LIMINAL RECIPROCITY AND FACTORIZATION STATISTICS

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ABSTRACT. Let $M_{d,n}(q)$ denote the number of monic irreducible polynomials in $\mathbb{F}_q[x_1, x_2, \ldots, x_n]$ of degree d. We show that for a fixed degree d, the sequence $M_{d,n}(q)$ converges q-adically to an explicitly determined rational function $M_{d,\infty}(q)$. Furthermore we show that the limit $M_{d,\infty}(q)$ is related to the classic necklace polynomial $M_{d,1}(q)$ by an involutive functional equation, leading to a phenomenon we call *liminal reciprocity*. The limiting first moments of factorization statistics for squarefree polynomials are expressed in terms of a family of symmetric group representations as a consequence of liminal reciprocity.

1. INTRODUCTION

Let $\mathbb{F}_q$ be a field with q elements. How many irreducible polynomials are there in $\mathbb{F}_q[x_1, x_2, \ldots, x_n]$ of total degree d? Let $M_{d,n}(q)$ denote the number of irreducible polynomials in $\mathbb{F}_q[x_1, x_2, \ldots, x_n]$ of total degree d which are monic with respect to some fixed monomial ordering ($M_{d,n}(q)$ is independent of which monomial ordering we choose.) When $n = 1$, $M_{d,1}(q)$ is given by the *dth necklace polynomial*

$$M_{d,1}(q) := \frac{1}{d} \sum_{e|d} \mu(d/e) q^e, \tag{1.1}$$

where μ is the Möbius function. There does not appear to be a simple analog of (1.1) for $M_{d,n}(q)$ when $n > 1$. In Lemma 2.1 we show that $M_{d,n}(q)$ is a recursively computable polynomial in q for all $n \geq 1$. The table below gives approximations of $M_{3,n}(q)$ for small n.

n	$M_{3,n}(q)$
1	$-\frac{1}{3}q + \frac{1}{3}q^3$
2	$-\frac{1}{3}q - \frac{1}{3}q^2 + \frac{1}{3}q^3 - q^5 - \frac{2}{3}q^6 + \ldots$
3	$-\frac{1}{3}q - \frac{1}{3}q^2 + q^4 + q^5 + \frac{1}{3}q^6 - q^7 + \ldots$
4	$-\frac{1}{3}q - \frac{1}{3}q^2 + \frac{2}{3}q^4 + 2q^5 + \frac{7}{3}q^6 + 2q^7 + \ldots$
5	$-\frac{1}{3}q - \frac{1}{3}q^2 + \frac{2}{3}q^4 + \frac{5}{3}q^5 + \frac{10}{3}q^6 + 4q^7 + \ldots$
6	$-\frac{1}{3}q - \frac{1}{3}q^2 + \frac{2}{3}q^4 + \frac{5}{3}q^5 + 3q^6 + 5q^7 + \ldots$
7	$-\frac{1}{3}q - \frac{1}{3}q^2 + \frac{2}{3}q^4 + \frac{5}{3}q^5 + 3q^6 + \frac{14}{3}q^7 + \ldots$

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The sequence of polynomials $M_{3,n}(q)$ appears to converge q-adically as the number of variables n increases. We show that for any degree d, the sequence of polynomials $M_{d,n}(q)$ converges q-adically to a rational function $M_{d,\infty}(q)$ as $n \to \infty$. The limit $M_{d,\infty}(q)$ is closely related to (1.1).

Theorem 1.1. *Let $M_{d,n}(q)$ be the number of irreducible degree d polynomials in $\mathbb{F}_q[x_1, x_2, \ldots, x_n]$ which are monic with respect to some fixed monomial ordering. Then $M_{d,n}(q)$ is a polynomial in q and for each $d \geq 1$ the sequence of polynomials $M_{d,n}(q)$ converges q-adically to the rational function*

$$M_{d,\infty}(q) := -\frac{1}{d} \sum_{e \mid d} \mu(d/e) \left(\frac{1}{1 - \frac{1}{q}} \right)^e .$$

In particular $M_{d,\infty}(q)$ satisfies the following functional equation,

$$M_{d,\infty}(q) = -M_{d,1}\left(\tfrac{1}{1-\frac{1}{q}}\right). \tag{1.2}$$

Furthermore the rate of convergence of $M_{d,n}(q)$ is bounded by the congruence

$$M_{d,n}(q) \equiv M_{d,\infty}(q) \bmod q^{n+1}.$$

Note that the q-adic convergence of a sequence of polynomials in q is equivalent to coefficientwise convergence in the ring of formal power series.

The fractional linear transformation $q \mapsto \frac{1}{1-\frac{1}{q}}$ is an involution, hence (1.2) is equivalent to

$$M_{d,1}(q) = -M_{d,\infty}\left(\tfrac{1}{1-\frac{1}{q}}\right).$$

We view this involutive functional equation relating irreducible polynomial counts in one and infinitely many variables as the first instance of a phenomenon which we call *liminal reciprocity*.

1.1. Liminal reciprocity for type polynomials.

Let $\mathrm{Poly}_{d,n}(\mathbb{F}_q)$ denote the set of polynomials in $\mathbb{F}_q[x_1, x_2, \ldots, x_n]$ of total degree d which are monic with respect to some fixed monomial ordering. Since the polynomial ring $\mathbb{F}_q[x_1, x_2, \ldots, x_n]$ has unique factorization, each element $f \in \mathrm{Poly}_{d,n}(\mathbb{F}_q)$ has a well-defined *factorization type*. The factorization type of a polynomial $f \in \mathrm{Poly}_{d,n}(\mathbb{F}_q)$ is the partition $\lambda \vdash d$ given by the degrees of the $\mathbb{F}_q$-irreducible factors of f.

The factorization type does not record the multiplicities of individual factors, only the degrees of the irreducible factors. For example, the polynomials x^2 and $x(x+1)$ both have factorization type $[1,1]$ since they each have two linear factors.

Definition. If $\lambda \vdash d$ is a partition, then the *λ-type polynomial* $T_{\lambda,n}(q)$ is the number of elements of $\mathrm{Poly}_{d,n}(\mathbb{F}_q)$ with factorization type λ. Similarly we define the *squarefree λ-type polynomial* $T^{\mathrm{sf}}_{\lambda,n}(q)$ to be the number of squarefree elements of $\mathrm{Poly}_{d,n}(\mathbb{F}_q)$ with factorization type λ. The type polynomials may be expressed in terms of $M_{d,n}(q)$ as

$$T_{\lambda,n}(q) = \prod_{j \geq 1} \left(\!\!\binom{M_{j,n}(q)}{m_j(\lambda)}\!\!\right) \qquad T^{\mathrm{sf}}_{\lambda,n}(q) = \prod_{j \geq 1} \binom{M_{j,n}(q)}{m_j(\lambda)},$$

where $m_j(\lambda)$ is the number of parts of λ of size j, $\binom{x}{m} = \frac{1}{m!}x(x-1)\cdots(x-m+1)$ is the usual binomial coefficient, and $\left(\!\binom{x}{m}\!\right) = \frac{1}{m!}x(x+1)\cdots(x+m-1)$. Recall that $\binom{x}{m}$ counts the number of subsets of size m in a set of size x and $\left(\!\binom{x}{m}\!\right)$ counts the number of subsets of size m with repetition in a set of size x.

It follows from Theorem 1.1 that the q-adic limits

$$T_{\lambda,\infty}(q) = \lim_{n\to\infty} T_{\lambda,n}(q) \qquad T^{\mathrm{sf}}_{\lambda,\infty}(q) = \lim_{n\to\infty} T^{\mathrm{sf}}_{\lambda,n}(q)$$

converge to rational functions. Our next result gives a version of liminial reciprocity for type polynomials.

