

Counting with Borel's Triangle

Yue Cai* and Catherine Yan†

Department of Mathematics, Texas A&M University, College Station, TX 77843

Abstract

Borel's triangle is an array of integers closely related to the classical Catalan numbers. In this paper we study combinatorial statistics counted by Borel's triangle. We present various combinatorial interpretations of Borel's triangle in terms of lattice paths, binary trees, and pattern avoiding permutations and matchings, and derive a functional equation that is useful in analyzing the involved structures.

Keywords. Catalan's triangle, Borel's triangle, marked Dyck paths, marked binary trees
MSC. 05A05, 05A19, 05A15

1 Introduction

In combinatorics, Catalan's triangle is a triangular array whose right boundary, as well as the row-sum, are the classical Catalan sequence. Entries in Catalan's triangle appear in many combinatorial structures, notably in lattice paths, plane trees and binary trees, triangulations, and parking functions. See the references at the On-line Encyclopedia of Integer Sequences (OEIS) [13] for the sequences A009766 and A033184, which are Catalan's triangle and its transpose, respectively.

Recently another triangular array that is closely related to Catalan's triangle has appeared in various studies in commutative algebra, combinatorics, and discrete geometry. It is the sequence A234950 in OEIS and is called *Borel's triangle*, which is related to pseudo-triangulations of point sets [1] and the Betti numbers of certain principal Borel ideals [9], and appears in Cambrian Hopf algebras [6], quantum physics [12], and permutation patterns [14]. In the second author's work of parking functions and parking distributions on trees, Borel's triangle gives the coefficients of certain generating functions on the nondecreasing parking functions [5, Section 3], which inspires the project on finding classes of objects that are counted by Borel's triangle and characterizing their combinatorial structures.

In this paper we study Borel's triangle from combinatorial and enumerative contexts. We start by recalling the preliminary results of Catalan's and Borel's triangles in Section 2. Then we describe combinatorial interpretations of Borel's triangle in terms of marked Catalan structures, vertex-marked binary trees, and combinatorial statistics of matchings avoiding certain pairs of patterns. This is the content of Section 3. In Section 4 we present a bijection between restricted Dumont permutations and marked Dyck paths; the latter is a typical structure enumerated by Borel's triangle. This bijection leads to a functional equation for Borel's triangle, which is used in the last section to analyze other structures counted by Borel's triangle.

*ycai@math.tamu.edu

†cyan@math.tamu.edu

2 Preliminary Results on Catalan's and Borel's Triangles

2.1 Catalan's Triangle

Catalan's triangle $\{C_{n,k} : 0 \leq k \leq n\}$ is the array defined by the recurrence $C_{n,k} = C_{n-1,k} + C_{n,k-1}$ for $0 < k < n$ and the boundary conditions $C_{n,0} = 1$ and $C_{n,n} = C_{n,n-1}$. The first seven rows of Catalan's triangle are given below.

$n \setminus k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	2				
3	1	3	5	5			
4	1	4	9	14	14		
5	1	5	14	28	42	42	
6	1	6	20	48	90	132	132

The entries in Catalan's triangle are often called ballot numbers, since $C_{n,k}$ counts the number of lattice paths in the coordinate plane from $(0,0)$ to (n,k) that do not go above the line $y = x$. Explicitly,

$$C_{n,k} = \frac{n-k+1}{n+1} \binom{n+k}{n}.$$

In particular, $C_{n,n} = \frac{1}{n+1} \binom{2n}{n}$ is the n -th Catalan number. The sum of entries in the n -th row is the $(n+1)$ -st Catalan number. The bivariate generating function $\mathcal{C}(t,x) = \sum_{n,k} C_{n,k} t^k x^n$ can be expressed as

$$\mathcal{C}(t,x) = \frac{C(tx)}{1-xC(tx)}, \tag{1}$$

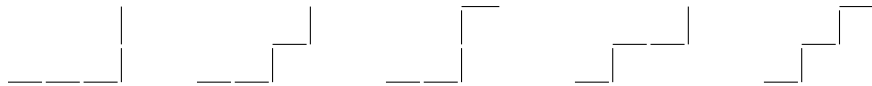
where $C(x)$ is the generating function for Catalan numbers, i.e.,

$$C(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n = \frac{1 - \sqrt{1-4x}}{2x}. \tag{2}$$

Catalan's triangle appears in countless places throughout enumerative combinatorics. A few examples are given in Theorem 1, which can be found directly or derived from the examples in OEIS A009766. As usual a *Dyck path of semi-length n* is a lattice path in the coordinate plane from $(0,0)$ to $(2n,0)$ consisting of n up-steps (along the vector $(1,1)$) and n down-steps (along the vector $(1,-1)$) such that the path never goes below the x -axis. A *parking function of length n* is a sequence of positive integers (a_1, \dots, a_n) such that $a_i \leq \sigma_i$ for some permutation $\sigma \in \mathfrak{S}_n$.

Theorem 1. *The entry $C_{n,k}$ of Catalan's triangle counts the following sets. The configurations with $n = 3$ and $k = 2$ are listed.*

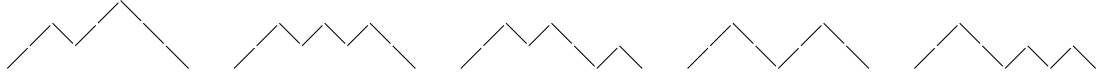
- (1) *Lattice paths in the coordinate plane from $(0,0)$ to (n,k) that never go above the line $y = x$.*



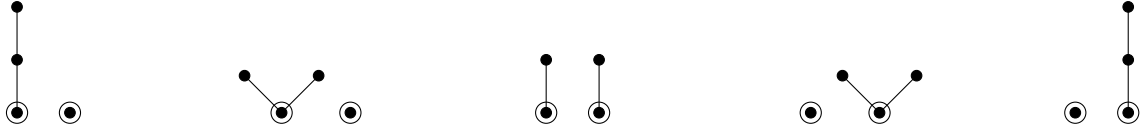
- (2) *Dyck paths of semi-length $n+1$ that have k up-steps (or down-steps) not at ground level. Equivalently, it is the set of Dyck paths of semi-length $n+1$ with $n+1-k$ returns to the x -axis, (not counting the starting point $(0,0)$.)*



(3) Dyck paths of semi-length $n + 1$ and having the first (or the last) peak at height $n - k + 1$.



(4) Unlabeled plane forests on $n + 1$ vertices such that there are $n + 1 - k$ planted plane trees.



(5) Nondecreasing parking functions of length $n + 1$ with maximal element $k + 1$.

1113 1123 1133 1223 1233

(6) Nondecreasing parking functions of length $n + 1$ with k unlucky “cars”, that is, the entries a_i such that $a_i \neq i$.¹

1114 1124 1133 1222 1223

(7) Nondecreasing parking functions of length $n + 1$ containing $n - k + 1$ ones.

