TILTING MODULES FOR THE AUSLANDER ALGEBRA OF $K[x]/(x^n)$

JAN GEUENICH

ABSTRACT. We construct an isomorphism between the partially ordered set of tilting modules for the Auslander algebra of $K[x]/(x^n)$ and the interval of rational permutation braids in the braid group on n strands. Hence, there are only finitely many tilting modules.

1. INTRODUCTION

In this note we classify all tilting modules for the Auslander algebra Λ_n of the truncated polynomial ring $K[x]/(x^n)$.

Brüstle, Hille, Ringel and Röhrle [BHRR99] characterized the classical tilting modules for Λ_n as those tilting modules that admit a Δ-filtration with respect to the unique quasi-hereditary structure. They also parameterized the basic classical tilting modules by the symmetric group $\mathcal{S}_n$, proving there are $c_n = n!$ of them.

Iyama and Zhang [IZ16] strengthened this result by constructing an anti-isomorphism from $\mathcal{S}_n$ viewed as a poset with the left weak order to the poset of classical tilting modules. More precisely, they associated with each $w \in \mathcal{S}_n$ an ideal I_w in Λ_n that is a classical tilting module.

Our main result is the following:

Theorem (Corollary 8.6)**.** *The tensor products $I_v \otimes_{\Lambda_n} I_w$ with $v, w \in \mathcal{S}_n$ are the basic tilting modules for Λ_n.*

In order to get a complete and irredundant list of the tilting modules and to determine the tilting poset, we will refine the previous statement.

For this, we consider the braid group $\mathcal{B}_n$ with generators $\mathfrak{s}_1, \ldots, \mathfrak{s}_{n-1}$ subject to the braid relations. The symmetric group $\mathcal{S}_n$ will be regarded as a subset of the braid group $\mathcal{B}_n$ where elements $w \in \mathcal{S}_n$ with reduced expression $(i_1, \ldots, i_\ell)$ are identified with $\underline{w} := \mathfrak{s}_{i_1} \cdots \mathfrak{s}_{i_\ell}$.

The right weak order on $\mathcal{S}_n$ then extends to a partial order on $\mathcal{B}_n$ such that $x \leq_R y$ if and only if $x^{-1}y$ can be written as a product of the generators $\mathfrak{s}_i$. The elements $x \in \mathcal{B}_n$ with the property $w_- \leq_R x \leq_R w_+$ now form the interval $[w_-, w_+]_R$ of so-called *rational permutation braids* where $w_- := \underline{w_0}^{-1}$, $w_+ := \underline{w_0}$ and w_0 is the longest element of $\mathcal{S}_n$.

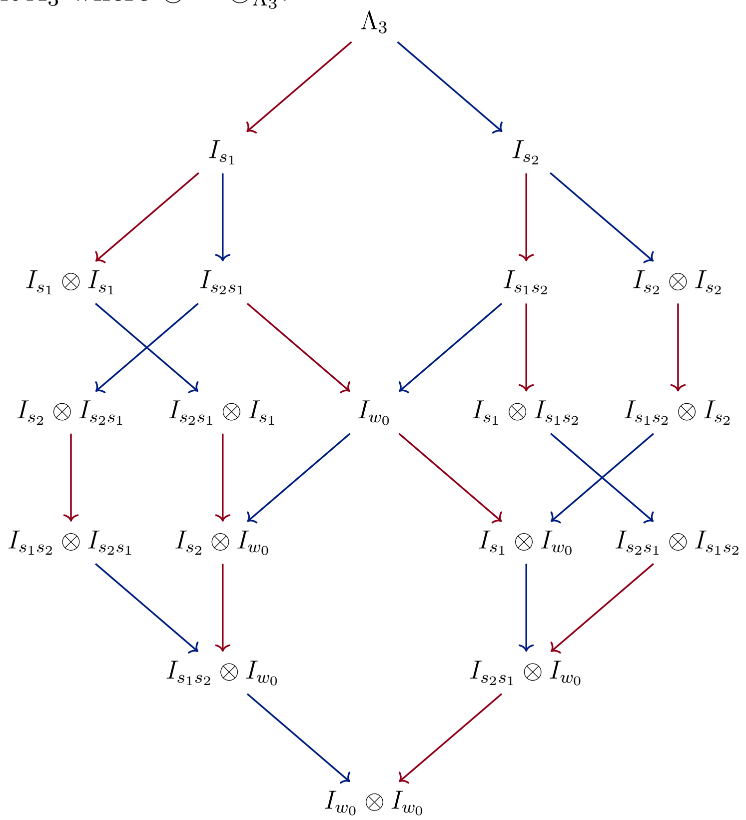
2010 *Mathematics Subject Classification.* 16G20, 16D90.
Key words and phrases. Auslander algebra, tilting theory, braid group.

Observing that the assignment $(v, w) \mapsto \underline{v}\,\underline{w}^{-1}$ defines a bijection between the set of pairs $(v, w) \in \mathcal{S}_n \times \mathcal{S}_n$ without common right descent and $[w_-, w_+]_R$ (see [DG17, Remark 5.9]), we can formulate the refined version of our result:

Theorem (Corollary 8.8)**.** *There is a poset isomorphism:*

$$[w_-, w_+]_R \longrightarrow \operatorname{tilt} \Lambda_n$$
$$\underline{v}\,\underline{w}^{-1} \longmapsto I_w \otimes_{\Lambda_n} I_{v^{-1}w_0}$$

To illustrate the theorem, we depict below the Hasse diagram of the poset $\operatorname{tilt} \Lambda_3$ where $\otimes = \otimes_{\Lambda_3}$:



As a consequence of the previous theorem, the number t_n of isomorphism classes of basic tilting modules for Λ_n equals the number of pairs of permutations in $\mathcal{S}_n$ without common right descent. These integers t_n form the sequence OEIS:A000275 and satisfy the recursive formula

$$t_0 = 1, \qquad t_n = \sum_{k=0}^{n-1} (-1)^{n+k+1} \binom{n}{k}^2 t_k \quad \text{for } n > 0.$$

The first few values of the sequences c_n and t_n are recorded below:

n	1	2	3	4	5	6	7
c_n	1	2	6	24	120	720	5040
t_n	1	3	19	211	3651	90921	3081513

2. Notation and terminology

For the remaining part of this paper we fix an algebraically closed field K and a finite-dimensional algebra Λ over K.

By a module we always mean a right module. The category of finite-dimensional Λ-modules is denoted by $\operatorname{mod}\Lambda$ and its bounded derived category by $\mathcal{D}(\Lambda) = \mathcal{D}^b(\operatorname{mod}\Lambda)$. We write $\operatorname{proj}\Lambda$ for the full subcategory of $\operatorname{mod}\Lambda$ consisting of projective modules and $\mathcal{K}(\Lambda) = \mathcal{K}^b(\operatorname{proj}\Lambda)$ for its bounded homotopy category. Given a complex $M \in \mathcal{D}(\Lambda)$ we denote by $\operatorname{add}(M)$ the smallest full additive subcategory and by $\operatorname{thick}(M)$ the smallest full triangulated subcategory of $\mathcal{D}(\Lambda)$ that contains all direct summands of M. We regard $\mathcal{K}(\Lambda)$ as a full subcategory of $\mathcal{D}(\Lambda)$. Objects of $\mathcal{D}(\Lambda)$ isomorphic to objects in $\mathcal{K}(\Lambda)$ are called *perfect*.

The *Hasse diagram* of a poset X is the quiver $Q(X)$ with vertex set X without parallel arrows such that there is an arrow $x \to z$ in $Q(X)$ if and only if $x > z$ and $x \geq y \geq z$ only for $y \in \{x, z\}$.

3. Background on tilting theory

We collect relevant definitions and results from tilting theory needed later. For details see [Ric91; Yek99; HU05a; HU05b; AI12].

A complex T in $\mathcal{K}(\Lambda)$ is *tilting* if $\operatorname{Hom}_{\mathcal{K}(\Lambda)}(T, T[q]) = 0$ for all $q \neq 0$ and $\operatorname{thick}(\Lambda) = \mathcal{K}(\Lambda)$. A module T in $\operatorname{mod}\Lambda$ or a complex T in $\mathcal{D}(\Lambda)$ is said to be *tilting* if T viewed as an object in $\mathcal{D}(\Lambda)$ is isomorphic to a tilting complex in $\mathcal{K}(\Lambda)$. By a *classical tilting module* over Λ we mean a tilting module $T \in \operatorname{mod}\Lambda$ with $\operatorname{proj.dim} T \leq 1$.

