the expansion of a binomial series, we obtain
$$\sum_{n=k}^{\infty} g_2(n,k)x^n = \sum_{j=0}^{\infty}\left[\sum_{i=0}^{k}(-1)^{i+j}\binom{k}{i}\frac{\prod_{t=0}^{j-1}(i-2t)}{2^{k-j}\cdot j!}\right]x^j.$$
Comparing coefficients of the same powers of x, we firstly obtain

Identity 3. If $n<k$, then
$$\sum_{i=0}^{k}(-1)^{i+n}\binom{k}{i}\prod_{t=0}^{n-1}(i-2t) = 0.$$

The case $n \geq k$ yields
$$g_2(n,k) = \frac{2^{n-k}}{n!}\sum_{i=1}^{k}(-1)^{i+n}\binom{k}{i}\prod_{t=0}^{n-1}(i-2t).$$

It is clear that $\prod_{t=0}^{n-1}(i-2t) = 0$, if i is even. If i is odd, we denote $i = 2j-1, (1 \leq j \leq \lfloor \frac{k+1}{2} \rfloor$. Hence

Proposition 2. The following formula holds:
$$g_2(n,k) = \frac{2^{n-k}}{n!}\sum_{j=1}^{\lfloor \frac{k+1}{2}\rfloor}(-1)^{j-1}\binom{k}{2j-1}\cdot(2j-1)!!\cdot(2n-2j-1)!!. \tag{6}$$

In particular, since $g_2(n,1) = f_1(n) = C_{n-1}$, we obtain the following result.

Corollary 4. For $n > 1$, we have

$$C_{n-1} = \frac{2^{n-1}}{n!} \cdot (2n-3)!!.$$

Remark 2. The preceding is the famous Euler's formula for the Catalan numbers.

We now prove that $g_2(n,k)$ satisfies a simple recurrence relation.

Proposition 3. For $1 \leq k < n$, the following recurrence holds:

$$g_2(n+1,k+1) = g_2(n+1,k+2) + g_2(n,k). \tag{7}$$

Proof. According to (1), we have

$$g_2(n+1,k+1) = \sum_{i_1+i_2+\cdots+i_{k+1}=n+1} C_{i_1-1} \cdot C_{i_2-1} \cdots C_{i_{k+1}-1}, \tag{8}$$

where the sum is taken over positive i_t.

Firstly, we extract the term obtained for $i_{k+1} = 1$. Since $C_{i_{k+1}-1} = C_0 = 1$, we obtain

$$\sum_{i_1+i_2+\cdots i_k=n} C_{i_1-1} \cdot C_{i_2-1} \cdots C_{i_k-1} = C_2(n,k),$$

which is the second term on the right-hand side in formula (3). It remains to calculate the sum on the right-hand side of Equation (8), when $i_{k+1} > 1$. We consider the equation

$$g_2(n+1,k+2) = \sum_{j_1+j_2+\cdots+j_{k+1}+j_{k+2}=n+1} C_{j_1-1} \cdot C_{j_2-1} \cdots C_{j_{k+1}-1} \cdot C_{j_{k+2}-1}.$$

Denote $j_{k+1} + j_{k+2} = i_{k+1} > 1$. This equation is fulfilled for the following pairs of (j_{k+1}, j_{k+2}):

$$\{(1, i_{k+1}-1), (2, i_{k+1}-2), \ldots, (i_{k+1}-1, 1)\}.$$

. We rearrange terms in the sum as follows:

$$g_2(n+1,k+2) = \sum_{j_1+\cdots+j_k+i_{k+1}=n+1} C_{j_1-1}C_{j_2-1} \cdots C_{j_k-1} \cdot \sum_{i=1}^{i_{k+1}-1} C_{i-1}C_{i_{k+1}-1-i}.$$

Segner's formula implies $\sum_{i=1}^{i_{k+1}-1} C_{i-1}C_{i_{k+2}-1-i} = C_{i_{k+1}-1}$. We thus obtain

$$g_2(n+1,k+2) = \sum_{j_1+\cdots+j_k+i_{k+1}=n+1} C_{j_1-1}C_{j_2-1} \cdots C_{j_k-1} \cdot C_{i_{k+1}-1},$$

for $i_{k+1} > 1$, which is the first term in Equation (7). □

We now prove that the following vertical recurrence holds:

Proposition 4. For $n, k > 1$, we have

$$g_2(n,k) = \sum_{i=k-1}^{n-2} g_2(n-1,i) + 1. \tag{9}$$

Proof. From Proposition 3, we obtain the following sequence of equations.

$$\begin{aligned} g_2(n+1,3) &= g_2(n+1,2) - g_2(n,1), \\ g_2(n+1,4) &= g_2(n+1,3) - g_2(n,2), \\ &\vdots \\ g_2(n+1,k+2) &= g_2(n+1,k+1) - g_2(n,k). \end{aligned}$$

Adding terms on the left-hand sides of these equations, and those on the right-hand sides, we obtain

$$\sum_{i=1}^{k} g_2(n,i) = g_2(n+1,2) - g_2(n+1,k+2).$$

Replacing n by $n-1$, and k by $k-2, (k>2)$, we obtain

$$g_2(n,2) = g_2(n,k) + \sum_{i=1}^{k-2} g_2(n-1,i).$$

In particular, for $k=n$, this equation becomes

$$g_2(n,2) = \sum_{i=1}^{n-2} g_2(n-1,i) + 1,$$

and the formula follows. □

We now derive a simpler explicit formula for $g_2(n,k)$.

Proposition 5. The following formula holds: $g_2(n,n)=1$, and

$$g_2(n,k) = \frac{k\prod_{i=1}^{n-k-1}(n+i)}{(n-k)!}, \tag{10}$$

otherwise. Equivalently,

$$g_2(n,k) = \frac{k}{n-k}\binom{2n-k-1}{n}, (n>k). \tag{11}$$

Proof. We use the recurrence (7). We have

$$\begin{aligned} g_2(n+1,k+1) - g_2(n,k) &= \frac{(n+2)\cdots(2n-k-1)\cdot[(k+1)(2n-k)-k(n+1)]}{(n-k)!} \\ &= \frac{(n+2)\cdots(2n-k-1)\cdot(k+2)(n-k)}{(n-k)!} \\ &= \frac{(k+2)(n+2)\cdots(2n-k-1)}{(n-k-1)!} = g_2(n+1,k+2). \end{aligned}$$

and the assertion follows by induction. □

From (9), we obtain the following identity:

Identity 4. For $n>k$, we have

$$\frac{k}{n-k}\cdot\binom{2n-k-1}{n} = \sum_{i=k-1}^{n-2}\frac{i}{n-i-1}\binom{2n-3-i}{n-1} + 1.$$

We denote by $A(n,k)$ the mirror triangle of $g_2(n,k)$. Hence, $A(n,k) = g_2(n, n-k+1)$.

Proposition 6. The triangle $A(n,k)$ satisfies the following conditions:

(1) $A(n,1) = 1, A(n,n) = C_{n-1}$.
(2) $A(n+1,k+1) = A(n+1,k) + A(n,k+1)$,
(3) $A(n,n-1) = C_{n-1}$.

Proof. (1) We have $A(n+1,1) = g_2(n+1,n+1) = f_0(1)^{n+1} = 1$. Also, $A(n,n) = g_2(n,1) = C_{n-1}$.

(2) We have $A(n+1,k+1) = g_2(n+1,n-k+1)$. Using Proposition 3 yields

$$A(n+1,k+1) = g_2(n+1,n-k+2) + g_2(n,n-k) = A(n+1,k) + A(n,k+1).$$

(3) We have $A(n,n-1) = g_2(n,2)$. According to (1), we have $g_2(n,2) = \sum_{i=1}^{n-2} C_{i-1}C_{n-i-2}$. Applying the Segner's formula yields $A(n,n-1) = C_{n-1}$.

□

Remark 3. We note that the triangle $A(n,k)$ is the Catalan triangle, considered in Koshy[3, Chapter 15]. The chapter is devoted to a family of binary words.

