# Enumeration of Łukasiewicz paths modulo some patterns

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### Abstract

For any pattern $\alpha$ of length at most two, we enumerate equivalence classes of Łukasiewicz paths of length $n \geq 0$ where two paths are equivalent whenever the occurrence positions of $\alpha$ are identical on these paths. As a byproduct, we give a constructive bijection between Motzkin paths and some equivalence classes of Łukasiewicz paths.

**Keywords:** Łukasiewicz path, Dyck path, Motzkin path, equivalence relation, patterns.

## 1  Introduction and notations

In the literature, lattice paths are widely studied. Their enumeration is a very active field in combinatorics, and they have many applications in other research domains as computer science, biology and physics [18, 19]. Dyck and Motzkin paths are the most often considered. This is partly due to the fact that they are respectively counted by the famous Catalan and Motzkin numbers (see A000108 and A001006 in the on-line encyclopedia of integer sequences [28]). Almost always, these paths are enumerated according to several parameters and statistics (see for instance [6, 14, 15, 17, 20, 21, 24, 25, 29] for Dyck paths and [4, 5, 7, 8, 16, 22, 26] for Motzkin paths). Also, many one-to-one correspondences have been found between lattice paths and some combinatorial objects such as Young tableaux, pattern avoiding permutations, bargraphs, RNA shapes and so on [30]. Recently a new approach has been introduced for studying statistics on lattice paths. It consists in determining the cardinality of the quotient set generated by an equivalence relation based on the positions of a given pattern: *two paths belong to the same equivalence class whenever the positions of*

*occurrences of a given pattern are identical on these paths.* Enumerating results are provided for the quotient sets of Dyck, Motzkin and Ballot paths for patterns of length at most three (see respectively [2], [3] and [13]). The purpose of this present paper is to extend these studies for Łukasiewicz paths that naturally generalizes Dyck and Motzkin paths. As a byproduct, we show how Motzkin paths are in one-to-one correspondence with some equivalence classes of Łukasiewicz paths.

Throughout this paper, a *lattice path* is defined by a starting point $P_0 = (0, 0)$, an ending point $P_n = (n, 0)$, consisting of steps lying in $S = \{(1, i), i \in \mathbb{Z}\}$, and never going below the $x$-axis. The *length* of a path is the number of its steps. We denote by $\epsilon$ the empty path, *i.e.*, the path of length zero. Constraining the steps to lie into $\{(1, 1), (1, -1)\}$ (resp. $\{(1, 1), (1, 0), (1, -1)\}$), we retrieve the well known definition of Dyck paths (resp. Motzkin paths). *Łukasiewicz paths* are obtained when the steps belong to $\{(1, i) \in S, i \geq -1\}$. We refer to [10, 23, 30, 32, 33] for some combinatorial studies on Łukasiewicz paths. Let $\mathcal{L}_n$, $\mathcal{D}_n$, $\mathcal{M}_n$, $n \geq 0$, respectively, be the sets of Łukasiewicz, Dyck and Motzkin paths of length $n$, and $\mathcal{L} = \cup_{n \geq 0} \mathcal{L}_n$, $\mathcal{D} = \cup_{n \geq 0} \mathcal{D}_n$, $\mathcal{M} = \cup_{n \geq 0} \mathcal{M}_n$. For convenience, we set $D = (1, -1)$, $F = (1, 0)$, $U = U_1 = (1, 1)$ and $U_i = (1, i)$ for $i \geq 2$. See Figure 1 for an illustration of Dyck, Motzkin and Łukasiewicz paths of length 18. Note that Łukasiewicz paths can be interpreted as an algebraic language of words $w \in \{x_0, x_1, x_2, \ldots\}^\star$ such that $\delta(w) = -1$ and $\delta(w') \geq 0$ for any proper prefix $w'$ of $w$ where $\delta$ is the map from $\{x_0, x_1, x_2, \ldots\}^\star$ to $\mathbb{Z}$ defined by $\delta(w_1 w_2 \ldots w_n) = \sum_{i=1}^n \delta(w_i)$ with $\delta(x_i) = i - 1$ (see [11, 27]).
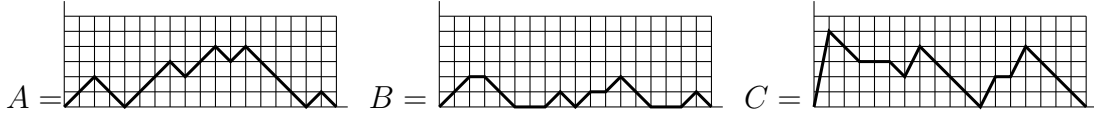


Figure 1: From left to right, we show a Dyck path $A = UUDDUUUDUUDUDDDDUD$, a Motzkin path $B = UUFDDFFUDUFUDDFFUD$ and a Łukasiewicz path $C = U_5DDFFDU_2DDDDU_2FU_2DDDD$.

Any non-empty Łukasiewicz path $L \in \mathcal{L}$ can be decomposed (see [9]) into one of the two following forms: (1) $L = FL'$ with $L' \in \mathcal{L}$, or (2) $L = U_k L_1 D L_2 D \ldots L_k D L'$ with $k \geq 1$ and $L_1, L_2, \ldots, L_k, L' \in \mathcal{L}$ (see Figure 2).
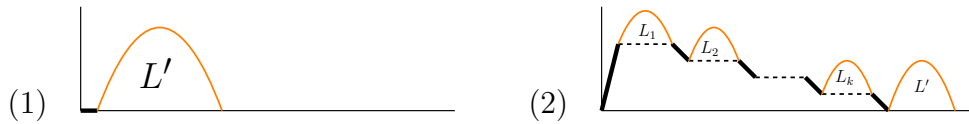


Figure 2: The two forms of the decomposition of a non-empty Łukasiewicz path.

Due to this decomposition, the generating function $L(x)$ for the cardinalities of the sets $\mathcal{L}_n$, $n \geq 0$, satisfies the functional equation $L(x) = 1 + xL(x) + \sum_{k \geq 1} x^{k+1} L(x)^{k+1}$, or equivalently, $L(x) = \frac{1}{1 - xL(x)}$. Then, $L(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$ and the coefficient of $x^n$ in the series

expansion of $L(x)$ is given by the $n$-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$ (see sequence A000108 in [28]).

A *pattern* of length one (resp. two) in a lattice path $L$ consists of one step (resp. two consecutive steps). We will say that an *occurrence* of a pattern is at *position* $i \geq 1$, in $L$ whenever the first step of this occurrence appears at the $i$-th step of the path. The *height* of an occurrence is the minimal ordinate reached by its points. For instance, the path $C = U_5DDF\mathbf{FD}U_2DDDDU_2FU_2DDDD$ (see Figure 1) contains one occurrence of the pattern $FD$ at position 5 and of height 2.

Following the recent studies [2, 3, 13], we define an equivalence relation on the set $\mathcal{L}$ for a given pattern $\alpha$: *two Łukasiewicz paths of the same length are $\alpha$-equivalent whenever the occurrences of the pattern $\alpha$ appear at the same positions in the two paths.* For instance, $UFFF\mathbf{FD}UDUDFFFFUDFF$ is $FD$-equivalent to the path $C$ in Figure 1 since the only one occurrence of the pattern $FD$ (in boldface) appear at the same position in the two paths. Note that the height of the occurrences of $\alpha$ does not involve in this definition.

In this paper, for any pattern $\alpha$ of length at most two, we consider the above equivalence relation on the set $\mathcal{L}$, and for each of them we provide the cardinality of the quotient set with respect to the length. Three general methods are used:

- ($M_0$) we prove that any $\alpha$-equivalence class contains at least one Motzkin path. Using $\mathcal{M}_n \subset \mathcal{L}_n$ for $n \geq 0$, we deduce that the number of $\alpha$-equivalence classes in $\mathcal{M}_n$ is equal to that of $\mathcal{L}_n$. Since the authors have already determined this number for $\mathcal{M}_n$ (see [3]), we can conclude,

- ($M_1$) we directly count the number of subsets $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$ of the possible occurrence positions of $\alpha$ in a Łukasiewicz path in $\mathcal{L}_n$,

- ($M_2$) we exhibit a one-to-one correspondence between a subset of Łukasiewicz paths (subset of representatives of the classes) and the set of equivalence classes by using combinatorial reasonings, and then, we evaluate algebraically the generating function for this subset.