Tilting complexes $T, T' \in \mathcal{D}(\Lambda)$ are *equivalent* if $\operatorname{add}(T) = \operatorname{add}(T')$. The set $\operatorname{tilt}^{\bullet}\Lambda$ of equivalence classes of tilting complexes in $\mathcal{D}(\Lambda)$ forms a poset under the relation

$$T \geq T' \;:\Leftrightarrow\; \operatorname{Hom}_{\mathcal{D}(\Lambda)}(T, T'[q]) = 0 \;\forall\, q > 0$$

as shown in [AI12, Theorem 2.11]. We denote by $\operatorname{tilt}\Lambda$ the subposet of tilting modules and by $\operatorname{tilt}_1\Lambda$ the subposet of classical tilting modules.

Remark 3.1. Set $X^{\perp} := \{Y \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^q_\Lambda(X, Y) = 0 \forall q > 0\}$. Then $T \geq T' \Leftrightarrow T^{\perp} \supseteq T'^{\perp}$ for all $T, T' \in \operatorname{tilt}\Lambda$ (see [HU05b, Lemma 2.1]).

Let $\Lambda^e = \Lambda^{\mathrm{op}} \otimes_K \Lambda$ be the enveloping algebra of Λ. A *two-sided tilting complex over* Λ is a complex $T \in \mathcal{D}(\Lambda^e)$ such that

$$T \otimes^{\mathbb{L}}_{\Lambda} \widetilde{T} \;\cong\; \Lambda \;\cong\; \widetilde{T} \otimes^{\mathbb{L}}_{\Lambda} T \quad \text{in } \mathcal{D}(\Lambda^e)$$

for some $\widetilde{T} \in \mathcal{D}(\Lambda^e)$. The set of isomorphism classes of two-sided tilting complexes over Λ is known as the *derived Picard group* $\operatorname{DPic}(\Lambda)$. It is a group with composition $- \otimes^{\mathbb{L}}_{\Lambda} -$ and identity Λ (see [Yek99]).

Theorem 3.2 ([Ric91])**.** *For every two-sided tilting complex T over Λ the functor pairs*

$$\mathcal{D}(\Lambda) \underset{-\otimes^{\mathbb{L}}_{\Lambda}\widetilde{T}}{\overset{-\otimes^{\mathbb{L}}_{\Lambda}T}{\rightleftarrows}} \mathcal{D}(\Lambda) \qquad\qquad \mathcal{D}(\Lambda^{\mathrm{op}}) \underset{\widetilde{T}\otimes^{\mathbb{L}}_{\Lambda}-}{\overset{T\otimes^{\mathbb{L}}_{\Lambda}-}{\rightleftarrows}} \mathcal{D}(\Lambda^{\mathrm{op}})$$

are quasi-inverse equivalences of triangulated categories.

In particular, $T_\Lambda \in \operatorname{tilt}^\bullet \Lambda$ and ${}_\Lambda T \in \operatorname{tilt}^\bullet \Lambda^{\mathrm{op}}$ are tilting complexes.

A complex T in $\mathcal{D}(\Lambda^e)$ is said to be *biperfect* if T_Λ is perfect in $\mathcal{D}(\Lambda)$ and ${}_\Lambda T$ is perfect in $\mathcal{D}(\Lambda^{\mathrm{op}})$. For every T in $\mathcal{D}(\Lambda^e)$ there are canonical morphisms $\Lambda \to \mathbb{R}\mathrm{Hom}_\Lambda(T,T)$ and $\Lambda \to \mathbb{R}\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(T,T)$ in $\mathcal{D}(\Lambda^e)$, the so-called *left-* and *right-multiplication maps* (see [Yek92, § 3]).

Theorem 3.3 ([Miy03, Proposition 1.8])**.** *Let $T \in \mathcal{D}(\Lambda^e)$ be biperfect. The complex T is a two-sided tilting complex over Λ if and only if both the left-multiplication map $\Lambda \to \mathbb{R}\mathrm{Hom}_\Lambda(T,T)$ and the right-multiplication map $\Lambda \to \mathbb{R}\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(T,T)$ are isomorphisms in $\mathcal{D}(\Lambda^e)$.*

The next result by Happel and Unger and its corollary will be crucial.

Theorem 3.4 ([HU05b, Theorem 2.2])**.** *For all modules $T, T' \in \operatorname{tilt}\Lambda$ with $T' > T$ there exists an arrow $T' \to T''$ in $Q(\operatorname{tilt}\Lambda)$ with $T'' \geq T$.*

A subquiver Q' of a quiver Q is *successor-closed* if ($v \in Q' \Rightarrow w \in Q'$) for all arrows $v \to w$ in Q.

Corollary 3.5. *If Q' is a finite successor-closed subquiver of $Q(\operatorname{tilt}\Lambda)$ with $\Lambda \in Q'$, then $Q' = Q(\operatorname{tilt}\Lambda)$.*

Proof. This follows from the proof of [HU05a, Corollary 2.2]. □

If Λ has finite global dimension, the concept of tilting modules coincides with the dual concept of cotilting modules (see [HU96, Lemma 1.3]). We continue with some results that are valid in this situation.

Lemma 3.6. *Assume $\operatorname{gl.dim}\Lambda < \infty$. There is a poset isomorphism*

$$\begin{aligned} \operatorname{tilt}\Lambda^{\mathrm{op}} &\longrightarrow (\operatorname{tilt}\Lambda)^{\mathrm{op}} \\ T &\longmapsto D(T) \end{aligned}$$

induced by the standard duality $D := \mathrm{Hom}_K(-,K) : \operatorname{mod}\Lambda^{\mathrm{op}} \to \operatorname{mod}\Lambda$.

Proof. Indeed, $D(T) \in \operatorname{tilt}\Lambda$ for every $T \in \operatorname{tilt}\Lambda^{\mathrm{op}}$ because Λ has finite global dimension. The map clearly is an isomorphism of posets. □

We denote by $\operatorname{f-res}\Lambda$ and $\operatorname{f-cores}\Lambda$ the posets of functorially finite subcategories of $\operatorname{mod}\Lambda$ that are resolving and coresolving, respectively. The poset structure is given by inclusion (see [AR91] for definitions). All subcategories are assumed to be closed under direct summands. Moreover, we abbreviate ${}^\perp Y := \{X \in \operatorname{mod}\Lambda \mid \mathrm{Ext}^q_\Lambda(X,Y) = 0 \;\forall q > 0\}$. The following characterization of tilting modules is often useful:

Theorem 3.7 ([AR91], [KS03, Corollary 0.3])**.** *Assume* $\operatorname{gl.dim}\Lambda < \infty$. *There is a commutative diagram of poset isomorphisms:*

$$\begin{array}{ccc} & \operatorname{tilt}\Lambda & \\ {\scriptstyle T\mapsto T^{\perp}} \swarrow & & \searrow {\scriptstyle T\mapsto {}^{\perp}(T^{\perp})} \\ \operatorname{f-cores}\Lambda & \longrightarrow & (\operatorname{f-res}\Lambda)^{\mathrm{op}} \end{array}$$

Moreover, $\operatorname{add}(T) = T^{\perp} \cap {}^{\perp}(T^{\perp})$ *for every* $T \in \operatorname{tilt}\Lambda$.

Still assuming $\operatorname{gl.dim}\Lambda < \infty$, we will end this background section with the remarkable observation that, whenever $\operatorname{tilt}\Lambda$ is finite, every self-orthogonal Λ-module is partial tilting and $\operatorname{tilt}\Lambda$ forms a lattice. This is a consequence of the following recent result by Iyama and Zhang:

Theorem 3.8 ([IZ18])**.** *If* $\operatorname{gl.dim}\Lambda < \infty$ *and* $\operatorname{tilt}\Lambda$ *is finite, all resolving and all coresolving subcategories of* $\operatorname{mod}\Lambda$ *are functorially finite.*

Proof. The implication (1) $\Rightarrow$ (2) of [IZ18, Theorem 3.2] and [KS03, Corollary 0.3] show that every resolving subcategory is functorially finite. Dually, since all cotilting modules are also tilting modules, every coresolving subcategory is functorially finite. □

Recall that $M \in \operatorname{mod}\Lambda$ is *self-orthogonal* if $\operatorname{Ext}^q_\Lambda(M, M) = 0 \;\forall\, q > 0$.

