Enumeration of super-strong Wilf equivalence classes of permutations

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Abstract

Super-strong Wilf equivalence classes in the symmetric group $\mathcal{S}_n$ on n letters were shown in [2] to be in bijection with pyramidal sequences of consecutive differences. In this article we enumerate the latter giving recursive formulae in terms of a two-dimensional analogue of the sequence of non-interval permutations. As a by-product, we give a recursively defined set of representatives of super-strong Wilf equivalence classes in $\mathcal{S}_n$.

1 Introduction

In this work we continue the study of *super-strong Wilf equivalence* on permutations in n letters that commenced in [2]. This notion was originally referred to as *strong Wilf equivalence* by S. Kitaev et al. in [3]. J. Pantone and V. Vatter in [4] used the term "super-strong Wilf" to distinguish this from a more general notion they defined and called strong Wilf equivalence.

First we recall some notation and definitions from [3]. Let $\mathbb{P}$ be the set of positive integers. For each $n, m \in \mathbb{P}$ with $m < n$ we let $[n] = \{1, 2, \ldots, n\}$ and $[m, n] = \{m, m+1, \ldots, n\}$. Let $\mathbb{P}^*$ be the set of words on the alphabet $\mathbb{P}$. Its elements are of the form $w = w_1 \cdots w_i \cdots w_n$, with $n \geq 0$ and $w_i \in \mathbb{P}$. If $n = 0$ then $w = \epsilon$, the *empty word*, whereas if $n \in \mathbb{P}$ and each letter w_i appears exactly once, w is a permutation in n letters. We let $|w|$ be the *length* n of the word w, $||w||$ be the *height* or *norm* of w defined as $||w|| = w_1 + \cdots + w_i + \cdots + w_n$ and $alph(w)$ be the set of distinct letters of $\mathbb{P}$ that occur in w. If $alph(w) = [n]$, the set of all permutations on n letters is denoted by $\mathcal{S}_n$.

Given $w, u \in \mathbb{P}^*$, we say that u is a *factor* of w if there exist words $s, v \in \mathbb{P}^*$ such that $w = suv$. If $s = \epsilon$ (resp. $v = \epsilon$) u is called a *prefix* (resp. *suffix*) of w. Consider the poset $(\mathbb{P}, \leq)$ with the usual order in $\mathbb{P}$. The *generalized factor order* on $\mathbb{P}^*$ is the partial order – also denoted by $\leq$ – obtained by letting $u \leq w$ if and only if there is a factor v of w such that $|u| = |v|$ and $u_i \leq v_i$, for each $i \in [|u|]$. The factor v is called an *embedding* of u in w. If the first element of v is the j-th element of w then the index j is called an *embedding index* of u into w. The *embedding index set* of u into w, or *embedding set* for brevity, is defined as the set of all embedding indices of u into w and is denoted by $Em(u, w)$.

Let now t, x be two commuting indeterminates. The *weight* of a word $w \in \mathbb{P}^*$ is defined as the monomial $wt(w) = t^{|w|}x^{||w||}$. A bijection $f : \mathbb{P}^* \to \mathbb{P}^*$ is called *weight-preserving* if the weight of w is preserved under f, i.e., $|f(w)| = |w|$ and $||f(w)|| = ||w||$, for every $w \in \mathbb{P}^*$.

Definition 1. ([3, Section 5]) Two words $u, v \in \mathbb{P}^*$ are called *super-strongly Wilf equivalent*, denoted $u \sim_{ss} v$, if there exists a weight-preserving bijection $f : \mathbb{P}^* \to \mathbb{P}^*$ such that $Em(u, w) = Em(v, f(w))$ for all $w \in \mathbb{P}^*$.

In [2] super-strongly Wilf equivalence classes in $\mathcal{S}_n$ were characterized using *sequences of consecutive differences* of permutations. The latter are defined as follows.

Definition 2. ([2, Definition 3]) Let $u \in \mathcal{S}_n$ and $s = s_1 \cdots s_i \cdots s_n = u^{-1}$. For $i = n-1$ down to 1 consider the proper suffix $s_i \cdots s_n$ of s and its alphabet set $\Sigma_i(s) = alph(s_i \cdots s_n) = \{s_i^{(i)}, \ldots, s_n^{(i)}\}$, where $s_i^{(i)} < \cdots < s_n^{(i)}$. We define $\Delta_i(s)$ to be the vector of *consecutive differences* in $\Sigma_i(s)$, i.e.,

$$\Delta_i(s) = (s_{i+1}^{(i)} - s_i^{(i)}, \ldots, s_n^{(i)} - s_{n-1}^{(i)}).$$

The sequence

$$p(s) = (\Delta_1(s), \Delta_2(s), \ldots, \Delta_{n-2}(s), \Delta_{n-1}(s))$$

has a pyramidal form and is called the *pyramid or sequence of consecutive differences* of $s \in \mathcal{S}_n$.