1122 1123 1124 1133 1134

2.2 Borel’s Triangle

Borel’s triangle $\{B_{n,k} : 0 \leq k \leq n\}$ is an array of numbers obtained from an invertible transformation to Catalan’s triangle by the equation

$$B_{n,k} = \sum_{s=k}^n \binom{s}{k} C_{n,s}. \quad (3)$$

Equivalently,

$$\sum_{k=0}^n B_{n,k} t^k = \sum_{k=0}^n C_{n,k} (1+t)^k,$$

and the bivariate generating function $\mathcal{B}(t, x) := \sum_{n,k \geq 0} B_{n,k} t^k x^n$ satisfies

$$\mathcal{B}(t, x) = \mathcal{C}(1+t, x) = \frac{\mathcal{C}((1+t)x)}{1 - x\mathcal{C}((1+t)x)}, \quad (4)$$

where $\mathcal{C}(x)$ is the Catalan generating function given in (2). A variation of Borel’s triangle, the OEIS sequence A094385, appeared in Barry’s study of generalized Pascal matrices defined by Riordan’s arrays [2]. Barry gave an explicit expression of $B_{n,k}$ as

$$B_{n,k} = \frac{1}{n+1} \binom{2n+2}{n-k} \binom{n+k}{n}. \quad (5)$$

The entries of $B_{n,k}$ for small values of n and k are listed below.

¹In an ordinary parking function (a_1, \dots, a_n) defined via a parking process [16], a car C_i is *lucky* if C_i is parked at its preferred space. For nondecreasing parking functions, the car C_i is lucky if and only if $a_i = i$.

$n \setminus k$	0	1	2	3	4	5	6
0	1						
1	2	1					
2	5	6	2				
3	14	28	20	5			
4	42	120	135	70	14		
5	132	495	770	616	252	42	
6	429	2002	4004	4368	2730	924	132

Computing the sum of the entries in each row of Borel’s triangle and adding an extra 1 at the beginning, we obtain the sequence 1, 1, 3, 13, 67, 381, 2307, 14589, This sequence is called the *generalized Catalan Number* and is denoted by $C(2, n)$ in OEIS (A064062). Explicitly, $C(2, n)$ is the sum of the entries in the $(n - 1)$ -st row of Borel’s triangle, i.e.,

$$C(2, i) = \sum_k B_{i-1, k} = \sum_k C_{i-1, k} 2^k, \quad (6)$$

and

$$\sum_{i \geq 0} C(2, i) x^i = \frac{1 + 2xC(2x)}{1 + x} = \frac{1}{1 - xC(2x)} = \frac{4}{3 + \sqrt{1 - 8x}}. \quad (7)$$

3 Combinatorial Interpretations of Borel’s Triangle

In this section we present various interpretations of Borel’s triangle and the generalized Catalan numbers in basic combinatorial structures, in particular, in Catalan structures, binary trees, and permutations.

3.1 Marked Catalan Structures

First we note that using the defining relation (3) , we can easily get a family of combinatorial interpretations of $B_{n, k}$ from those of $C_{n, k}$. Assume that we have a discrete structure P with a combinatorial statistic $\alpha : P \rightarrow \mathbb{N}$ counting some special features of P . If $C_{n, k}$ counts the set of such structures with parameter n (usually indicates the size) for which $\alpha(P) = k$, then the generalized Catalan number $C(2, n)$ counts the set of pairs $\{(P, S)\}$ in which P is a structures with parameter n and S is a subset of the special features of P , and $B_{n, k}$ counts those pairs $\{(P, S)\}$ with $|S| = k$. We say that the special features in S are *marked*, and the set of pairs (P, S) a *marked Catalan structure*, which are the most basic structures counted by $B_{n, k}$. We will establish other combinatorial interpretations of $B_{n, k}$ by constructing bijections to a marked Catalan structure.

Each item in Theorem 1 gives a marked Catalan structure. In particular, in this paper we would frequently use the following one, which comes from item (2) of Theorem 1.

Theorem 2. *The entry $B_{n, k}$ of Borel’s triangle counts the set of (D, S) where D is a Dyck path of semi-length $n + 1$ and S consists of k up-steps of D , none of which is at ground level. Consequently, the generalized Catalan number $C(2, n)$ counts all the pairs (D, S) where D is a Dyck path of semi-length $n + 1$ and S is a set of up-steps of D not at ground level.*

By symmetry of Dyck paths, the “up-steps” in the previous statement can be replaced by “down-steps”.

3.2 Binary trees with Marked Vertices

In [10] Francisco et al. gave four interpretations of the entries of Borel's triangle; two of them are in terms of marked binary trees, which are shown to be counted by $B_{n,k}$ via bijections to two structures in commutative algebra and discrete geometry, namely, the EK-symbols of the smallest Borel ideal of $K[x_1, x_2, \dots, x_n]$ containing the monomial $x_1 x_2 \cdots x_n$, and the pointed pseudotriangulations. The two interpretations of marked binary trees are different from the marked Catalan structures in the preceding subsection as each set contains binary trees with k marked vertices and $n + 1$ unmarked vertices, and hence the total number of vertices in the binary trees is $n + k + 1$. Our goal is to find more direct, combinatorial proofs for these two sets of marked binary trees.

First we recall the definition of marked binary trees from [10]. A *binary tree* is a rooted plane tree in which each vertex has at most two children, which are referred to as the *left child* and the *right child*. A vertex of a binary tree is a *leaf* if it has no children, and it is a *branching vertex* if it has two children.

Definition 1. Let T be a binary tree. The *rightmost* leaf of T is the last leaf of T in the preorder depth-first traversal, i.e., the leaf obtained by starting at the root and descending, taking a right edge whenever possible. Let X be a set of leaves of T not containing the rightmost leaf. Then the pair (T, X) is called a *leaf-marked binary tree*. We say that the vertices in X are *marked*, and the other vertices *unmarked*.

If B is a set of branching vertices of T , we call the pair (T, B) a *branch-marked binary tree*.

Theorem 3. [10] *The entry $B_{n,k}$ of Borel's triangle counts the following sets.*

- (a) *Leaf-marked binary trees with $n + 1$ unmarked vertices and k marked leaves.*
- (b) *Branch-marked binary trees with $n + 1$ unmarked vertices and k marked branching vertices.*

As a corollary, the generalized Catalan number $C(2, n)$ counts the number of leaf-marked binary trees with n unmarked vertices or branch-marked binary trees with n unmarked vertices. (There is no constraint on the number of marked vertices.)

In this section we give a bijective proof for (a). A proof of (b) is presented in Section 5 using a tree decomposition and a functional equation of Borel's triangle. We remark that an easy bijection between the sets in (a) and (b) is given in [10], which is based on the observation that any binary tree with k branching vertices must have $k + 1$ leaves, hence one can associate the branching vertices bijectively to the set of all leaves except the rightmost one. This bijection, although very simple, does not give much insight on the structure of branching vertices.

Let U represents an up-step and D represents a down-step in a Dyck path. A Dyck path P of semi-length n is represented as a $\{U, D\}$ -sequence of length $2n$ with exactly n U 's and n D 's. Any such a sequence must end with a sequence of down-steps, which we call the *last run* of down steps of P . A *UDD-pattern* is a subword of P consists of three consecutive letters U, D, D . Let S be a set of *UDD-patterns* of a Dyck path P in which no pattern contains any down-steps in the last run of down-steps of P . We say that the pair (P, S) is a *Dyck path with marked UDD-patterns*. We shall prove that Dyck paths with marked *UDD-patterns* are also counted by Borel's triangle, and then we construct a bijection between the set of leaf-marked binary trees and that of Dyck paths with marked *UDD-patterns*.