Corollary 3.9. *If* $\operatorname{gl.dim}\Lambda < \infty$ *and* $\operatorname{tilt}\Lambda$ *is finite, every self-orthogonal module in* $\operatorname{mod}\Lambda$ *is a direct summand of a tilting module.*

Proof. Use Theorem 3.8 and the dual of [AR91, Proposition 5.12]. □

Corollary 3.10. *If* $\operatorname{gl.dim}\Lambda < \infty$ *and* $\operatorname{tilt}\Lambda$ *is finite,* $\operatorname{tilt}\Lambda$ *is a lattice. In this case, the meet* $T \wedge T'$ *of elements* T, T' *in* $\operatorname{tilt}\Lambda$ *is given by*

$$(T \wedge T')^{\perp} \;=\; T^{\perp} \cap T'^{\perp}\,.$$

Proof. Theorem 3.8 shows that $\operatorname{f-res}\Lambda$ admits meets, since the intersection of resolving subcategories is resolving. Analogously, $\operatorname{f-cores}\Lambda$ admits meets. Now apply Theorem 3.7. □

4. The Auslander algebra of $K[x]/(x^n)$

From now on let $\Lambda = \Lambda_n$ be the path algebra over K of the quiver

$$1 \;\underset{\alpha_1}{\overset{\beta_2}{\rightleftarrows}}\; 2 \;\underset{\alpha_2}{\overset{\beta_3}{\rightleftarrows}}\; \cdots\cdots \;\underset{\alpha_{n-1}}{\overset{\beta_n}{\rightleftarrows}}\; n$$

modulo the relations $\alpha_1\beta_2$ and $\alpha_i\beta_{i+1} - \beta_i\alpha_{i-1}$ for all $1 < i < n$.

Then Λ is the Auslander algebra of $K[x]/(x^n)$, i.e. the endomorphism algebra of the direct sum of the n indecomposable $K[x]/(x^n)$-modules $K[x]/(x^i)$ with $1 \le i \le n$. Its classical tilting, support τ-tilting and exceptional modules were investigated in [BHRR99; IZ16; HP17].

Remark 4.1. Some basic properties of Λ are collected in [HP17, § 1]. For us it is important to know that $\operatorname{gl.dim}\Lambda \le 2$ and $e_n\Lambda$ is a projective-injective Λ-module.

5. Classical tilting modules

The classical tilting modules for the Auslander algebra $\Lambda = \Lambda_n$ are classified in [BHRR99]. An explicit anti-isomorphism between the poset $\operatorname{tilt}_1 \Lambda$ and the symmetric group $\mathcal{S} = \mathcal{S}_n$ on n letters with the left weak order was established by Yusuke Tsujioka in his Master's thesis. We will recall this classification as presented in [IZ16] below.

For $1 \leq i < n$ we denote by $s_i \in \mathcal{S}$ the transposition of i and $i+1$. The *length* of $w \in \mathcal{S}$ is

$$\ell(w) \;:=\; \sharp\{(i,j) \,|\, 1 \leq i < j \leq n, w(i) > w(j)\}\,.$$

A sequence $(i_1, \ldots, i_\ell)$ in $\{1, \ldots, n-1\}$ is said to be a *reduced expression* for w if $w = s_{i_1} \cdots s_{i_\ell}$ and $\ell = \ell(w)$. The *left weak order* $\geq_L$ and the *right weak order* $\geq_R$ on $\mathcal{S}$ are defined by:

$$w \geq_L v \quad :\Leftrightarrow \quad \ell(w) = \ell(v) + \ell(wv^{-1})$$
$$w \geq_R v \quad :\Leftrightarrow \quad \ell(w) = \ell(v) + \ell(v^{-1}w)$$

Each of these two orders turns $\mathcal{S}$ into a lattice with maximal element

$$w_0 \;:=\; \left(\begin{smallmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{smallmatrix}\right) \;\in\; \mathcal{S}\,.$$

Observe that the arrows in the Hasse diagram $Q(\mathcal{S}, \geq_L)$ are $s_i v \to v$ for $v \in \mathcal{S}$ and $1 \leq i < n$ with $s_i v >_L v$ and the assignment $w \mapsto w^{-1}$ yields an isomorphism of posets $(\mathcal{S}, \geq_R) \to (\mathcal{S}, \geq_L)$.

For $1 \leq i < n$ let I_i be the ideal $\Lambda(1 - e_i)\Lambda$ in Λ. More generally, we define for all $w \in \mathcal{S}$

$$I_w \;:=\; I_{i_1} \cdots I_{i_\ell}$$

where $\underline{i} = (i_1, \ldots, i_\ell)$ is any reduced expression for w. For a proof why the ideal I_w in Λ only depends on w and not on the particular choice of the reduced expression $\underline{i}$ see [IZ16, Proposition 3.15].

Theorem 5.1 ([IZ16, Theorem 3.18])**.** *There are poset isomorphisms:*

$$\begin{array}{ccc} (\mathcal{S}, \geq_L) & \longrightarrow & (\operatorname{tilt}_1 \Lambda)^{\mathrm{op}} \\ w & \longmapsto & I_w \end{array} \qquad\qquad \begin{array}{ccc} (\mathcal{S}, \geq_R) & \longrightarrow & (\operatorname{tilt}_1 \Lambda^{\mathrm{op}})^{\mathrm{op}} \\ w & \longmapsto & I_w \end{array}$$

Theorem 5.2 ([IZ16, Theorem 3.5])**.** *For every element $w \in \mathcal{S}$ both the left-multiplication map $\Lambda \to \operatorname{End}_\Lambda(I_w)$ and the right-multiplication map $\Lambda^{\mathrm{op}} \to \operatorname{End}_{\Lambda^{\mathrm{op}}}(I_w)$ are isomorphisms of algebras.*

6. The derived Picard group

The goal of this section is to construct a homomorphism from the braid group $\mathcal{B} = \mathcal{B}_n$ on n strands to the derived Picard group $\operatorname{DPic}(\Lambda)$.

Proposition 6.1. *For every element $w \in \mathcal{S}$ the ideal I_w is a two-sided tilting complex over Λ.*

Proof. It is $\mathbb{R}\mathrm{Hom}_\Lambda(T,T) = \mathrm{End}_\Lambda(T)$ and $\mathbb{R}\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(T,T) = \mathrm{End}_{\Lambda^{\mathrm{op}}}(T)$ for $T = I_w$ by Theorem 5.1. Now use Theorems 3.3 and 5.2. $\square$

Before stating this section's main result, we recall properties of braid groups. Details can be found for example in [KT08, Chapter 6].

Let $\mathcal{F} = \mathcal{F}_n$ be the free group with generators $\mathfrak{s}_i$ for $1 \le i < n$. The braid group $\mathcal{B}$ is by definition the quotient of $\mathcal{F}$ by the relations

$$(\star)\qquad \begin{aligned} \mathfrak{s}_i\mathfrak{s}_j &= \mathfrak{s}_j\mathfrak{s}_i && \text{for } 1 \le i < i+1 < j < n,\\ \mathfrak{s}_i\mathfrak{s}_{i+1}\mathfrak{s}_i &= \mathfrak{s}_{i+1}\mathfrak{s}_i\mathfrak{s}_{i+1} && \text{for } 1 \le i < i+1 < n. \end{aligned}$$

We denote by $\mathcal{B}_+$ the monoid generated by $\mathfrak{s}_i$ with $1 \le i < n$ subject to the same relations $(\star)$. It is well-known that the canonical monoid morphism $\mathcal{B}_+ \to \mathcal{B}$ is injective. In this way, $\mathcal{B}_+$ becomes a subset of $\mathcal{B}$. With $\mathcal{B}_- := (\mathcal{B}_+)^{-1}$ we then have

$$\mathcal{B} \;=\; \mathcal{B}_+\mathcal{B}_- \;=\; \mathcal{B}_-\mathcal{B}_+\,.$$

The *length* of an element $x = \mathfrak{s}_{i_1}\cdots\mathfrak{s}_{i_\ell}$ in $\mathcal{B}_+$ is the integer $\ell(x) = \ell$.

