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THE GEOMETRY OF SOME FIBONACCI IDENTITIES IN THE HOSOYA TRIANGLE

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Abstract. In this paper we explore some Fibonacci identities that are interpreted geometrically in the Hosoya triangle. Specifically we explore a generalization of the Cassini and Catalan identities from a geometric point of view. We also extend some properties present in the Pascal triangle to the Hosoya triangle.

1. Introduction

In 1976 Hosoya [5] introduced his triangle, then called the Fibonacci triangle. It is a triangular array where the entries are products of Fibonacci numbers. Later Koshy [7] changed the name of this triangle from the Fibonacci triangle to the Hosoya triangle. Since then several geometric properties of the Hosoya triangle have already been discovered and published. See for example articles by Flórez *et al.*, Hosoya, and Koshy, [2,3,5,7].

Some properties of Fibonacci numbers that were known algebraically, now have a geometric interpretation. The classical proofs of most well-known identities that we study here are based on mathematical induction, however in this paper, we provide geometrical proofs of those identities using the properties of the Hosoya triangle (with some inductive steps in certain cases), therefore making the proofs more visual. The tools here can be extended to prove other classic identities. In addition, we extend some properties that are well-known in the Pascal triangle to the Hosoya triangle (see the book by Green and Hamburg [4]). The hockey stick property is one of the well-known properties that we successfully extend to the Hosoya triangle. The T-stick property in Pascal triangle gives rise to a triangular property here. This is due to the definition of the Hosoya triangle.

The Hosoya triangle is a great tool to represent Fibonacci identities geometrically. In this paper we also study some other geometric properties that the Hosoya triangle has. For example we give a geometric proof of the Cassini, Catalan, and Johnson identities.

We have found that if a rectangle is given in a Hosoya triangle, then the differences of two of its corners is equal to the difference of the remaining corners. This fundamental property allows us to have geometrical proofs of several identities.

The symmetry present in the Hosoya triangle helps us explore several patterns, and many identities. The rectangle property gives rise to other geometrical configurations and therefore, more identities associated with those configurations.

2. The Hosoya triangle and its coordinate system

The construction presented in this section can be found in articles by Flórez *et al.* and Hosoya [2, 5]. A similar construction is also present in a book by Koshy [7]. The *Hosoya sequence* $\{H(r,k)\}_{r,k\geq 0}$ is defined using the double recursion

$$H(r,k) = H(r-1,k) + H(r-2,k) \quad \text{and} \quad H(r,k) = H(r-1,k-1) + H(r-2,k-2)$$

with initial conditions $H(0,0)=0$; $H(1,0)=0$; $H(1,1)=0$; $H(2,1)=1$, where $r>1$ and $0\leq k\leq r-1$. This sequence gives rise to the *Hosoya triangle*, where the entry in position k (taken

						$H(0,0)$						
					$H(1,0)$		$H(1,1)$					
				$H(2,0)$		$H(2,1)$		$H(2,2)$				
			$H(3,0)$		$H(3,1)$		$H(3,2)$		$H(3,3)$			
		$H(4,0)$		$H(4,1)$		$H(4,2)$		$H(4,3)$		$H(4,4)$		
	$H(5,0)$		$H(5,1)$		$H(5,2)$		$H(5,3)$		$H(5,4)$		$H(5,5)$	
$H(6,0)$		$H(6,1)$		$H(6,2)$		$H(6,3)$		$H(6,4)$		$H(6,5)$		$H(6,6)$

Table 1. Hosoya triangle $\mathcal{H}$.

from left to right) of the rth row is equal to $H(r,k)$ (see Tables 1 and 7, and Sloane [8] at A058071). For simplicity in this paper we use $\mathcal{H}$ to denote the Hosoya triangle.

Proposition 1 ([5,7]). $H(r,k) = F_k F_{r-k}$.

Proposition 1 gives rise to another coordinate system (see also Flórez *et al.*, [2,3]). If P is a point in $\mathcal{H}$, then it is clear that there are two unique positive integers r and k such that $P = H(r,k)$ with $k \leq r$. From $H(r,k) = F_k F_{r-k}$ it is easy to see that an nth *diagonal* in $\mathcal{H}$ is the collection of all Fibonacci numbers multiplied by $F_n(x)$. For example, from Table 1 we can see that the diagonal $H(3,0)$, $H(4,1)$, $H(5,2)$, $H(6,3)$, $H(7,4)$, $H(8,5), \dots$ is equal to the diagonal $0, 2, 2, 4, 6, 10, 16, 26, 42, \dots$ in Table 7 which results from multiplying the Fibonacci sequences by $F_3 = 2$.

3. Geometric properties in the Hosoya triangle

A parallel configuration of points in the Hosoya triangle is called a *ladder configuration*, for simplicity we are going to refer to this as a ladder. A *rung* is the set of points on a line intersecting both parallel configurations of the ladder. See Figures 1(a), 2, 6, and 12. The *length* of rung is the difference of its end points. The *absolute length* of a rung is the absolute value of its length.

In this section we use the ladder configuration to explore geometric and algebraic properties in the Hosoya triangle. The properties here in this paper can be easily extended to the Hosoya polynomial triangle (see Flórez *et al.* [1]).

We first prove a lemma that will be helpful in proving several results in this paper.

If L is a horizontal ladder in $\mathcal{H}$ where its rungs have exactly two points, then a rung sum is a Fibonacci number and it is the same for every rung.