Proposition 4. *The entry $B_{n,k}$ of Borel's triangle counts the number of Dyck paths of semi-length $n + k + 1$ with k marked *UDD-patterns*.*

Proof. We show that the number of Dyck paths of semi-length $n + k + 1$ with k marked UDD -patterns can be computed by the formula $\sum_{s=k}^n \binom{s}{k} C_{n,s}$. Then the claim follows from Eq. (3).

By item (3) of Theorem 1, $C_{n,s}$ is the number of Dyck paths of semi-length $n + 1$ with the last peak at height $n + 1 - s$. Any such a Dyck path has s down-steps not in the last run. If $s \geq k$, we can pick k down-steps not in the last run and replace each of them with a marked UDD . This results in a Dyck path of semi-length $n + k + 1$ with k marked UDD -patterns. Conversely, from any Dyck path of semi-length $n + k + 1$ with k marked UDD s that are disjoint from the last run of down steps, replacing each marked UDD with D would yield a Dyck path of semi-length $n + 1$ with at least k down steps not in the last run. ■

Theorem 5. *There is a bijection between the set of leaf-marked binary trees with $n + 1$ unmarked vertices and k marked leaves and the set of Dyck paths of semi-length $n + k + 1$ with k marked UDD -patterns.*

Proof. A full binary tree is a binary tree in which every vertex has either 0 or 2 children. Given a binary tree T with $n + k + 1$ vertices, first we turn it into a full binary tree \mathcal{T} by adding left and/or right children to any non-branching vertex so that every vertex of T has two children. Any mark in T remains unchanged. In the full binary tree \mathcal{T} the vertices with two children are called *internal vertices*, and \mathcal{T} contains $n + k + 1$ internal vertices and $n + 2 + k$ leaves. Clearly one can recover T from \mathcal{T} by removing all the leaves.

Both the set of full binary trees with m internal vertices and the set of Dyck paths of semi-length m are counted by the m -th Catalan number, and there are well-known bijections between these two sets. One bijection goes as follows. Given a full binary tree \mathcal{T} , we start at the root and traverse in the preorder traversal. In the process record “ U ” for each internal vertex and “ D ” for each leaf of \mathcal{T} , except for the rightmost leaf of \mathcal{T} . This gives a $\{U, D\}$ -sequence that corresponds to a Dyck path of semi-length $n + k + 1$, see [15, Corollary 6.2.3].

A leaf of T corresponds to an internal vertex of \mathcal{T} for which both children in \mathcal{T} are leaves. Under the above bijection it corresponds to a UDD -pattern in the word representation of the Dyck path except for the rightmost leaf of T which corresponds to the last peak of the Dyck path. Hence if S is a set of leaves of T not containing the rightmost one, then under the above map it becomes a set of UDD -patterns not containing any down steps in the last run. ■

Figure 1 shows a leaf-marked binary tree, its corresponding full binary tree and Dyck path with marked UDD -patterns.

Let $a_{n,k}$ be the number of binary trees on n vertices with $k + 1$ leaves, and $P_{n,k}$ be the number of leaf-marked binary trees on n vertices with k marked leaves. Then the arrays $\{a_{n,k} : n \geq 1, 0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor\}$ and $\{P_{n,k} : n \geq 1, 0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor\}$ are closely related to the Catalan’s and Borel’s triangles. In particular, $P_{n,k} = B_{n-1-k,k}$. The entries of $\{a_{n,k}\}$ and $\{P_{n,k}\}$ for small values of n and k are listed below.

$a_{n,k}$	k=0	1	2	3
n=0	1			
1	1			
2	2			
3	4	1		
4	8	6		
5	16	24	2	
6	32	80	20	

$P_{n,k}$	k=0	1	2	3
n=1	1			
2	2			
3	5	1		
4	14	6		
5	42	28	2	
6	132	120	20	
7	429	495	135	5

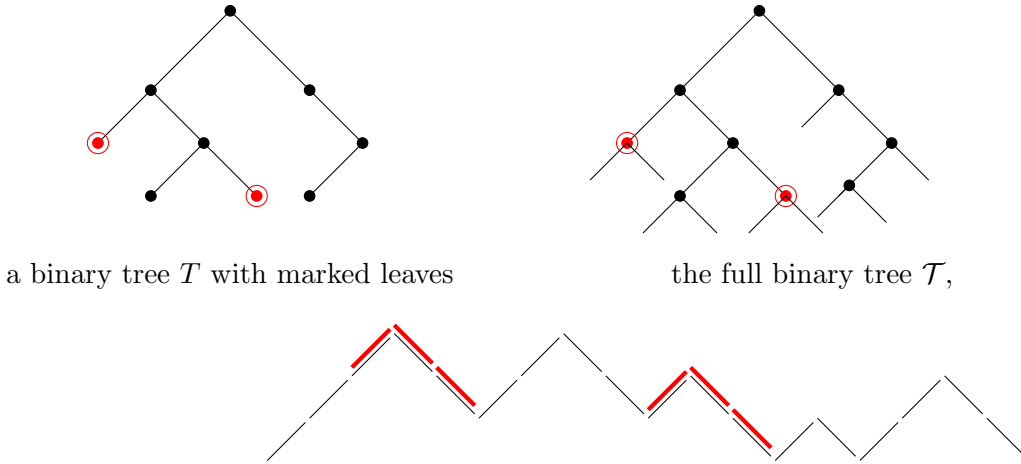


Figure 1: A leaf-marked binary tree, its corresponding full binary tree, and the Dyck path with marked UDD -patterns.

The array $\{a_{n,k}\}$ is the sequence A091894 in OEIS, whose values are given by

$$a_{n,k} = 2^{n-2k-1} \binom{n-1}{2k} C_k$$

for $n \geq 1$ and whose row-sum is the Catalan sequence. The array $\{P_{n,k}\}$ can be obtained from Borel's triangle by reading the entries diagonally from lower left to upper right. Let $\mathcal{A}(t, x) = \sum_{n \geq 1, k \geq 0} a_{n,k} t^k x^n$ and $\mathcal{P}(t, x) = \sum_{n \geq 1, k \geq 0} P_{n,k} t^k x^n$ be the bivariate generating functions, then we have

$$\mathcal{P}(t, x) = \mathcal{A}(1+t, x) \quad \text{and} \quad \mathcal{P}(t, x) = x\mathcal{B}(tx, x).$$

From these equations we can derive

$$\mathcal{A}(t, x) = x\mathcal{C}(1-x+tx, x),$$

where $\mathcal{C}(t, x)$ and $\mathcal{B}(t, x)$ are the bivariate generating functions for Catalan's and Borel's triangles, respectively. Explicitly, one can compute that

$$\mathcal{A}(t, x) = \frac{1 - 2x - \sqrt{1 - 4x + 4x^2 - 4x^2t}}{2tx},$$

which can also be derived directly from the binary tree structures.