Theorem 1.2 (Liminal reciprocity)**.** *Let λ be a partition and let $\ell(\lambda) = \sum_{j\geq 1} m_j(\lambda)$ be the number of parts of λ. The following identities hold in $\mathbb{Q}(q)$,*

$$T_{\lambda,\infty}(q) = (-1)^{\ell(\lambda)} T^{\mathrm{sf}}_{\lambda,1}\left(\frac{1}{1-\frac{1}{q}}\right)$$

$$T^{\mathrm{sf}}_{\lambda,\infty}(q) = (-1)^{\ell(\lambda)} T_{\lambda,1}\left(\frac{1}{1-\frac{1}{q}}\right)$$

These identities are involutive in the sense that we can swap the ∞ and 1 subscripts to get equivalent statements. The new feature we see in Theorem 1.2 is the relationship between squarefree polynomials and general polynomials of a given factorization type. This connection is closely related to Stanley's *combinatorial reciprocity phenomenon* [13].

1.2. **Liminal first moments of squarefree factorization statistics.** We call a function P defined on $\mathrm{Poly}_{d,n}(\mathbb{F}_q)$ a *factorization statistic* if $P(f)$ depends only on the factorization type of f. Recently we demonstrated a surprising connection between the first moments of factorization statistics on the set of univariate polynomials ($n = 1$) and the symmetric group representation theoretic structure of the cohomology of point configurations in Euclidean space. See Section 3 for precise definitions.

Theorem 1.3 ([9, Thm. 2.4, Thm. 2.5])**.** *Let P be a factorization statistic, and let ψ_d^k, ϕ_d^k be the characters of the S_d-representations $H^{2k}(\mathrm{PConf}_d(\mathbb{R}^3),\mathbb{Q})$ and $H^k(\mathrm{PConf}_d(\mathbb{R}^2),\mathbb{Q})$ respectively. Then*

$$(1) \qquad \sum_{f\in\mathrm{Poly}_{d,1}(\mathbb{F}_q)} P(f) = \sum_{k=0}^{d-1} \langle P, \psi_d^k\rangle q^{d-k}$$

$$(2) \qquad \sum_{f\in\mathrm{Poly}^{\mathrm{sf}}_{d,1}(\mathbb{F}_q)} P(f) = \sum_{k=0}^{d-1} (-1)^k \langle P, \phi_d^k\rangle q^{d-k},$$

where $\langle P, Q\rangle = \frac{1}{d!}\sum_{\tau\in S_d} P(\tau)Q(\tau)$ is the standard inner product of class functions on S_d.

The squarefree case of Theorem 1.3 is originally due to Church, Ellenberg, and Farb [6, Prop. 4.1]. The general case was shown by the author [9] using different methods which also gave a new proof of the squarefree case. Theorem 1.3 provides a bridge between the arithmetic and combinatorics of factorization statistics on one hand and the geometry and representation theory of configuration spaces on the other.

Computations suggest there are not direct analogs of Theorem 1.3 for $n > 1$. However, an analog does emerge in the liminal squarefree case.

Theorem 1.4. *Let P be a factorization statistic, and let σ_d^k be the character of the S_d-representation*

$$\Sigma_d^k = \bigoplus_{j=k}^{d-1} \mathrm{sgn}_d \otimes H^{2j}(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q})^{\oplus \binom{j}{k}}. \tag{1.3}$$

Then for each n the first moment $\sum_{f \in \mathrm{Poly}_{d,n}^{\mathrm{sf}}(\mathbb{F}_q)} P(f)$ is a polynomial in q and

$$\lim_{n\to\infty} \sum_{f \in \mathrm{Poly}_{d,n}^{\mathrm{sf}}(\mathbb{F}_q)} P(f) = \frac{1}{(1-q)^d} \sum_{k=0}^{d-1} (-1)^k \langle P, \sigma_d^k \rangle q^{d-k},$$

where the limit is taken q-adically.

Since the limit in Theorem 1.4 is taken q-adically, the representation theoretic interpretation of first moments manifests for sufficiently large n. For example, let L be the *linear factor* statistic where $L(f)$ is the number of linear factors of f; the following table shows approximations of the first moment of L on $\mathrm{Poly}_{3,n}^{\mathrm{sf}}(\mathbb{F}_q)$ scaled by $(1-q)^3$.

n	$(1-q)^3 \sum_{f \in \mathrm{Poly}_{3,n}^{\mathrm{sf}}(\mathbb{F}_q)} L(f)$
1	$q - 5q^2 + 10q^3 - 10q^4 + 5q^5 - q^6$
2	$q - 4q^2 + 2q^3 + 7q^4 - 6q^5 - 3q^6 + 2q^7 + q^8 + q^9 - q^{10}$
3	$q - 4q^2 + 3q^3 - q^4 + 7q^5 - 6q^6 - 3q^8 + 3q^9 - q^{11} + q^{12} + q^{14} - q^{15}$
4	$q - 4q^2 + 3q^3 - q^5 + 7q^6 - 6q^7 - 3q^{10} + 3q^{11} - q^{16} + q^{17} + q^{20} - q^{21}$
5	$q - 4q^2 + 3q^3 - q^6 + 7q^7 - 6q^8 - 3q^{12} + 3q^{13} - q^{22} + q^{23} + q^{27} - q^{28}$

This table suggests that

$$\sum_{f \in \mathrm{Poly}_{3,n}^{\mathrm{sf}}(\mathbb{F}_q)} L(f) = \frac{q - 4q^2 + 3q^3 + O(q^{n+1})}{(1-q)^3}.$$

From Theorem 1.4 we conclude that

$$\langle L, \sigma_3^2 \rangle = 1 \quad \langle L, \sigma_3^1 \rangle = 4 \quad \langle L, \sigma_3^0 \rangle = 3.$$

Note that these inner products are positive integers—this reflects that L, viewed as a class function of the symmetric group, is the character of the standard permutation representation.

1.3. **Related work.** Carlitz [4, 5] studied the asymptotic behavior of $M_{d,n}(q)$ for $n \geq 1$. His investigations were subsequently refined and extended in [1, 7, 8, 14, 15]. Our Theorem 1.1 may be interpreted as a result on the q-adic asymptotics of $M_{d,n}(q)$ as $n \to \infty$. The q-adic convergence of $M_{d,n}(q)$ and the determination of the limit appear to be new.

The liminal reciprocity identities (Theorem 1.1 and Theorem 1.2) were discovered empirically. We do not know the proper context for these results. The proof of the liminal reciprocity for type polynomials (Theorem 1.2) passes through a well-known example of Stanley's *combinatorial reciprocity phenomenon* [13, Ex. 1.1]. Combinatorial reciprocity is a family of dualities between related combinatorial problems which concretely manifests as functional equations similar in form to our liminal reciprocity identities. However, the precise relationship between liminal and combinatorial reciprocity remains unclear. We would be interested to know of other examples of liminal reciprocity.

The relationship between the liminal first moments of squarefree factorization statistics and representations of the symmetric group parallels our results in [9]. Church, Ellenberg, and Farb [6] established the connection between first moments of squarefree factorization statistics for univariate polynomials and the cohomology of point configurations in $\mathbb{R}^2$ in a formula they call the *twisted Grothendieck-Lefschetz formula* for squarefree polynomials. One application of their result is to deduce the asymptotic stability of first moments as a consequence of *representation stability*. We extend this connection to general univariate polynomials in [9, Thm. 2.7]. However, this connection does not extend to liminal first moments; the family of representations Σ_d^k does not exhibit representation stability.