Lemma 2. *In $\mathcal{H}$, for every $k, j \geq 0$ and $j + i + 1 \leq k$ it holds that*

$$F_j F_{k-j} + F_{j+1} F_{k-j+1} = F_{j+i} F_{k-j-i} + F_{j+i+1} F_{k-j-i+1} = F_{k+1},$$

or equivalently, $H(k,j) + H(k+2, j+1) = H(k, j+i) + H(k+2, j+i+1) = F_{k+1}$.

Proof. First, we take two consecutive rungs of L forming a square (see Figure 1(a)). Observe that each diagonal (slash and backslash) of this square has three points –two corner points and one inner point–. Those two diagonals intersect in the inner point p. From the recursive definition of the entries of $\mathcal{H}$ and the point p, it is easy to see that the difference of the two corner points of the backslash diagonal of the square is equal to the difference of the corner points of the slash diagonal of the square. This implies that sum of the points in any two consecutive rungs have the same value. Using an inductive argument we can extend the result for any two arbitrary rungs. Since this is true for any rung of L, it is true for the first rung on the left where one of the two points is zero and the other is the Fibonacci number $F_{k+1} = H(k+2, 1)$. □

An alternate (technical) proof can be found using the recursive definition of the Hosoya triangle and first proving that $H(r,k) + H(r+2, k+1) = H(r, k+1) + H(r+2, k+2)$.

All horizontal rung in a vertical ladder in $\mathcal{H}$ have the same length except by the order of their measure (see Figures 1(b) and 2). This result is formally stated as Proposition 3.

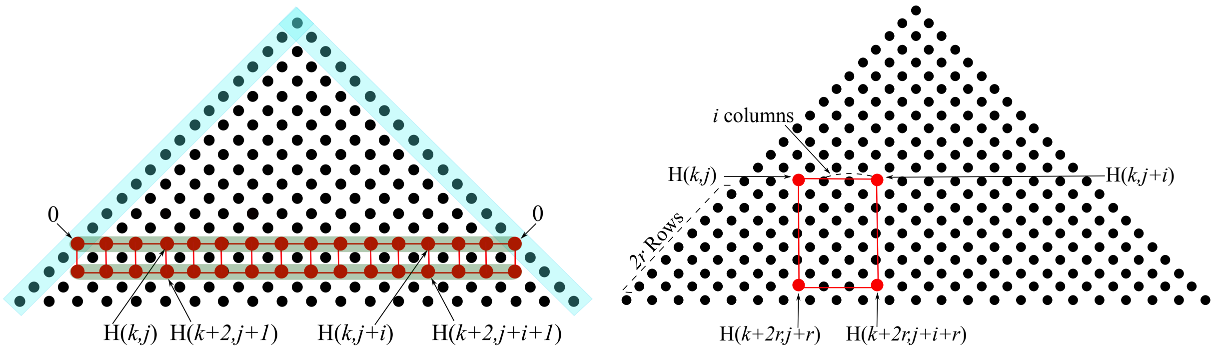

FIGURE 1. Rectangle property and the rung sum of a ladder.

Proposition 3 (Rectangle Property)**.** *In $\mathcal{H}$ it holds that*

$$F_jF_{k-j} - F_{j+i}F_{k-j-i} = (-1)^{r+1}(F_{j+r}F_{k+r-j} - F_{j+i+r}F_{k+r-j-i}) = (-1)^{r+1}F_iF_{k-2j-i},$$

or equivalently,

$$H(k,j) - H(k,j+i) = (-1)^{r+1}(H(k+2r,j+r) - H(k+2r,j+i+r)) = (-1)^{r+1}H(k-2j,i).$$

Proof. From Figure 2 and Lemma 2 we can observe that

$$|a_0 - b_0| = |a_1 - b_1| = |a_2 - b_2| = \cdots = |a_i - b_i|.$$

From Figure 2 we can see that, in particular, if we take $a_0 = 0$, then $b_0 = H(k-2j,i)$. □

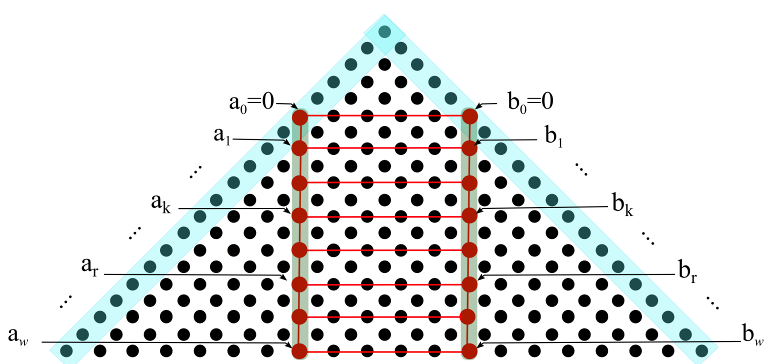

FIGURE 2. Every rung in any vertical ladder in $\mathcal{H}$ have the same length.

The following result provides several identities in the Hosoya triangle. In particular it shows that the alternating sum of the points in a horizontal rung of a vertical ladder and the sum of the points in the vertical rungs of horizontal ladder is a constant provided that the rung has even number of points in each case. We also see that the absolute length of each rung in a horizontal ladder is the same if there are odd number of points in the rungs. Finally, if the ladders are oblique (see Figure 6), then the absolute length of each rung is the absolute length of the first rung multiplied by a Fibonacci number and the sum of the points in the oblique rungs equals the sum of the points in the second rung multiplied by a Fibonacci number.

We may also use Proposition 1 to give an algebraic reinterpretation of the results mentioned above in terms of Fibonacci numbers.