3.3 Matchings Avoiding Pair of Patterns

In [3, Section 6], Bloom and Elizalde gave nice results on pattern avoiding matchings. In particular, the number of pattern avoiding matchings from their class I is the generalized Catalan number. This suggests that there may be certain combinatorial statistics on those matchings that are enumerated by $B_{n,k}$. In this subsection we give explicit description of such statistics.

For $2n$ points $1, 2, \dots, 2n$ on a horizontal line, we represent a *matching* on these $2n$ points by drawing n arcs between pairs of points. A *crossing* is formed by two arcs (i, j) and (k, ℓ) if $i < k < j < \ell$, and a *nesting* is formed by two arcs (i, j) and (k, ℓ) where $i < k < \ell < j$. Denote

by \mathcal{M}_n the set of all matchings on $2n$ elements. For a matched pair (i, j) with $i < j$, we call i an *opener* and j a *closer*.

A *full rook placement* is a pair (R, F) where F is a Ferrers board and R is a set of rooks placed in F such that each row and column of F contains exactly one rook. Denote by \mathcal{R}_F the set of all full rook placements on a given F , and \mathcal{R}_n the set of all full rook placements on Ferrers boards having n nonempty rows and columns. Throughout this section, a Ferrers board is left- and bottom-aligned, and we will use the name *rook placement* to refer to a full rook placement.

The sets \mathcal{M}_n and \mathcal{R}_n are related by the following natural bijection, see [7, 11]. Given a rook placement $F \in \mathcal{R}_n$, label its border from top-left to lower-right by numbers $1, 2, \dots, 2n$. If a rook has column labeled s and row labeled t , draw an arc that connects points s and t in the matching. Conversely, for a matching $M \in \mathcal{M}_n$, draw a path from $(0, n)$ to $(n, 0)$ by reading the vertices of M in increasing order, and each opener corresponds to an east step while each closer corresponds to a south step. The resulting path will be the border of the Ferrers board F . A matched pair of (s, t) then corresponds to a rook in F that appears in column associated to s and row associated to t . See the example in Figure 2 for an illustration.

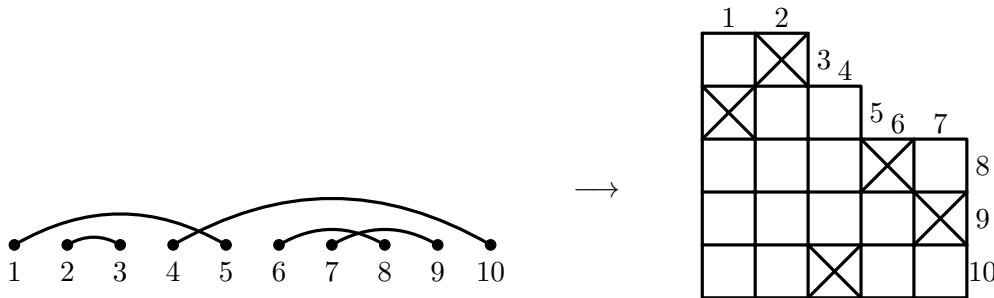


Figure 2: A bijection between matchings and rook placements.

To better describe the location of a rook, we consider the lower-left corner of a Ferrers board F as the origin $(0, 0)$ and label the columns of F from left to right as $1, 2, \dots, n$. The rows of F are labeled from bottom to top in the same manner. Then we say a rook is in the position (i, j) if it appears in column i and row j . Note that this labeling is different from the one in Figure 2.

With above bijection and coordinates for rooks, we can translate the terms nesting and crossing to rook placements.

Definition 2. Given a rook placement (R, F) , if two rooks appear in positions (i, j) and (k, ℓ) , then they form a *nesting pair* if $i < k$ and $j < \ell$, that is, the rook (i, j) is to the southwest of rook (k, ℓ) . They form a *crossing pair* if $i > k$ and $j < \ell$, that is, the rook (i, j) is to the southeast of rook (k, ℓ) and F contains a square in the position (i, ℓ) .

Next we define pattern avoiding matchings and rook placements.

Definition 3. Let $M \in \mathcal{M}_n$ be a matching, then it avoids a pattern $\sigma \in \mathfrak{S}_k$ if there are no $2k$ vertices $i_1 < i_2 < \dots < i_{2k}$ such that M contains all pairs of $(i_r, i_{2k+1-\sigma(r)})$ for $1 \leq r \leq k$.

Equivalently, a rook placement (R, F) avoids a pattern $\sigma \in \mathfrak{S}_k$ if it does not contain a square subboard of size $k \times k$ such that the restriction of F on the subboard is $\{(r, \sigma(r)) : 1 \leq r \leq k\}$.

For a pair of patterns $\{\sigma, \tau\}$ let $\mathcal{M}_n(\sigma, \tau)$ be the set of all matchings of $[2n]$ that avoid the patterns σ and τ , and $\mathcal{R}_n(\sigma, \tau)$ be the corresponding set of rook placements. Then we have:

Theorem 6. *The entry $B_{n,k}$ of Borel's triangle enumerates the following sets.*

- (1) $(123, 213)$ -avoiding matchings on $2(n+1)$ elements with k nestings.
- (2) $(231, 321)$ -avoiding matchings on $2(n+1)$ elements with k crossings.

Figure 3 gives an example of rook placements of patterns 123 and 213 respectively.

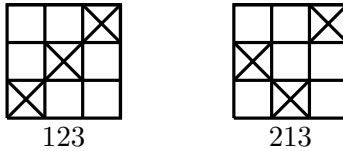


Figure 3: Rook placements of patterns 123 and 213 respectively.

Proof. We prove (1) by constructing a bijection between rook placements in $\mathcal{R}_{n+1}(123, 213)$ with k nesting pairs and Dyck paths of semilength $n+1$ with k marked down-steps that are not at ground level.

For any rook placement $F \in \mathcal{R}_{n+1}(123, 213)$, its boundary is a Dyck path rotated 45° , so that down-steps are south steps on the boundary of F . To mark this Dyck path, if $r_1 = (i, j)$ and $r_2 = (k, \ell)$ is a nesting pair where $i < k$ and $j < \ell$, mark the south step of the Dyck path corresponding to row ℓ . Since the rook placement is 123- and 213-avoiding, each r_2 corresponds to exactly one r_1 , that is, r_2 contributes to only one nesting pair and a down-step will be marked at most once. Thus k nesting pairs of rooks give k marked down-steps in the Dyck path.