Example 1. ([2, Example 4]) Let $u = 21365874$. Then $s = u^{-1} = 21385476$. The sequence of differences for s is the following:

$$\begin{gathered}
\Delta_7(s) = (1) \\
\Delta_6(s) = (2, 1) \\
\Delta_5(s) = (1, 1, 1) \\
\Delta_4(s) = (1, 1, 1, 1) \\
\Delta_3(s) = (1, 1, 1, 1, 1) \\
\Delta_2(s) = (2, 1, 1, 1, 1, 1) \\
\Delta_1(s) = (1, 1, 1, 1, 1, 1, 1)
\end{gathered}$$

The main result of [2] is the following.

Theorem 1. *([2, Theorem 3]) Let $u, v \in \mathcal{S}_n$ and $s = u^{-1}, t = v^{-1}$. Then $u \sim_{ss} v$ if and only if $\Delta_i(s) = \Delta_i(t)$, for each $i \in [2, n-1]$, i.e., if and only if $p(s) = p(t)$.*

Observing the way pyramidal sequences of consecutive differences of a permutation are constructed, we can see that the transition between two consecutive vectors follows one of three simple steps (see [2, Lemma 3]), namely: we either decompose a number into two summands or we add an extra number on the left or on the right of the vector.

To enumerate such structures it is more convenient to leave aside their connections to permutations and focus on these simple rules that can indeed construct all possible pyramidal sequences of this type.

Definition 3. A *pyramidal sequence of vectors* is a sequence of the form

$$p = (\Delta_1, \ldots, \Delta_i, \Delta_{i+1}, \ldots, \Delta_{n-1}),$$

where each Δ_i is a sequence of $n-i$ positive integers such that $\Delta_1 = (\underbrace{1, 1, \ldots, 1}_{n-1})$ and if $\Delta_i = (d_1, d_2, \ldots, d_{n-i-1}, d_{n-i})$ we have the following three options for Δ_{i+1}:

$$\Delta_{i+1} = \begin{cases} (d_1, \ldots, d_{k-1}, d_k + d_{k+1}, d_{k+2}, \ldots, d_{n-i}), \text{ for some } k \in [n-i-1], \text{or} \\ (d_2, \ldots, d_{n-i-1}, d_{n-i}), \quad \text{or} \\ (d_1, d_2, \ldots, d_{n-i-1}). \end{cases}$$

It is important to note that if $\Delta_i = (\underbrace{d, d, \ldots, d}_{n-i})$ for some $d \in \mathbb{P}$, the second and third options coincide.

Let $\mathit{\Pi}_n$ denote the set of all pyramidal sequences of the above form. It is evident that given a permutation $s \in \mathcal{S}_n$, its pyramidal sequence $p(s)$ is of the above form. Conversely, given an element $p \in \mathit{\Pi}_n$ we can construct a permutation $s \in \mathcal{S}_n$ such that $p = p(s)$. This construction is similar to the one in [2, Example 5].

It is helpful to view the above definition in the following way. Suppose that we originally have n walls which define $n-1$ chambers with one ball in each one of them. This is precisely the situation in Δ_1. Then at each step the transition from Δ_i to Δ_{i+1} can be visualized by a removal of one wall. If this wall is internal, the balls at its left and right chamber will all be concentrated at one unified chamber. On the other hand, if this wall is external, all corresponding balls to its left (if it is a right wall) or to its right (resp. if it is a left one) will be removed. This combinatorial game ends when all the original $n-1$ balls will be removed. We want to enumerate the number of ways that this can be done, considering that two moves are different if they result to a different set-up of chambers and balls.

For a subset $X = \{x_1 < x_2 < \cdots < x_{k-1} < x_k\}$ of $[n]$, let

$$\Delta(X) = (x_2 - x_1, \ldots, x_k - x_{k-1})$$

be the vector of consecutive differences in X. For a given vector of differences $\Delta = (d_1, \ldots, d_{k-1})$ we set

$$\overline{x_1}(\Delta) = \{x_1, x_1 + d_1, \ldots, x_1 + d_1 + \cdots + d_{k-1}\}.$$

Note that when $\Delta = \Delta(X)$ then $\overline{x_1}(\Delta) = X$.

2 Prefixes of generalized non-interval permutations

A word of length $l \geq 2$ is called *periodic* when its vector of consecutive differences is equal to $(\underbrace{d, d, \ldots, d}_{l-1})$, for some $d \in \mathbb{P}$.

Definition 4. For $i \in [n-2]$, we define the set $\mathcal{D}_{i,n}$ as the set of words u of length i which appear as non-empty prefixes of permutations in $\mathcal{S}_n$ whose remaining $(n-i)$-lettered suffix is periodic and furthermore this index i is the smallest one attaining this form of periodicity.

For $n = 3$ we clearly have $\mathcal{D}_{1,3} = \{1, 2, 3\}$. For $n \geq 4$ we may construct $\mathcal{D}_{i,n}$ recursively as follows:

1. For $i = 1$ we have $\mathcal{D}_{1,n} = \{1, n\}$.
2. For $i \geq 2$, $\mathcal{D}_{i,n}$ is the set of prefixes u of length i of permutations in $\mathcal{S}_n$ such that
$$\Delta([n] \setminus alph(u)) = (\underbrace{d, d, \ldots, d}_{n-i-1}), \text{ for some } d \in \mathbb{P},$$
and no proper prefix u' of u of length $1 \leq j < i$ lies in $D_{j,n}$.