The results in [9] are expressed in terms of expected values of factorization statistics. In this paper we focus on first moments as they lead to a cleaner statement for Theorem 1.4. The translation between expected values and first moments is simply a factor of q^d for general polynomials, but is more subtle for squarefree polynomials as it affects the family of characters determining the coefficients. The equivalence between Theorem 1.3 (2) and [9, Thm. 2.5] follows from [10, Prop. 4.2]. Alternatively, Theorem 1.3 (2) may be found as stated in [6, Prop. 4.1].

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2. Hypersurface factorization statistics

Let $\mathbb{F}_q$ be a finite field. Fix some monomial ordering on $\mathbb{F}_q[x_1, x_2, \ldots, x_n]$ and let $\mathrm{Poly}_{d,n}(\mathbb{F}_q)$ be the set of all degree d polynomials in $\mathbb{F}_q[x_1, x_2, \ldots, x_n]$ which are monic with respect to the monomial ordering. Note that the size of $\mathrm{Poly}_{d,n}(\mathbb{F}_q)$ is independent of the choice of monomial ordering. For each $m \geq 1$ let $\mathrm{Poly}_{d,n}^m(\mathbb{F}_q) \subseteq \mathrm{Poly}_{d,n}(\mathbb{F}_q)$ be the subset of those polynomials with all factors of multiplicity at most m. There is a filtration

$$\mathrm{Poly}_{d,n}^1(\mathbb{F}_q) \subseteq \mathrm{Poly}_{d,n}^2(\mathbb{F}_q) \subseteq \mathrm{Poly}_{d,n}^3(\mathbb{F}_q) \subseteq \ldots \subseteq \mathrm{Poly}_{d,n}(\mathbb{F}_q),$$

and $\mathrm{Poly}^1_{d,n}(\mathbb{F}_q) = \mathrm{Poly}^{\mathrm{sf}}_{d,n}(\mathbb{F}_q)$ is the set of the squarefree polynomials.

Recall that $\mathbb{F}_q[x_1, x_2, \ldots, x_n]$ is a unique factorization domain, hence every element of $\mathrm{Poly}_{d,n}(\mathbb{F}_q)$ can be uniquely factored into a product of irreducible monic polynomials. We define the *factorization type* of $f \in \mathrm{Poly}_{d,n}(\mathbb{F}_q)$ to be the partition of d given by the degrees of the irreducible factors of f. If λ is a partition of d, then we write $\mathrm{Poly}_{\lambda,n}(\mathbb{F}_q)$ for the set of all $f \in \mathrm{Poly}_{d,n}(\mathbb{F}_q)$ with factorization type λ. For $m \geq 1$, let $\mathrm{Poly}^m_{\lambda,n}(\mathbb{F}_q) := \mathrm{Poly}^m_{d,n}(\mathbb{F}_q) \cap \mathrm{Poly}_{\lambda,n}(\mathbb{F}_q)$. If $\lambda = [d]$ is the partition with one part, we write $\mathrm{Irr}_{d,n}(\mathbb{F}_q) := \mathrm{Poly}_{[d],n}(\mathbb{F}_q)$ for the set of monic, irreducible, degree d polynomials.

Lemma 2.1 shows that the cardinality of each of the sets of polynomials just defined is given by a polynomial in the size of the field q.

Lemma 2.1. *For any $d, n \geq 1$,*

(1) $|\mathrm{Poly}_{d,n}(\mathbb{F}_q)| = P_{d,n}(q)$*, where*

$$P_{d,n}(q) = \frac{q^{\binom{d+n}{n}} - q^{\binom{d+n-1}{n}}}{q-1} = q^{\binom{d+n-1}{n}} \frac{q^{\binom{d+n-1}{n-1}} - 1}{q-1}.$$

(2) $M_{d,n}(q)$ *is a polynomial of q with rational coefficients.*

(3) *For every $m \geq 1$ and every partition $\lambda \vdash d$,*

$$|\mathrm{Poly}^m_{\lambda,n}(\mathbb{F}_q)| = T^m_{\lambda,n}(q) := \prod_{j \geq 1} \binom{M_{j,n}(q) + \min\{m, m_j(\lambda)\} - 1}{m_j(\lambda)},$$

where $m_j(\lambda)$ is the number of parts of λ of size j. In particular, when $m = 1$ we have

$$|\mathrm{Poly}^{\mathrm{sf}}_{\lambda,n}(\mathbb{F}_q)| = T^{\mathrm{sf}}_{\lambda,n}(q) := \prod_{j \geq 1} \binom{M_{j,n}(q)}{m_j(\lambda)}.$$

(4) *For every partition $\lambda \vdash d$,*

$$|\mathrm{Poly}_{\lambda,n}(\mathbb{F}_q)| = T_{\lambda,n}(q) := \prod_{j \geq 1} \left(\!\!\binom{M_{j,n}(q)}{m_j(\lambda)}\!\!\right),$$

where $\left(\!\binom{x}{m}\!\right) := \binom{x+m-1}{m}$ is the number of subsets with repetition of size m chosen from an x element set.

Proof. (1) There are $q^{\binom{d+n}{n}}$ polynomials in n variables of degree at most d. Hence there are $q^{\binom{d+n}{n}} - q^{\binom{d+n-1}{n}}$ polynomials in n variables of degree exactly d. If we choose a monomial order, every degree d polynomial has a nonzero leading coefficient. Therefore the total number of degree d monic polynomials in n variables is

$$|\mathrm{Poly}_{d,n}(\mathbb{F}_q)| = \frac{q^{\binom{d+n}{n}} - q^{\binom{d+n-1}{n}}}{q-1}.$$

(2) We proceed by induction on d to show that

$$|\mathrm{Irr}_{d,n}(\mathbb{F}_q)| = M_{d,n}(q)$$

for some polynomial $M_{d,n}(x) \in \mathbb{Q}[x]$. If $d = 1$, then all polynomials are irreducible, hence

$$|\mathrm{Irr}_{1,n}(\mathbb{F}_q)| = |\mathrm{Poly}_{1,n}(\mathbb{F}_q)| = q^n,$$

So $M_{1,n}(q) = q^n$. Suppose our claim were true for all degrees less than $d > 1$. Counting the number of polynomials with factorization type λ directly we find

$$|\mathrm{Poly}_{\lambda,n}(\mathbb{F}_q)| = \prod_{j \geq 1} \left(\!\!\binom{|\mathrm{Irr}_{j,n}(\mathbb{F}_q)|}{m_j(\lambda)}\!\!\right). \tag{2.1}$$

By unique factorization there is a decomposition

$$\mathrm{Poly}_{d,n}(\mathbb{F}_q) = \bigsqcup_{\lambda \vdash d} \mathrm{Poly}_{\lambda,n}(\mathbb{F}_q),$$

hence

$$P_{d,n}(q) = |\mathrm{Irr}_{d,n}(\mathbb{F}_q)| + \sum_{\substack{\lambda \vdash d \\ \lambda \neq [d]}} |\mathrm{Poly}_{\lambda,n}(\mathbb{F}_q)|.$$

If $\lambda \neq [d]$, then all parts j of λ are smaller than d, hence by our inductive hypothesis we have $|\mathrm{Irr}_{j,n}(\mathbb{F}_q)| = M_{j,n}(q)$ for all such j. Thus

$$|\mathrm{Irr}_{d,n}(\mathbb{F}_q)| = M_{d,n}(q) := P_{d,n}(q) - \sum_{\substack{\lambda \vdash d \\ \lambda \neq [d]}} \prod_{j \geq 1} \left(\!\!\binom{M_{j,n}(q)}{m_j(\lambda)}\!\!\right).$$

Finally, (3) and (4) follow from (2.1) and (2). □

For ease of reference we collect the definitions of the polynomials appearing in Lemma 2.1.