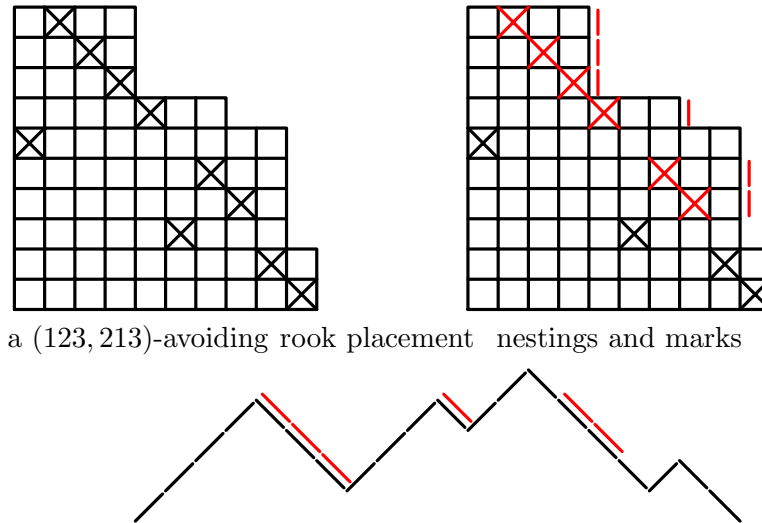
Next we need to show these marks are not at ground level. Assume the mark of row ℓ corresponds to a down-step at ground level, that is, this step has the lower endpoint on the diagonal $x + y = n$. Consider the subboard F' obtained by deleting all rows below r_2 . Since r_2 lies in a row on the diagonal, F' contains m columns and m rows. Since F is a full rook placement, F' must be a full rook placement. But r_1 is to the southwest of r_2 , this means at least one column in F' must be empty, a contradiction. Thus r_2 can not be a rook in a row on the diagonal, and the resulting Dyck path is one where all marked down-steps are not at ground level.

To see the inverse, consider a Dyck path as the border of a Ferrers board F . Place rooks in F from the top row to bottom in the following manner: If a row corresponds to an unmarked down-step in the Dyck path, place a rook in the leftmost empty column. If a row corresponds to a marked down-step, place a rook in the second leftmost empty column in F so that there is one empty column to its left. Since this down-step is not at ground level, there are more up-steps than down-steps in the segment of the Dyck path before this step, that is, there are more columns than rows in the Ferrers board prior to this row. Since we place rooks by rows, there is always an empty column to the left of a rook corresponding to a marked down-step.

Since the rooks are placed from top to bottom, for any rook r in this placement, there is at most one rook that is to the southwest of it, hence it is $(123, 213)$ -avoiding. Moreover, the rooks $r_1 = (i, j)$ and $r_2 = (k, \ell)$ with $i < k$ is a nesting pair if and only if r_2 is in a marked row and r_1 is in the first unmarked row below it, thus marks in the Dyck path correspond to nesting pairs in the rook placement and hence this is a bijection. See Figure 4 for an example.

The proof of (2) is similar, we find a bijection between $(231, 321)$ -rook placements and Dyck path with marked up-steps that are not at ground level.

For any rook placement $F \in \mathcal{R}_{n+1}(231, 321)$, its border is a rotated Dyck path in which up-steps are east-steps. If two rooks $r_1 = (i, j)$ and $r_2 = (k, \ell)$ is a crossing pair and r_1 is to the northwest



a $(123, 213)$ -avoiding rook placement nestings and marks

Figure 4: An example of $(123, 213)$ -avoiding rook placement and its corresponding Dyck path where the marked down-steps are not at ground level.

of r_2 , mark the east step of the Dyck path corresponding to the column of r_2 . Conversely, given a Dyck path with marked up-steps, use it as the border of a Ferrers board F . Place the rooks in F from right to left in the following manner: If the column corresponding to an east step is not marked, place a rook in the highest empty row. If the column corresponding to an east step is marked, place a rook in the second highest empty row such that there is an empty row above the rook.

By a similar argument as in the proof of (1), we can show that this map is well-defined and bijective, moreover, the resulting rook placement is $(231, 321)$ -avoiding. The details are left to the readers. See Figure 5 for an example. ■

In [3, Section 6] the authors gave seven other families of pattern avoiding matchings that are enumerated by the generalized Catalan number. For each family, we can define a statistic $s(M)$ for $M \in \mathcal{M}_{n+1}(\sigma, \tau)$ such that the subset $S_{n,m} := \{M \in \mathcal{M}_{n+1}(\sigma, \tau) : s(M) = m\}$ has cardinality $B_{n,m}$. We describe in each case the statistic $s(M)$ in Table 1 without proofs.

4 Restricted Dumont Permutation and Marked Dyck paths

Dumont permutations were introduced by Dumont [8] as a combinatorial interpretation for Genocchi numbers, which is a multiple of Bernoulli numbers. Surprisingly, the sequence of generalized Catalan numbers also appears in the enumeration of restricted Dumont permutations. Using generating functions Burnstein proved in [4, Theorem 3.1] that the n -th generalized Catalan number counts the number of Dumont permutations of the first kind with length $2n$ that avoid patterns $(2413, 3142)$. In this section we construct a bijection between those restricted Dumont permutations and Dyck paths with marked down-steps not at ground level. Our construction leads to a functional equation for the generalized Catalan numbers and Borel's triangle, which provides a useful tool to analyze the structures counted by Borel's triangle, (c.f. Section 5).

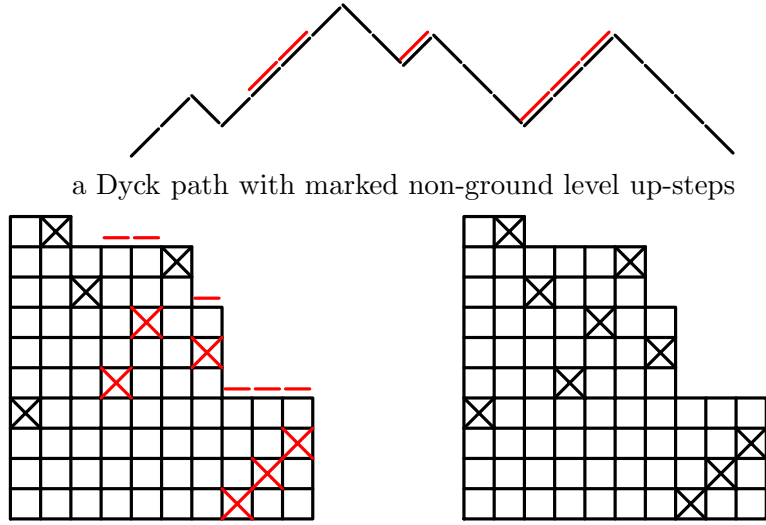


Figure 5: An example of $(231, 321)$ -avoiding rook placement and its corresponding Dyck path with marked up-steps that are not at ground level.

Patterns (σ, τ)	$s(M)$ for $M \in \mathcal{M}_{n+1}(\sigma, \tau)$
$\{231, 312\}$	$ \{(i, j) : \text{there exists some pair } (k, \ell) \text{ such that } k < i < j < \ell, \text{ and there is no pair } (r, s) \text{ where } r < i < s < j.\} $
$\{132, 213\}$	$ \{(i, j) : \text{there exists some pair } (k, \ell) \text{ such that } k < i < j < \ell.\} $
$\{213, 231\}$	$ \{(i, j) : \text{there exists some pair } (k, \ell) \text{ such that } k < i < j < \ell, \text{ and there is pair } (i-1, r) \text{ where } i < r < j.\} $
$\{132, 312\}$	$ \{(i, j) : \text{there exists some pair } (k, \ell) \text{ such that } k < i < j < \ell.\} $
$\{321, 312\}$	$ \{(i, j) : \text{there exists some pair } (k, \ell) \text{ such that } i < k < j < \ell.\} $
$\{213, 312\}$	$ \{(i, j) : \text{there exists some pair } (k, \ell) \text{ such that } k < i < j < \ell, \text{ and there is pair } (r, j+1) \text{ where } i < r < j.\} $
$\{132, 231\}$	$ \{(i, j) : \text{there exists some pair } (k, \ell) \text{ such that } k < i < j < \ell.\} $

Table 1: Seven cases of pattern avoiding matchings on $2(n+1)$ elements such that $|\{M \in \mathcal{M}_{n+1}(\sigma, \tau) : s(M) = m\}| = B_{n,m}$.