Theorem 4. *In the Hosoya triangle $\mathcal{H}$ these hold,* *(1) If $r, k > 0$, $j \geq 0$ and $0 < 2n-1 \leq k$ for some n, then*

$$\left|\sum_{t=0}^{2n-1} (-1)^t H(k, j+t)\right| = \left|\sum_{t=0}^{2n-1} (-1)^i H(k+2r, j+t+r)\right|.$$

Equivalently, $\left|\sum_{t=0}^{2n-1} (-1)^t F_{j+t} F_{k-j-t}\right| = \left|\sum_{t=0}^{2n-1} (-1)^i F_{j+r+t} F_{k+r-j-t}\right|$.

(2) If $m, k > 0$, $j \geq 0$ and $0 < 2n-1 \leq k$ for some n, then

$$\sum_{t=0}^{2n-1} H(k+2t, j+t) = \sum_{t=0}^{2n-1} H(k+2t, j+m+t).$$

Equivalently, $\sum_{t=0}^{2n-1} F_{j+t} F_{k+t-j} = \sum_{t=0}^{2n-1} F_{j+m+t} F_{k+t-j-m}$.

(3) If i a positive odd number, then

$$H(k+2i, j+i) - H(k, j) = H(k+2i, j+n+i) - H(k, j+n) = H(k+2i, i),$$

Equivalently, $\det \begin{bmatrix} F_{j+i} & F_j \\ F_{k-j} & F_{k+i-j} \end{bmatrix} = \det \begin{bmatrix} F_{j+n+i} & F_{j+n} \\ F_{k-j-n} & F_{k+i-j-n} \end{bmatrix} = \det \begin{bmatrix} F_i & 0 \\ 0 & F_{k+i} \end{bmatrix}$.

(4) If $r, k,$ and j are positive integers with $r \geq j$, then

$$H(r+k, j) - H(r, j) = F_j (H(r+k-j+1, 1) - H(r-j+1, 1)).$$

Equivalently, $\det \begin{bmatrix} F_j & F_{r-j} \\ F_{r+k-j} & F_j \end{bmatrix} = F_j \left(F_{r+k-j} - F_{r-j}\right)$.

(5) If $i, j,$ and k are positive integers, then

$$\sum_{i=0}^{m} H(k+i, j+i) = F_{k-j} \sum_{i=0}^{m} H(j+i+1, 1).$$

Proof. We prove part (1) for two consecutive rungs. The general case follows easily using an inductive argument, so we omit it. From Figure 3 and Lemma 2 we have

$$c_0 + d_0 = c_1 - d_1 = c_2 - d_2 = \cdots = c_{2n-1} - d_{2n-1}.$$

This implies that $|\sum_{i=0}^{2n-1} (-1)^i c_i| = |\sum_{i=0}^{2i-1} (-1)^i d_i|$.

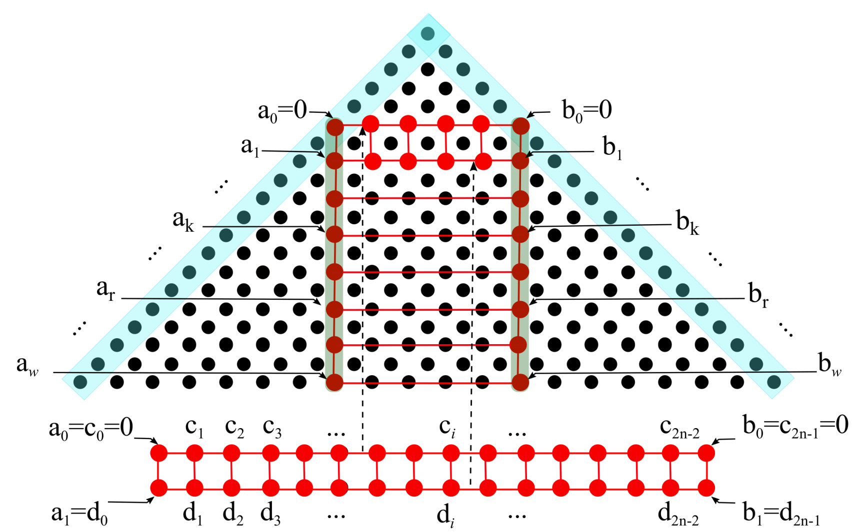

FIGURE 3. The alternating sum of the points in a rung is the same.

Proof of part (2). From Figure 4 and Lemma 2 we have

$$c_0 + c_1 = d_0 - d_1; \quad c_2 - c_3 = d_2 - d_3; \quad \dots \quad c_{2i-2} - c_{2i-1} = d_{2i-2} - d_{2i-1}.$$

This leads to the conclusion.

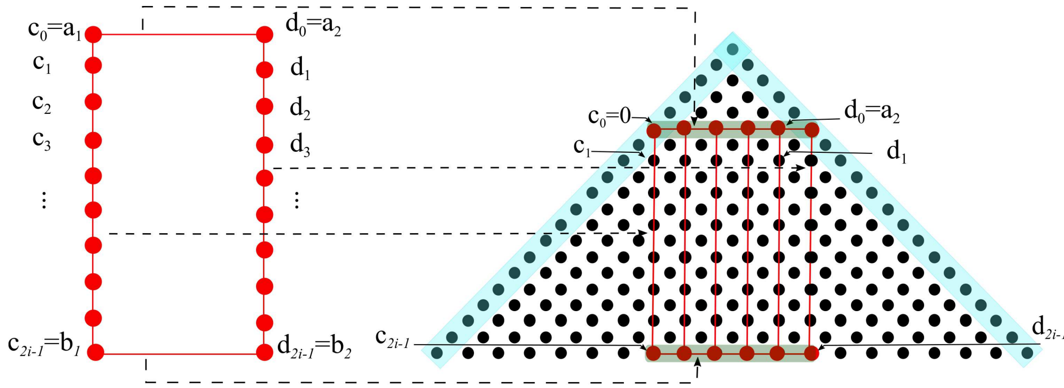

Figure 4. Rungs with even number of points have the same sum.