Definition 2.2. *Let $d, n \geq 1$ and $\lambda \vdash d$, then*

$$\begin{aligned}
P_{d,n}(q) &= \frac{q^{\binom{d+n}{n}} - q^{\binom{d+n-1}{n}}}{q-1} = q^{\binom{d+n-1}{n}} \frac{q^{\binom{d+n-1}{n-1}} - 1}{q-1} \\
M_{d,n}(q) &= |\mathrm{Irr}_{d,n}(\mathbb{F}_q)| = |\mathrm{Poly}_{[d],n}(\mathbb{F}_q)| \\
T_{\lambda,n}(q) &= |\mathrm{Poly}_{\lambda,n}(\mathbb{F}_q)| = \prod_{j \geq 1} \left(\!\!\binom{M_{j,n}(q)}{m_j(\lambda)}\!\!\right) \\
T^m_{\lambda,n}(q) &= |\mathrm{Poly}^m_{\lambda,n}(\mathbb{F}_q)| = \prod_{j \geq 1} \binom{M_{j,n}(q) + \min(m, m_j(\lambda)) - 1}{m_j(\lambda)} \\
T^{\mathrm{sf}}_{\lambda,n}(q) &= T^1_{\lambda,n}(q) = |\mathrm{Poly}^{\mathrm{sf}}_{\lambda,n}(\mathbb{F}_q)| = \prod_{j \geq 1} \binom{M_{j,n}(q)}{m_j(\lambda)} \\
P^m_{d,n}(q) &= |\mathrm{Poly}^m_{d,n}(\mathbb{F}_q)| = \sum_{\lambda \vdash d} T^m_{\lambda,n}(q),
\end{aligned}$$

where d represents ***d****egree, n the* ***n****umber of variables, and m the maximum* ***m****ultiplicity of a factor.*

There is a well-known formula [12, Cor. 2.1] for $M_{d,1}(q)$ given by counting elements in $\mathbb{F}_{q^d}$ by the field they generate,

$$M_{d,1}(q) = \frac{1}{d}\sum_{e\mid d}\mu(e)q^{d/e}. \tag{2.2}$$

The value of $M_{d,1}(k)$ for an integer $k \geq 1$ has a combinatorial interpretation as the number of aperiodic necklaces made with beads of k colors. For this reason, $M_{d,1}(q)$ is known as the dth *necklace polynomial*. There is no apparent analog of (2.2) nor a combinatorial interpretation for $M_{d,n}(k)$ when $n > 1$. Instead $M_{d,n}(q)$ may be computed inductively as in the proof of Lemma 2.1:

$$\begin{aligned} M_{1,n}(q) &= P_{1,n}(q) = q^n \\ M_{d,n}(q) &= P_{d,n}(q) - \sum_{\substack{\lambda\vdash d\\ \lambda\neq[d]}} T_{\lambda,n}(q). \end{aligned}$$

Our next result shows that all the polynomials listed in Definition 2.2 converge q-adically to rational functions as the number of variables n tends to infinity.

Theorem 2.3. *Let $d \geq 1$. Then,*

(1) *The sequence $P_{d,n}(q)$ converges q-adically to*

$$P_{d,\infty}(q) = \lim_{n\to\infty} P_{d,n}(q) = \begin{cases} -\frac{1}{1-\frac{1}{q}} & d = 1 \\ 0 & d > 1. \end{cases}$$

(2) *For every $m \geq 1$ the sequence $P^m_{d,n}(q)$ converges q-adically to*

$$P^m_{d,\infty}(q) = \lim_{n\to\infty} P^m_{d,n}(q) = \begin{cases} -\left(\frac{1}{1-\frac{1}{q}}\right)^k & d = (m+1)(k-1)+1 \\ \left(\frac{1}{1-\frac{1}{q}}\right)^k & d = (m+1)k \\ 0 & d \not\equiv 0,1 \bmod m+1. \end{cases}$$

In particular, if $m = 1$, then

$$P^{\mathrm{sf}}_{d,\infty}(q) = (-1)^d\left(\frac{1}{1-\frac{1}{q}}\right)^{\lfloor \frac{d+1}{2}\rfloor}.$$

(3) *For all partitions $\lambda \vdash d$ and $m \geq 1$ the sequences $M_{d,n}(q)$, $T_{\lambda,n}(q)$, and $T^m_{\lambda,n}(q)$ converge q-adically to rational functions as $n \to \infty$. Furthermore,*

$$\begin{aligned} T_{\lambda,\infty}(q) &= \prod_{j\geq 1}\left(\!\!\binom{M_{j,\infty}(q)}{m_j(\lambda)}\!\!\right) \\ T^{\mathrm{sf}}_{\lambda,\infty}(q) &= \prod_{j\geq 1}\binom{M_{j,\infty}(q)}{m_j(\lambda)}. \end{aligned}$$

Proof. (1) From Lemma 2.1 we have

$$P_{d,n}(q) = q^{\binom{d+n-1}{n}} \frac{q^{\binom{d+n-1}{n-1}} - 1}{q-1}.$$

For $d = 1$ this simplifies to

$$P_{1,n}(q) = q\frac{q^n - 1}{q-1}.$$

Since $\lim_{n\to\infty} q^n = 0$ in the q-adic topology, it follows that

$$P_{1,\infty}(q) = \lim_{n\to\infty} q\frac{q^n-1}{q-1} = -\frac{q}{q-1} = -\frac{1}{1-\frac{1}{q}}.$$

If $d > 1$, then $\binom{d+n-1}{n} \to \infty$ as $n \to \infty$. Thus

$$P_{d,\infty}(q) = \lim_{n\to\infty} q^{\binom{d+n-1}{n}} \frac{q^{\binom{d+n-1}{v-1}} - 1}{q-1} = 0.$$

(2) Consider the generating functions

$$Z(T_n^m, t) = \sum_{d\geq 0}\sum_{\lambda\vdash d} T_{\lambda,n}^m(q)t^d = \sum_{d\geq 0} P_{d,n}^m(q)t^d$$
$$Z(T_n, t) = \sum_{d\geq 0}\sum_{\lambda\vdash d} T_{\lambda,n}(q)t^d = \sum_{d\geq 0} P_{d,n}(q)t^d.$$

From unique factorization in $\mathbb{F}_q[x_1, x_2, \ldots, x_n]$ we have the following product formulas

$$Z(T_n^m, t) = \prod_{j\geq 1}(1 + t^j + t^{2j} + \ldots + t^{mj})^{M_{j,n}(q)} = \prod_{j\geq 1}\left(\frac{1-t^{(m+1)j}}{1-t^j}\right)^{M_{j,n}(q)}$$
$$Z(T_n, t) = \prod_{j\geq 1}\left(\frac{1}{1-t^j}\right)^{M_{j,n}(q)}.$$

Hence $Z(T_n, t) = Z(T_n, t^{m+1})Z(T_n^m, t)$. Taking a coefficientwise q-adic limit as $n \to \infty$ we have by (1) that

$$1 - \frac{1}{1-\frac{1}{q}}t = Z(T_\infty, t) = Z(T_\infty, t^{m+1})Z(T_\infty^m, t) = \left(1 - \frac{1}{1-\frac{1}{q}}t^{m+1}\right)\sum_{d\geq 0} P_{d,\infty}^m(q)t^d.$$

Comparing coefficients we conclude that

$$P_{d+m+1,\infty}^m(q) = \frac{1}{1-\frac{1}{q}} P_{d,\infty}^m(q)$$

for all $d \geq 0$, together with the initial values

$$P_{0,\infty}^m(q) = 1$$
$$P_{1,\infty}^m(q) = -\frac{1}{1-\frac{1}{q}}$$
$$P_{d,\infty}^m(q) = 0 \text{ for } 1 < d \leq m.$$

Then (2) follows by induction.