Set $d_{i,n} = |\mathcal{D}_{i,n}|$.

Example 2. For $n = 5$, by definition $\mathcal{D}_{1,5} = \{1, 5\}$. To calculate the set of prefixes $\mathcal{D}_{2,5}$, observe that the only possible periodic vectors of differences in this case are $(1, 1)$ and $(2, 2)$. These correspond to the sets $\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}$ and $\{1, 3, 5\}$, respectively. All possible prefixes with letters in the complements of the above sets are $45, 54; 15, 51; 12, 21$ and $24, 42$, respectively. The prefixes $54, 15, 51$, and 12 are rejected since they have a proper prefix in $\mathcal{D}_{1,5}$. Hence, $\mathcal{D}_{2,5} = \{21, 24, 42, 45\}$ and $d_{2,5} = 4$.

Definition 5. For $i \in [n-2]$, a *trapezoidal sequence of vectors* is a sequence of the initial parts $(\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_{i+1})$ of an element in Π_n such that $\Delta_{i+1} = (\underbrace{d, d, \ldots, d}_{n-i-1})$, for some $d \in \mathbb{P}$ and there is no $j \in [2, i]$ such that $\Delta_j = (\underbrace{e, e, \ldots, e}_{n-j})$, for some $e \in \mathbb{P}$. Let $\mathit{\Delta}_{i,n}$ denote the set of all such trapezoidal sequences.

The following result allows us to enumerate the prefixes $\mathcal{D}_{i,n}$ instead of $\mathit{\Delta}_{i,n}$.

Proposition 1. *Let $n \in \mathbb{N}$ and $i \in [2, n-2]$. There is a bijection between the set of prefixes $\mathcal{D}_{i,n}$ and the set $\mathit{\Delta}_{i,n}$.*

Proof. Suppose that $u = u_1 \ldots u_i \in D_{i,n}$. We construct a unique element in $\mathit{\Delta}_{i,n}$ in the following way: First, we define sets X_j, $j \in [i+1]$, inductively as:

$$X_1 = [n] \text{ and } X_{j+1} = X_j \setminus \{u_j\}, \text{ for } j \in [i].$$

The image of u is then defined to be

$$\phi(u) = (\Delta_1, \ldots, \Delta_j, \ldots, \Delta_{i+1}), \text{ where } \Delta_j = \Delta(X_j).$$

By construction and in view of Definitions 4 and 5, $\phi(u) \in \mathit{\Delta}_{i,n}$.

For the reverse direction, consider an element $L = (\Delta_1, \ldots, \Delta_j, \Delta_{j+1}, \ldots, \Delta_{i+1})$ in $\Delta_{i,n}$. Suppose that $\Delta_{j+1} = (d_1, \ldots, d_{n-j-1})$ and $\Delta_j = (e_0, e_1, \ldots, e_{n-j-1})$. We define sets Y_j inductively as follows. Set $Y_1 = [n]$. For $j \in [2, i]$, let $y_j = \min(Y_j)$ and set

$$Y_{j+1} = \begin{cases} \overline{y_j}(\Delta_{j+1}), \text{ if } \Delta_{j+1} \neq (e_1, \ldots, e_{n-j-1}) \\ \overline{y_j + e_0}(\Delta_{j+1}), \text{ if } \Delta_{j+1} = (e_1, \ldots, e_{n-j-1}). \end{cases}$$

We then define $\psi(L) = u_1 \cdots u_j \cdots u_i$, where $\{u_j\} = Y_{j+1} \setminus Y_j$.

To show that ϕ and ψ are inverses of each other it suffices to demonstrate inductively on j that $Y_j = X_j$.

For the one direction, given the set X_j, for $j \in [k]$, we argue as follows. For $j = 1$ we immediately get $Y_1 = X_1 = [n]$. Assuming that $Y_k = X_k$, and thus $y_k = min(Y_k) = min(X_k) = x_k$, we have to show that $Y_{k+1} = X_{k+1}$. If $\Delta_{k+1} \neq (e_1, \ldots, e_{n-k-1})$, then $Y_{k+1} = \overline{y_k}(\Delta_{k+1}) = \overline{x_k}(\Delta_{k+1})$. Now since $\Delta_{k+1} = \Delta(X_{k+1})$ we get $\overline{x_k}(\Delta_{k+1}) = X_{k+1}$ and the result follows.

If, on the other hand, $\Delta_{k+1} = (e_1, \ldots, e_{n-k-1})$, then $Y_{k+1} = \overline{y_k + e_0}(\Delta_{k+1}) = \overline{x_k + e_0}(\Delta_{k+1})$. Clearly $y_k \notin Y_{k+1}$ and in fact we have $\{y_k\} = Y_k \backslash Y_{k+1}$ and furthermore $y_k = u_k$. Since $X_{k+1} = X_k \setminus \{u_k\}$, we obtain $X_{k+1} = Y_{k+1}$.