The paper is organized as follows. In Section 2, we consider the case of patterns $\alpha$ studied using method ($M_0$), *i.e.*, $\alpha \in \{U, UU, UD, UF, DU, FU\}$. In Section 3, we focus on these ones that can be dealt using method ($M_1$), *i.e.*, $\alpha \in \{F, D, FD, DF, DD\}$. In Section 4, we complete our study by the remaining cases which are obtained using method ($M_2$). We refer to Table 1 for an exhaustive list of our enumerative results.

## 2    Modulo $\alpha \in \{U, UU, UD, UF, DU, FU\}$

In this section, we focus on the patterns that can be dealt using method ($M_0$).

**Lemma 1** *For $n \geq 0$, let $L$ be a Łukasiewicz path in $\mathcal{L}_n$ and $\alpha \in \{U, UU, UD, UF, DU, FU\}$. Then, there exists a Motzkin path $M \in \mathcal{M}_n$ such that $M$ and $L$ are $\alpha$-equivalent.*

*Proof.* Let us assume that $\alpha \in \{U, UU, UD\}$. Any non-empty Łukasiewicz path can be decomposed into one of the two following forms: (*i*) $L = FL'$ with $L' \in \mathcal{L}$, and (*ii*) $L =$

| Pattern $\alpha$ | Sequence | Sloane | $a_n, 1 \le n \le 10$ | Method |
|---|---|---|---|---|
| $U$ | $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ | A001405 | $1, 2, 3, 6, 10, 20, 35, 70, 126, 252$ | |
| $UU$ | $\frac{1-2x+x^2-\sqrt{(x^2+1)(1-3x^2)}}{2x(-1+2x-x^2+x^3)}$ | $A191385$ | $1, 1, 1, 2, 3, 5, 7, 12, 18, 31$ | $M_0$ |
| $UD$ | Fibonacci | A005251 | $1, 2, 3, 5, 8, 13, 21, 34, 55, 89$ | |
| $UF, FU$ | $\frac{2}{1-2x-\sqrt{1-4x^3}}$ | A165407 | $1, 1, 2, 3, 4, 7, 11, 16, 27, 43$ | |
| $DU$ | Shift of Fibonacci | A000045 | $1, 1, 1, 2, 3, 5, 8, 13, 21, 34$ | |
| $F$ | $2^n - n$ | A000325 | $1, 2, 5, 12, 27, 58, 121, 248, 503, 1014$ | |
| $D$ | $2^{n-1}$ | A011782 | $1, 1, 2, 4, 8, 16, 32, 64, 128, 256$ | $M_1$ |
| $FD, DF$ | Fibonacci | A005251 | $1, 1, 2, 3, 5, 8, 13, 21, 34, 55$ | |
| $DD$ | $\frac{1-x}{1-2x+x^2-x^3}$ | A005251 | $1, 1, 2, 4, 7, 12, 21, 37, 65, 114$ | |
| $U_k$ | Motzkin | A001006 | $1, 2, 4, 9, 21, 51, 127, 323, 835, 2188$ | |
| $FF$ | $\frac{1-3x+4x^2-5x^3+7x^4-7x^5+6x^6-3x^7+x^8}{(1-2x+x^2-x^3)(1-x)^2}$ | New | $1, 2, 2, 5, 9, 17, 32, 59, 107, 192$ | |
| $FU_k, U_kF$ | $\frac{1-x+2x^2+\sqrt{1-2x+x^2-4x^3}}{1-2x-3x^3+(1-x+x^2)\sqrt{1-2x+x^2-4x^3}}$ | A023431 | $1, 1, 2, 4, 7, 13, 26, 52, 104, 212$ | $M_2$ |
| $U_kD$ | $\frac{1-x+x^2+\sqrt{1-2x-x^2-2x^3+x^4}}{1-2x-x^3+(1-x)\sqrt{1-2x-x^2-2x^3+x^4}}$ | A292460 | $1, 2, 4, 8, 17, 37, 82, 185, 423, 978$ | |
| $DU_k$ | $\frac{1-x-x^2-2x^3+\sqrt{1-2x-x^2-2x^3+x^4}}{1-2x-x^3+(1-x)\sqrt{1-2x-x^2-2x^3+x^4}}$ | A004148 | $1, 1, 1, 2, 4, 8, 17, 37, 82, 185$ | |

Table 1: Number of $\alpha$-equivalence classes for Łukasiewicz paths. The last three sequences are recorded in OEIS [28] as generalized Catalan sequences.

$U_k L_1 D L_2 D \ldots L_k D L'$ with $k \ge 1$ and $L_1, L_2, \ldots, L_k, L' \in \mathcal{L}$. For $n \ge 0$, we recursively define a map $\phi$ from $\mathcal{L}_n$ to $\mathcal{M}_n$ as follows:

$$
\begin{cases}
\phi(\epsilon) & = \epsilon, \\
\phi(FL') & = F\phi(L'), \\
\phi(UL_1DL') & = U\phi(L_1)D\phi(L'), \\
\phi(U_k L_1 D L_2 D \ldots L_k D L') & = F\phi(L_1)F\phi(L_2)F\ldots\phi(L_k)F\phi(L') \qquad \text{for } k \ge 2.
\end{cases}
$$

Clearly, $\phi(L)$ is a Motzkin path in $\mathcal{M}_n$, and whenever $\alpha \in \{U, UU, UD\}$ the occurrence positions of $\alpha$ in $L$ and $\phi(L)$ are identical. Then, the equivalence class of $L$ contains a Motzkin path $\phi(L)$.

Let us assume that $\alpha = DU$. Any non-empty Łukasiewicz path $L$ can be written as follows:

$$L = K_0 \prod_{i=1}^{r} (DU)^{a_i} K_i$$

with $r \ge 0$, $a_i \ge 1$ for $1 \le i \le r$, and where $K_i$, $0 \le i \le r$, are some parts that do not contain any pattern $DU$. Note that $K_0$ and $K_r$ necessarily contain at least one step. From

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$L \in \mathcal{L}_n$, we define the Motzkin path

$$M = UF^{b_0-1}\left(\prod_{i=1}^{r-1}(DU)^{a_i}F^{b_i}\right)(DU)^{a_r}DF^{b_r-1} \in \mathcal{M}_n$$

where $b_i = |K_i|$ for $0 \leq i \leq r$. Since the occurrence positions of $DU$ in $L$ and $M$ are identical, $M$ is a Motzkin path in the same class as $L$.

Let us assume that $\alpha \in \{FU, UF\}$. Any non-empty Łukasiewicz path $L$ can be written as follows:

$$L = K_0 \prod_{i=1}^{r} \alpha^{a_i} K_i$$

with $r \geq 0$, $a_i \geq 1$ for $1 \leq i \leq r$, and where $K_i$, $0 \leq i \leq r$, are some parts that do not contain any pattern $\alpha$. From $L \in \mathcal{L}_n$, we define the Motzkin path

$$M = F^{b_0}\prod_{i=1}^{r} \alpha^{a_i}D^{c_i}F^{b_i-c_i} \in \mathcal{M}_n$$

where $b_0 = |K_0|$, and for $1 \leq i \leq r$, $b_i = |K_i|$ and $c_i = \min\{b_i, a_i + \sum_{j=1}^{i-1}(a_j - c_j)\}$. Less formally, $K_0$ is replaced with $F^{|K_0|}$, and for $i$ from 1 to $r$, $K_i$ is replaced with $D^{c_i}F^{|K_i|-c_i}$ where the value $c_i$ is the maximal number of down steps $D$ that can be placed so that $M$ remains a lattice path. This ensures that $M$ has the same occurrence positions of $\alpha$ as $L$, which means that $M$ is a Motzkin path in the same class as $L$. $\qquad\square$

Using Lemma 1 and the fact that $\mathcal{M} \subset \mathcal{L}$, we directly deduce the following theorem.

**Theorem 1** *For $\alpha \in \{U, UU, UD, UF, DU, FU\}$ and $n \geq 0$, the number of $\alpha$-equivalence classes in $\mathcal{L}_n$ also is that of $\mathcal{M}_n$.*

Since the authors have already determined the number of $\alpha$-equivalence classes in $\mathcal{M}_n$, we refer to their paper [3] for a detailed description of the different proofs, and we report the results in Table 1.