Definition 4. A Dumont permutation of the first kind is a permutation π of even length, i.e., $\pi \in \mathfrak{S}_{2n}$, of which each even entry is followed by a descent and each odd entry is followed by an ascent or is at the end of the permutation. In other words, for every $i = 1, 2, \dots, 2n$,

$$\begin{aligned}\pi(i) \text{ is even} &\implies i < 2n \text{ and } \pi(i) > \pi(i+1), \\ \pi(i) \text{ is odd} &\implies \pi(i) < \pi(i+1) \text{ or } i = 2n.\end{aligned}$$

Denote by \mathcal{D}_{2n}^1 the set of Dumont permutations of the first kind of length $2n$. For example, $\mathcal{D}_2^1 = \{21\}$ and $\mathcal{D}_4^1 = \{2143, 3421, 4213\}$. Note that in a Dumont permutation of the first kind, 21 must appear together, and the permutation always ends with an odd number.

REMARK. Dumont permutations of odd lengths can be defined similarly by the conditions in Definition 4. To meet the condition on the odd entries, the largest entry $2n+1$ must appear at the end. Hence \mathcal{D}_{2n+1}^1 can be obtained simply by adjoining $2n+1$ to the end of each permutation in \mathcal{D}_{2n}^1 and $|\mathcal{D}_{2n+1}^1| = |\mathcal{D}_{2n}^1|$.

Let S be a set of permutations and $\mathcal{D}_{2n}^1(S)$ be the set of Dumont permutations of length $2n$ that avoid patterns in S , that is, there is no subsequence that is order isomorphic to any pattern in S . Burnstein analyzed the structures and enumerated Dumont permutations avoiding a certain pattern or sets of patterns of length 3 and 4. One of his results relates to the n -th generalized Catalan number $C(2, n)$ defined via equations (6) and (7).

Theorem 7 (Burnstein).

$$|\mathcal{D}_{2n}^1(2413, 3142)| = C(2, n).$$

For example, $\mathcal{D}_4^1(2413, 3142) = \mathcal{D}_4^1$, but $|\mathcal{D}_6^1| = 17$ while $|\mathcal{D}_6^1(2413, 3142)| = 13$ since there are four Dumont permutations of the first kind and length 6 that contain patterns 2413 or 3142. More precisely, they are $\{\mathbf{364215}, \mathbf{421563}, \mathbf{436215}, \mathbf{421635}\}$, where an occurrence of the forbidden patterns is boldfaced.

We will prove Theorem 7 by constructing a bijection ρ from the set $\mathcal{D}_{2n}^1(2413, 3142)$ to the set of Dyck paths of semi-length n with marked down-steps not at ground level. First we need to understand the structure of the permutations in $\mathcal{D}_{2n}^1(2413, 3142)$. The following decomposition is also the idea behind Burnstein's generating function argument. To construct the desired bijection, we prove more properties for the decomposition and correct a typo² in [4].

Let $\pi \in \mathcal{D}_{2n}^1(2413, 3142)$. Assume the last entry of π is $2k-1$ for some $k \in [1, n]$. Then π can be decomposed into blocks of the form

$$\cdots A_i B_i A_{i-1} B_{i-1} \cdots A_1 B_0 (2k) A_0 (2k-1), \quad (8)$$

where A_0 is the part between entries $2k$ and $2k-1$, and A_i, B_i are blocks of consecutive entries such that each entry in A_i ($i \geq 1$) is less than $2k-1$ and each entry in B_i ($i \geq 0$) is greater than $2k$. The blocks $A_1, B_1 \cdots$ are nonempty but A_0 and B_0 may be empty.

Lemma 8. *If A_0 is nonempty, then each entry in A_0 is less than $2k-1$.*

Proof. If A_0 is nonempty, since $2k$ is followed by a descent, the entry i immediately after $2k$ is less than $2k-1$. If there is another entry j in A_0 that is larger than $2k$, then the four entries $2k, i, j, 2k-1$ form a pattern of 3142, a contradiction. ■

Lemma 9. *1. For any $i \geq 0$, assume $a \in A_i$ and $b \in A_{i+1}$. Then $a > b$.*

²The block form (8) appeared in [4], but the term $(2k)$ was written as $(2k-2)$.

2. For any $i \geq 0$, assume $a \in B_i$ and $b \in B_{i+1}$. Then $a < b$.

Proof. For (1), between a and b there is always an element c which is larger than or equal to $2k$. If $b > a$, then the terms a, c, b and the last entry $2k - 1$ of π form a pattern of 2413, a contradiction. Part (2) is similar, ■

Lemma 10. *All the sets $|A_i|$ and $|B_j|$ are of even size.*

Proof. From the previous lemma each of A_i and B_j contains a set of consecutive integers. Lemma 10 follows from the fact that in each A_i , the smallest element is followed by an ascent, hence it must be odd. Similarly in each B_i , the largest element is followed by a descent and hence must be even. ■

For a string W of distinct integers, let $\tau(W)$ be the permutation of length $|W|$ that is order isomorphic to W . For a permutation $\pi = a_1 a_2 \cdots a_n$, let $c(\pi)$ be the complement of π , i.e., the permutation $b_1 b_2 \cdots b_n$ where $b_i = n + 1 - a_i$.

Lemma 11. *For all i , $\tau(A_i)$ and $c(\tau(B_i))$ are Dumont permutations of the first kind that avoid $\{2413, 3142\}$.*

Proof. The statement for A_i is obvious. For each B_i , the sequence avoids $\{2413, 3142\}$, has an even length, and ends with an even entry. Hence taking the complement of $\tau(B_i)$, we obtain a Dumont permutation of the first kind that avoids $\{c(2413), c(3142)\} = \{3142, 2413\}$. ■

Conversely, one checks that any permutation of the form (8) and satisfies the above lemmas is a Dumont permutation of the first kind avoiding $\{2413, 3142\}$.

Now we are ready to describe the bijection ρ from $\mathcal{D}_{2n}^1(2413, 3142)$ to the set of Dyck paths of semi-length n with marked down-steps not at ground level. Our construction is defined recursively.

For $n = 1$, the only permutation in \mathcal{D}_2^1 is 21. Let $\rho(21)$ be the unique Dyck path UD of semi-length 1.

For $n \geq 2$, assume that $\pi \in \mathcal{D}_{2n}^1(2413, 3142)$ has form (8).