The rule $\mathfrak{s}_i \mapsto s_i$ induces a surjective group morphism $\mathcal{B} \to \mathcal{S}, x \mapsto \overline{x}$, whose kernel is generated by the elements $\mathfrak{s}_i^2$. Furthermore, we can and will regard the symmetric group $\mathcal{S}$ as a subset $\mathcal{S}_+$ of $\mathcal{B}_+$ by identifying each element $w = s_{i_1}\cdots s_{i_\ell} \in \mathcal{S}$ of length ℓ with $\underline{w} := \mathfrak{s}_{i_1}\cdots\mathfrak{s}_{i_\ell} \in \mathcal{B}_+$. We write w_+ for $w_0 \in \mathcal{S}$ when considered as the element $\underline{w_0}$ of $\mathcal{S}_+$.

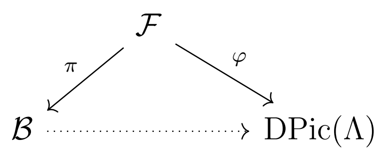
Remark 6.2. The pair $(\mathcal{B}_+, w_+)$ is a *comprehensive Garside monoid* in the sense of [KT08, Theorem 6.20] and the canonical map $\mathcal{B}_+ \to \mathcal{B}$ is the embedding into its group of fractions.

After the preparation, we get to the promised result:

Proposition 6.3. *The map $\mathcal{S} \to \mathrm{DPic}(\Lambda)$ given by $w \mapsto I_w$ extends to a group homomorphism:*

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathrm{DPic}(\Lambda) \\ x & \longmapsto & T_x \end{array}$$

Proof. Consider the diagram



where π and φ are the group morphisms given by $\mathfrak{s}_i \mapsto \mathfrak{s}_i$ and $\mathfrak{s}_i \mapsto I_i$, respectively. For all reduced expressions $(i_1, \ldots, i_\ell)$ for $w \in \mathcal{S}$ we have by [IZ16, Propositions 3.17] in $\mathrm{DPic}(\Lambda)$

$$\varphi\big(\mathfrak{s}_{i_1}\cdots\mathfrak{s}_{i_\ell}\big) \;=\; I_{i_1} \otimes^{\mathbb{L}}_\Lambda \cdots \otimes^{\mathbb{L}}_\Lambda I_{i_\ell} \;=\; I_w\,.$$

It follows that φ factors over π because $\mathcal{B}$ is defined by relations $v = w$ with $v = \mathfrak{s}_{i_1} \cdots \mathfrak{s}_{i_\ell}, w = \mathfrak{s}_{j_1} \cdots \mathfrak{s}_{j_\ell} \in \mathcal{F}$ where $(i_1, \ldots, i_\ell)$ and $(j_1, \ldots, j_\ell)$ are reduced expressions for the element $\overline{\pi(v) = \pi(w)} \in \mathcal{S}$. □

7. Tilting complexes

Composing the map $\mathcal{B}_+ \to \mathrm{DPic}(\Lambda)$ from Proposition 6.3 with the canonical map $\mathrm{DPic}(\Lambda) \to \mathrm{tilt}^\bullet \Lambda$ yields a map $\mathcal{B}_+ \to \mathrm{tilt}^\bullet \Lambda$. In this section we discuss why this map becomes an anti-morphism of posets when endowing $\mathcal{B}_+$ with the right divisibility order. Furthermore, we show that it preserves covering relations.

The *right-divisibility order* $\geq_L$ and the *left-divisibility order* $\geq_R$ are extensions of $\geq_L$ and $\geq_R$ from $\mathcal{S}_+$ to $\mathcal{B}$ where for $v, w \in \mathcal{B}$:

$$y \geq_L x \quad :\Leftrightarrow \quad yx^{-1} \in \mathcal{B}_+$$
$$y \geq_R x \quad :\Leftrightarrow \quad x^{-1}y \in \mathcal{B}_+$$

Proposition 7.1. *There is a morphism of strict posets:*

$$\begin{array}{ccc} (\mathcal{B}_+, >_L) & \longrightarrow & (\mathrm{tilt}^\bullet \Lambda)^{\mathrm{op}} \\ x & \longmapsto & T_x \end{array}$$

Proof. It suffices to verify $T_x > T_{\mathfrak{s}_i x}$ for every $x \in \mathcal{B}_+$ and $1 \leq i < n$. Theorem 5.1 shows $\Lambda > I_i$ and we get $T_x = \Lambda \otimes^{\mathbb{L}}_\Lambda T_x > I_i \otimes^{\mathbb{L}}_\Lambda T_x = T_{\mathfrak{s}_i x}$ in $\mathrm{tilt}^\bullet \Lambda$ with Theorem 3.2 and Proposition 6.3. □

The following fact is a variation of [IZ16, Lemma 4.3]:

Lemma 7.2. *For all $1 \leq i < n$ we have a short exact sequence*

$$0 \to e_i\Lambda \xrightarrow{\iota = \left(\begin{smallmatrix} \alpha_{i-1}\cdot \\ \beta_{i+1}\cdot \end{smallmatrix}\right)} e_{i-1}\Lambda \oplus e_{i+1}\Lambda \xrightarrow{\pi = (\,-\beta_i\cdot\ \alpha_i\cdot\,)} e_i I_i \to 0$$

in $\mathrm{mod}\,\Lambda$ *where ι is a minimal left and π a minimal right* $\mathrm{add}((1-e_i)\Lambda)$-*approximation and by convention $e_0 := 0$.*

Proof. This short exact sequence is the minimal projective resolution of $e_i I_i = \mathrm{rad}(e_i\Lambda)$. For $j \neq i$, applying $\mathrm{Hom}_\Lambda(-, e_j\Lambda)$ and $\mathrm{Hom}_\Lambda(e_j\Lambda, -)$ yields exact sequences:

$$\mathrm{Hom}_\Lambda(e_{i-1}\Lambda \oplus e_{i+1}\Lambda, e_j\Lambda) \xrightarrow{\iota^*} \mathrm{Hom}_\Lambda(e_i\Lambda, e_j\Lambda) \longrightarrow \mathrm{Ext}^1_\Lambda(e_i I_i, e_j\Lambda)$$

$$\mathrm{Hom}_\Lambda(e_j\Lambda, e_{i-1}\Lambda \oplus e_{i+1}\Lambda) \xrightarrow{\pi_*} \mathrm{Hom}_\Lambda(e_j\Lambda, e_i I_i) \longrightarrow \mathrm{Ext}^1_\Lambda(e_j\Lambda, e_i\Lambda)$$

Letting S_i be the simple Λ^e-module given by the short exact sequence $0 \to I_i \to \Lambda \to S_i \to 0$ we have with [IZ16, Lemma 3.6]

$$\mathrm{Ext}^1_\Lambda(e_i I_i, e_j\Lambda) \;\cong\; \mathrm{Ext}^2_\Lambda(S_i, e_j\Lambda) \;\cong\; e_j\Lambda \otimes_\Lambda S_i \;=\; 0\,.$$

Clearly, $\mathrm{Ext}^1_\Lambda(e_j\Lambda, e_i\Lambda) = 0$, too. Therefore ι and π are $\mathrm{add}((1-e_i)\Lambda)$-approximations. Both of them are minimal by [AR91, Proposition 1.1], since neither $e_i I_i$ nor $e_i\Lambda$ is a direct summand of $e_{i-1}\Lambda \oplus e_{i+1}\Lambda$. □

The next lemma will be essential to determine $Q(\operatorname{tilt}\Lambda)$.