# 3    Modulo $\alpha \in \{F, D, FD, DF, DD\}$

In this section, we focus on the patterns that can be dealt with method $(M_1)$ which consists in counting directly the possible subsets of occurrence positions of the pattern in a Łukasiewicz path.

**Theorem 2** *The number of $F$-equivalence classes in $\mathcal{L}_n$, $n \geq 0$, is given by $2^n - n$ (see sequence A000325 in [28]).*

*Proof.* Let $L$ be a Łukasiewicz path of length $n \geq 1$, and let $1 \leq i_1 < i_2 < \ldots < i_\ell \leq n$, $0 \leq \ell \leq n$, be the sequence of occurrence positions of $F$ in $L$. Since a Łukasiewicz path cannot contain exactly $n-1$ occurrences of $F$, we have $\ell \neq n-1$. Now, let us prove that for any $\ell \neq n-1$, $0 \leq \ell \leq n$, and for any sequence $1 \leq i_1 < i_2 < \ldots < i_\ell \leq n$, there exists $L \in \mathcal{L}_n$ where the positions of its flats are exactly $i_1, \ldots, i_\ell$. We distinguish two cases: $n-\ell$ is odd (different from one), and $n-\ell$ is even.

If $n - \ell \neq 1$ is odd, then we define the Łukasiewicz path $K = U_2DD(UD)^k$ of length $n - \ell$ (that is $k = \frac{n-\ell-3}{2}$); otherwise, we define the Łukasiewicz path $K = (UD)^{\frac{n-\ell}{2}}$.

For these two cases, we consider the Łukasiewicz path $L$ (of length $n$) obtained by inserting $\ell$ flats in $K$ so that the positions of flats in $L$ are given by $i_1, \ldots, i_\ell$.

Then, any increasing sequence $1 \leq i_1 < i_2 < \ldots < i_\ell \leq n$, $0 \leq \ell \leq n$ and $n - \ell \neq 1$, corresponds to the positions of $F$ in a Łukasiewicz path. Since there are $2^n - n$ possible such sequences, the proof is completed. $\qquad\square$

**Theorem 3** *The number of D-equivalence classes in $\mathcal{L}_n$, $n \geq 0$, is given by $2^{n-1}$ (see sequence A011782 in [28]).*

*Proof.* Let $L$ be a Łukasiewicz path of length $n \geq 1$, and let $1 \leq i_1 < i_2 < \ldots < i_\ell \leq n$, $0 \leq \ell \leq n$, be the sequence of occurrence positions of $D$ in $L$. Since the first step of a Łukasiewicz path cannot be a down step $D$, we have $0 \leq \ell \leq n-1$ and $i_1 \neq 1$. Now, let us prove that for any $\ell$, $0 \leq \ell \leq n-1$, and for any sequence $2 \leq i_1 < i_2 < \ldots < i_\ell \leq n$, there exists $L \in \mathcal{L}_n$ where the positions of its down steps are exactly $i_1, \ldots, i_\ell$.

We define the Łukasiewicz path $L$ as follows:

$$L = U_\ell F^{i_1-2} \prod_{j=1}^{\ell} DF^{i_{j+1}-i_j-1},$$

where $i_{\ell+1} = n + 1$.

This definition ensures that the down steps appear on positions $i_1, i_2, \ldots, i_\ell$. Therefore, any increasing sequence $2 \leq i_1 < i_2 < \ldots < i_\ell \leq n$, $0 \leq \ell \leq n-1$, corresponds to the sequence of positions of $D$ in a Łukasiewicz path. Since there are $2^{n-1}$ possible such sequences, the proof is completed. $\qquad\square$

**Theorem 4** *The number of FD-equivalence (resp. DF-equivalence) classes in $\mathcal{L}_n$, $n \geq 0$, is given by the Fibonacci number defined by $f_0 = 1$, $f_1 = 1$, $f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ (see sequence A005251 in [28]).*

*Proof.* Let $L$ be a Łukasiewicz path of length $n \geq 1$. For $\alpha \in \{FD, DF\}$, let $1 \leq i_1 < i_2 < \ldots < i_\ell \leq n$, $0 \leq \ell \leq n$, be the sequence of occurrence positions of $\alpha$ in $L$. Since a Łukasiewicz path cannot contain two occurrences of $\alpha$ at two adjacent positions $i$ and $i+1$, we necessarily have $i_j - i_{j-1} > 1$ for $j \geq 2$. A pattern $\alpha$ cannot appear at position one, which implies $i_1 \geq 2$. Also, a pattern $\alpha$ cannot appear at position $n$, which implies $i_\ell \leq n-1$. Now, let us prove that for any $\ell$, $0 \leq \ell \leq n$, and for any sequence $2 \leq i_1 < i_2 < \ldots < i_\ell \leq n-1$

with $i_j - i_{j-1} > 1$ for $j \geq 2$, there exists $L \in \mathcal{L}_n$ where the positions of its $\alpha$ are exactly $i_1, \ldots, i_\ell$. We set $I = \{i_1, \ldots, i_\ell\}$. So we define the Łukasiewicz path $L$ as follows:

$$L = U_\ell F^{i_1 - 2} \prod_{j=1}^{\ell} \alpha F^{i_{j+1} - i_j - 2},$$

where $i_{\ell+1} = n + 1$.

This definition ensures that the down steps appear on positions $i_1, i_2, \ldots, i_\ell$. Therefore, any increasing sequence $2 \leq i_1 < i_2 < \ldots < i_\ell \leq n - 1$, $0 \leq \ell \leq n$ with $i_j - i_{j-1} > 1$ for $j \geq 2$, corresponds to the sequence of positions of $\alpha$ in a Łukasiewicz path. It is well known (see [31] for instance) that such a sequence is enumerated by the Fibonacci number $f_n$ defined by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. $\qquad \square$

**Theorem 5** *The number of DD-equivalence classes in $\mathcal{L}_n$, $n \geq 0$, is given by the n-th term $g_n$ of the sequence defined by $g_0 = 1$, $g_1 = 1$, $g_2 = 1$, $g_3 = 2$ and $g_n = g_{n-1} + g_{n-2} + g_{n-4}$ for $n \geq 4$ (see sequence A005251 in [28]).*

*Proof.* Let $L$ be a Łukasiewicz path of length $n \geq 1$, and let $1 \leq i_1 < i_2 < \ldots < i_\ell \leq n$, $0 \leq \ell \leq n$, be the sequence of occurrence positions of $DD$ in $L$. Whenever a Łukasiewicz path contains two occurrences of $DD$ at two positions $i$ and $i + 2$, $L$ necessarily have an occurrence of $DD$ at position $i + 1$, that is $i_{j+1} \neq i_j + 2$. A pattern $DD$ cannot appear at position one, which implies $i_1 \geq 2$. Also, a pattern $DD$ cannot appear at position $n$, which implies $i_\ell \leq n - 1$. Now, let us prove that for any $\ell$, $0 \leq \ell \leq n$, and for any sequence $2 \leq i_1 < i_2 < \ldots < i_\ell \leq n-1$ satisfying $i_j - i_{j-1} \neq 2$ for $j \geq 2$, there exists $L \in \mathcal{L}_n$ where the positions of its $DD$ are exactly $i_1, \ldots, i_\ell$. We set $I = \{i_1, \ldots, i_\ell\}$, $I^+ = \{i_1 + 1, \ldots, i_\ell + 1\}$. We consider the unique partition $G_1, G_2, \ldots, G_r$, $r \geq 1$, of $[1, n+1] \backslash (I \cup I^+)$ such that $G_i$, $1 \leq i \leq r$, is a maximal non empty interval satisfying $\max G_i < \min G_{i+1}$ for $i \leq r - 1$. So we define the Łukasiewicz path $L$ as follows:

$$L = U_b F^{i_1 - 2} D^{a_1} \left( \prod_{j=2}^{r-1} F^{g_j} D^{a_j} \right) F^{|G_r| - 1}$$

where $b = |I \cup I^+| - 1$ is the number of down steps $D$ in $L$, $a_j = \min G_{j+1} - \max G_j - 1$ and $g_j = |G_j|$ for $1 \leq j \leq r - 1$. Less formally, we place occurrences $DD$ on positions $i_1, \ldots, i_\ell$, we start the path with $U_b$ where $b$ is the number of down steps $D$, and we place flat steps anywhere else.