Proof of part (3). Using Figure 5 we define x as the sum of all points between c_0 and d_0, including both of them, and let y be the sum of all points between c_1 and d_1, including both of them. Note that z_1 and z_2 in Figure 5 represent sets with an even number of points. Now we have $a_0 + x$, $a_1 + y$, $b_0 + x$, and $b_1 + y$ are sums of an even number of points. Therefore, by part (2) we have

$$a_0 + x = a_1 + y \text{ and } b_0 + x = b_1 + y.$$

From this it is easy to see that $b_0 - a_0 = b_1 - a_1$. If in particular we take $a_0 = H(k, 0)$, we have $b_0 = H(k + 2i, i)$.

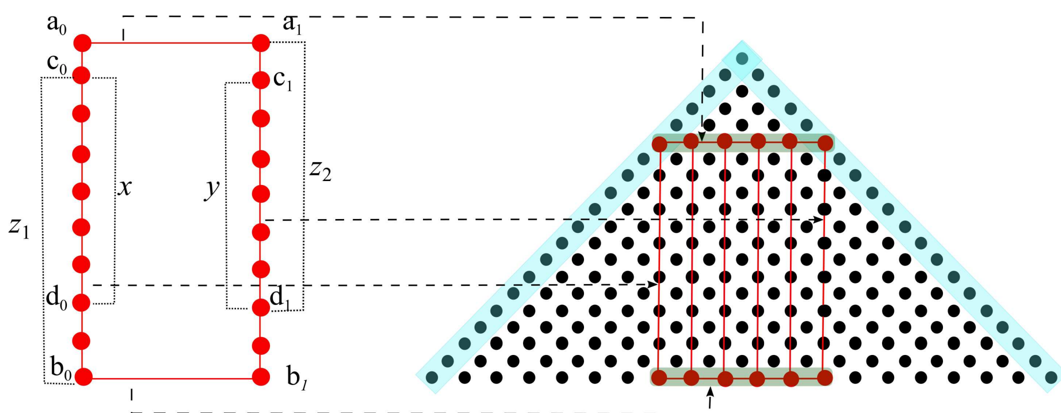

Figure 5. Rungs with odd number of points have the same length.

Proof of parts (4) and (5). Using the coordinate system described in Section 2 and Figure 6 we can see that the points in the first right-up of the ladder are of the form $F_k F_i$ where F_k is fixed and the points in the second right-up are of the form $F_{k+j} F_i$ where F_{k+j} is fixed. Therefore, the points in the r-th rungs are $F_k F_r$, $F_{k+1} F_r$, $F_{k+2} F_r, \dots,$ $F_{k+j} F_r$. To prove part (4) we first note that the length of any rung is given by $F_{k+j} F_r - F_k F_r = F_r(F_{k+j} - F_k)$ where $F_{k+j} - F_k$ is the length of the first rung. The proof of part (5) follows by adding the points of a rung. Thus,

$$F_k F_r + F_{k+1} F_r + F_{k+2} F_r + \cdots + F_{k+j} F_r = F_r \left(F_k + F_{k+1} + F_{k+2} \cdots + F_{k+j} \right).$$

Note that $(F_k + F_{k+1} + F_{k+2} \cdots + F_{k+j})$ is the sum of points of the second rung. □

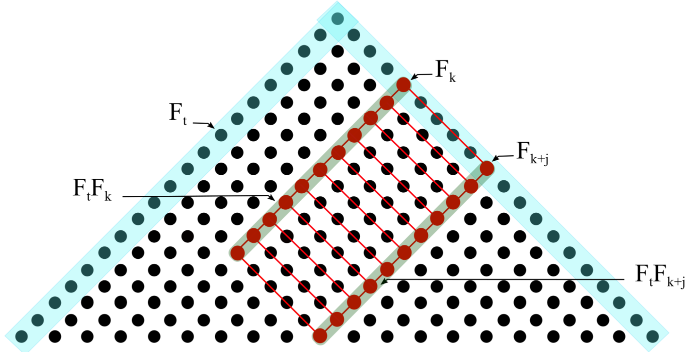

FIGURE 6. The sum of the points of the rungs are proportionally related.

In the next part we give a geometric interpretation in the Hosoya triangle of the Cassini, Catalan, and Johnson identities (see [6]).

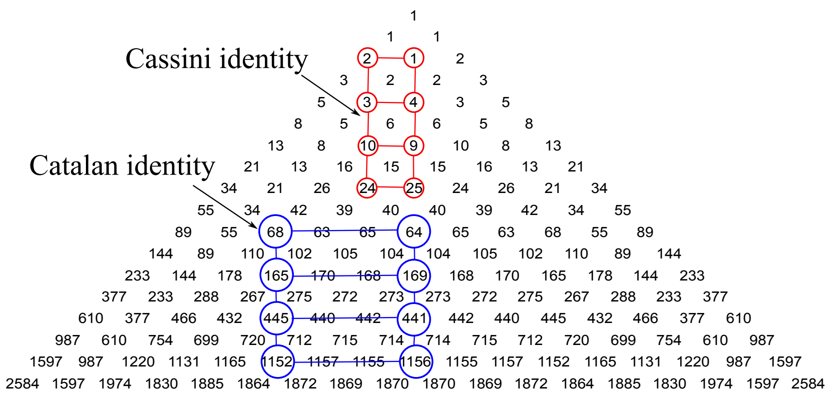

FIGURE 7. Geometry of the Cassini and Catalan identities.