For the other direction our argument is reversed starting with the given set Y_j. □

The enumeration of super-strong Wilf equivalence classes is achieved using the numbers $d_{i,n}$ as follows.

Theorem 2. *The number s_n of distinct super-strong Wilf equivalence classes of $\mathcal{S}_n$ is given by the recursive formula*

$$s_n = s_{n-1} + \sum_{i=2}^{n-2} d_{i,n} \cdot s_{n-i}.$$

Proof. Let $\mathcal{T}_{i,n} = \{(\Delta_1, \dots \Delta_i, \Delta_{i+1}, \dots, \Delta_{n-1}) \in \mathit{\Pi}_n : (\Delta_1, \dots \Delta_i, \Delta_{i+1}) \in \mathit{\Delta}_{i,n}\}$, for $i \in [n-2]$. We clearly have

$$\mathit{\Pi}_n = \mathcal{T}_{1,n} \sqcup \mathcal{T}_{2,n} \sqcup \cdots \sqcup \mathcal{T}_{i,n} \sqcup \cdots \sqcup \mathcal{T}_{n-2,n}. \tag{2.1}$$

Observe that $\mathit{\Delta}_{1,n}$ consists of just one element, namely (Δ_1, Δ_2), where $\Delta_1 = (\underbrace{1, 1, \dots, 1}_{n-1})$ and $\Delta_2 = (\underbrace{1, 1, \dots, 1}_{n-2})$. Then there is an immediate bijective correspondence between $\mathit{\Pi}_{n-1}$ and $\mathcal{T}_{1,n}$, therefore $|\mathcal{T}_{1,n}| = s_{n-1}$. Now let $i \in [2, n-2]$. Consider a pyramidal sequence in $\mathcal{T}_{i,n}$. Then there exists a $d \in [n-1]$ such that $\Delta_{i+1} = (\underbrace{d, d, \dots, d}_{n-i-1})$ and for all $j \in [2, i]$ there exists no e such that $\Delta_j = (\underbrace{e, e, \dots, e}_{n-j})$.

Our first claim is that all entries in Δ_k for $k \in [i+1, n-1]$ will be multiples of d. Indeed, suppose that $\Delta_k = (d_1, \dots, d_{n-k})$, where by induction it is assumed that $d|d_i$, for $l \in [n-k]$. If Δ_{k+1} is equal either to $(d_2, \dots, d_{n-k})$ or $(d_1, \dots, d_{n-k-1})$, then the result follows immediately. On the other hand, if $\Delta_{k+1} = (d_1, \dots, d_{m-1}, d_m + d_{m+1}, d_{m+2}, \dots, d_{n-k})$ it is enough to show that $d|(d_m + d_{m+1})$ which follows inductively from $d|d_m$ and $d|d_{m+1}$.

We define a map $\tau_{i,n} : \mathcal{T}_{i,n} \to \mathit{\Delta}_{i,n} \times \mathit{\Pi}_{n-i}$ as

$$\tau_{i,n}(\Delta_1, \dots, \Delta_i, \Delta_{i+1}, \dots, \Delta_{n-1}) = \big((\Delta_1, \dots, \Delta_i, \Delta_{i+1}), \frac{1}{d} \cdot (\Delta_{i+1}, \dots, \Delta_{n-1})\big), \tag{2.2}$$

where $\Delta_{i+1} = (\underbrace{d, d, \dots, d}_{n-i-1})$. Our previous claim ensures that $\tau_{i,n}$ is well defined. Furthermore, it is a bijection whose inverse is the map $\rho_{i,n} : \mathit{\Delta}_{i,n} \times \mathit{\Pi}_{n-i} \to \mathcal{T}_{i,n}$ defined as

$$\rho_{i,n}\big((\Delta_1, \dots, \Delta_i, \Delta_{i+1}), (\Delta'_1, \dots \Delta'_{n-i-1})\big) = (\Delta_1, \dots, \Delta_i, \Delta_{i+1}, d \cdot \Delta'_1, \dots d \cdot \Delta'_{n-i-1}),$$

where $\Delta_{i+1} = (\underbrace{d, d, \dots, d}_{n-i-1})$. It follows that $|\mathcal{T}_{i,n}| = d_{i,n} \cdot s_{n-i}$, for $i \in [2, n-2]$. $\square$

Let τ be a permutation of a set of distinct numbers. Then $red(\tau)$ is the *reduced form* of τ, i.e., the permutation obtained by replacing the smallest entry of τ by 1, the second smallest by 2 and so on.