For instance, if $n = 14$ and $I = \{2, 3, 7, 10, 13\}$, then we have $[1, 15] \backslash (I \cup I^+) = \{1, 5, 6, 9, 12, 15\}$ $b = 15 - 6 = 9$ and $G_1 = \{1\}$, $G_2 = \{5, 6\}$, $G_3 = \{9\}$, $G_4 = \{12\}$, $G_5 = \{15\}$, which induces $L = U_9 DDDFFDDFDDFDD$.

Therefore, any increasing sequence $2 \leq i_1 < i_2 < \ldots < i_\ell \leq n - 1$, $0 \leq \ell \leq n$ with $i_j - i_{j-1} \neq 2$ for $j \geq 2$, corresponds to the set of positions of $DD$ in a Łukasiewicz path. It is already known (see [1] for instance) that such sequences are enumerated by the general term $g_n$ of A005251 in [28] defined by $g_0 = 1$, $g_1 = 1$, $g_2 = 1$, $g_3 = 2$ and $g_n = g_{n-1} + g_{n-2} + g_{n-4}$ for $n \geq 4$. $\qquad \square$

# 4   Other patterns

In this section, we consider the equivalence relation on $\mathcal{L}$ where two paths $L$ and $L'$ belong to the same class whenever for any $k \geq 1$, the occurrence positions of $U_k$ (resp. $DU_k$, $U_kD$, $FU_k$, $U_kF$) are the same in $L$ and $L'$. Also, we study the $FF$-equivalence relation in $\mathcal{L}$. For all these cases, we use the method $(M_2)$ that consists in exhibiting subsets of representatives of equivalence classes, and determining algebraically their cardinalities.

## 4.1   Modulo the up steps $U_k$, $k \geq 1$

In this part, we assume that two Łukasiewicz paths $L$ and $L'$ of the same length are $U_k$-equivalent whenever for any $k \geq 1$, $L$ and $L'$ have the same positions of $U_k$.

Let $\mathcal{B}$ be the set of Łukasiewicz paths without any flat steps at positive height. For instance, we have $U_3DDDFUD \in \mathcal{B}$ and $U_3FDDDUD \notin \mathcal{B}$. Let $\bar{\mathcal{B}} \subset \mathcal{B}$ be the set of Łukasiewicz paths without any flat steps.

**Lemma 2** *There is a bijection between $\mathcal{B}$ and the set of $U_k$-equivalence classes of $\mathcal{L}$.*

*Proof.* Let $L$ be a non-empty Łukasiewicz path in $\mathcal{L}$. Let us prove that there exists a Łukasiewicz path $L' \in \mathcal{B}$ (with the same length as $L$) such that $L$ and $L'$ are equivalent. We write

$$L = K_0 \prod_{i=1}^{r} \alpha_i K_i$$

with $r \geq 0$, where $K_i$ is a part that does not contain any up steps for $0 \leq i \leq r$, and $\alpha_i \in \{U_k, k \geq 1\}$ for $1 \leq i \leq r$. From $L \in \mathcal{L}$, we define the Łukasiewicz path

$$L' = F^{b_0} \prod_{i=1}^{r} \alpha_i D^{c_i} F^{b_i - c_i}$$

with $b_0 = |K_0|$, and for $1 \leq i \leq r$, $b_i = |K_i|$, $c_i = \min\{b_i, a_i + \sum_{j=1}^{i-1}(a_j - c_j)\}$ where $\alpha_i = U_{a_i}$. Less formally, $K_0$ is replaced with $F^{|K_0|}$ and for $i$ from 1 to $r$, $K_i$ is replaced with $D^{c_i}F^{|K_i|-c_i}$ where $c_i$ is the maximal number of down steps that can be placed so that $L'$ remains a Łukasiewicz path. Clearly, $L'$ belongs to $\mathcal{B}$ (it does not contain any flat at positive height), and for any $k \geq 1$ the occurrence positions of $U_k$ are the same as for $L$, *i.e.*, $L' \in \mathcal{B}$ is in the same class as $L$. For instance, if $L = U_3DUDFFFFUUDDFFDDFF$, then we obtain $L' = U_3DUDDDFFUUDDFFFFFF$ (see Figure 3 for an illustration of this example).

Since the positions of the up steps $U_k$, $k \geq 1$, remain fixed inside a class, and that any flat of $L' \in \mathcal{B}$ lies necessarily on the $x$-axis, there are no other paths in $\mathcal{B}$ in the same class as $L$. The proof is completed. □

**Theorem 6** *The generating function for the set of $U_k$-equivalence classes of $\mathcal{L}$ with respect to the length is given by*
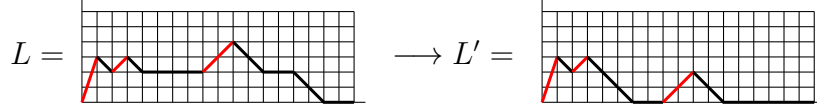
$$\frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2},$$

Figure 3: Illustration of the example described in the proof of Lemma 2.

*which generates the Motzkin numbers ([A001006](#) in [28]).*

*Proof.* Using Lemma 2, it suffices to obtain the generating function $B(x)$ for the set $\mathcal{B}$. A non-empty Łukasiewicz path $L \in \mathcal{B}$ can be written either $L = FL'$ where $L' \in \mathcal{B}$, or $L = U_k L_1 D L_2 D \ldots L_k D L'$ for $k \geq 1$ where and $L_1, L_2, \ldots, L_k \in \bar{\mathcal{B}}$ are some Łukasiewicz paths without flats, and $L' \in \mathcal{B}$. So we obtain the functional equation $B(x) = 1 + xB(x) + xB(x)\sum_{k \geq 0} x^k \bar{B}(x)^k$ where $\bar{B}(x)$ is the generating function for the set $\bar{\mathcal{B}}$ of Łukasiewicz paths without flats. Using the classical decomposition of a Łukasiewicz path, $\bar{B}(x)$ satisfies $\bar{B}(x) = 1 + \sum_{k \geq 2} x^k \bar{B}(x)^k$, or equivalently $\bar{B}(x) = \frac{1}{(1+x)(1-x\bar{B}(x))}$. A simple calculation provides the result. $\qquad\square$

Let us define recursively a map $\psi$ from $\mathcal{L}$ to the set of Motzkin paths $\mathcal{M}$ as follows:

$$\begin{cases} \psi(\epsilon) & = \epsilon, \\ \psi(FL) & = F\psi(L), \\ \psi(U_k L_1 D L_2 D \ldots L_k D L) & = U\psi(L_1)F\psi(L_2)F \ldots \psi(L_k)D\psi(L), \end{cases}$$

where $L, L_1, L_2, \ldots, L_k$ are are some Łukasiewicz paths. See Figure 4 for an illustration of the bijection $\psi$. For instance, the image by $\psi$ of $U_4FU_2DFDDDU_2U_1DDDFDDFU_2FDU_2DDD$ is $UFUFFDFFUUDFDFFDFUFFUFDD$. Obviously, the map $\psi$ preserves the length of the paths.
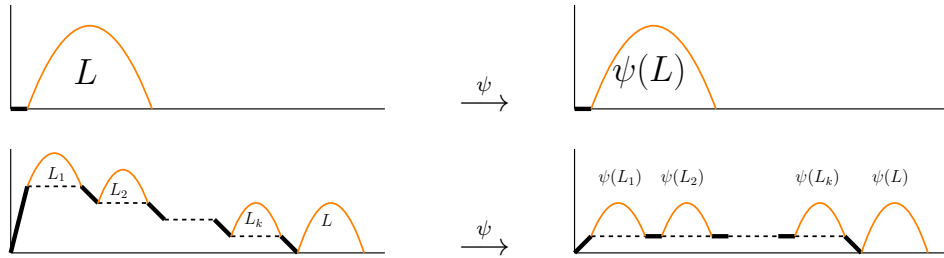


Figure 4: Illustration of the map $\psi$ from $\mathcal{L}$ to $\mathcal{M}$.

We easily deduce the two following facts.