The length of a rung in a vertical ladder in $\mathcal{H}$ gives rise to the Cassini identity, if one of the uprights is located in a central vertical line of $\mathcal{H}$ and the rung has exactly two points (see Figure 7). Thus,

$$H(2k,k) - H(2k,k-1) = (-1)^{k-1}.$$

The same type of ladder as above also gives rise to the Catalan identity. Thus,

$$H(2k,k) - H(2k,k-j) = (-1)^{k-j} H(2j,j).$$

The length of a rung in a vertical ladder in $\mathcal{H}$ gives rise to the d'Ocagne identity if one of them is located on the top of the ladder, and the coordinates of the end points of the second rung are $H(k+j+1,j)$ and $H(k+j+1,k)$. Thus,

$$H(k+j+1,k) - H(k+j+1,j) = (-1)^j H(k-j+1,k-j).$$

The length of a rung in a vertical ladder in $\mathcal{H}$ gives rise to the Johnson identity. Thus, if $k+j = r+i$ and $i < j$, then for every $l \le i$ it holds that

$$\begin{aligned} H(k+j,j) - H(r+i,i) &= (-1)^l (H(k+j-2l,j-l) - H(r+i-2l,i-l)) \\ &= (-1)^i H(k+j-2i,j-i). \end{aligned}$$

As a corollary of Theorem 4 part 3 we have that if the points a_i and b_i in the Hosoya triangle are as in Figure 8 (e), then $(a_j + a_{j+1}) - (b_j + b_{j+1})$ is a constant. This property is analogous to a property in Pascal's triangle that yields the Catalan numbers (see [9]).

Proposition 5. *Let a, b, j be positive integers with $j \leq \min\{a, b\}$. If $A(F_{a-j}, F_{a+j})$, $B(F_{b-j}, F_{b+j})$ and $C(F_a, F_b)$ are points in the Cartesian plane, then*

(1) the line passing through A and B is parallel to the line passing through $(0, 0)$ and C. Thus,

$$\frac{F_a}{F_b} = \frac{F_{a-j} - F_{a+j}}{F_{b-j} - F_{b+j}}.$$

(2) The triangle with base $F_{a+j} - F_{a-j}$ and height F_b has same area as the triangle with base $F_{b+j} - F_{b-j}$ and height F_a.

The proof of Proposition 5 is easy using Proposition 3, therefore we omit it.

The configuration depicted in Figure 8 part (a) is called a *zigzag*. The configuration depicted in Figure 8 part (b) is called a *left zigzag*. The configuration depicted in Figure 8 part (c) is called a *right zigzag*. The configuration depicted in Figure 8 part (d) is called a *long zigzag*. There should be a finite number of points in any zigzag configuration.

The configuration depicted in Figure 11 part (a) is called a *braid*. The configuration depicted in Figure 11 part (b) is called a *left braid*. The configuration depicted in Figure 11 part (c) is called a *right braid*. There should be a finite number of points in any braid configuration.

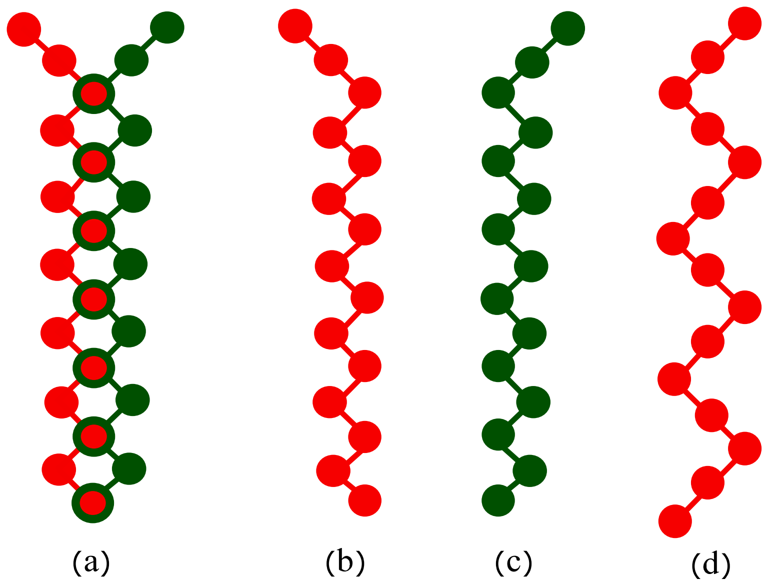

FIGURE 8. Zigzag configurations.

Corollary 6. *The sum of alternating points of a long zigzag configuration in $\mathcal{H}$ starting from its second point is equal to the difference of the last point and the first point of the zigzag configuration. Moreover, any column of points forming a rectangle with the column of alternating points has the same sum (see Figure 9(a) and Figure 8 part (d)). More precisely, if $a, b, c,$ and d are positive integers such that $a + c = b + d$, then for every positive integer k it holds that*

$$\sum_{j=0}^{2k-1} F_{a+j}F_{c+j} = \sum_{j=0}^{2k-1} F_{b+j}F_{d+j} = \begin{cases} F_{k+n_e}^2 - F_{n_e}^2 & \text{if } a+b \text{ is odd} \\ F_{n_e+k}F_{n_e+k-1} - F_{n_e}F_{n_e-1} & \text{if } a+b \text{ is even,} \end{cases}$$

where $n_e = \lfloor (a + b)/2 \rfloor$.

Proof. We prove that the sum of alternating points of a long zigzag configuration in $\mathcal{H}$ starting from its second point is equal to the difference of the last point with the first point of the long zigzag. The last part of this corollary follows from Theorem 4 part (2).