- [Case 1] If π is of the form $(2n)A_0(2n - 1)$, then $\rho(\pi) = U\rho(A_0)D$.
- [Case 2] If the first non-empty block (from left) of π is B_i , then $\rho(\pi)$ is the juxtaposition of $\rho(c(\tau(B_i)))$ and $\rho(\pi - B_i)$, where $\pi - B_i$ is the permutation obtained from π by removing B_i .
- [Case 3] If the first non-empty block of π is A_i for some $i \geq 1$, then let

$$\rho(\pi) = \rho(\tau(A_i)) \sqcup \rho(\tau(\pi - A_i)) \tag{9}$$

where $\pi - A_i$ is the permutation obtained from π by removing A_i , and for two Dyck paths P_1 and P_2 with marked steps, the operation $P_1 \sqcup P_2$ is a new Dyck path with marked steps defined by the following rules.

- (i) Add a mark to the last down-step of P_1 to get \tilde{P}_1 .
- (ii) Assume that the primitive decomposition of path P_2 is $P_2 = Q_1 Q_2 \cdots Q_j$, where each Q_i ($1 \leq i \leq j$) is a primitive Dyck path, i.e., Dyck path that does not touch the x -axis except at the endpoints. If $j \geq 2$, change Q_{j-1} to \tilde{Q}_{j-1} by add a mark to the last down-step of Q_{j-1} .

(iii) Assume that $Q_j = UR_jD$ where Q_j is the last primitive factor of P_2 . Set

$$P_1 \sqcup P_2 = U\tilde{P}_1Q_1 \cdots Q_{j-2}\tilde{Q}_{j-1}R_jD.$$

Example 12. For $n = 2$, there are three permutations in $\mathcal{D}_4^1(2413, 3142)$. The permutation 4213 is of Case 1 and $\rho(4213) = UUDD$. The permutation 3421 is of Case 2 and $\rho(3421) = UDUD$. The last permutation 2143 is of Case 3, and hence $\rho(2143) = (UD) \sqcup (UD) = UU\hat{D}D$, where \hat{D} means the corresponding down step is marked. See Figure 6, where the marked down step is represented by a red line along the step.

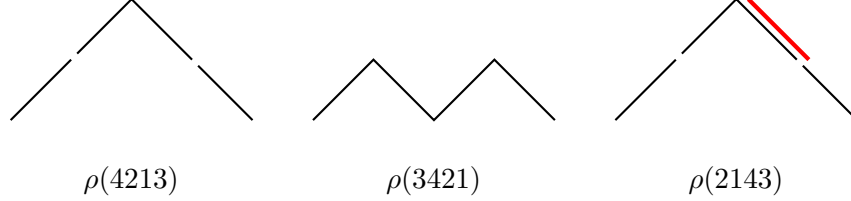


Figure 6: Bijection from $\mathcal{D}_4^1(2413, 3142)$ to marked Dyck paths.

To see that ρ is invertible and hence a bijection, we note some special properties of the preceding construction.

Proposition 13. 1. if π is of the form $(2n)A_0(2n-1)$, then $\rho(\pi)$ is a primitive Dyck path with no marked down-steps at level 0 or 1.

2. If the first non-empty block of π is B_i , then $\rho(\pi)$ is not primitive.

3. If the first non-empty block of π is A_i for some $i \geq 1$, then $\rho(\pi)$ is a primitive Dyck path with at least one marked down-step at level 1.

– If $\rho(\pi)$ has exactly one marked down-step at level 1, then π is of the form $A_1(2n)A_0(2n-1)$.

– If $\rho(\pi)$ has at least two marked down-steps at level 1, then B_{i-1} is not empty.

Given a Dyck path P of length $2n$ with marked down-steps not at ground level, we can decide which case it comes from by using Proposition 13. Then the original permutation can be recovered by the recursive structure. Case 1 is straightforward. For Case 2, the last primitive factor of P corresponds to $\pi - B_i$, and the other part corresponds to $\tau(B_i)$, from which we can easily get B_i . For Case 3, the path P is primitive. The part to the first marked down-step at level 1 allows us to recover A_i , and the following part until the second marked down-step at level 1 (if there is one) allows us to recover P_2 and hence $\tau(\pi - A_i)$. This proves that ρ is invertible.

We do not have a refinement of Theorem 7 in terms of Borel's triangle since we do not have a clean description of the statistic on Dumont permutations that corresponds to the marked down-steps in Dyck paths. Nevertheless, the bijection ρ allows us to count the number of primitive Dumont permutations.

Definition 5. We say a Dumont permutation $\pi = \pi(1)\pi(2) \cdots \pi(2n)$ is decomposable if there is an even number $2i$ with $i < n$ such that any term in $\pi(1), \pi(2), \dots, \pi(2i)$ is greater than any term in $\pi(2i+1), \dots, \pi(2n)$. In other words, $\pi(1), \dots, \pi(2i)$ is a permutation of $2n - 2i + 1, \dots, 2n$. If a Dumont permutation is not decomposable, we say that it is primitive.

Corollary 14. *The number of primitive Dumont permutations in $\mathcal{D}_{2n}^1(2413, 3142)$ is $2^{n-1}C_{n-1}$, where C_{n-1} is the $(n-1)$ -st Catalan number.*

Proof. By Proposition 13, a permutation π in $\mathcal{D}_{2n}^1(2413, 3142)$ is primitive if and only if the image $\rho(\pi)$ is a primitive Dyck path with possibly some marked down-steps not at ground level. There are C_{n-1} primitive Dyck paths, each with exactly $n-1$ down-steps not at ground level, hence 2^{n-1} ways to mark those down-steps. \blacksquare

5 Functional Equations of Borel's Triangle

The bijection proof of Theorem 7 leads to a functional equation for the bivariate generating function of the entries in Borel's triangle. Let $f_{0,0} = 1$ and $f_{n,k}$ be the number of Dyck paths of semi-length n with k marked down-steps not at ground level. We have $f_{n,k} = B_{n-1,k}$ for $n \geq 1$ and $F(t, x) = 1 + xB(t, x)$, where $F(t, x) := \sum_{n,k \geq 0} f_{n,k} t^n x^k$ and $B(t, x)$ is given in Eq.(4).

Using the block-decomposition (8) and writing $F(t, x)$ as F , we get

$$\begin{aligned} F &= 1 + xF^2 + xF \sum_{i=1}^{\infty} t^{2i-1} ((F-1)^{2i-1} + (F-1)^{2i}) \\ &\quad + xF \cdot t(F-1) \sum_{i=1}^{\infty} t^{2i-1} ((F-1)^{2i-1} + (F-1)^{2i}). \end{aligned} \quad (10)$$

In the above formula, xF^2 is the contribution of those Dumont permutations in $\mathcal{D}_{2n}^1(2413, 3142)$ which are of the form $B_0(2k)A_0(2k-1)$, where B_0 and A_0 may be empty. The first summation on the right-hand side is the contribution from all those Dumont permutations with $B_0 = \emptyset$ and $A_1 \neq \emptyset$, and the last summation is the contribution from those with $B_0 \neq \emptyset$ and $A_1 \neq \emptyset$. Simplifying Eq.(10) we obtain the following proposition, where the special case when $t = 1$ is given in [4].

Theorem 15. *The bivariate generating function $F = F(t, x)$ satisfies the equation*

$$F = 1 + xF^2 \cdot \frac{1}{1 - t(F-1)} \quad (11)$$

It is easy to derive a quadratic equation from (11)

$$(x+t)F^2 - (2t+1)F + t + 1 = 0,$$

whose root with $F(0,0) = 1$ is

$$F(t, x) = \frac{2t+1 - \sqrt{1-4x(1+t)}}{2(x+t)} = \frac{1}{1 - xC((1+t)x)}, \quad (12)$$

which agrees with (4) by a simple algebraic manipulation.