**Fact 1** *If $L, L' \in \mathcal{L}$, then $LL' \in \mathcal{L}$ and we have $\psi(LL') = \psi(L)\psi(L')$.*

**Fact 2** *There is a one-to-one correspondence between:*

(a) *steps $\{U_k, k \geq 1\}$ in $L$ and steps $U$ in $\psi(L)$;*

(b) *$\{U_k U_\ell, k, \ell \geq 1\}$ in $L$ and $UU$ in $\psi(L)$;*

(c) *steps $F$ on the $x$-axis in $L$ and steps $F$ on the $x$-axis in $\psi(L)$;*

(d) *$\{U_k D, k \geq 2\} \cup \{U_k F, k \geq 1\}$ in $L$ and $UF$ in $\psi(L)$;*

(e) *peaks $UD$ in $L$ and peaks $UD$ in $\psi(L)$.*

**Theorem 7** *For any $n \geq 0$, the map $\psi$ induces a bijection from $\mathcal{B}_n$ to $\mathcal{M}_n$.*

*Proof.* We proceed by induction on $n$. Obviously, for $n = 0$ we have $\psi(\epsilon) = \epsilon$. We assume that $\psi$ is a bijection from $\mathcal{B}_k$ to $\mathcal{M}_k$, $0 \leq k \leq n$, and we prove the result for $n + 1$. Using the enumerating result of Theorem 6, it suffices to prove that $\psi$ is surjective. So, let $M$ be a Motzkin path in $\mathcal{M}_{n+1}$. We distinguish two cases: $(i)$ $M = FM'$ with $M' \in \mathcal{M}_n$, and $(ii)$ $M = UM'DM''$ where $M'$ and $M''$ are two Motzkin paths in $\mathcal{M}$.

$(i)$ Using the recurrence hypothesis, there is $L' \in \mathcal{B}_n$ such that $M' = \psi(L')$. So, the Łukasiewicz path $L = FL'$ lies into $\mathcal{B}_{n+1}$ and satisfies $\psi(L) = M$ which proves that $M$ belongs to the image by $\psi$ of $\mathcal{B}_{n+1}$.

$(ii)$ We suppose $M = UM'DM''$. We can uniquely write $M' = M_0 \prod_{i=1}^r FM_i$ with $r \geq 0$ and where $M_i$ is a (possibly empty) Motzkin path without flat $F$ on the $x$-axis. Using the recurrence hypothesis, there are $B_0, B_1, \ldots, B_r \in \mathcal{B}$ such that $\psi(B_i) = M_i$, $0 \leq i \leq r$. Also let $B \in \mathcal{B}$ such that $\psi(B) = M''$. Since $B_i$ (resp. $B$) belongs to $\mathcal{B}$, it does not contain any flat at positive height. Since $M_i = \psi(B_i)$, Fact 2(c) implies that $B_i$ does not contain any flat on the $x$-axis. So, $B_i$ does not contain any flat steps. So, let us define

$$L = U_{r+1} B_0 D \left( \prod_{i=1}^r B_i D \right) B.$$

Clearly, $L$ lies in $\mathcal{B}_{n+1}$ and satisfies $\psi(L) = M$; then, $M$ belongs to the image by $\psi$ of $\mathcal{B}_{n+1}$.
The map $\psi$ from $\mathcal{B}_n$ to $\mathcal{M}_n$ is a bijection. $\qquad \square$

## 4.2   Modulo $U_k D$ for $k \geq 1$, and $U_k F$ for $k \geq 1$

For a given $\alpha \in \{D, F\}$, we define the $U_k\alpha$-equivalence in $\mathcal{L}$ as follows: two Łukasiewicz paths $L$ and $L'$ of the same length are $U_k\alpha$-equivalent whenever for any $k \geq 1$, $L$ and $L'$ have the same positions of $U_k\alpha$.

Let $\mathcal{C} \subset \mathcal{B}$ be the set of Łukasiewicz paths without any flat steps at positive height and such that any up step $U_k$, $k \geq 1$, is immediately followed by a down step $D$. For instance, we have $U_3DDDFUD \in \mathcal{C}$ and $U_3FDDDUD \notin \mathcal{C}$. Let $\bar{\mathcal{C}} \subset \mathcal{C}$ be the set of Łukasiewicz paths without flats in $\mathcal{C}$.

**Lemma 3** *There is a bijection between $\mathcal{C}$ and the set of $U_kD$-equivalence classes of $\mathcal{L}$.*

*Proof.* Let $L$ be a non-empty Łukasiewicz path in $\mathcal{L}$. Let us prove that there exists a Łukasiewicz path $L' \in \mathcal{C}$ (with the same length as $L$) such that $L$ and $L'$ belong to the same class. We write

$$L = K_0 \prod_{i=1}^{r} \left( U_{k_i} D K_i \right),$$

where $k_i \geq 1$ for $1 \leq i \leq r$, and $K_0, K_1, K_2, \ldots, K_r$, $r \geq 0$, are some parts (possibly empty) without pattern $U_kD$ for any $k \geq 1$.

We define the Łukasiewicz path

$$L' = F^{b_0} \prod_{i=1}^{r} \left( U_{k_1} D D^{a_i} F^{b_i - a_i} \right),$$

with $b_i = |K_i|$, $0 \leq i \leq r$, and for $1 \leq i \leq r$, $a_i = \min\{b_i, k_i - 1 + \sum_{j=1}^{i-1}(k_j - 1 - a_j)\}$. Less formally, $K_0$ is replaced with $F^{|K_0|}$, and $K_i$ is replaced with $D^{a_i} F^{b_i - a_i}$ where the value $a_i$, $1 \leq i \leq r$, is the maximal number of down steps that can be placed between the two occurrences $U_{k_i}D$ and $U_{k_{i+1}}D$ so that $L'$ remains a lattice path. Clearly, $L' \in \mathcal{C}$ and $L'$ belongs to the same class as $L$.

For instance, from $L = U_3DUDFFFFUUDDFFDDFF$, we obtain the path $L' = U_3DUDDDFFFUDFFFFFFFF$ (see Figure 5 for an illustration of this example).

Now we will prove that any $U_kD$-equivalence class contains at most one element in $\mathcal{C}$. For a contradiction, let $L$ and $L'$ be two different Łukasiewicz paths in $\mathcal{C}$ belonging to the same class. We write $L = QR$ and $L' = QS$ where $R$ and $S$ start with two different steps. Since $L$ and $L'$ lie in the same class, the two first steps of $R$ and $S$ cannot be $U_kD$ for $k \geq 1$. Moreover, since $L$ (resp. $L'$) lies into $\mathcal{C}$, the two first steps of $R$ (resp. $S$) cannot constituted a pattern $U_kF$ for $k \geq 1$. Then, $R$ and $S$ cannot start with any up step $U_k$, $k \geq 1$.

Without loss of generality, let us assume that the first step of $R$ is a down step $D$ and then, the first step of $S$ is a flat step $F$. This means that the last point of $Q$ has its ordinate equal to zero (otherwise $L'$ could not belong to $\mathcal{C}$). As the first step of $R$ is $D$, the height of this step is $-1$ which gives a contradiction and completes the proof. $\qquad\square$
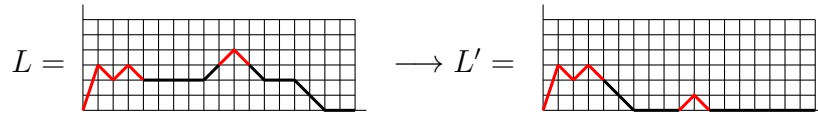


$$L = \quad \longrightarrow L' =$$

Figure 5: Illustration of the example described in the proof of Lemma 3.

**Theorem 8** *The generating function for the set of $U_kD$-equivalence classes of $\mathcal{L}$ with respect to the length is given by*

$$\frac{1 - x + x^2 + \sqrt{1 - 2x - x^2 - 2x^3 + x^4}}{1 - 2x - x^3 + (1 - x)\sqrt{1 - 2x - x^2 - 2x^3 + x^4}},$$

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which generates the generalized Catalan sequence defined by $g_0 = 1$, and $g_{n+1} = g_n + \sum_{k=1}^{n-1} g_k g_{n-1-k}$ for $n \geq 0$ (see A004148 in [28]).