Suppose $p_1, p_2, p_3, \dots, p_{n-2}, p_{n-1}, p_n$ are the points of the long zigzag ordered from top to bottom where p_1 is the first point on the top and p_n is the last point in the bottom (see Figure 9(a)). We want to show that $p_2 + p_4 + p_6 + \cdots + p_{n-1} = p_n - p_1$. From definition of $\mathcal{H}$ on page 1 we know that $H(r,k) = H(r-1,k) + H(r-2,k)$ and $H(r,k) = H(r-1,k-1) + H(r-2,k-2)$. This implies that

$$\begin{aligned} p_3 &= p_1 + p_2, & (1)\\ p_5 &= p_3 + p_4, & (2)\\ p_7 &= p_5 + p_6, & (3)\\ &\vdots \\ p_n &= p_{n-2} + p_{n-1}. & (4) \end{aligned}$$

Substituting Equation (1) into $p_2 + p_4 + p_6 + \cdots + p_{n-1}$ we obtain

$$p_2 + p_4 + p_6 + \cdots + p_{n-1} = -p_1 + p_3 + p_4 + p_6 + \cdots + p_{n-1}.$$

Substituting Equation (2) into the right side of this equality, we obtain

$$p_2 + p_4 + p_6 + \cdots + p_{n-1} = -p_1 + p_5 + p_6 + \cdots + p_{n-1}.$$

Substituting Equation (3) into the right side side of this equality, we obtain

$$p_2 + p_4 + p_6 + \cdots + p_{n-1} = -p_1 + p_7 + p_8 + \cdots + p_{n-1}.$$

We systematically keep making these substitutions to obtain

$$p_2 + p_4 + p_6 + \cdots + p_{n-1} = -p_1 + p_n.$$

This completes the proof of the corollary. □

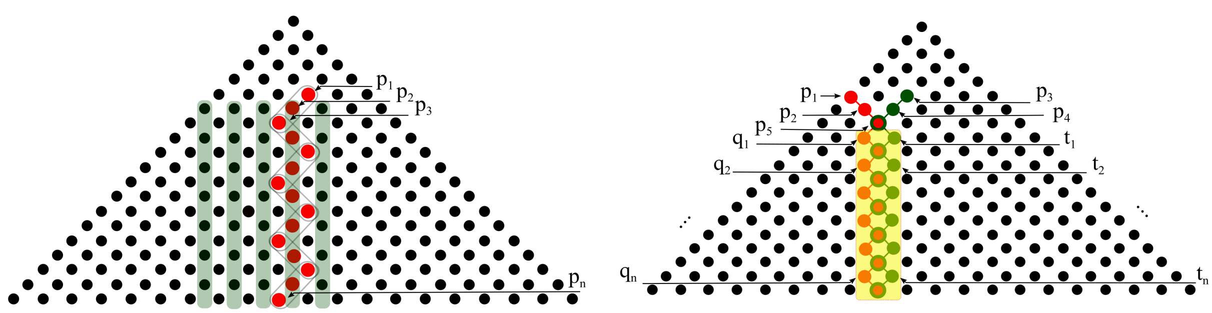

FIGURE 9. Zigzag Configurations.

Theorem 7 (Zigzag property)**.** *If a zigzag configuration with $6k+5$ points holds in $\mathcal{H}$, then the sum of all points in its left zigzag is equal to the sum of all points in its right zigzag (see Figure 8 and Figure 9(b)).*

Proof. From Figure 9(b) and definition of $\mathcal{H}$ on page 1 it is easy to see that $p_1 + p_2 = p_3 + p_4 = p_5$. Since the zigzag configuration has $6k+5$ points, there are remaining $6k$ points distributed in three vertical equal sets (see Figure 9(b) for the labeling of points). So, every set has an even number of points. This and Theorem 4 part 2 imply that $q_1 + q_1 + \cdots + q_n = t_1 + t_1 + \cdots + t_n$. This completes the proof. □

As a corollary of Theorem 7 we can see that the Hockey Stick property seen in Figure 10(a) and originally found in the Pascal triangle (see [9]), also holds in $\mathcal{H}$. Thus, the sum of all points in the shaft of a hockey stick is equal to the point on the blade of the hockey stick. The blade of the hockey stick is going to the left or to the right depending on the numbers of points that are on the shaft. If we consider the hockey stick configuration on one side of the Hosoya triangle ($\mathcal{H}$), then (by the symmetry of $\mathcal{H}$) when the same configuration is represented on the other side of $\mathcal{H}$, the blade of the hockey stick changes direction (from left to right or from right to left). We use $s_1, s_2, \cdots, s_i$ to represent the points on the shaft of the hockey stick. We use b_L and b_R to represent the point on the blade of the hockey stick. We use b_L to indicate that it is on the left side of the hockey stick and b_R indicates that it is on right side of the hockey stick. (See Figure 10(a).)

Corollary 8 (Hockey Stick property)**.** *Let $s_1, s_2, \cdots, s_i$ be the points on the shaft of the hockey stick where s_1 is a point on one of the edges of $\mathcal{H}$. If b_t is the point on the blade of the hockey stick, with $t \in \{L, R\}$ (see Figure 10(a)), then*

(1) $s_1 + s_2 + \cdots + s_{2n} = b_L$, if the hockey stick is on the left side of $\mathcal{H}$,
(2) $s_1 + s_2 + \cdots + s_{2n} = b_R$, if the hockey stick is on the right side of $\mathcal{H}$,
(3) $s_1 + s_2 + \cdots + s_{2n+1} = b_R$, if the hockey stick is on the left side of $\mathcal{H}$,
(4) $s_1 + s_2 + \cdots + s_{2n+1} = b_L$, if the hockey stick is on the right side of $\mathcal{H}$,
(5) $s_1 + s_2 + \cdots + s_i = b_L = b_R$ for every i, if the hockey stick is in the center of $\mathcal{H}$.