We define $\mathcal{E}_{i,n}$ to be

$$\mathcal{E}_{i,n} = \begin{cases} \{1\}, & i = 1 \\ \mathcal{D}_{i,n}, & i \in [2, n-2]. \end{cases}$$

Corollary 1. *A set of super-strong Wilf equivalence classes representatives in $\mathcal{S}_n$ is described recursively by the set of the inverses of*

$$\mathcal{R}_n = \{u \cdot v \quad : \quad u \in \mathcal{E}_{i,n};\ red(v) \in \mathcal{R}_{n-i};\ i \in [n-2]\},$$

where $red(v)$ is the reduced form of v.

Proof. In view of Theorem 2, the cardinality of $\mathcal{R}_n$ is the correct one. Also note that for a fixed $u \in \mathcal{E}_{i,n}$, the set $alph(v)$ is immediately determined. For a such a given set $alph(v)$ and a particular permutation $\tau \in \mathcal{R}_{n-i}$, there exists a unique suffix v such that $\tau = red(v)$.

Consider two elements in $\mathcal{R}_n$, namely $w = u \cdot v$ and $w' = u' \cdot v'$. We will prove inductively on n that if $w \neq w'$ then $w^{-1} \sim_{ss} w'^{-1}$. To do that it suffices to show that the corresponding pyramidal sequences of differences are not the same.

If $u \neq u'$ then the corresponding pyramidal sequences for w and w' will differ in the part corresponding to u and u', respectively. On the other hand, if $u = u'$ then $v \neq v'$ and by induction the corresponding pyramidal sequences of that part are not the same. $\square$

Let $s_{j,n}$ be the number of super-strong Wilf equivalence classes of order 2^j in $\mathcal{S}_n$, where $j \in [n-1]$. Note that $s_{0,n} = 0$.

Theorem 3.

$$s_{j,n} = s_{j-1,n-1} + \sum_{k=2}^{n-j-1} d_{k,n} \cdot s_{j,n-k}.$$

Proof. Let $\mathcal{C}_{j,n}$ be the set of all pyramidal sequences in Π_n with corresponding super-strong Wilf equivalence class of order 2^j. From (2.1) it clearly follows that

$$\mathcal{C}_{j,n} = (\mathcal{C}_{j,n} \cap \mathcal{T}_{1,n}) \sqcup (\mathcal{C}_{j,n} \cap \mathcal{T}_{2,n}) \sqcup \cdots \sqcup (\mathcal{C}_{j,n} \cap \mathcal{T}_{i,n}) \sqcup \cdots \sqcup (\mathcal{C}_{j,n} \cap \mathcal{T}_{n-2,n}). \quad (2.3)$$

First observe that the exponent j is equal to the number of transitions from $\Delta_k = (\underbrace{d, d, \ldots, d}_{n-k})$ to $\Delta_{k+1} = (\underbrace{d, d, \ldots, d}_{n-k-1})$, for $k \in [n-1]$. Note that for $k = n-1$ by convention Δ_{k+1} is the empty vector.

Let $i \in [2, n-1]$. Consider a pyramidal sequence $(\Delta_1, \dots \Delta_{n-1}) \in \mathcal{T}_{i,n}$. Observe that for $k \in [i]$ there are no transitions of the aforementioned form, therefore restricting the bijection $\tau_{i,n}$ in (2.2) to $\mathcal{C}_{j,n}$ the only contribution to the exponent j comes from the part $(\Delta_{i+1}, \dots, \Delta_{n-1})$. But the number of transitions there is defined to be equal to $s_{j,n-i}$.

Now let $i = 1$. Observe that there is a transition from Δ_1 to $\Delta_2 = (\underbrace{1, 1, \dots, 1}_{n-2})$ that raises the exponent of the order of the equivalence class by one, hence to get the desired exponent j we need $j - 1$ additional transitions on the upper part of the pyramid; these are precisely $s_{j-1,n-1}$ and the result follows. ◻

In view of the above results, to calculate s_n and $s_{j,n}$ we need a formula for the coefficients $d_{i,n}$.

Theorem 4. *Let $n \geq 4$. For a given $i \in [n-2]$ set $m = n-i-1$ and let $q_{l,m}$ and $r_{l,m}$ be the unique quotient and remainder, respectively, of the Euclidean division of of an arbitrary integer l with m. Then we have the following recursive formula for the $d_{i,n}$*

$$\sum_{k=1}^{i} \frac{q_{n-k,m}}{2} \cdot (r_{n-k,m} + i - k + 1) \cdot d_{k,n} \cdot (i-k)! = \frac{q_{n,m}}{2} \cdot (r_{n,m} + i + 1) \cdot i! \qquad (2.4)$$

Proof. Let $p_{i,n}$ be the number of all prefixes of length i of permutations in $\mathcal{S}_n$ with corresponding suffix an m-periodic word, i.e., a word of the form $a \dots (a+jd) \dots (a+md)$, for a suitable $a \in [n]$.

One way to calculate $p_{i,n}$ is to count first all m-periodic words in $[n]$ and multiply each one of them with the $i!$ choices of the remaining prefix letters. For this purpose let $q_{n,m}$ and $r_{n,m}$ be the unique quotient and remainder respectively of the Euclidean division of n with m. It is straightforward to see that $d \in [q_{n,m}]$ and $p_{i,n} = \sum_{d=1}^{q_{n,m}} (n - jm)$. It follows that the number of all m-periodic words is equal to $\frac{q_{n,m}}{2}(n + r_{m,n} - m) = \frac{q_{n,m}}{2}(r_{n,m} + i + 1)$. In this way $p_{i,n}$ is shown to be equal to the number on the right hand side of the formula of our theorem.