*Proof.* Using Lemma 3, it suffices to obtain the generating function $C(x)$ for the set $\mathcal{C}$. A non-empty Łukasiewicz path $L \in \mathcal{C}$ can be written either $L = FL'$ where $L' \in \mathcal{C}$, or $L = U_k DL_1 DL_2 D \ldots L_{k-1} DL'$ for $k \geq 1$ where and $L_1, L_2, \ldots, L_{k-1}$ are some Łukasiewicz paths without flats in $\bar{\mathcal{C}}$, and $L' \in \mathcal{C}$. So we obtain the functional equation $C(x) = 1 + xC(x) + x^2 C(x) \sum_{k \geq 0} x^k \bar{C}(x)^k$ where $\bar{C}(x)$ is the generating function for the set $\bar{\mathcal{C}}$ of Łukasiewicz paths without flats in $\mathcal{C}$. Using the classical, decomposition of a Łukasiewicz path, we have $\bar{C}(x) = 1 + x \sum_{k \geq 1} x^k \bar{C}(x)^k$. A simple calculation provides the result. $\square$

**Theorem 9** *For any $n \geq 0$, the map $\psi$ induces a bijection from $\mathcal{C}_n$ to the set of Motzkin paths in $\mathcal{M}_n$ that avoid the pattern $UU$.*

*Proof.* Theorem 7 ensures that $\psi$ is a bijection from $\mathcal{B}_n$ to $\mathcal{M}_n$ for $n \geq 0$. We have $\mathcal{C} \subset \mathcal{B}$, and the paths in $\mathcal{C}$ are those in $\mathcal{B}$ that avoid the patterns $U_k U_\ell$, $k, \ell \geq 1$ and $U_k F$, $k \geq 1$. Using Fact 2(b,d,e), the map $\psi$ transforms occurrences of $U_k U_\ell$, $k, \ell \geq 1$, into occurrences of $UU$, occurrences $U_k D$, $k \geq 2$, and $U_k F$, $k \geq 1$ into occurrences of $UF$, and occurrences of $UD$ into occurrences of $UD$. Then, the image by $\psi$ of $\mathcal{C}_n$ is the subset of Motzkin paths in $\mathcal{M}_n$ that does not contain any pattern $UU$. $\square$

Let $\mathcal{E} \subset \mathcal{L}$ be the set of Łukasiewicz paths such that any up step $U_k$, $k \geq 1$, is immediately followed by a flat step $F$, and any flat step $F$ of positive height belongs to a pattern $U_k F$, $k \geq 1$. For instance, we have $U_3 DDDFUFD \notin \mathcal{E}$ and $U_3 FDDFDUD \in \mathcal{E}$. Let $\bar{\mathcal{E}} \subset \mathcal{E}$ be the set of Łukasiewicz paths in $\mathcal{E}$ without flat step on the $x$-axis.

**Lemma 4** *There is a bijection between $\mathcal{E}$ and the set of $U_k F$-equivalence classes of $\mathcal{L}$.*

*Proof.* The proof is obtained *mutatis mutandis* as for Lemma 3 by replacing $U_k D$ with $U_k F$. Let $L$ be a non-empty Łukasiewicz path in $\mathcal{L}$. Let us prove that there exists a Łukasiewicz path $L' \in \mathcal{E}$ (with the same length as $L$) such that $L$ and $L'$ belong to the same class.

We write

$$L = K_0 \prod_{i=1}^r \left( U_{k_i} F K_i \right),$$

where $k_i \geq 1$, $1 \leq i \leq r$, and $K_0, K_1, K_2, \ldots, K_r$, $r \geq 0$, are some parts (possibly empty) without pattern $U_k F$ for any $k \geq 1$.

We define the Łukasiewicz path

$$L' = F^{b_0} \prod_{i=1}^r \left( U_{k_1} F D^{a_i} F^{b_i - a_i} \right),$$

with $b_i = |K_i|$, $0 \leq i \leq r$, and for $1 \leq i \leq r$, $a_i = \min\{b_i, k_i + \sum_{j=1}^{i-1}(k_j - a_j)\}$. Less formally, $K_0$ is replaced with $F^{|K_0|}$, and $K_i$ is replaced with $D^{a_i} F^{b_i - a_i}$ where the value $a_i$, $1 \leq i \leq r$,

is the maximal number of down steps that can be placed between the two occurrences $U_{k_i}F$ and $U_{k_{i+1}}F$ so that $L'$ remains a lattice path. Clearly, $L' \in \mathcal{E}$ and $L'$ belongs to the same class as $L$.

Now we will prove that any $U_kF$-equivalence class contains at most one element in $\mathcal{E}$. For a contradiction, let $L$ and $L'$ be two different Łukasiewicz paths in $\mathcal{E}$ belonging to the same class. We write $L = QR$ and $L' = QS$ where $R$ and $S$ start with two different steps. Since $L$ and $L'$ lie in the same class, the two first steps of $R$ and $S$ cannot be $U_kF$ for $k \geq 1$. So, since $L$ (resp. $L'$) lies into $\mathcal{E}$, any up step is followed by a flat step, which means that the first step of $R$ (resp. $S$) cannot be $U_k$, $k \geq 1$. Then, $R$ and $S$ cannot start with any up step $U_k$, $k \geq 1$.

Without loss of generality, let us assume that the first step of $R$ is a down step $D$ and then, the first step of $S$ is a flat step $F$. Note that the first step of $S$ is necessarily on the $x$-axis. This means that the last point of $Q$ has its ordinate equal to zero (otherwise $L'$ could not belong to $\mathcal{E}$). As the first step of $R$ is $D$, the height of this step is $-1$ which gives a contradiction and completes the proof. $\qquad\square$

**Theorem 10** *The generating function for the set of $U_kF$-equivalence classes of $\mathcal{L}$ with respect to the length is given by*

$$\frac{1 - x + 2x^2 + \sqrt{1 - 2x + x^2 - 4x^3}}{1 - 2x - 3x^3 + (1 - x + x^2)\sqrt{1 - 2x + x^2 - 4x^3}},$$

*which generates the generalized Catalan sequence defined by $h_0 = 1$, and for $n \geq 0$, $h_{n+1} = h_n + \sum_{k=0}^{n-2} h_k h_{n-2-k}$ (see A023431 in [28]).*

*Proof.* Using Lemma 4, it suffices to obtain the generating function $E(x)$ for the set $\mathcal{E}$. A non-empty Łukasiewicz path $L \in \mathcal{E}$ can be written either $L = FL'$ where $L' \in \mathcal{E}$, or $L = U_kFL_1DL_2D\ldots L_kDL'$ for $k \geq 1$ where and $L_1, L_2, \ldots, L_k$ are some Łukasiewicz paths in $\bar{\mathcal{E}}$, and $L' \in \mathcal{E}$. So we obtain the functional equation $E(x) = 1 + xE(x) + x^3E(x)\sum_{k\geq 0} x^k \bar{E}(x)^{k+1}$ where $\bar{E}(x)$ is the generating function for the set $\bar{\mathcal{E}}$. Using the classical decomposition of a Łukasiewicz path, we have $\bar{E}(x) = 1 + \sum_{k\geq 3} x^k \bar{E}(x)^{k-1}$. A simple calculation provides the result. $\qquad\square$

**Theorem 11** *For $n \geq 0$, the map $\psi$ is a bijection from $\mathcal{E}_n$ to the subset $\mathcal{M}'_n$ of Motzkin paths in $\mathcal{M}_n$ that avoid $UU$ and $UD$.*

*Proof.* We proceed by induction on $n$. Obviously, for $n = 0$, we have $\psi(\epsilon) = \epsilon$. For $0 \leq k \leq n$, we assume that $\psi$ is a bijection from $\mathcal{E}_k$ to the subset $\mathcal{M}'_k$ and we prove the result for $n + 1$. Since the set of length $n$ Motzkin paths avoiding $UU$ and $UD$ is enumerated by the value $h_n$ defined in Theorem 10 (see A023431 in [28]), it suffices to prove that $\psi$ is surjective. So, let $M$ be a Motzkin path in $\mathcal{M}'_{n+1}$. We distinguish two cases: (i) $M = FM'$ with $M' \in \mathcal{M}'_n$, and (ii) $M = UM'DM''$ where $M$ and $M'$ are two Motzkin paths in $\mathcal{M}'$.