(3) It suffices to prove that for every $d \geq 1$ the sequence $M_{d,n}(q)$ converges q-adically to a rational function, the other claims follow by the explicit formuls given in Definition 2.2 and continuity. Recall the recursive formulas for $M_{d,n}(q)$ used in the proof of Lemma 2.1. For all $d, n \geq 1$,

$$M_{1,n}(q) = P_{1,n}(q)$$
$$M_{d,n}(q) = P_{d,n}(q) - \sum_{\substack{\lambda \vdash d \\ \lambda \neq [d]}} \prod_{j \geq 1} \left(\!\!\binom{M_{j,n}(q)}{m_j(\lambda)}\!\!\right).$$

Taking q-adic limits as $n \to \infty$ we have by (1) that

$$M_{1,\infty}(q) = P_{1,\infty}(q) = -\frac{1}{1-\frac{1}{q}},$$
$$M_{d,\infty}(q) = -\sum_{\substack{\lambda \vdash d \\ \lambda \neq [d]}} \prod_{j \geq 1} \left(\!\!\binom{M_{j,\infty}(q)}{m_j(\lambda)}\!\!\right).$$

It follows by induction that $M_{d,\infty}(q)$ is a rational function of q for all $d \geq 1$. □

There is a surprising relationship between the number of irreducible polynomials in one variable $M_{d,1}(q)$ and the limit $M_{d,\infty}(q)$ which gives us an explicit formula for $M_{d,\infty}(q)$. This relationship takes the form of an involutive functional equation which we call *liminal reciprocity*.

Theorem 2.4 (Liminal reciprocity)**.** *For all $d \geq 1$,*

$$M_{d,\infty}(q) = -M_{d,1}\left(\tfrac{1}{1-\frac{1}{q}}\right).$$

More explicitly,

$$M_{d,\infty}(q) = -\frac{1}{d}\sum_{e \mid d} \mu(d/e)\left(\frac{1}{1-\frac{1}{q}}\right)^e.$$

Proof. Recall the generating function $Z(T_n, t)$ used in the proof of Theorem 2.3 (2),

$$Z(T_n, t) = \sum_{d \geq 0} P_{d,n}(q) t^d = \prod_{j \geq 1}\left(\frac{1}{1-t^j}\right)^{M_{j,n}(q)}$$

Taking the coefficientwise q-adic limit as $n \to \infty$ gives, by Theorem 2.3 (1), that

$$1 - \tfrac{1}{1-\frac{1}{q}} t = \prod_{d \geq 1}\left(\frac{1}{1-t^d}\right)^{M_{d,\infty}(q)}. \tag{2.3}$$

Taking logarithms and expanding as power series in t,

$$-\sum_{k \geq 1}\left(\frac{1}{1-\frac{1}{q}}\right)^k \frac{t^k}{k} = \sum_{d \geq 1}\sum_{m \geq 1} M_{d,\infty}(q)\frac{t^{md}}{m}.$$

Comparing coefficients of $\frac{t^k}{k}$ yields the identity,

$$-\left(\frac{1}{1-\frac{1}{q}}\right)^k = \sum_{d|k} dM_{d,\infty}(q).$$

Therefore by Möbius inversion,

$$M_{d,\infty}(q) = -\frac{1}{d}\sum_{e|d}\mu(d/e)\left(\frac{1}{1-\frac{1}{q}}\right)^e.$$

Recall that

$$M_{d,1}(q) = \frac{1}{d}\sum_{e|d}\mu(d/e)q^e.$$

Hence we conclude

$$M_{d,\infty}(q) = -M_{d,1}\left(\tfrac{1}{1-\frac{1}{q}}\right).$$

□

The rate of convergence of $M_{d,n}(q)$ can be determined from the proof of Theorem 2.4.

Corollary 2.5. *For all* $d \geq 1$*,*

$$M_{d,n}(q) \equiv M_{d,\infty}(q) \bmod q^{n+1}.$$

Proof. Recall that

$$P_{d,n}(q) = q^{\binom{d+n-1}{n}}\frac{q^{\binom{d+n-1}{n-1}}-1}{q-1}.$$

Since $\binom{d+n-1}{n} \geq n+1$ for $d \geq 2$ and

$$P_{1,n}(q) = \frac{q^{n+1}-q}{q-1},$$

it follows that

$$\sum_{d\geq 0} P_{d,n}(q)t^d \equiv 1 - \tfrac{1}{1-\frac{1}{q}}t \bmod q^{n+1}.$$

Thus

$$1 - \tfrac{1}{1-\frac{1}{q}}t \equiv \prod_{d\geq 1}\left(\frac{1}{1-t^d}\right)^{M_{d,n}(q)} \bmod q^{n+1},$$

and (2.3) implies

$$M_{d,n}(q) \equiv M_{d,\infty}(q) \bmod q^{n+1}.$$

□

Notice that the fractional linear transformation $q \longmapsto \frac{1}{1-\frac{1}{q}}$ is an involution. Thus Theorem 2.4 is equivalent to

$$M_{d,1}(q) = -M_{d,\infty}\left(\tfrac{1}{1-\frac{1}{q}}\right).$$

Our next result combines the liminal reciprocity between $M_{d,1}(q)$ and $M_{d,\infty}(q)$ with the *combinatorial reciprocity* identity

$$\binom{-x}{m} = (-1)^m \left(\!\!\binom{x}{m}\!\!\right), \tag{2.4}$$

to deduce a striking relationship between factorization statistics of polynomials when $n = 1$ and $n = \infty$.

Theorem 2.6 (Liminal reciprocity)**.** *For any partition λ, let $\ell(\lambda) = \sum_{j\geq 1} m_j(\lambda)$ denote the number of parts of λ. Then*

$$T^{\mathrm{sf}}_{\lambda,\infty}(q) = (-1)^{\ell(\lambda)} T_{\lambda,1}\left(\tfrac{1}{1-\frac{1}{q}}\right),$$
$$T_{\lambda,\infty}(q) = (-1)^{\ell(\lambda)} T^{\mathrm{sf}}_{\lambda,1}\left(\tfrac{1}{1-\frac{1}{q}}\right).$$

Proof. By Theorem 2.3 (3), Theorem 2.4, and the combinatorial reciprocity identity (2.4) we have

$$\begin{aligned} T^{\mathrm{sf}}_{\lambda,\infty}(q) &= \prod_{j\geq 1} \binom{M_{j,\infty}(q)}{m_j(\lambda)} \\ &= \prod_{j\geq 1} \binom{-M_{j,1}\left(\frac{1}{1-\frac{1}{q}}\right)}{m_j(\lambda)} \\ &= \prod_{j\geq 1} (-1)^{m_j(\lambda)} \left(\!\!\binom{M_{j,1}\left(\frac{1}{1-\frac{1}{q}}\right)}{m_j(\lambda)}\!\!\right) \\ &= (-1)^{\ell(\lambda)} T_{\lambda,1}\left(\tfrac{1}{1-\frac{1}{q}}\right). \end{aligned}$$

The second identity follows from a parallel computation noting that (2.4) is equivalent to

$$\left(\!\!\binom{-x}{m}\!\!\right) = (-1)^m \binom{x}{m}.$$

□

The liminal reciprocity identity

$$T^{\mathrm{sf}}_{\lambda,\infty}(q) = (-1)^{\ell(\lambda)} T_{\lambda,1}\left(\tfrac{1}{1-\frac{1}{q}}\right)$$

relates the limiting number of squarefree polynomials with factorization type λ in $\mathbb{F}_q[x_1, x_2, \ldots, x_n]$ as $n \to \infty$ to the number of polynomials $\mathbb{F}_q[x]$ with factorization type λ with no restrictions on factor multiplicity. This relationship is, to us, rather mysterious. It would be interesting to find a conceptual explanation for this relationship between infinite and one dimensional factorization statistics.