An alternative counting method is to start from the prefixes themselves. Consider a prefix u of length i such that the remaining $n - i$ lettered suffix is a periodic word. Then there exists a unique $k \in [i]$ such that the prefix u' of u lies in $\mathcal{D}_{k,n}$. For this particular k-lettered prefix u' let us count the number of all m-periodic words in the remaining $n - k$ letters. As before, it is straightforward to see that this is precisely equal to $\frac{q_{n-k,m}}{2}(n - k + r_{n-k,m} - m) = \frac{q_{n-k,m}}{2}(r_{n-k,m} + i - k + 1)$. Now as the choices for the suffix of u' in u are $(i - k)!$ due to the remaining $i - k$ letters in u, and as the number of prefixes u' is equal to $d_{k,n}$, $p_{i,n}$ is also equal to the sum appearing on the left hand side of our formula. $\square$

Remark 1. The aforementioned numbers $d_{i,n}$ are related to the number a_n of *non-secable* or *non-interval* permutations. These are all permutations $s = s_1 s_2 \cdots s_n$ of size $n \geq 2$ such that any prefix $s_1 \cdots s_l$ of length $2 \leq l < n$ is not, up to order, equal to the interval $[k, l+k-1]$ [1, 4.4]. This is the sequence $2, 2, 8, 44, 296, 2312, 20384, \ldots$ (also known as $|b_n|$, where b_n is Sequence A077607 of [5]) with recurrence formula

$$\sum_{k=1}^{i} a_{k+1} \cdot (i-k+1)! = (i+1)!.$$

Let $i < \lfloor \frac{n}{2} \rfloor$. Recall that $m = n - i - 1$. Then it follows that $m \geq \lceil \frac{n}{2} \rceil - 1$, so that $q_{n,m} = q_{n-k,m} = 1$ and $r_{n,m} = n - m$, $r_{n-k,m} = (n-k) - m$, for $k \in [i]$. Substituting in (2.4) we obtain

$$\sum_{k=1}^{i} d_{k,n} \cdot (i-k+1)! = (i+1)!.$$

It follows that $d_{k,n} = a_{k+1}$, for all $k < \lfloor \frac{n}{2} \rfloor$, since $d_{1,n} = a_2 = 2$.

This equality of cardinalities is not a mere coincidence. There is actually a deeper connection between $\mathcal{D}_{i,n}$ and the corresponding non-interval permutations.

Proposition 2. *There is a bijection between the set of prefixes $\mathcal{D}_{k,n}$, for $k < \lfloor \frac{n}{2} \rfloor$ and the set $\mathcal{A}_{k+1}$ of all non-interval permutations of length $k+1$.*

Proof. Let $u = u_1 u_2 \cdots u_k$ be a prefix in $\mathcal{D}_{k,n}$. Let $a = \min([n] \setminus alph(u))$. It is more convenient to find a bijection from $\mathcal{D}_{k,n}$ to the set $\mathcal{B}_{k+1}$ of all permutations $b = b_1 b_2 \cdots b_k b_{k+1}$ of size $k+1 \geq 2$ such that any suffix $b_{k-l+2} \cdots b_{k+1}$ of length $2 \leq l < k+1$ is not, up to order, equal to the interval $[k, l+k-1]$. This is due to the fact that $\mathcal{B}_{k+1}$ is clearly equipotent to $\mathcal{A}_{k+1}$, via the bijection $w \mapsto \tilde{w}$.

Then we define a map $\phi : \mathcal{D}_{k,n} \longrightarrow \mathcal{B}_{k+1}$ as $\phi(u) = red(ua)$, where $red(\tau)$ is the *reduced form* of τ, i.e., the permutation obtained by replacing the smallest entry of τ by 1, the second smallest by 2 and so on.

For the reverse direction we define a map $\psi : \mathcal{B}_{k+1} \longrightarrow \mathcal{D}_{k,n}$ as $\psi(b_1 \cdots b_k b_{k+1}) = v_1 \cdots v_k$, where

$$v_i = \begin{cases} b_i, & b_i < b_{k+1} \\ b_i + (n-k-1), & b_i > b_{k+1} \end{cases}$$

Let $u = u_1 \ldots u_k \in \mathcal{D}_{k,n}$. Observe that since $k < \lfloor n/2 \rfloor$ the vector of consecutive differences of $[n] \setminus alph(u)$ is necessarily equal to $(\underbrace{1, 1, \ldots, 1}_{n-k-1})$, hence the word $\pi =$