($i$) Using the recurrence hypothesis, there is $L' \in \mathcal{E}_n$ such that $M' = \psi(L')$. So, the Łukasiewicz path $L = FL' \in \mathcal{E}_{n+1}$ satisfies $\psi(L) = M$ which proves that $M$ belongs to the image by $\psi$ of $\mathcal{E}_{n+1}$.

($ii$) We suppose $M = UM'DM''$ with $M', M'' \in \mathcal{M}'$. Since $M \in \mathcal{M}'_{n+1}$, we have $M' \neq \epsilon$ and $M'$ does not start with $U$, which implies that $M'$ starts with $F$. Using the recurrence hypothesis, there are $L' \in \mathcal{E}$ and $L'' \in \mathcal{E}$ such that $\psi(L') = M'$ and $\psi(L'') = M''$. Since $M'$ starts with a flat step, $L'$ also starts with a flat step. So, $L = UL'DL''$ belongs to $\mathcal{E}_{n+1}$ and satisfies $\psi(L) = M$ which proves that $\psi$ from $\mathcal{B}_n$ to $\mathcal{M}_n$ is bijective. $\qquad\square$

## 4.3   Modulo $FU_k$ for $k \geq 1$, and $DU_k$ for $k \geq 1$

For a given $\alpha \in \{D, F\}$, the $\alpha U_k$-equivalence in $\mathcal{L}$ is defined as follows: two Łukasiewicz paths $L$ and $L'$ of the same length are $\alpha U_k$-equivalent whenever for any $k \geq 1$, $L$ and $L'$ have the same positions of $\alpha U_k$.

Let $\xi$ be the map from $\mathcal{L}$ to himself defined by $\xi(L)$ is obtained from $L$ by replacing any occurrence $U_k F$ by an occurrence $FU_k$ for $k \geq 1$. It is straightforward to verify that $\xi$ induces a bijection $\bar{\xi}$ between the set of $U_k F$-equivalence classes and the set of $FU_k$-equivalence classes. Then, Theorem 12 is directly deduced from Theorem 10.

**Theorem 12** *The generating function for the set of $FU_k$-equivalence classes of $\mathcal{L}$ with respect to the length also is the generating function given in Theorem 10.*

Let $\mathcal{L}'_n$, $n \geq 2$, be the set of Łukasiewicz paths of length $n$ starting by $U$ and ending by $D$. For $n \geq 0$, we define the bijection $\theta$ from $\mathcal{L}_n$ to $\mathcal{L}'_{n+2}$ as follows: $\theta(L)$ is obtained from $L$ by replacing any occurrence $U_k D$ by an occurrence $DU_k$ for $k \geq 1$, and by adding a step $U$ at the beginning and a step $D$ at the ending. It is straightforward to verify that $\theta$ induces a bijection $\bar{\theta}$ between the set of $U_k D$-equivalence classes of $\mathcal{L}_n$ and the set of $DU_k$-equivalence classes of $\mathcal{L}'_{n+2}$. Then, Theorem 13 is deduced from Theorem 8.

**Theorem 13** *The generating function for the set of $DU_k$-equivalence classes of $\mathcal{L}$ with respect to the length is given by*

$$\frac{1 - x - x^2 - 2x^3 + \sqrt{1 - 2x - x^2 - 2x^3 + x^4}}{1 - 2x - x^3 + (1 - x)\sqrt{1 - 2x - x^2 - 2x^3 + x^4}},$$

*which generates the generalized Catalan sequence defined by $u_0 = u_1 = u_2 = 1$, and for $n \geq 3$ $u_n = g_{n-2}$ where $g_n$ is defined in Theorem 8. (see A004148 in [28]).*

## 4.4   Modulo $FF$

Let $\mathcal{F}$ be the set constituted of the union of $\{\epsilon, F\}$ with the set of Łukasiewicz paths containing at most one up step $U_k$, $k \geq 1$ and such that any flat step $F$ is contained into a pattern $FF$. For instance, $FFU_3DFFDDFFDFF \in \mathcal{F}$ and $FFU_3FFDDFDFFF \notin \mathcal{F}$.

**Lemma 5** *There is a bijection between $\mathcal{F}$ and the set of $FF$-equivalence classes of $\mathcal{L}$.*

*Proof.* Let $L$ be a non-empty Łukasiewicz path in $\mathcal{L}$. Let us prove that there exists a Łukasiewicz path $L' \in \mathcal{F}$ (with the same length as $L$) such that $L$ and $L'$ belong to the same class. We write

$$L = K_1 F^{a_1} K_2 F^{a_2} K_3 \ldots K_r F^{a_r} K_{r+1},$$

with $r \geq 0$ and $a_i \geq 2$ for $1 \leq i \leq r$, such that $K_1, K_2, \ldots, K_r, K_{r+1}$ are some parts without pattern $FF$, $k \geq 1$, and $K_2, \ldots, K_r$ are not empty and do not have any $F$ in first and last position, and $K_1$ has no flat in last position, and $K_{r+1}$ has no flat in first position.

If $L = F^n$ with $n \geq 0$, then its equivalence class is reduced to a singleton. Now let us assume that $L \neq F^n$. We distinguish two cases: (1) $K_1$ is not empty, and (2) $K_1$ is empty which means that $L = F^{a_1} K_2 F^{a_2} K_3 \ldots K_r F^{a_r} K_{r+1}$.

In case (1), we define the Łukasiewicz path

$$L' = U_b D^{b_1 - 1} F^{a_1} D^{b_2} F^{a_2} D^{b_3} \ldots D^{b_r} F^{a_r} D^{b_{r+1}}$$

with $b_i = |K_i|$, $1 \leq i \leq r+1$, and $b = b_1 - 1 + \sum_{i=2}^{r+1} b_i$.

In case (2), we define the Łukasiewicz path

$$L' = F^{a_1} U_b D^{b_2 - 1} F^{a_2} D^{b_3} \ldots D^{b_r} F^{a_r} D^{b_{r+1}}$$

with $b_i = |K_i|$, $2 \leq i \leq r+1$, and $b = b_2 - 1 + \sum_{i=3}^{r+1} b_i$.

Less formally, we obtained $L'$ from $L$ by replacing any $K_i$ (excepted the first) with a run of down steps $D^{|K_i|}$, and by replacing the first $K_i$ ($K_1$ or $K_2$ according to the case (1) or (2)) with $U_b D^{|K_1|-1}$ (or $U_b D^{|K_2|-1}$) where the up step $U_b$ balances all down steps in $L'$, *i.e.*, $b$ is the number of down steps in $L'$. Clearly, $L' \in \mathcal{F}$ and $L'$ belongs to the same class as $L$. For instance, if $L = U_2 DFFU_2 DDFFU_3 DDFFDDFF$, then $L' = U_9 DFFDDDFFDDDFFDDFF$ (see Figure 6).

The definition of $\mathcal{L}$ implies that there is only one path of $\mathcal{F}$ in the same class as $L$, which completes the proof. $\square$
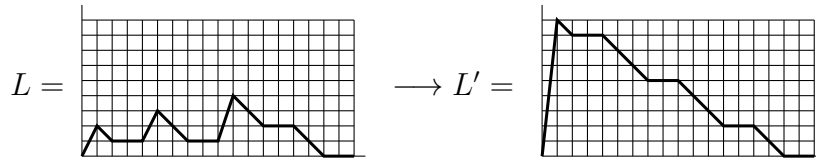


Figure 6: Illustration of the example described in the proof of Lemma 5.

**Theorem 14** *The generating function for the set of $FF$-equivalence classes of $\mathcal{L}$ with respect to the length is given by*

$$\frac{1 - 3x + 4x^2 - 5x^3 + 7x^4 - 7x^5 + 6x^6 - 3x^7 + x^8}{(1 - 2x + x^2 - x^3)(1 - x)^2}.$$

*(Note that the associated sequence does not yet appear in [28]).*

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*Proof.* Using Lemma 5, it suffices to obtain the generating function $F(x)$ for the set $\mathcal{F}$. A non-empty Łukasiewicz path $L \in \mathcal{F}$ can be written either $(i)$ $L = F^k$ for $k \geq 0$, or $(ii)$ $L = F^{i_0} U_k F^{i_1} D^{j_1} F^{i_2} D^{j_2} \ldots F^{i_\ell} D^{j_\ell} F^{i_{\ell+1}}$ with $\ell \geq 1$, $i_0 = 0$ or $i_0 \geq 2$, $i_1 = 0$ or $i_1 \geq 2$, $i_{\ell+1} = 0$ or $i_{\ell+1} \geq 2$, $i_m \geq 2$ for $2 \leq m \leq \ell$, and $j_m \geq 1$ for $1 \leq m \leq \ell$.