3. LIMINAL FIRST MOMENTS OF SQUAREFREE FACTORIZATION STATISTICS

A *factorization statistic* is a function P defined on $\mathrm{Poly}_{d,n}(\mathbb{F}_q)$ such that $P(f)$ only depends on the factorization type of $f \in \mathrm{Poly}_{d,n}(\mathbb{F}_q)$. Equivalently, P is a function defined on the partitions of the degree d, or as a class function of the symmetric group S_d. Recently we [9] determined explicit formulas for the first moments of factorization statistics on $\mathrm{Poly}_{d,1}(\mathbb{F}_q)$ and $\mathrm{Poly}^{\mathrm{sf}}_{d,1}(\mathbb{F}_q)$ in terms of the characters of symmetric group representations related to the cohomology of point configurations in Euclidean space.

Theorem 3.1 ([9, Thm. 2.4, Thm. 2.5], [6, Prop. 4.1])**.** *Let P be a factorization statistic, and let ψ_d^k, ϕ_d^k be the characters of the S_d-representations $H^{2k}(\mathrm{PConf}_d(\mathbb{R}^3),\mathbb{Q})$ and $H^k(\mathrm{PConf}_d(\mathbb{R}^2),\mathbb{Q})$ respectively. Then*

$$(1) \quad \sum_{f\in \mathrm{Poly}_{d,1}(\mathbb{F}_q)} P(f) = \sum_{k=0}^{d-1} \langle P, \psi_d^k\rangle q^{d-k}$$

$$(2) \quad \sum_{f\in \mathrm{Poly}^{\mathrm{sf}}_{d,1}(\mathbb{F}_q)} P(f) = \sum_{k=0}^{d-1} (-1)^k \langle P, \phi_d^k\rangle q^{d-k},$$

where $\langle P, \psi_d^k\rangle = \frac{1}{d!}\sum_{\tau\in S_d} P(\tau)\psi_d^k(\tau)$ is the standard inner product of class functions on S_d.

The identity (2) was first shown by Church, Ellenberg, and Farb [6, Prop. 4.1] using algebro-geometric methods including the Grothendieck-Lefschetz trace formula. They called this identity the *twisted Grothendieck-Lefschetz formula*. We gave a new proof in [9, Thm. 2.5] using a generating function argument. Our results in [9] are stated in terms of expected values instead of first moments; this distinction has little effect in the $\mathrm{Poly}_{d,1}(\mathbb{F}_q)$ case, but does change the family of representations in the squarefree case $\mathrm{Poly}^{\mathrm{sf}}_{d,1}(\mathbb{F}_q)$. This version of (2) appears in [6, Prop. 4.1].

The next result combines Theorem 3.1 with liminal reciprocity to express the limiting first moments of squarefree factorization statistics in terms of characters of symmetric group representations.

Theorem 3.2. *Let P be a factorization statistic, and let σ_d^k be the character of the S_d-representation*

$$\Sigma_d^k = \bigoplus_{j=k}^{d-1} \mathrm{sgn}_d \otimes H^{2j}(\mathrm{PConf}_d(\mathbb{R}^3),\mathbb{Q})^{\oplus\binom{j}{k}}. \tag{3.1}$$

Then

$$\lim_{n\to\infty} \sum_{f\in \mathrm{Poly}^{\mathrm{sf}}_{d,n}(\mathbb{F}_q)} P(f) = \frac{1}{(1-q)^d}\sum_{k=0}^{d} (-1)^k \langle P, \sigma_d^k\rangle q^{d-k}.$$

Theorem 3.2 follows from the following representation theoretic interpretation of the liminal squarefree type polynomials $T^{\mathrm{sf}}_{\lambda,\infty}(q)$. Recall that for a partition λ

the liminal squarefree type polynomial $T^{\mathrm{sf}}_{\lambda,\infty}(q)$ is defined by

$$T^{\mathrm{sf}}_{\lambda,\infty}(q) = \lim_{n\to\infty} T^{\mathrm{sf}}_{\lambda,n}(q),$$

where $T^{\mathrm{sf}}_{\lambda,n}(q)$ is the number of monic squarefree polynomials in $\mathbb{F}_q[x_1, x_2, \ldots, x_n]$ with factorization type λ.

Theorem 3.3. *Let $\lambda \vdash d$ be a partition, and let σ_d^k be the character of the S_d-representation Σ_d^k defined in* (3.1). *Then*

$$T^{\mathrm{sf}}_{\lambda,\infty}(q) = \frac{1}{z_\lambda(1-q)^d}\sum_{k=0}^{d-1}(-1)^k\sigma_d^k(\lambda)q^{d-k},$$

where $z_\lambda = \prod_{j\geq 1} j^{m_j(\lambda)}m_j(\lambda)!$ is the number of permutations in S_d commuting with a permutation of cycle type λ.

Proof. Let ψ_d^k be the character of the S_d-representation $H^{2k}(\mathrm{PConf}_d(\mathbb{R}^3),\mathbb{Q})$. In [9, Thm. 2.1] we showed that for all partitions $\lambda \vdash d$,

$$T_{\lambda,1}(q) = \frac{1}{z_\lambda}\sum_{k=0}^{d-1}\psi_d^k(\lambda)q^{d-k}.$$

Thus, by Theorem 2.6 we have

$$\begin{aligned}
T^{\mathrm{sf}}_{\lambda,\infty}(q) &= (-1)^{\ell(\lambda)}T_{\lambda,1}\left(\tfrac{1}{1-\frac{1}{q}}\right)\\
&= \frac{1}{z_\lambda}\sum_{j=0}^{d-1}(-1)^{\ell(\lambda)}\psi_d^j(\lambda)\left(\frac{1}{1-\frac{1}{q}}\right)^{d-j}\\
&= \frac{1}{z_\lambda(1-q)^d}\sum_{j=0}^{d-1}(-1)^{d-\ell(\lambda)}\psi_d^j(\lambda)q^{d-j}(q-1)^j\\
&= \frac{1}{z_\lambda(1-q)^d}\sum_{j=0}^{d-1}\mathrm{sgn}_d(\lambda)\psi_d^j(\lambda)q^{d-j}\sum_{k=0}^{j}(-1)^k\binom{j}{k}q^{j-k}\\
&= \frac{1}{z_\lambda(1-q)^d}\sum_{k=0}^{d-1}(-1)^k\left(\sum_{j=k}^{d}\binom{j}{k}\mathrm{sgn}_d(\lambda)\psi_d^j(\lambda)\right)q^{d-k}\\
&= \frac{1}{z_\lambda(1-q)^d}\sum_{k=0}^{d-1}(-1)^k\sigma_d^k(\lambda)q^{d-k}.
\end{aligned}$$

□

We now prove Theorem 3.3.

Proof. Since P depends only on factorization type we have

$$\lim_{n\to\infty}\sum_{f\in\mathrm{Poly}^{\mathrm{sf}}_{d,n}(\mathbb{F}_q)}P(f) = \lim_{n\to\infty}\sum_{\lambda\vdash d}P(\lambda)T^{\mathrm{sf}}_{\lambda,n}(q) = \sum_{\lambda\vdash d}P(\lambda)T^{\mathrm{sf}}_{\lambda,\infty}(q).$$

Then Theorem 3.3 implies

$$\begin{aligned}\sum_{\lambda \vdash d} P(\lambda) T^{\mathrm{sf}}_{\lambda,\infty}(q) &= \sum_{\lambda \vdash d} \frac{1}{z_\lambda (1-q)^d} \sum_{k=0}^{d} (-1)^k P(\lambda) \sigma_d^k(\lambda) q^{d-k} \\ &= \frac{1}{(1-q)^d} \sum_{k=0}^{d} (-1)^k \sum_{\lambda \vdash d} \frac{P(\lambda)\sigma_d^k(\lambda)}{z_\lambda} q^{d-k} \\ &= \frac{1}{(1-q)^d} \sum_{k=0}^{d} (-1)^k \langle P, \sigma_d^k \rangle q^{d-k}.\end{aligned}$$

□

The coefficients of $T^{\mathrm{sf}}_{\lambda,1}(q)$ also have representation theoretic interpretations, which suggests that we might hope for a version of Theorem 3.3 for the limiting first moments of factorization statistics on $\mathrm{Poly}_{d,n}(\mathbb{F}_q)$. However, computations show that the coefficients of $T_{\lambda,\infty}(q)$ are determined by virtual characters, unlike those of $T^{\mathrm{sf}}_{\lambda,\infty}(q)$. Since this is what we would expect for an arbitrary class function valued in $\frac{1}{z_\lambda}\mathbb{Z}$ we do not pursue it.