$u_1 \cdots u_k a(a+1)\cdots(a+n-k-1) \in \mathcal{S}_n$. This implies that if $red(ua) = b_1 \cdots b_k b_{k+1}$, then we must have

$$b_i = \begin{cases} u_i, & u_i < a \\ u_i - (n-k-1), & u_i > a, \end{cases} \tag{2.5}$$

for $i \in [k]$ and $b_{k+1} = a$. We will first show that $red(ua) \in \mathcal{B}_{k+1}$. Suppose, for the sake of contradiction, that this is not the case. Then there exists an index $j \in [k]$ such that $\{b_j, \ldots b_{k+1}\} = \{a-l, \ldots, a-1, a, a+1, \ldots, a+r\}$, for some $l, r \in [k-j]$. Observe that we have $l + r = k - j$. In view of (2.5) we have that

$$\{u_j, \ldots, u_{k+1}\} = \{a - l. \ldots, a-1, a, a+(n-k), \ldots, a+r+(n-k-1)\}. \tag{2.6}$$

Combining the form of the permutation π and (2.6) we obtain that $u_1 \cdots u_{j-1} \in \mathcal{D}_{j-1,n}$, which contradicts the definition of $\mathcal{D}_{k,n}$. □

Example 3. Let us calculate the number s_{10} of super-strong Wilf equivalence classes of $\mathcal{S}_{10}$. In view of Theorem 2 we need to calculate the numbers $d_{i,10}$, for $i \in [8]$. Since $\lfloor 10/2 \rfloor = 5$, Proposition 2 immediately yields $d_{1,10} = a_2 = 2$, $d_{2,10} = a_3 = 2$, $d_{3,10} = a_4 = 8$ and $d_{4,10} = a_5 = 44$.

To evaluate $d_{5,10}$ set $i = 5$ and $m = 4$ in (2.4) and since $q_{10,4} = q_{9,4} = q_{8,4} = 2$; $q_{7,4} = q_{6,4} = 1$; $r_{10,4} = r_{6,4} = 2$, $r_{9,4} = 1$, $r_{8,4} = 0$ and $r_{7,4} = 3$, we obtain

$$\frac{2}{2}\cdot(1+5)\cdot 2\cdot 4! + \frac{2}{2}\cdot(0+4)\cdot 2\cdot 3! + \frac{1}{2}\cdot(3+3)\cdot 8\cdot 2! + \frac{1}{2}\cdot(2+2)\cdot 44\cdot 1! + d_{5,10} = \frac{2}{2}\cdot(2+6)\cdot 5!.$$

It follows that $d_{5,10} = 488$.

To evaluate $d_{6,10}$ set $i = 6$ and $m = 3$ in (2.4) and since $q_{10,3} = q_{9,3} = 3$; $q_{8,3} = q_{7,3} = q_{6,3} = 2$; $q_{5,3} = 1$, $r_{10,3} = r_{7,4} = 1$; $r_{9,3} = r_{6,3} = 0$ and $r_{8,3} = r_{5,3} = 2$, we obtain

$$\frac{3}{2}\cdot(0+6)\cdot 2\cdot 5! + \frac{2}{2}\cdot(2+5)\cdot 2\cdot 4! + \frac{2}{2}\cdot(1+4)\cdot 8\cdot 3! + \frac{2}{2}\cdot(0+3)\cdot 44\cdot 2! \\ + \frac{1}{2}\cdot(2+2)\cdot 488\cdot 1! + d_{6,10} = \frac{3}{2}\cdot(1+7)\cdot 6!.$$

It follows that $d_{6,10} = 4,664$.

To evaluate $d_{7,10}$ set $i = 7$ and $m = 2$ in (2.4). Calculating all the necessary quotients and remainders of the divisions with 2 we obtain

$$\frac{4}{2}\cdot(1+7)\cdot 2\cdot 6! + \frac{4}{2}\cdot(0+6)\cdot 2\cdot 5! + \frac{3}{2}\cdot(1+5)\cdot 8\cdot 4! + \frac{3}{2}\cdot(0+4)\cdot 44\cdot 3! \\ + \frac{2}{2}\cdot(1+3)\cdot 488\cdot 2! + \frac{2}{2}\cdot(0+2)\cdot 4,664\cdot 1! + d_{7,10} = \frac{5}{2}\cdot(0+8)\cdot 7!.$$

It follows that $d_{7,10} = 58,336$.

Finally, for $d_{8,10}$ set $i = 8$ and $m = 1$ in (2.4). Then we immediately obtain

$$2 \cdot 9! + 2 \cdot 8! + 8 \cdot 7! + 44 \cdot 6! + 488 \cdot 5! + 4,664 \cdot 4! + 58,336 \cdot 3! + d_{8,10} = 10!,$$

which yields $d_{8,10} = 1,114,944$.

Substituting the above values of $d_{i,10}$, $i \in [2,8]$ and the values $s_2 = 1$, $s_3 = 2$, $s_4 = 8$, $s_5 = 40$, $s_6 = 256$, $s_7 = 1,860$, $s_8 = 15,580$ and $s_9 = 144,812$ for $i < 10$ to the recursive formula of Theorem 2 we finally obtain $s_{10} = 1,490,564$.