The generating function for the Łukasiewicz paths satisfying $(i)$ is given by $\frac{1}{1-x}$.

For Łukasiewicz paths satisfying $(ii)$, we give the generating function for each part of $L$, and we multiply them:

- For $F^{i_0}$, with $i_0 = 0$ or $i_0 \geq 2$, the generating function is $1 + \frac{x^2}{1-x}$;
- For $F^{i_{\ell+1}}$, with $i_{\ell+1} = 0$ or $i_{\ell+1} \geq 2$, the generating function is $1 + \frac{x^2}{1-x}$;
- For $U_k F^{i_1} D^{j_1}$, with $i_1 = 0$ or $i_1 \geq 2$ and $j_1 \geq 1$, the generating function is $x(1 + \frac{x^2}{1-x})\frac{x}{1-x}$;
- For $F^{i_2} D^{j_2} \ldots F^{i_\ell} D^{j_\ell}$, with $i_m \geq 2$, and $j_m \geq 1$, the generating function is $\frac{1}{1 - \frac{x^3}{(1-x)^2}}$.

Considering all these cases, we deduce:

$$F(x) = \left(1 + \frac{x^2}{1-x}\right)^3 x^2 (1-x)^{-1} \left(1 - \frac{x^3}{(1-x)^2}\right)^{-1} + (1-x)^{-1}$$

which completes the proof. $\square$

# 5    Concluding remarks

Extending recent works on Dyck and Motzkin paths [2, 3], the goal of this paper is to calculate the number of Łukasiewicz paths modulo the positions of a given pattern, *i.e.* the number of possible sets $I = \{i_1, i_2, \ldots, i_k\}$ where $i_1, i_2, \ldots, i_k$ are the occurrence positions of the pattern in Łukasiewicz paths. Can one do the same study for other lattice paths such as meanders, bridges and excursions, or Schroeder and Riordan paths?

For a pattern $\alpha \in \{F, D, FD, DF, DD\}$, we have characterized the possible sets $I$ of positions of $\alpha$ in a Łukasiewicz path. More generally, is it possible to characterize these sets for other patterns? From our study, we can deduce a lower bound for the maximal cardinality of a class by calculating the average of cardinalities of the classes, *i.e.*, the total number of Łukasiewicz paths divided by the number of classes. Is it possible to calculate the exact value of the maximal cardinality for a class, and for which set $I$ it is reached? Also, it would be interesting to study some properties of the number of Łukasiewicz paths (of a given length) having $I$ as set of positions of the pattern. One can think this number is a polynomial with respect to the length $n$. If this is true, then we could give properties of these coefficients and roots, which would be a counterpart for lattice paths of the study of descent polynomial on the symmetric group $S_n$ (see MacMahon [12]).

# References

[1] R. Austin, R.K. Guy. Binary sequences without isolated ones. *Fib. Quart.*, 16(1978), 84-86.

[2] J.-L. Baril, A. Petrossian. Equivalence of Dyck paths modulo some statistics. *Discrete Math.*, 338(2015), 655-660.

[3] J.-L. Baril, A. Petrossian. Equivalence classes of Motzkin paths modulo a pattern of length at most two. *Journal of Integer Sequences*, 18(2015), Article1199-1205.

[4] E. Barcucci, R. Pinzani, R. Sprugnoli. The Motzkin family. *Pure Math. Appl. Ser. A*, 2(34)(1991), 249-279.

[5] C. Brennan, S. Mavhungu. Peaks and valleys in Motzkin paths. *Quaestiones Mathematicae*, 33(2)(2010), 171-188.

[6] E. Deutsch. Dyck path enumeration. *Discrete Math.*, 204(1999), 167-202.

[7] R. Donaghey, L.W. Shapiro. Motzkin numbers. *J. Combin. Theory Ser. A*, 23(1977), 291-301.

[8] D. Drake, R. Gantner. Generating functions for plateaus in Motzkin paths. *J. Chungcheong Math. Society*, 25(2012), 475-489.

[9] P. Flajolet, R. Sedgewick. Analytic Combinatorics. *Cambridge University Press*, 2009.

[10] I.M. Gessel, S. Ree. Lattice paths and Faber polynomials. *Advances in Combinatorial Methods and Applications to Probability and Statistics*, Birkhauser Verlag, Boston, 1997.

[11] M. Lothaire. Combinatorics on Words. *Encyclopedia of Mathematics and Its Applications*, Vol. 17, Addison-Wesley, Reading, Massachusetts, 1983, Chapter 11, p. 219.

[12] P.A. MacMahon. Combinatory analysis, Vol.I,II. Dover Phoenix Editions, Dover Publications Inc., Mineola, NY, 2004.

[13] K. Manes, A. Sapounakis, I. Tasoulas, and P. Tsikouras. Equivalence classes of ballot paths modulo strings of length 2 and 3. *Discrete Mathematics*, 339(10)(2016), 2557-2572.

[14] T. Mansour. Statistics on Dyck paths. *J. Integer Sequences*, 9(2006), 06.1.5.

[15] T. Mansour. Counting peaks at height $k$ in a Dyck path. *J. Integer Sequences*, 5(2002), 02.1.1.

[16] T. Mansour, M. Schork, Y. Sun. Motzkin numbers of higher rank: Generating function and explicit expression. *J. Integer Sequences*, 10(2007), 07.7.4.

[17] D. Merlini, R. Sprugnoli, M.C. Verri. Some statistics on Dyck paths. *J. Statis. Plann. Inference*, 101(2002), 211-227.

[18] S.G. Mohanty. Lattice Path Counting and Applications. *Academic Press*, New York, 1979.

[19] T.V. Narayana. Lattice Path Combinatorics With Statistical Applications. *Math. Expositions*, 23, Univ. of Toronto Press, Toronto, 1979.

[20] A. Panayotopoulos, A. Sapounakis. On the prime decomposition of Dyck paths. *J. Combin. Math. Combin. Comput.*, 40(2002), 33-39.

[21] P. Peart, W.J. Woan. Dyck paths with no peaks at height $k$. *J. Integer Sequences*, 4(2001), 01.1.3.

[22] H. Prodinger, S. Wagner. Minimal and maximal plateau lengths in Motzkin paths. *DMTCS Proceedings*, AH(2007), 353-362.

[23] G.N. Raney. Functional composition patterns and power series reversion. *Trans. Amer. Math. Soc.*, 94(1960), 411-451.

[24] A. Sapounakis, I. Tasoulas, P. Tsikouras. Counting strings in Dyck paths. *Discrete Math.*, 307(23)(2007), 2909-2924.

[25] A. Sapounakis, I. Tasoulas, P. Tsikouras. Some strings in Dyck paths. *Australasian J. Combin.*, 39(2007), 49-72.

[26] A. Sapounakis, P. Tsikouras. Counting peaks and valleys in $k$-colored Motzkin words. *Electron. J. Comb.*, 12(2005), Article 12.

[27] M.P. Schützenberger. Le théorème de Lagrange selon G.N. Raney. *Séminaire IRIA, Rocquencourt*, (1971), 199-205.

[28] N.J.A. Sloane: The On-line Encyclopedia of Integer Sequences, available electronically at http://oeis.org.

[29] Y. Sun. The statistic "number of udu's" in Dyck paths. *Discrete Math.*, 287(2004), 177-186.

[30] R.P. Stanley. Enumerative Combinatorics, Vol. 2, Cambridge University Press, 1999.

[31] S. Vajda. Fibonacci and Lucas Numbers, and the Golden Section, Theory and Applications, Ellis Horwood Ltd., Chichester, 1989.

[32] A. Varvak. Lattice path encodings in a combinatorial proof of a differential identity. *Discrete Math.*, 308(2008), 5834-5840.

[33] G. Viennot. Une théorie combinatoire des polynômes orthogonaux généraux. *Notes of lectures given at University of Quebec in Montreal*, 1983.