In [9] we pose the question of finding a geometric interpretation of Theorem 3.1 which explains the connection between the configuration space $\mathrm{PConf}_d(\mathbb{R}^3)$ and factorization statistics of degree d polynomials over $\mathbb{F}_q$. Going further, we would like to know any conceptual explanation for Theorem 3.3, be it geometric or combinatorial. The sequence of representations Σ_d^k is unfamiliar to us; some basic properties are collected below in Proposition 3.5 with the hope that they may be recognized by the reader.

The representation theoretic interpretation of the coefficients of $T^{\mathrm{sf}}_{\lambda,\infty}(q)$ was discovered empirically by the author pursuing generalizations of squarefree splitting measures to multivariate polynomials. It was in the course of trying to establish this connection with representation theory that the liminal reciprocity and all the results of [9] were found.

3.1. **Example.** We demonstrate the liminal reciprocity identity of Theorem 2.6 by computing the expected value of the sign statistic sgn_d on degree d univariate polynomials $\mathrm{Poly}_{d,1}(\mathbb{F}_q)$ and the limiting expected value of sgn_d on squarefree degree d polynomials $\mathrm{Poly}^{\mathrm{sf}}_{d,\infty}(\mathbb{F}_q)$.

Let sgn_d be the sign character of S_d. Note that $\mathrm{sgn}_d(\lambda) = (-1)^d(-1)^{\ell(\lambda)}$, where $\ell(\lambda) = \sum_{j\geq 1} m_j(\lambda)$ is the number of parts of λ. Recall that $P_{d,n}(q) = |\mathrm{Poly}_{d,n}(\mathbb{F}_q)|$ and $P^{\mathrm{sf}}_{d,n}(q) = |\mathrm{Poly}^{\mathrm{sf}}_{d,n}(\mathbb{F}_q)|$.

Proposition 3.4. *Let* $d \geq 1$.

(1) *The expected value* $E_{d,1}(\mathrm{sgn}_d)$ *of the sign statistic on the set* $\mathrm{Poly}_{d,1}(\mathbb{F}_q)$ *is given by*

$$E_{d,1}(\mathrm{sgn}_d) := \frac{1}{P_{d,1}(q)} \sum_{f \in \mathrm{Poly}_{d,1}(\mathbb{F}_q)} \mathrm{sgn}_d(f) = \frac{1}{q^{\lfloor d/2 \rfloor}}.$$

(2) *The limiting expected value* $E_{d,\infty}^{\text{sf}}(\text{sgn}_d)$ *of the sign statistic on the set* $\text{Poly}_{d,n}^{\text{sf}}(\mathbb{F}_q)$ *as* $n \to \infty$ *is given by*

$$E_{d,\infty}^{\text{sf}}(\text{sgn}_d) := \lim_{n\to\infty} \frac{1}{P_{d,n}^{\text{sf}}(q)} \sum_{f\in\text{Poly}_{d,n}^{\text{sf}}(\mathbb{F}_q)} \text{sgn}_d(f) = \left(\frac{1}{1-\frac{1}{q}}\right)^{\lfloor d/2\rfloor},$$

where the limit is taken $1/q$*-adically.*

Proof. (1) Since $\text{sgn}_d(f)$ depends only on the factorization type of f we have

$$\sum_{f\in\text{Poly}_{d,1}(\mathbb{F}_q)} \text{sgn}_d(f) = \sum_{\lambda\vdash d} \text{sgn}(\lambda) T_{\lambda,1}(q).$$

Theorem 2.6 gives the identity

$$(-1)^{\ell(\lambda)} T_{\lambda,1}(q) = T_{\lambda,\infty}^{\text{sf}}\left(\tfrac{1}{1-\frac{1}{q}}\right),$$

from which we deduce for each $d \geq 1$

$$\begin{aligned}\sum_{\lambda\vdash d} \text{sgn}(\lambda) T_{\lambda,1}(q) &= \sum_{\lambda\vdash d} (-1)^d (-1)^{\ell(\lambda)} T_{\lambda,1}(q)\\ &= \sum_{\lambda\vdash d} (-1)^d T_{\lambda,\infty}^{\text{sf}}\left(\tfrac{1}{1-\frac{1}{q}}\right)\\ &= (-1)^d P_{d,\infty}^{\text{sf}}\left(\tfrac{1}{1-\frac{1}{q}}\right).\end{aligned}$$

Theorem 2.3 (2) tells us

$$P_{d,\infty}^{\text{sf}}(q) = (-1)^d \left(\tfrac{1}{1-\frac{1}{q}}\right)^{\lfloor \frac{d+1}{2}\rfloor}.$$

Thus,

$$\sum_{\lambda\vdash d} \text{sgn}_d(\lambda) T_{\lambda,1}(q) = (-1)^d P_{d,\infty}^{\text{sf}}\left(\tfrac{1}{1-\frac{1}{q}}\right) = q^{\lfloor \frac{d+1}{2}\rfloor}.$$

Since $P_{d,1}(q) = q^d$ and $d - \lfloor (d+1)/2 \rfloor = \lfloor d/2 \rfloor$ it follows that

$$E_{d,1}(\text{sgn}_d) = \frac{1}{P_{d,1}(q)} \sum_{f\in\text{Poly}_{d,1}(\mathbb{F}_q)} \text{sgn}(f) = \frac{1}{q^{\lfloor d/2\rfloor}}.$$

(2) For each $n \geq 1$,

$$E_{d,n}^{\text{sf}}(\text{sgn}_d) := \frac{1}{P_{d,n}^{\text{sf}}(q)} \sum_{f\in\text{Poly}_{d,n}^{\text{sf}}(\mathbb{F}_q)} \text{sgn}_d(f) = \frac{1}{P_{d,n}^{\text{sf}}(q)} \sum_{\lambda\vdash d} \text{sgn}(\lambda) T_{\lambda,n}^{\text{sf}}(q).$$

Taking a limit as $n \to \infty$,

$$E_{d,\infty}^{\text{sf}}(\text{sgn}_d) = \frac{1}{P_{d,\infty}^{\text{sf}}(q)} \sum_{\lambda\vdash d} \text{sgn}_d(\lambda) T_{\lambda,\infty}^{\text{sf}}(q).$$

Theorem 2.6 gives us

$$(-1)^{\ell(\lambda)} T_{\lambda,\infty}^{\text{sf}}(q) = T_{\lambda,1}\left(\tfrac{1}{1-\frac{1}{q}}\right).$$

Therefore,

$$\begin{aligned}\sum_{\lambda\vdash d}\mathrm{sgn}_d(\lambda)T^{\mathrm{sf}}_{\lambda,\infty}(q) &= \sum_{\lambda\vdash d}(-1)^d(-1)^{\ell(\lambda)}T^{\mathrm{sf}}_{\lambda,\infty}(q)\\ &= \sum_{\lambda\vdash d}(-1)^d T_{\lambda,1}\left(\tfrac{1}{1-\frac{1}{q}}\right)\\ &= (-1)^d\left(\tfrac{1}{1-\frac{1}{q}}\right)^d.\end{aligned}$$