Using the same reasoning we calculate all the numbers s_n, for $n \in [12]$. The calculation is based on the numbers $d_{i,n}$, for $1 \leq i \leq 10$ and $3 \leq n \leq 12$ (see Table 2 in the Appendix).

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Appendix

$i\backslash n$	3	4	5	6	7	8	9	10	11	12
1	3	2	2	2	2	2	2	2	2	2
2		6	4	2	2	2	2	2	2	2
3			24	16	14	8	8	8	8	8
4				168	100	80	68	44	44	44
5					1,212	712	500	488	416	296
6						10,824	6,376	4,664	3,704	3,512
7							103,992	58,336	43,592	33,152
8								1,114,944	630,544	444,992
9									12,907,824	7,167,802
10										162,773,970

Table 1: The numbers $d_{i,n}$ for $1 \leq i \leq 10$ and $3 \leq n \leq 12$

n	1	2	3	4	5	6	7	8	9	10	11	12
s_n	1	1	2	8	40	256	1,860	15,580	144,812	1,490,564	16,758,972	205,029,338

Table 2: The numbers s_n for $1 \leq n \leq 12$

$j\backslash n$	2	3	4	5	6	7	8	9	10	11	12
1	1	1	6	28	196	1,452	12,632	119,744	1,260,432	14,389,600	178,692,748
2		1	1	10	46	330	2,416	21,216	197,120	2,067,024	23,263,418
3			1	1	12	62	442	3,204	28,276	262,080	2,707,296
4				1	1	14	72	546	3,992	34,680	318,408
5					1	1	16	82	630	4,744	41,108
6						1	1	18	92	718	5,412
7							1	1	20	102	810
8								1	1	22	112
9									1	1	24
10										1	1
11											1

Table 3: The numbers $s_{j,n}$ for $1 \leq j \leq 11$ and $2 \leq n \leq 12$

n	R_n
3	123, 213
4	1234, 1324
	2134, 2314, 2413, 3124, 3214, 3412
5	12345, 12435, 13245, 13425, 13524, 14235, 14325, 14523;
	21345, 21435; 24135, 24315; 42135, 42315; 45123, 45213;
	23145, 23415, 23514, 25134, 25314, 25413, 31245, 31425, 31524, 32145, 32415, 32514, 34125, 34215, 34512, 35124, 35214, 35412, 41235, 41325, 41523, 43125, 43215, 43512
6	123456, 123546, 124356, 124536, 124635, 125346, 125436, 125634, 132456, 132546, 135246, 135426, 153246, 153426, 156234, 156324, 134256, 134526, 134625, 136245, 136425, 136524, 142356, 142536, 142635, 143256, 143526, 143625, 145236, 145326, 145623, 146235, 146325, 146523, 152346, 152436, 152634, 154236, 154326, 154623;
	213456, 213546, 214356, 214536, 214635, 215346, 215436, 215634; 561234, 561324, 562134, 562314, 562413, 563124, 563214, 563412;
	231456, 231546; 246135, 246315; 261345, 261435; 264135, 264315; 312456, 312546; 315246, 315426; 321456, 321546; 351246, 351426; 426135, 426315; 456123, 456213; 462135, 462315; 465123, 465213; 513246, 513426; 516234, 516324; 531246, 531426; 546123, 546213;
	234156, 234516, 234615; 235146, 235416, 235614; 236145, 236415, 236514; 241356, 241536, 241635; 243156, 243516, 243615; 245136, 245316, 245613; 251346, 251436, 251634; 253146, 253416, 253614; 254136, 254316, 254613; 256134, 256314, 263145; 263415, 263514, 265134; 314256, 314526, 314625; 316245, 316425, 316524; 324156, 324516, 324615; 325146, 325416, 325614; 326145, 326415, 326514; 341256, 341526, 341625; 342156, 342516, 342615; 345126, 345216, 345612; 346125, 346215, 346512; 352146, 352416, 352614; 354126, 354216, 354612; 356124, 356214, 356412; 361245, 361425, 361524; 362145, 362415, 362514; 364125, 364215, 364512; 365124, 365214, 365412; 412356, 412536, 412635; 413256, 413526, 413625; 415236, 415326, 415623; 416235, 416325, 416523; 421356, 421536, 421635; 423156, 423516, 423615; 425136, 425316, 425613; 431256, 431526, 431625; 432156, 432516, 432615; 435126, 435216, 435612; 436125, 436215, 436512; 451236, 451326, 451623; 452136, 452316, 452613; 453126, 453216, 453612; 461235, 461325, 461523; 463125, 463215, 463512; 512346, 512436, 512634; 514236, 514326, 514623; 526134, 526314, 526413; 524135, 524316, 524613; 523146, 523416, 523614; 521346, 521436, 521634; 532146, 532416, 532614; 534126, 534216, 534612; 536124, 536214, 536412; 541236, 541326, 541623; 542136, 542316, 542613; 543126, 543216, 543612;

Table 4: The sets R_n for $3 \leq n \leq 6$.