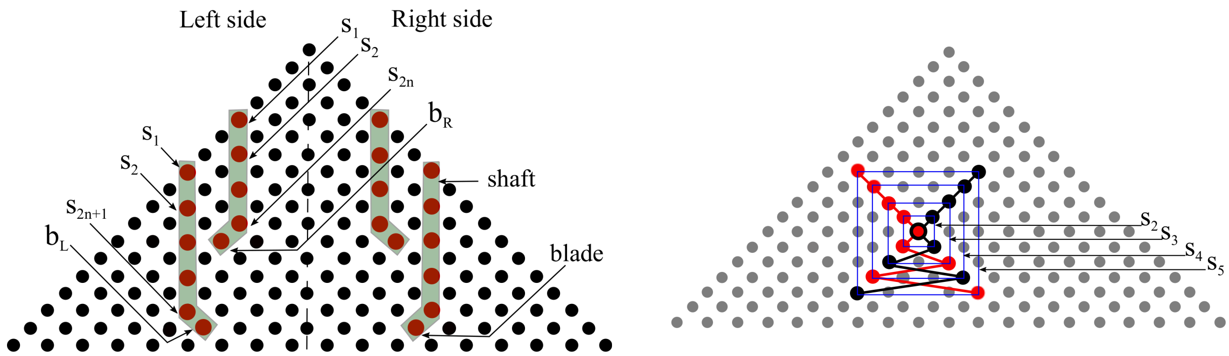

FIGURE 10. Hockey Stick and Braid configurations.

Proposition 9 (Braid property)**.** *If the braid configuration holds in $\mathcal{H}$, then the sum of all points in the left braid is equal to the sum of all points in the right braid (see Figure 11 and Figure 10(b))*

Proof. In the Figure 10(b) we observe that the left braid configuration is formed by all left corner points of the squares of even side length and all corner points in backslash diagonal in the squares of odd length (see for example S_2, S_3, S_4, and S_5 in Figure 10(b)). The right braid configuration is formed, similarly, by all right corner points of the squares of even side length and all points in the slash diagonal in the squares of odd side length.

From Proposition 3 it is easy to deduce that in a square configuration in $\mathcal{H}$ with even side length, it holds that sum of two vertical corner points is equal to the sum of the remaining corner points. If the square configuration has odd side length, then it holds that the sum of two corner points in a diagonal of the square is equal to the sum of the remaining corner points in the other diagonal. Using this property and Figure 10(b) we can see that the sum of left corner points of the innermost square S_2 is equal to the sum of its right corner points. We now observe that the square S_3 has odd side length. Therefore, the sum of the corner points in the slash diagonal equals the corner points in the backslash diagonal. The square S_4 satisfies the property that the sum of the vertical left corner points equals the right corner points. The square S_5 satisfies the property that the sum of the corner points in the slash diagonal equals the remaining corner points in the backslash diagonal. We can continue this process inductively as long as it is required by the braid configuration embedded in $\mathcal{H}$. From this, the Figure 10(b), and the observation given in the first paragraph it is easy to obtain the conclusion of the proposition. □

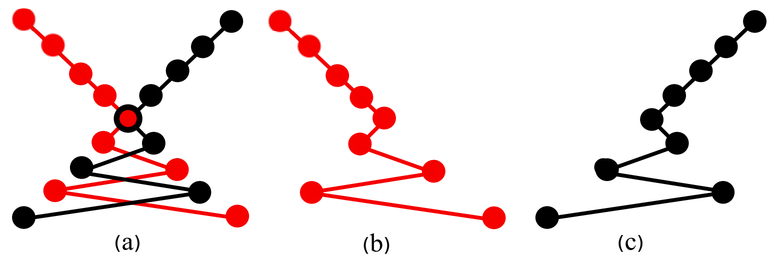

Figure 11. Braid configurations.

For any square S in Figure 10(b) it holds that the sum of the corner points in one diagonal of S is the additive inverse of the sum of the corner points in the remaining diagonal of S. Therefore, using all squares in Figure 10(b) it holds that

$$\sum_{k=0}^{l} H(n-k, m-k) + \sum_{k=1}^{l}(-1)^k H(n+k, m+k) = \sum_{k=0}^{l} H(n-k, m) + \sum_{k=1}^{l}(-1)^k H(n+k, m).$$

If in Figure 10(b) we eliminate the common point –the point that is the intersection of left braid and right braid– we obtain

$$F_{n-m}\sum_{k=1}^{l}(F_{m-k} + (-1)^k F_{m+k}) = F_m \sum_{k=1}^{l}(F_{n-m-k} + (-1)^k F_{n-m+k}).$$

This with $r = n - m$ implies

$$\sum_{k=1}^{l} \frac{F_{m-k} + (-1)^k F_{m+k}}{F_m} = \sum_{k=1}^{l} \frac{F_{r-k} + (-1)^k F_{r+k}}{F_r}.$$

So,

$$\sum_{k=1}^{l} \frac{F_{m-k} + (-1)^k F_{m+k}}{F_m} = \begin{cases} F_l + F_{l-2} + 1 & \text{if } l \text{ is odd} \\ 5F_{l'-1}F_{l'} + 1 + (-1)^{l'} & \text{if } l = 2l'. \end{cases}$$

4. Properties of the Pascal triangle in the Hosoya triangle

In this section we extend a few properties from the Pascal triangle to the Hosoya triangle. These properties of the Pascal triangle may be found in [4].