Since $P^{\mathrm{sf}}_{d,\infty}(q) = (-1)^d\left(\frac{1}{1-\frac{1}{q}}\right)^{\lfloor (d+1)/2\rfloor}$ and $d-\lfloor(d+1)/2\rfloor = \lfloor d/2\rfloor$ we conclude that

$$E^{\mathrm{sf}}_{d,\infty}(\mathrm{sgn}_d) = \frac{1}{P^{\mathrm{sf}}_{d,\infty}(q)}\sum_{\lambda\vdash d}\mathrm{sgn}_d(\lambda)T^{\mathrm{sf}}_{\lambda,\infty}(q) = \left(\frac{1}{1-\frac{1}{q}}\right)^{\lfloor d/2\rfloor}.$$

□

Note that Theorem 3.1 (1) tells us that

$$E_{d,1}(\mathrm{sgn}_d) = \sum_{k=0}^{d-1}\frac{\langle \mathrm{sgn}_d, \psi^k_d\rangle}{q^k}.$$

Comparing this with Proposition 3.4 (1) it follows that $H^{2k}(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q})$ has a one dimensional sgn_d component when $k = \lfloor d/2\rfloor$ and no sgn_d component for any other value of k.

The sign function sgn_d is closely related to the *Liouville function* λ studied by Carlitz [2, 3] in the context of polynomials in $\mathbb{F}_q[x]$. In particular, if $f(x) \in \mathrm{Poly}_{d,1}(\mathbb{F}_q)$

$$\lambda(f) = (-1)^d\mathrm{sgn}_d(f).$$

Carlitz [2, (ii) pg. 121][3, Sec. 3] computes the first moment of the Liouville function using zeta functions. Proposition 3.4 may also be deduced from his result. We thank Ofir Gorodetsky for bringing this work to our attention.

3.2. **The S_d-representations Σ^k_d.** Theorem 3.2 relates the limiting first moments of factorization statistics on squarefree polynomials with a family of symmetric group representations Σ^k_d. Recall that

$$\Sigma^k_d = \bigoplus_{j=k}^{d-1}\mathrm{sgn}_d\otimes H^{2j}(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q})^{\oplus\binom{j}{k}}.$$

We conclude with Proposition 3.5 which records some observations about the representations Σ^k_d.

Proposition 3.5. *Let σ^k_d be the character of Σ^k_d. Then*

(1) *The dimension of Σ^k_d is*

$$\dim\Sigma^k_d = \sum_{i=k}^{d-1}\begin{bmatrix} d\\ d-i\end{bmatrix}\binom{i}{i-k},$$

where $\left[{m \atop n}\right]$ *is an unsigned Stirling number of the first kind.*

(2) *The representation*

$$\bigoplus_{k=0}^{d-1} \Sigma_d^k$$

has dimension $(2d-1)!! = (2d-1)(2d-3)\cdots 3 \cdot 1$.

(3) Σ_d^0 *is isomorphic to the regular representation* $\mathbb{Q}[S_d]$.

Note that the sequence $\dim \Sigma_d^k$ appears as A088996 in the *Online Encyclopedia of Integer Sequences* [11].

Proof. (1) The dimension of a representation is given by evaluating its character on the identity, hence

$$\dim \Sigma_d^k = \sigma_d^k([1^d]).$$

Theorem 3.3 implies that

$$T^{\mathrm{sf}}_{[1^d],\infty}(q) = \frac{1}{d!(1-q)^d}\sum_{k=0}^{d-1}(-1)^k \sigma_d^k([1^d])q^{d-k}.$$

On the other hand, we may compute $T^{\mathrm{sf}}_{[1^d],\infty}(q)$ directly as

$$T^{\mathrm{sf}}_{[1^d],\infty}(q) = \binom{M_{d,\infty}(q)}{d} = \binom{-\frac{1}{1-\frac{1}{q}}}{d}.$$

The binomial coefficient $\binom{x}{d}$ expands as a polynomial in x in terms of the unsigned Stirling numbers of the first kind,

$$\binom{x}{d} = \frac{1}{d!}\sum_{k=0}^{d-1}(-1)^k \left[{d \atop d-k}\right] x^{d-k}.$$

Thus,

$$\begin{aligned} T^{\mathrm{sf}}_{[1^d],\infty}(q) &= \frac{1}{d!}\sum_{i=0}^{d-1}(-1)^i \left[{d \atop d-i}\right]\left(-\frac{1}{1-\frac{1}{q}}\right)^{d-i} \\ &= \frac{1}{d!(1-q)^d}\sum_{i=0}^{d-1}(-1)^i \left[{d \atop d-i}\right] q^{d-i}(1-q)^i \\ &= \frac{1}{d!(1-q)^d}\sum_{i=0}^{d-1}\sum_{j=0}^{i}(-1)^{i+j} \left[{d \atop d-i}\right]\binom{i}{j} q^{d-(i-j)}. \end{aligned}$$

Let $k = i - j$ and write the sum in terms of i and k to get

$$T^{\mathrm{sf}}_{[1^d],\infty}(q) = \frac{1}{d!(1-q)^d}\sum_{k=0}^{d-1}(-1)^k \left(\sum_{i=k}^{d-1}\left[{d \atop d-i}\right]\binom{i}{i-k}\right) q^{d-k}.$$

Comparing coefficients in our two expressions for $T^{\mathrm{sf}}_{[1^d],\infty}(q)$ we conclude that

$$\dim \Sigma_d^k = \sigma_d^k([1^d]) = \sum_{i=k}^{d-1} \begin{bmatrix} d \\ d-i \end{bmatrix} \binom{i}{i-k}.$$

(2) Let ψ_d^k be the character of $H^{2k}(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q})$. Then using the definition of Σ_d^k and switching the order of summation we have

$$\sum_{k=0}^{d-1} \sigma_d^k([1^d]) = \sum_{k=0}^{d-1} \sum_{j=k}^{d} \binom{j}{k} \psi_d^j([1^d]) = \sum_{j=0}^{d-1} \sum_{k=0}^{j} \binom{j}{k} \psi_d^j([1^d]) = \sum_{j=0}^{d-1} 2^j \psi_d^j([1^d]).$$

Note that by Theorem 3.1 (1),

$$\sum_{j=0}^{d-1} \frac{\psi_d^j([1^d])}{q^j} = d! \frac{T_{[1^d],1}(q)}{q^d} = \frac{d!}{q^d} \binom{q+d-1}{d}. \tag{3.2}$$

Evaluating (3.2) at $q = \frac{1}{2}$ implies

$$\sum_{j=0}^{d-1} 2^j \psi_d^j([1^d]) = 2^d d! \binom{d-\frac{1}{2}}{d} = (2d-1)(2d-3)\cdots 3 \cdot 1 = (2d-1)!!.$$

Therefore $\dim \bigoplus_{k=0}^{d} \Sigma_d^k = (2d-1)!!$.

(3) By definition we have

$$\Sigma_d^0 \cong \mathrm{sgn}_d \otimes \bigoplus_{j=0}^{d-1} H^{2j}(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q}).$$

In [9, Thm. 2.8] we showed that

$$\bigoplus_{j=0}^{d-1} H^{2j}(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q}) \cong \mathbb{Q}[S_d],$$

where $\mathbb{Q}[S_d]$ is the regular representation. The claim follows from

$$\mathrm{sgn}_d \otimes \mathbb{Q}[S_d] \cong \mathbb{Q}[S_d].$$

□

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