Lemma 7.3. *For all $x \in \mathcal{B}_+$ and $1 \leq i < n$ we have a triangle*

$$e_i T_x \xrightarrow{\ \iota\ } e_{i-1}T_x \oplus e_{i+1}T_x \xrightarrow{\ \pi\ } e_i T_{\mathfrak{s}_i x} \longrightarrow \cdot$$

in $\mathcal{D}(\Lambda)$ where ι is a minimal left and π a minimal right $\operatorname{add}((1-e_i)T_x)$*-approximation. Furthermore, $e_j T_{\mathfrak{s}_i x} = e_j T_x$ for all $1 \leq j \leq n$ with $j \neq i$.*

Proof. Apply $- \otimes^{\mathbb{L}}_\Lambda T_x$ to the triangle $e_i\Lambda \to e_{i-1}\Lambda \oplus e_{i+1}\Lambda \to e_i I_i \to \cdot$ induced by the sequence in Lemma 7.2 and to the identities $e_j I_i = e_j \Lambda$. Then use Theorem 3.2. □

Corollary 7.4. *There is an arrow $T_x \longrightarrow T_{\mathfrak{s}_i x}$ in the quiver $Q(\operatorname{tilt}^\bullet \Lambda)$ for all $x \in \mathcal{B}_+$ and $1 \leq i < n$.*

Proof. Use Lemma 7.3 and [AI12, Theorem 2.35]. □

For $x \in \mathcal{B}_+$ and $1 \leq i \leq n$ it makes sense to refer to $e_i T_x$ as the *i-th summand* of T_x, since by Theorem 3.2

$$\dim_K \operatorname{End}_{\mathcal{D}(\Lambda)}(e_i T_x) \ = \ \dim_K \operatorname{End}_\Lambda(e_i\Lambda) \ = \ i\,.$$

We write $T_x \xrightarrow{i} T_{\mathfrak{s}_i x}$ for an arrow $T_x \longrightarrow T_{\mathfrak{s}_i x}$ in $Q(\operatorname{tilt}^\bullet \Lambda)$ to emphasize the fact that it corresponds to mutating the i-th summand.

We close this section with an interesting observation that will enable us to determine the possible dimension vectors of tilting modules for Λ. For this purpose, let $V = K_0(\Lambda)$ be the Grothendieck group of $\mathcal{D}(\Lambda)$. The symmetric group $\mathcal{S}$ acts on V^n via

$$s_i \cdot (\ldots, v_{i-1}, v_i, v_{i+1}, \ldots) \ = \ (\ldots, v_{i-1}, v_{i-1} - v_i + v_{i+1}, v_{i+1}, \ldots)$$

with $v_0 := 0$. For each $x \in \mathcal{B}_+$ we define $d(T_x)$ as the element in V^n whose i-th component is the equivalence class of the i-th summand $e_i T_x$. It can be computed by the following formula:

Lemma 7.5. *$d(T_x) = \overline{x} \cdot d(\Lambda)$ for all $x \in \mathcal{B}_+$.*

Proof. The case $x = 1$ is trivial. Otherwise we write $x = \mathfrak{s}_i y$ for some i and $y \in \mathcal{B}_+$. Let $u := d(T_x)$. By induction we have $v := d(T_y) = \overline{y} \cdot d(\Lambda)$. With Lemma 7.3 we get $u_i = v_{i-1} - v_i + v_{i+1}$ and $u_j = v_j$ for all $j \neq i$. We conclude $u = s_i \cdot v = \overline{\mathfrak{s}_i} \cdot (\overline{y} \cdot d(\Lambda)) = \overline{x} \cdot d(\Lambda)$. □

Corollary 7.6. *The set $\{d(T_x) \,|\, x \in \mathcal{B}_+\} = \{d(I_w) \,|\, w \in \mathcal{S}\}$ is finite.*

8. Tilting modules

In this section we finally classify the tilting modules for the Auslander algebra Λ and determine the poset structure of $\operatorname{tilt}\Lambda$. We begin with four lemmata that serve as the main steps of the classification's proof.

Lemma 8.1. *$T_{\underline{v}\,\underline{w}} = I_v \otimes^{\mathbb{L}}_\Lambda I_w \cong I_v \otimes_\Lambda I_w \in \operatorname{tilt}\Lambda$ for all $v, w \in \mathcal{S}$.*

Proof. The short exact sequence $0 \to I_w \to \Lambda \to \Lambda/I_w \to 0$ shows that $\mathrm{Tor}_q^\Lambda(I_v, I_w) \cong \mathrm{Tor}_{q+1}^\Lambda(I_v, \Lambda/I_w)$ for all $q > 0$. Since $\operatorname{proj.dim}(I_v)_\Lambda \leq 1$, we see that $I_v \otimes_\Lambda^{\mathbb{L}} I_w \cong I_v \otimes_\Lambda I_w$ is a module. □

We use the notation $[a,b]_L$ for the interval $\{x \in \mathcal{B} \mid a \leq_L x \leq_L b\}$ and define the interval $[a,b]_R$ similarly. Let $\mathcal{S}_- := (\mathcal{S}_+)^{-1}$ and $w_- := (w_+)^{-1}$. The elements of

$$[w_-, w_+] := [w_-, w_+]_L = [w_-, w_+]_R = \mathcal{S}_+\mathcal{S}_- = \mathcal{S}_-\mathcal{S}_+$$

are the *rational permutation braids* studied in [DG17, Proposition 4.3].

It is not hard to see that for $x \in [w_-, w_+]$ the module $T_{w_+x} \in \operatorname{tilt}\Lambda^{\mathrm{op}}$ corresponds to $T_{x^{-1}w_+} \in \operatorname{tilt}\Lambda$ under the isomorphism from Lemma 3.6. We prove a special case:

Lemma 8.2. $T_{w_+^2} = D(\Lambda)$ *in* $\operatorname{tilt}\Lambda$.

Proof. Let $(i_1, \dots, i_\ell)$ be a reduced expression for w_0. By Theorem 5.1 there are paths

$$\Lambda_\Lambda \xrightarrow{i_\ell} \cdots \xrightarrow{i_1} (I_{w_0})_\Lambda \qquad \text{in } Q(\operatorname{tilt}\Lambda) \text{ and}$$

$$_\Lambda\Lambda \xrightarrow{i_1} \cdots \xrightarrow{i_\ell} {}_\Lambda(I_{w_0}) \qquad \text{in } Q(\operatorname{tilt}\Lambda^{\mathrm{op}}).$$

Now, $(I_{w_0})_\Lambda$ is the unique module $T_\Lambda \in \operatorname{tilt}\Lambda$ with $\operatorname{proj.dim} T_\Lambda \leq 1$ and $\operatorname{inj.dim} T_\Lambda \leq 1$ (see [BHRR99]). Similarly, ${}_\Lambda(I_{w_0})$ is the unique module ${}_\Lambda T \in \operatorname{tilt}\Lambda^{\mathrm{op}}$ with $\operatorname{proj.dim} {}_\Lambda T \leq 1$ and $\operatorname{inj.dim} {}_\Lambda T \leq 1$. Because of Lemma 3.6 we must have $(I_{w_0})_\Lambda \cong D({}_\Lambda(I_{w_0}))$ such that there is a path

$$\Lambda \xrightarrow{i_\ell} \cdots \xrightarrow{i_1} I_{w_0} \xrightarrow{i_\ell} \cdots \xrightarrow{i_1} D(\Lambda)$$

in $Q(\operatorname{tilt}\Lambda)$. Corollary 7.4 now implies $T_{w_+^2} = D(\Lambda)$ in $\operatorname{tilt}\Lambda$. □

The next insight is a consequence of Voigt's lemma.

Lemma 8.3. *The set* $\mathbb{T} = \{T_x \mid x \in \mathcal{B}_+\} \cap \operatorname{tilt}\Lambda$ *is finite.*

Proof. Let $X = \{\dim_K T_x \mid T_x \in \mathbb{T}\}$. On the one hand, for each $d \in X$ the set $\{T_x \in \mathbb{T} \mid \dim_K T_x = d\}$ is finite because of [HS01, Corollary 9]. On the other hand, Corollary 7.6 implies $X \subseteq \{\dim_K I_w \mid w \in \mathcal{S}\}$, so X is finite, too. This proves the claim. □

We formulate one last lemma before turning to the classification.