Comparing result which is obtained in this Chapter, and our result, we obtain the following result:

Identity 5. The following sets has the same number of elements:

(1) The number of Dyck paths of semilength $n-1$ having hills in two colors, of which $n-k$ hills in color 2.
(2) The number of binary words of length $n+k-2$ having $n-1$ ones and $k-1$ zeros and no initial segment has more zeros than ones.

We also give a bijective proof.

Proof. In a Dyck word of semilength $n-1$ with $n-k$ hills of color 2, we replace each hill of color 2 by 1. Between two hills of color 2 are the standard Dyck paths, which we interpret as binary words having the same number of zeros and ones, and no initial segment having more zeros that ones. In this way we obtain a binary words having $n-1$ ones and $k-1$ zeros, and no initial segment has more zeros then ones. It is clear that this correspondence is injective.

Conversely, if w is a binary word satisfying 2. Scanning from left to right, starting from the last zero, we find interval consisting of the same numbers of ones and zeros. This interval produces a standard Dyck path. If there ones behind the last zero, we replace each of them by the hill of color 2.

Continuing the same procedure, we obtain Dyck path of semilength $n-1$ having $n-k$ hills of color 2. This correspondence is also bijective.

□

4. Catalan triangle $g_3(n,k)$

From Theorem 1, we obtain

Corollary 5. (1) The value of $g_3(n,k)$ is the number of Dyck paths of semilength $n-1$ hills i three colors, of which $k-1$ hills in color 3.

(2) The number of Dyck paths of semilngth $n-1$ having hills in three colors equals $f_3(n)$.

We derive an explicit formula for $g_3(n,k)$.

Proposition 7. We have

$$g_3(n,k) = \frac{k}{n}\binom{2n}{n-k}. \tag{12}$$

Proof. According to (2), we have

$$g_3(n,k) = \sum_{i=k}^{n} \binom{i-1}{k-1} g_2(n,i).$$

Using (10) yields

$$g_3(n,k) = \sum_{i=k}^{n} \binom{i-1}{k-1} \frac{i\cdot(n+1)\cdot(n+2)\cdots(2n-i-1)}{(n-i)!}$$

Using the identity $i \cdot \binom{i-1}{k-1} = k\binom{i}{k}$ implies

$$g_3(n,k) = \frac{k}{n}\cdot\sum_{i=k}^{n} \binom{i}{k} \frac{n(n+1)\cdot(n+2)\cdots(2n-i-1)}{(n-i)!},$$

that is

$$g_3(n,k) = \frac{k}{n}\cdot\sum_{i=k}^{n} \binom{i}{k}\binom{2n-i-1}{n-1}.$$

Hence, our statement is equivalent to the following binomial identity. □

Identity 6. We have

$$\binom{2n}{n+k} = \sum_{i=k}^{n} \binom{i}{k}\binom{2n-i-1}{n-1}. \tag{13}$$

Proof. We prove the identity combinatorially. We count subsets of $n+k$ elements of the set width $2n$ element after the position of the $(k+1)$th element. If $i+1$ is the position of the $k+1$th element of the set then we have $\binom{i}{k}$ elements before and $\binom{2n-i-1}{n-1}$ after this element. Since $k \le i \le n$ the identity is true. □

Remark 4. The Catalan triangle $g_3(n,k)$ is defined by Shapiro [4].

Taking into account his original combinatorial interpretation, we obtain the following:

Identity 7. The following sets has the same number of elements:

(1) The number of nonintersecting lattice paths in the first quadrant at the distance k.
(2) The number of Dyck paths of semilength $n-1$ having hills in three colors, of which $k-1$ hills are of color 3.

Proof. Dyck paths consider here have a simple recurrence, and it is easy to see that $g_3(n,k)$ satisfies this recurrence. □

Remark 5. This array is also considered in Koshy[3, Chapter 14].

We derive one more relation between $g_2(n,k)$ and $g_3(n,k)$.