If we construct an oblique (backslash) ladder with horizontal rungs of length two (see Figure12), then the ladder gives rise to generalized Fibonacci numbers. Thus, adding the two points of each rung of the ladder gives rise to a sequence of second order which is a generalized Fibonacci sequence. Recall that the generalized Fibonacci sequence is given by $G_n = G_{n-1} + G_{n-2}$ with $G_1 = a$ and $G_2 = b$. Here we note that a and b are the points in the first rung of the oblique ladder, (see Figure12). In particular, a and b are consecutive Fibonacci numbers with $a > b$. Some of the ladders give rise to certain sequences found in Sloane [8]. In particular, with $a = 1$ and $b = 1$ we obtain the Fibonacci number sequence A000045. If $a = 2$ and $b = 1$, we obtain the Lucas number sequence A000032. With $a = 3$, $b = 2$, we obtain sequence A013655 which is a sequence where each

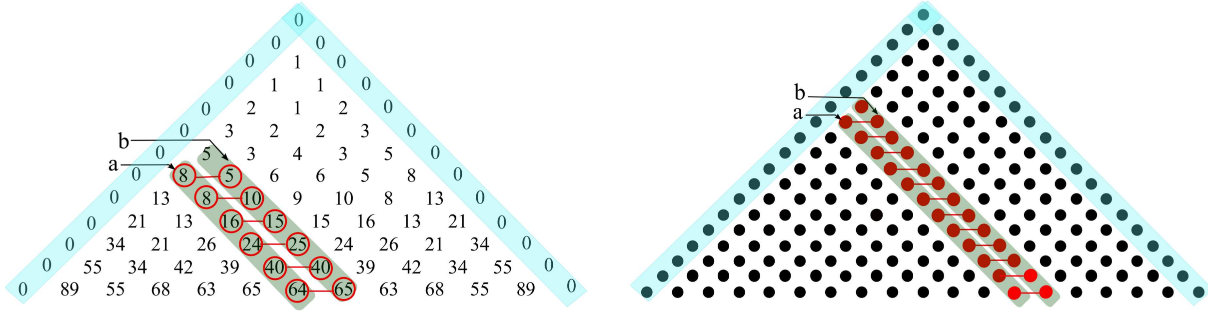

FIGURE 12. Generalized Fibonacci.

term is obtained by adding a Fibonacci and a Lucas number. The sequence A206610 is obtained with $a = 13$ and $b = 8$.

If we consider two consecutive rungs of an oblique ladder –that has horizontal rungs– we obtain a *rhombus property* (see Figure 13(b)). This property is an extension of a similar property found in Pascal's triangle (see [4]).

Let us take two consecutive rungs of the ladder, for example the $(2n-1)$-th and $(2n)$-th rungs. Then, differences of a cross multiplication is always a constant. In reality, the cross multiplication is the determinant of the numbers present in the rungs of the oblique ladder (see Figure 13(b)). In particular, here the determinant is the product of consecutive Fibonacci numbers.

Algebraically, using the definition of coordinates (see Section 2) for each point in the rungs of the oblique ladder, we obtain that for $n, r > 0$

$$\begin{vmatrix} H(n,r) & H(n,r+1) \\ H(n+1,r) & H(n+1,r+1) \end{vmatrix} = (-1)^{n-r+1} F_r F_{r+1}.$$

An additional configuration that yields a geometry and an identity we can explore is a *triangle configuration* as seen in Figure 13(a).

If we take the triangle configuration as in Figure 13 part (a) then $a+b-c$ is a Fibonacci number. Note that a and b are points constituting the top oblique side of the triangle and the points b and c are points along the same vertical line at a distance two from each other. The triangle may be oriented to the left or to the right, the orientation does not change this result. In particular, if $a = H(n+1, r-1)$, $b = H(n, r)$, and $c = H(n+2, r+1)$ then $a + b - c = F_{2r-n+1}$.

The algebraic proof of this property is easy using the definition of the coordinates of each entry $H(r, k)$ of the Hosoya triangle.

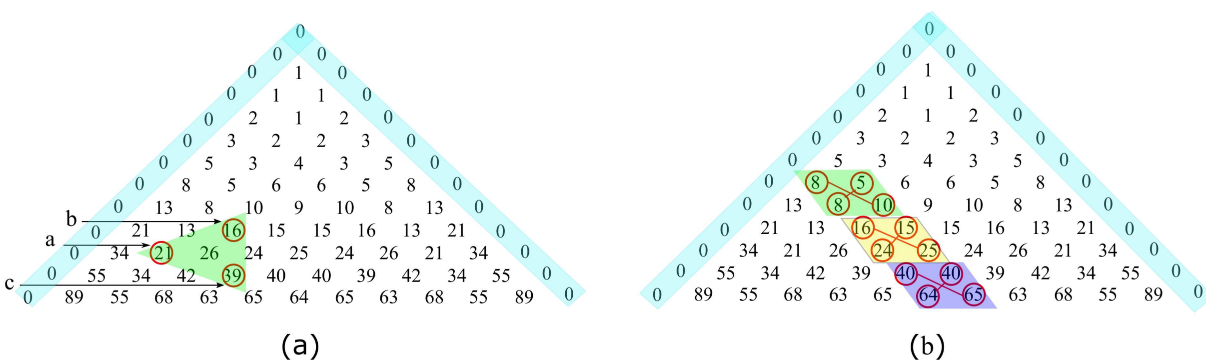

FIGURE 13. Triangle and rhombus properties.

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