Lemma 8.4. *Let Q' be the full subquiver of $Q = Q(\operatorname{tilt}\Lambda)$ spanned by $\mathbb{T}$. Then $Q' = Q$ and every arrow in this quiver is of the form $T_x \xrightarrow{i} T_{\mathfrak{s}_i x}$ for some $x \in \mathcal{B}_+$ and $1 \leq i < n$.*

Proof. We show that Q' is a successor-closed subquiver of Q so that, using Lemma 8.3, Corollary 3.5 is applicable: Let $T_x \to T$ be an arrow in Q for some $x \in \mathcal{B}_+$. According to [HU05b, § 1], there exist $1 \leq i \leq n$, an indecomposable Λ-module Y such that $T = (1 - e_i)T_x \oplus Y$ and a short exact sequence $0 \to e_i T_x \xrightarrow{\iota} E \to Y \to 0$ in which ι is a minimal left $\operatorname{add}((1-e_i)T_x)$-approximation. Given that the projective-injective module $e_n\Lambda$ appears as a summand of every tilting module, we have $e_n T_x \cong e_n\Lambda$, so $i \neq n$. Thus $Y \cong e_i T_{\mathfrak{s}_i x}$ and $T = T_{\mathfrak{s}_i x}$ by Lemma 7.3. □

Now we are ready to prove our main result.

Theorem 8.5. *There is a poset isomorphism:*

$$\begin{array}{ccc} [1, w_+^2]_L & \longrightarrow & (\operatorname{tilt}\Lambda)^{\mathrm{op}} \\ x & \longmapsto & T_x \end{array}$$

Proof. The map is well-defined by Lemma 8.1 because $[1, w_+^2]_L = \mathcal{S}_+\mathcal{S}_+$. According to Proposition 7.1 it is a morphism of posets.

(1) *For all $T_x, T_y \in \operatorname{tilt}\Lambda$ with $x, y \in \mathcal{B}_+$ and $T_x \geq T_y$ there is $z \in \mathcal{B}_+$ with $T_{zx} = T_y$:* Given a path $T_x = T_{x_0} \to \cdots \to T_{x_\ell}$ in Q with $T_{x_\ell} \geq T_y$ and $x_k = \mathfrak{s}_{i_k} \cdots \mathfrak{s}_{i_1} x$ for all $0 \leq k \leq \ell$, either $T_{x_\ell} = T_y$ or by Theorem 3.4 and Lemma 8.4 there is an arrow $T_{x_\ell} \to T_{x_{\ell+1}}$ in Q with $T_{x_{\ell+1}} \geq T_y$ and $x_{\ell+1} = \mathfrak{s}_{i_{\ell+1}} x_\ell$ for some $i_{\ell+1}$. If our claim were false, we would get an infinite path $T_{x_0} \to \cdots \to T_{x_\ell} \to \cdots$ in contradiction to Lemma 8.3.

(2) *For all $T \in \operatorname{tilt}\Lambda$ and $x, y \in \mathcal{B}_+$ with $T_x = T = T_y$ we have $x = y$:* Our argument uses induction on $\ell = \min\{\ell(x), \ell(y)\}$ and follows [AM17, Lemma 6.4]. If $\ell = 0$, we have $T = \Lambda$, so $x = 1 = y$ by Proposition 7.1. Otherwise we can write $x = x'\mathfrak{s}_i$, $y = y'\mathfrak{s}_j$ for some i, j and $x', y' \in \mathcal{B}_+$. Let $\mathfrak{s}_{ij}$ be the join of the elements $\mathfrak{s}_i$ and $\mathfrak{s}_j$ in the lattice $(\mathcal{S}_+, \geq_L)$, i.e.

$$\mathfrak{s}_{ij} = \begin{cases} \mathfrak{s}_i = \mathfrak{s}_j & \text{if } i = j, \\ \mathfrak{s}_i\mathfrak{s}_j = \mathfrak{s}_j\mathfrak{s}_i & \text{if } |i-j| > 1, \\ \mathfrak{s}_i\mathfrak{s}_j\mathfrak{s}_i = \mathfrak{s}_j\mathfrak{s}_i\mathfrak{s}_j & \text{if } |i-j| = 1. \end{cases}$$

Then $T_{\mathfrak{s}_{ij}}^\perp = T_{\mathfrak{s}_i}^\perp \cap T_{\mathfrak{s}_j}^\perp$ with [IZ16, Theorem 4.12], [IRTT15, Remark 1.13] and [AIR14]. Now $T_{\mathfrak{s}_i} \geq T$ and $T_{\mathfrak{s}_j} \geq T$ because of $x \geq_L \mathfrak{s}_i$ and $y \geq_L \mathfrak{s}_j$. Hence, $T_{\mathfrak{s}_{ij}} \geq T$ by Remark 3.1. So by (1) there is $z \in \mathcal{B}_+$ with $T_{z\mathfrak{s}_{ij}} = T$. Consequently, $T_{z_i} = T_{x'}$ and $T_{z_j} = T_{y'}$ for $z_i = z\mathfrak{s}_{ij}\mathfrak{s}_i^{-1}$ and $z_j = z\mathfrak{s}_{ij}\mathfrak{s}_j^{-1}$ by Proposition 6.3. Without loss of generality we may assume $\ell(x) = \ell$ so that $z_i = x'$ by induction. Because of $\ell(z_j) = \ell(z_i) = \ell - 1$ induction also gives $z_j = y'$. Thus $x = z\mathfrak{s}_{ij} = y$.

(3) *Injectivity:* Follows immediately from (2).

(4) *Surjectivity:* By Lemma 8.4 it suffices to check for each $x \in \mathcal{B}_+$ with $T_x \in \operatorname{tilt}\Lambda$ that $x \in [1, w_+^2]_L$. Firstly, we have $T_x \geq D(\Lambda) = T_{w_+^2}$ due to Lemma 8.2. Secondly, there exists $z \in \mathcal{B}_+$ with $T_{zx} = T_{w_+^2}$ by (1). Finally, we conclude $zx = w_+^2$ with (2), so $x \in [1, w_+^2]_L$. □

Corollary 8.6. *The tensor products $I_v \otimes_\Lambda I_w$ with $v, w \in \mathcal{S}$ are the basic tilting modules for Λ.*

Recall that a Λ-module E is called *exceptional* if it is self-orthogonal and $\operatorname{End}_\Lambda(E) = K$.

Corollary 8.7. *The modules* $e_1(I_v \otimes_\Lambda I_w)$ *with* $v, w \in \mathcal{S}$ *are the exceptional modules for* Λ.

Proof. Use $\dim_K \operatorname{End}_\Lambda(e_i(I_v \otimes_\Lambda I_w)) = i$ and Corollaries 3.9 and 8.6. □

Corollary 8.8. *There is a poset isomorphism:*

$$\begin{aligned} [w_-, w_+]_R &\longrightarrow \operatorname{tilt}\Lambda \\ x \quad &\longmapsto T_{x^{-1}w_+} \end{aligned}$$

Proof. Use Theorem 8.5 and the fact that $x \mapsto x^{-1}w_+$ defines an anti-isomorphism $[w_-, w_+]_R \to [1, w_+^2]_L$ of posets. □

Remark 8.9. The poset isomorphism from Corollary 8.8 restricts to an isomorphism $[1, w_+]_R \to \operatorname{tilt}_1 \Lambda$.

Next, we strengthen Theorem 8.5 by describing the *simplicial complex of tilting modules* $\Sigma(\Lambda)$ combinatorially. Recall from [Ung07] that $\Sigma(\Lambda)$ is by definition the abstract simplicial complex whose r-dimensional faces are the sets $\{M_0, \ldots, M_r\}$ of isomorphism classes of indecomposable Λ-modules with the property that $M_0 \oplus \cdots \oplus M_r$ is a direct summand of a tilting module for Λ. The vertex set of $\Sigma(\Lambda)$ is by Corollary 3.9 the set of isomorphism classes of indecomposable self-orthogonal Λ-modules.

We define $\mathcal{V} = \mathcal{V}_n$ as the set $[1, w_+^2]_L \times \{1, \ldots, n\}$ modulo the equivalence relation $\sim$ generated by $(\mathfrak{s}_j x, i) \sim (x, i)$ for $\mathfrak{s}_j x >_L x$ and $j \neq i$. Let $\Sigma = \Sigma_n$ be the $(n-1)$-dimensional abstract simplicial complex with

$$\big\{\{(x, i_0), \ldots, (x, i_r)\} \in \mathcal{V}^{r+1} \mid 1 \leq i_0 < \cdots < i_r \leq n\big\}$$

as its set of r-dimensional faces.