Eq.(11) is useful to determine structures counted by Borel's triangle. Next we describe tree decompositions for two kinds of marked binary trees and prove that they are counted by Borel's triangle. The first one is the branch-marked binary trees in Theorem 3(b). Figure 7 gives examples of these two decompositions, where the red circle means a mark.

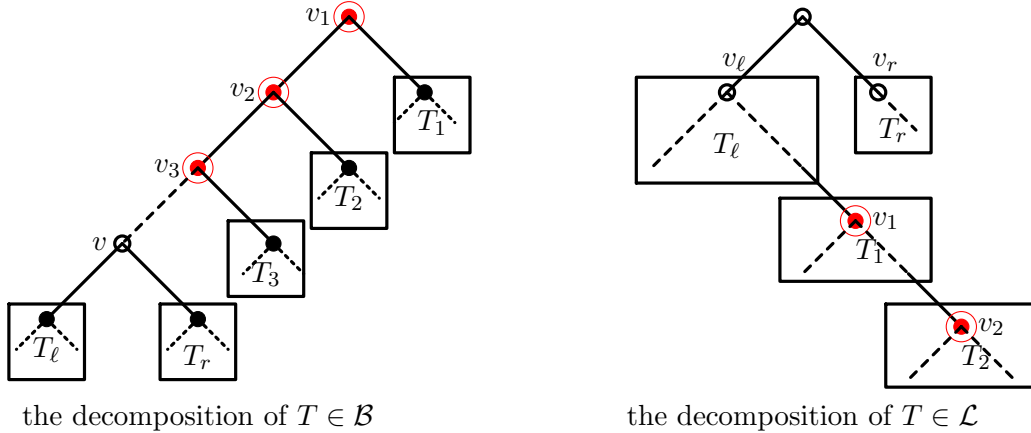


Figure 7: Two tree structures that are enumerated by Borel's triangle.

Proof of Theorem 3(b). Consider binary trees where only the branching vertices can be marked, denote the set of all such binary trees by \mathcal{B} . Let $T \in \mathcal{B}$ and let v be the first vertex on the left spine of T that is not marked by depth-first search. There exists such a vertex since the last vertex on the left spine is a leaf and can not be marked. Denote by T_ℓ and T_r the left and right subtrees of v respectively, then they are both branching-labeled binary trees. Now denote the ancestors of v as v_1, v_2, \dots by depth, every v_i is marked and has a right child. Denote the right subtree of v_i by T_i , again we have $T_i \in \mathcal{B}$. While T_ℓ and T_r may be empty, all T_i 's must be nonempty, so if we let $b_{n,k}$ be the number of binary trees $T \in \mathcal{B}$ with k marked branching vertices and n unmarked vertices, the generating function $B(t, x) = \sum_{n,k \geq 0} b_{n,k} t^k x^n$ also satisfies the relation.

$$B = 1 + x \cdot B^2 \cdot \frac{1}{1 - t(B - 1)}.$$

Together with $b_{1,0} = 1$, $b_{2,0} = 2$ and $b_{2,1} = 1$, we have $b_{n,k} = B_{n-1,k}$ and the set \mathcal{B} is enumerated by (the shifted) Borel's triangle. ■

Our second example is another kind of vertex-marked binary trees.

Theorem 16. *The entry $B_{n,k}$ of Borel's triangle counts the set of binary trees on $n + 1$ vertices with k marked vertices such that there is no marked vertices on the right spine. Here the right spine consists of the root and any vertex u for which all the vertices on the path from the root to u is the right-child of its parent node.*

Proof. First we consider \mathcal{L} , the set of all binary trees where the marked vertices are not on its right spine. For any binary tree $T \in \mathcal{L}$, let r be the root of T and v_ℓ, v_r be the left and right children of r respectively, we can find all the marked vertices on the right spine of v_ℓ , name them by depth as v_1, v_2, \dots . In particular, if v_ℓ is marked, then $v_\ell = v_1$. Now remove the edges $(r, v_\ell), (r, v_r)$ and all edges $(p(v_i), v_i)$, here $p(v_i)$ denotes the parent of v_i , we obtain a forest. Let the tree rooted at v_r in the forest be T_r , the tree rooted at v_ℓ be T_ℓ and the tree rooted at v_i be T_i . While T_r and T_ℓ might be empty, T_i always contain v_i and thus is not empty. Moreover, T_r and T_ℓ have no marked vertices on right spine, that is, $T_r, T_\ell \in \mathcal{L}$. Each T_i has its root v_i marked but no other vertex on the right spine is marked.

Denote by $\ell_{n,k}$ the number of binary trees $T \in \mathcal{L}$ with k marked vertices and n vertices in total. Hence $\ell_{1,0} = 1$, $\ell_{2,0} = 2$ and $\ell_{2,1} = 1$. From the above decomposition, we conclude that the generating function $L(t, x) = \sum_{n,k \geq 0} \ell_{n,k} t^k x^n$ satisfies the following equation

$$L = 1 + x \cdot L^2 \cdot \frac{1}{1 - t(L - 1)}.$$

which yields $\ell_{n,k} = f_{n,k} = B_{n-1,k}$. ■

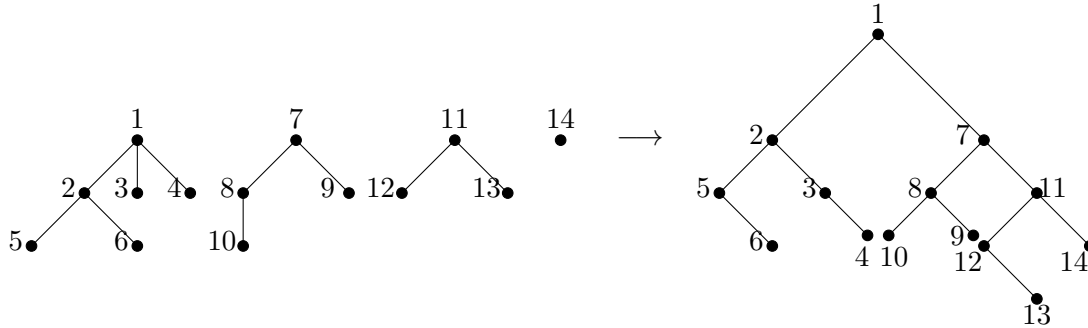


Figure 8: The canonical map from plane forests to binary trees

REMARK. The vertex-marked binary trees in Theorem 16 can also be obtained from the basic marked Catalan structure obtained from item (4) of Theorem 1, which is the set of unlabeled plane forest on $n + 1$ vertices with k marked non-root vertices. To get vertex-marked binary trees, one uses the canonical bijection from plane forests on $n + 1$ vertices to binary trees on $n + 1$ vertices, where for any node of a plane forest, its first child becomes the left child, and its first sibling on the right becomes the right child. Under this bijection the root of the plane forest becomes the nodes in the right spine the binary tree. See Figure 8 for an illustration.

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