Proposition 8. We have

$$g_2(n,k)=\sum_{i=0}^{k-1}\binom{k}{i}g_3(n-k,k-i). \tag{14}$$

Proof. Using [2, Proposition 2], we obtain

$$g_2(n,k)=\sum_{i_1+i_2+\cdots+i_k=n}C_{i_1-1}\cdot C_{i_2-1}\cdots C_{i_k-1},$$

where the sum is taken over positive $i_t,(t=1,\ldots,k)$. Replacing $i_t-1=j_t,(t=1,2,\ldots,k)$ we obtain

$$g_2(n,k)=\sum_{j_1+j_2+\cdots+j_k=n-k}C_{j_1}\cdot C_{j_2}\cdots C_{j_k},$$

where the sum is taken over nonnegative j_t. Note that in the case $k=n$ we have $g_2(n,n)=1$. We consider the case $k<n$. Assume that there are $i,(0\leq i\leq k-1)$ of j_t which are equal 0. Then

$$g_2(n,k)=\sum_{i=0}^{k-1}\binom{k}{i}\cdot\sum_{s_1+s_2+\cdots+s_{k-i}=n-k}C_{s_1}\cdot C_{s_2}\cdots C_{s_{k-i}},$$

where $s_t>0,(t=1,2,\ldots,k-i)$ and $k-i\geq n-k$. According Equation (**??**), we have

$$\sum_{s_1+s_2+\cdots+s_{k-i}=n-k}C_{s_1}\cdot C_{s_2}\cdots C_{s_{k-i}}=g_3(n-k,k-i),$$

which proves the statement. □

We next derive an explicit formula for $f_3(n)$.

Proposition 9. The following formula holds:

$$f_3(n)=\binom{2n-1}{n}.$$

Proof. We have

$$\begin{aligned}
f_3(n)&=\frac{1}{n}\sum_{k=1}^{n}k\binom{2n}{n-k}=\frac{1}{n}\sum_{k=0}(n-k)\binom{2n}{k}\\
&=\sum_{k=0}^{n}\binom{2n}{k}-\sum_{k=1}^{n}\frac{k}{n}\cdot\frac{2n}{k}\cdot\binom{2n-1}{k-1}\\
&=1+\sum_{k=1}^{n}\left[\binom{2n-1}{k}+\binom{2n-1}{k-1}\right]-2\sum_{k=1}^{n}\binom{2n-1}{k-1}\\
&=\sum_{k=0}^{n}\binom{2n-1}{k}-\sum_{k=0}^{n-1}\binom{2n-1}{k}\\
&=\binom{2n-1}{n}.
\end{aligned}$$

□

Remark 6. A combinatorial proof of this equation is given in [4, Proposition 3.1].

The preceding proof means that all results in our paper depend only on the fundamental properties of Fine and Catalan numbers.

5. Catalan array $g_4(n,k)$

This case is considered in [2, Section 4], where the following results are obtained.

Proposition 10. (1)

$$g_4(n,k) = \frac{2^{n-k}}{n!}\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}\cdot\prod_{j=0}^{n-1}(i+2j). \tag{15}$$

(2) The value of $g_4(n,k)$ is the number of ternary words of length $2n-1$, having $k-1$ letters equal to 2, and in all binary subwords the number of ones is greater by 1 than the number of zeros. Also, each 2 is both preceded and followed by a binary subword.

(3) The value of $f_4(n)$ is the number of ternary words of length $2n-1$ in which 2 is preceded and followed by a binary subword in which the number of ones is greater by 1 than the number of zeros.

As a consequence, we have the following Euler-type identities:

Identity 8. The following sets have the same number of elements.

(1) The set of Dyck paths of semilength $n-1$ having hills in four colors, of which $k-1$ in color 4.

(2) The set of ternary words of length $2n-1$, having $k-1$ letters equal to 2, and in all binary subwords the number of ones is greater by 1 than the number of zeros. Also, each 2 is both preceded and followed by a binary subword.

Identity 9. (1) The set of Dyck paths of semilength $n-1$ having hills in four color.

(2) The set of ternary words of length $2n-1$, such that in all binary subwords the number of ones is greater by 1 than the number of zeros, and each 2 is both preceded and followed by a binary subword.