Theorem 8.10. *There is an isomorphism* $\Sigma \to \Sigma(\Lambda)$ *of abstract simplicial complexes given by the assignment* $(x, i) \mapsto e_i T_x$.

Proof. The assignment defines a surjective simplicial map by Lemma 7.3 and Theorem 8.5. To prove that it yields an isomorphism, it is enough to check its injectivity. For this, assume $e_i T_x \cong U \cong e_i T_y$ for some vertex U of $\Sigma(\Lambda)$. We will show $(x, i) \sim (y, i)$. Let $T_x \wedge T_y$ be the meet of T_x and T_y in $\operatorname{tilt}\Lambda$. By (1) in the proof of Theorem 8.5 we can choose $x' = \mathfrak{s}_{j_\ell} \cdots \mathfrak{s}_{j_1}$ with $T_{x'x} = T_x \wedge T_y$. Then $T_x \geq T_{x_r} \geq T_x \wedge T_y$ for every $x_r = \mathfrak{s}_{j_r} \cdots \mathfrak{s}_{j_1} x$ with $0 \leq r \leq \ell$. Using Remark 3.1 and Corollary 3.10 we conclude $T_x^\perp \supseteq T_{x_r}^\perp \supseteq T_x^\perp \cap T_y^\perp$ and then also ${}^\perp(T_x^\perp) \subseteq {}^\perp(T_{x_r}^\perp)$. Thus

$$U \;\in\; \operatorname{add}(T_x) \cap \operatorname{add}(T_y) \;\subseteq\; (T_x^\perp \cap T_y^\perp) \cap {}^\perp(T_x^\perp) \;\subseteq\; \operatorname{add}(T_{x_r})$$

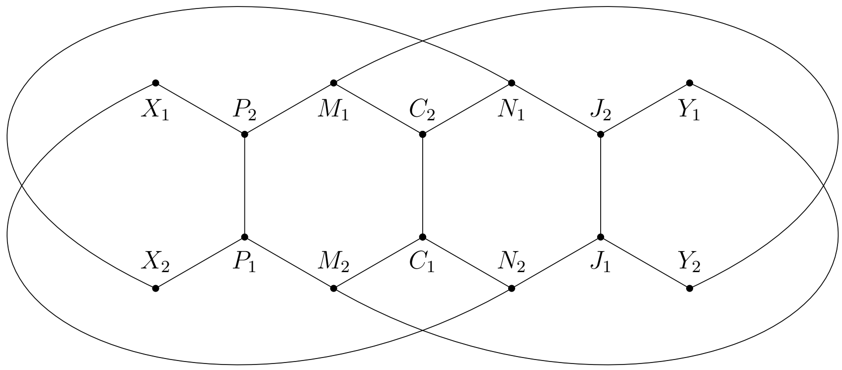
by Theorem 3.7, so $e_i T_{x_r} \cong U$ for all $0 \leq r \leq \ell$. It follows $i \notin \{j_1, \ldots, j_\ell\}$ by Lemma 7.3 and therefore $(x, i) \sim (x'x, i)$. Analogously, there is y' with $T_x \wedge T_y = T_{y'y}$ and $(y, i) \sim (y'y, i)$. But then $x'x = y'y$ because of $T_{x'x} = T_x \wedge T_y = T_{y'y}$ and Theorem 8.5. Consequently, $(x, i) \sim (y, i)$. □

The *boundary* $\partial\Delta$ of a pure-dimensional abstract simplicial complex Δ is the subcomplex spanned by all its faces of codimension one that are contained in precisely one facet of Δ. Note that

$$\Sigma \;=\; \partial\Sigma \;\dot\cup\; \{(1,n)\} \;\dot\cup\; \big\{F \cup \{(1,n)\} \,|\, F \in \partial\Sigma\big\}\,.$$

Therefore Σ is completely determined by its boundary.

Example 8.11. The geometric realization of $\partial\Sigma(\Lambda_3)$ looks as follows:



Its vertices are the following self-orthogonal modules where $\otimes = \otimes_{\Lambda_3}$:

$$\begin{array}{lllll} P_1 = e_1\Lambda_3 & M_1 = e_1 I_{s_1} & C_1 = e_1 I_{w_0} & N_1 = D(I_{s_1}e_1) & J_1 = D(\Lambda_3 e_1) \\ P_2 = e_2\Lambda_3 & M_2 = e_2 I_{s_2} & C_2 = e_2 I_{w_0} & N_2 = D(I_{s_2}e_2) & J_2 = D(\Lambda_3 e_2) \end{array}$$

$$\begin{array}{ll} X_1 = e_1 I_{s_1} \otimes I_{s_1} & Y_1 = D(I_{s_1} \otimes I_{s_1} e_1) \\ X_2 = e_2 I_{s_2} \otimes I_{s_2} & Y_2 = D(I_{s_2} \otimes I_{s_2} e_2) \end{array}$$

Returning to the general case, let $p_{n,i}$ be the number of isomorphism classes of modules occurring as i-th summand of some tilting module T_x for Λ_n. By Theorem 8.10 we have

$$p_{n,i} \;=\; \#\{(x,i) \in \mathcal{V}_n\}\,.$$

Remark 8.12. Computations for small n suggest that $p_{n,i}$ is the integer $a_{n+1,i+1}$ in OEIS:A046802 such that $p_n := \sum_{1\le i\le n} p_{n,i}$ would be one less than the number of arrangements of an n-element set (OEIS:A000522):

n	$p_{n,1}$	$p_{n,2}$	$p_{n,3}$	$p_{n,4}$	$p_{n,5}$	p_n
1	1	-	-	-	-	1
2	3	1	-	-	-	4
3	7	7	1	-	-	15
4	15	33	15	1	-	64
5	31	131	131	31	1	325

Since the first summands of the tilting modules T_x are the exceptional modules, this would be in line with [HP17] who proved that the number of isomorphism classes of exceptional modules for Λ_n is $2^n - 1$. In fact, checking $p_{n,1} = 2^n - 1$ would give an alternative proof of their result.

9. Exceptional sequences

In this final section we will relate the action of the braid group on full exceptional sequences in $\mathcal{D}(\Lambda)$, which was studied in [HP17], to its action on $\mathrm{DPic}(\Lambda)$ by left multiplication (via the map in Proposition 6.3). For simplicity, we only consider full exceptional sequences of modules.

The set of all sequences $(E_1, \ldots, E_n)$ of isomorphism classes of exceptional Λ-modules with $\mathrm{Ext}^q_\Lambda(E_j, E_i) = 0$ for all $q \geq 0$ and $i < j$ will be denoted by $\operatorname{exs}\Lambda$ in what follows.

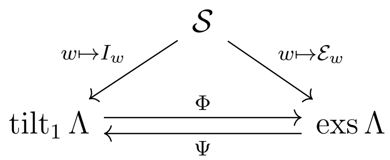
Remark 9.1. For $0 \leq i < n$ and $i^* := n - i$ let Δ_{i^*} be the *standard module* given by the short exact sequence

$$(\Delta) \qquad 0 \longrightarrow e_i\Lambda \xrightarrow{\beta_{i+1}\cdot} e_{i+1}\Lambda \longrightarrow \Delta_{i^*} \longrightarrow 0\,.$$

Applying the functor $- \otimes^{\mathbb{L}}_\Lambda I_w$ for $w \in \mathcal{S}$ yields a short exact sequence

$$0 \longrightarrow e_i I_w \longrightarrow e_{i+1} I_w \longrightarrow \mathcal{E}_{w,i^*} \longrightarrow 0$$

with $\mathcal{E}_{w,i^*} := \Delta_{i^*} \otimes_\Lambda I_w \cong \Delta_{i^*} \otimes^{\mathbb{L}}_\Lambda I_w$ (compare the proof of Lemma 8.1).
Then $\mathcal{E}_w := (\mathcal{E}_{w,1}, \ldots, \mathcal{E}_{w,n}) \in \operatorname{exs}\Lambda$ because of $(\Delta_1, \ldots, \Delta_n) \in \operatorname{exs}\Lambda$ and Theorem 3.2. In view of Theorem 5.1 and [HP17, Proposition 3.4] we have a commutative diagram of bijections:



Let us prove that Φ and Ψ commute with mutation, i.e. $\mathrm{L}_{i^*}(\mathcal{E}_w) = \mathcal{E}_{s_i w}$ whenever $s_i w >_L w$. By Theorem 3.2 and Proposition 6.3 it suffices to check this for $w = 1$. Then the left mutation

$$\mathrm{L}_{i^*}(\mathcal{E}_1) \;=\; (\ldots, \Delta_{i^*-1}, \mathrm{L}_{\Delta_{i^*}}(\Delta_{i^*+1}), \Delta_{i^*}, \Delta_{i^*+2}, \ldots)$$

is given by the canonical triangle

$$\mathbb{R}\mathrm{Hom}_\Lambda(\Delta_{i^*}, \Delta_{i^*+1}) \otimes_K \Delta_{i^*} \longrightarrow \Delta_{i^*+1} \longrightarrow \mathrm{L}_{\Delta_{i^*}}(\Delta_{i^*+1}) \longrightarrow \cdot \;\;.$$

Taking cohomology, $\mathrm{L}_{\Delta_{i^*}}(\Delta_{i^*+1})$ is seen to be the middle term of a short exact sequence $0 \to S_i \to \mathrm{L}_{\Delta_{i^*}}(\Delta_{i^*+1}) \to \Delta_{i^*} \to 0$ that corresponds to a non-zero element of the one-dimensional vector space $\mathrm{Ext}^1_\Lambda(\Delta_{i^*}, S_i)$. An easy calculation that uses $e_i I_i = \mathrm{rad}(e_i\Lambda)$ and $e_j I_i = e_j\Lambda$ for $j \neq i$ now shows $\mathcal{E}_{s_i} = \mathrm{L}_{i^*}(\mathcal{E}_1)$.

Remark 9.2. Every tilting module $T = I_v \otimes_\Lambda I_w$ with $v, w \in \mathcal{S}$ gives rise to an exceptional sequence $\mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_n)$ in $\mathcal{D}(\Lambda)$ where

$$\mathcal{E}_{i^*} \;:=\; \cdots \longrightarrow \underset{-2}{0} \longrightarrow \underset{-1}{e_i T} \xrightarrow{\beta_{i+1}\cdot} \underset{0}{e_{i+1} T} \longrightarrow \underset{1}{0} \longrightarrow \cdots \;\;.$$

This follows by applying $- \otimes^{\mathbb{L}}_\Lambda T$ to the triangle induced by (Δ).

Acknowledgments

First of all, I would like to thank Julia Sauter for raising the question of whether the tilting modules for the Auslander algebra Λ_n can be classified. I am grateful to Baptiste Rognerud for pointing me to [DG17]. I thank both of them and Biao Ma for interesting discussions.

References

[AIR14] T. Adachi, O. Iyama, and I. Reiten, "τ-tilting theory," *Compos. Math.*, vol. 150, no. 3, pp. 415–452, 2014. DOI: 10.1112/S0010437X13007422. arXiv: 1210.1036 [math.RT].

[AI12] T. Aihara and O. Iyama, "Silting mutation in triangulated categories," *J. Lond. Math. Soc. (2)*, vol. 85, no. 3, pp. 633–668, 2012. DOI: 10.1112/jlms/jdr055. arXiv: 1009.3370 [math.RT].

[AM17] T. Aihara and Y. Mizuno, "Classifying tilting complexes over preprojective algebras of Dynkin type," *Algebra Number Theory*, vol. 11, no. 6, pp. 1287–1315, 2017. DOI: 10.2140/ant.2017.11.1287. arXiv: 1509.07387 [math.RT].

[AR91] M. Auslander and I. Reiten, "Applications of contravariantly finite subcategories," *Adv. Math.*, vol. 86, no. 1, pp. 111–152, 1991. DOI: 10.1016/0001-8708(91)90037-8.

[BHRR99] T. Brüstle, L. Hille, C. M. Ringel, and G. Röhrle, "The Δ-filtered modules without self-extensions for the Auslander algebra of $k[T]/(T^n)$," *Algebr. Represent. Theory*, vol. 2, no. 3, pp. 295–312, 1999. DOI: 10.1023/A:1009999006899.

[DG17] F. Digne and T. Gobet, "Dual braid monoids, Mikado braids and positivity in Hecke algebras," *Math. Z.*, vol. 285, no. 1-2, pp. 215–238, 2017. DOI: 10.1007/s00209-016-1704-z. arXiv: 1508.06817 [math.GR].

[HU96] D. Happel and L. Unger, "Modules of finite projective dimension and cocovers," *Math. Ann.*, vol. 306, no. 3, pp. 445–457, 1996. DOI: 10.1007/BF01445260.

[HU05a] D. Happel and L. Unger, "On a partial order of tilting modules," *Algebr. Represent. Theory*, vol. 8, no. 2, pp. 147–156, 2005. DOI: 10.1007/s10468-005-3595-2.

[HU05b] D. Happel and L. Unger, "On the quiver of tilting modules," *J. Algebra*, vol. 284, no. 2, pp. 857–868, 2005. DOI: 10.1016/j.jalgebra.2004.11.007.

[HP17] L. Hille and D. Ploog, "Exceptional sequences and spherical modules for the Auslander algebra of $k[x]/(x^t)$," version 2, 2017. arXiv: 1709.03618 [math.RT].

[HS01] B. Huisgen-Zimmermann and M. Saorín, "Geometry of chain complexes and outer automorphisms under derived equivalence," *Trans. Amer. Math. Soc.*, vol. 353, no. 12, pp. 4757–4777, 2001. DOI: 10.1090/S0002-9947-01-02815-X.

[IRTT15] O. Iyama, I. Reiten, H. Thomas, and G. Todorov, "Lattice structure of torsion classes for path algebras," *Bull. Lond. Math. Soc.*, vol. 47, no. 4, pp. 639–650, 2015.
DOI: 10.1112/blms/bdv041. arXiv: 1312.3659 [math.RT].

[IZ16] O. Iyama and X. Zhang, "Classifying τ-tilting modules over the Auslander algebra of $K[x]/(x^n)$," version 1, 2016.
arXiv: 1602.05037 [math.RT].

[IZ18] O. Iyama and X. Zhang, "Tilting modules over Auslander-Gorenstein algebras," version 2, 2018.
arXiv: 1801.04738 [math.RT].

[KT08] C. Kassel and V. Turaev, *Braid groups*. Springer, New York, 2008, vol. 247, pp. xii+340.
DOI: 10.1007/978-0-387-68548-9.

[KS03] H. Krause and Ø. Solberg, "Applications of cotorsion pairs," *J. London Math. Soc. (2)*, vol. 68, no. 3, pp. 631–650, 2003.
DOI: 10.1112/S0024610703004757.

[Miy03] J.-I. Miyachi, "Recollement and tilting complexes," *J. Pure Appl. Algebra*, vol. 183, no. 1-3, pp. 245–273, 2003.
DOI: 10.1016/S0022-4049(03)00072-0.

[OEIS] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences.* Online: https://oeis.org/.

[Ric91] J. Rickard, "Derived equivalences as derived functors," *J. London Math. Soc. (2)*, vol. 43, no. 1, pp. 37–48, 1991.
DOI: 10.1112/jlms/s2-43.1.37.

[QPA17] The QPA-team, *QPA – quivers, path algebras and representations – a GAP package, version 1.27*, 2017.
Online: https://folk.ntnu.no/oyvinso/QPA/.

[Ung07] L. Unger, "Combinatorial aspects of the set of tilting modules," in *Handbook of tilting theory*, vol. 332, Cambridge Univ. Press, Cambridge, 2007, pp. 259–277.
DOI: 10.1017/CBO9780511735134.010.

[Yek99] A. Yekutieli, "Dualizing complexes, Morita equivalence and the derived Picard group of a ring," *J. London Math. Soc. (2)*, vol. 60, no. 3, pp. 723–746, 1999.
DOI: 10.1112/S0024610799008108. arXiv: math/9810134 [math.RA].

[Yek92] A. Yekutieli, "Dualizing complexes over noncommutative graded algebras," *J. Algebra*, vol. 153, no. 1, pp. 41–84, 1992.
DOI: 10.1016/0021-8693(92)90148-F.

Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld
E-mail address: jan.geuenich@math.uni-bielefeld.de