6. Some explicit formulas and identities

From (3) and the fact that, for each integer p, we have

$$L_n^p = \left(p^{i-j}\binom{i-1}{j-1}\right)_{n\times n},$$

a mutually connection among different $g_m(n,k)$ is easy to obtain.

Up to now, we have no an explicit formulas for $g_1(n,k)$.

In matrix form, we have $G_1(n) = G_3(n)L_n^{-2}$. Hence, the following equation holds:

$$g_1(n,k) = \frac{k}{n}\cdot\sum_{i=k}^{n}(-2)^{i-k}\binom{i}{k}\binom{2n}{n-i}. \tag{16}$$

Since $g_1(n,1) = \mathbb{F}_n$, we have the following identity for the Fine numbers:

Identity 10.

$$\mathbb{F}_n = \frac{1}{n} \cdot \sum_{i=1}^{n} (-2)^{i-1} \cdot i \cdot \binom{2n}{n-i}.$$

We next prove the following binomial identity.

Identity 11.

$$\binom{2n-2}{n-1} = \sum_{i=1}^{n} (-1)^{i-1} i \cdot \binom{2n}{n-i}.$$

Proof. We have

$$f_1(n) = \sum_{k=1}^{n} g_1(n,k) = \frac{1}{n} \sum_{k=1}^{n} \sum_{i=k}^{n} k(-2)^{i-k} \binom{i}{k} \binom{2n}{n-i}.$$

Changing the order of summation yields

$$f_1(n) = \frac{1}{n} \sum_{i=1}^{n} \binom{2n}{n-i} \cdot \sum_{k=1}^{i} k(-2)^{i-k} \binom{i}{k}.$$

□

Next, we have

$$\sum_{k=1}^{i} k(-2)^{i-k} \binom{i}{k} = i \sum_{t=0}^{i-1} (-2)^t \binom{i-1}{t} = (-1)^{i-1} i.$$

Next, we write g_2, g_3, g_4 as alternating sums.

Since $G_2(n) = G_3(n) \cdot L_n^{-1}$, we have

$$g_2(n,k) = \frac{k}{n} \cdot \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} \cdot \binom{2n}{n-i}. \tag{17}$$

Comparing this equation and (11), we obtain the following identity:

Identity 12. For $k > 0$, we have

$$\binom{2n-k-1}{n} = \frac{n-k}{k} \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} \binom{2n}{n-i}.$$

Also, since $g_2(n,1) = C_{n-1}$, we obtain the following identity for the Catalan numbers.

Identity 13.

$$nC_{n-1} = \sum_{i=1}^{n} (-1)^{i-1} i \cdot \binom{2n}{n-i}.$$

We finish with two exotic identities. The first one consists of eigth items: a sum, a product, two integers, a rising factorial, a falling factorial, and two binomial coefficients.

Identity 14.

$$k \cdot \prod_{i=1}^{n-k-1} (n+i) = (n-k-1)! \cdot \sum_{i=0}^{k-1} (k-i) \binom{k}{i} \binom{2n-2k}{n+2k-i}.$$

The identity is derived from (14).

From $G_4(n) = G_3(n)L_n$, we obtain the identity consisting of ten items: an integer, two sums, a power of -1, a power of 2, a falling factorial, a rising factorial, and three binomial coefficients.

Identity 15.

$$\sum_{i=k}^{n} i\binom{i-1}{k-1}\binom{2n}{n-i} = \frac{2^{n-k}}{(n-1)!}\sum_{i=1}^{k}(-1)^{k+i}\binom{k}{i} i(i+2)\cdots(i+2n-2).$$

References

[1] M. Janjić, Some Formulas for Numbers of Restricted Words, *J. Integer Seq.* **20** (2017), Article 17.6.5

[2] M. Janjić, Pascal Triangle and Restricted Words, arXiv:1705.02479.

[3] T. Koshy, *Catalan numbers with applications*, Oxford Univ. Press, (2009).

[4] L. W. Shapiro, A Catalan triangle, *Discrete Math.* 14 (1976) 8390.

[5] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, http://oeis.org.

[6] R. Stanley, *Catalan Numbers*, Cambridge University Press, (2015).